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Dimension and Embedding Theorems for Geometric Lattices

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Let **G** be an n-dimensional geometric lattice. Suppose that $1 \leqslant e, f \leqslant n-1$, $e+f \geqslant n$, but e and f are not both $n-1$. Then, in general, there are $E, F \in \mathbf{G}$ with $\dim E = e$, $\dim F = f$, $E \vee F = 1$, and $\dim E \wedge F = e+f-n-1$; any exception can be embedded in an n-dimensional modular geometric lattice **M** in such a way that joins and dimensions agree in **G** and **M**, as do intersections of modular pairs, while each point and line of **M** is the intersection (in **M**) of the elements of **G** containing it.

1. INTRODUCTION

The following situation has been considered in several recent papers [8, 10, 11, 14], although lattices were not explicitly mentioned. A finite geometric lattice **G** was given in which, for each point p, the elements $\geqslant p$ formed a projective space of dimension $d \geqslant 2$. Moreover, it was assumed that, for each i, all i-spaces had the same number of points. When $d \geqslant 3$, it was deduced that **G** must be a projective or affine space; when a highly transitive automorphism group was available, the same conclusion was obtained for $d = 2$, with one exception: the Witt space $\mathbf{W}_{22}$ associated with the Mathieu group M_{22} [17, 18]. A natural extension of this idea was used in [11] and [14] to also characterize the Witt spaces $\mathbf{W}_{23}$ and $\mathbf{W}_{24}$.

We will prove similar results, without assuming the finiteness of **G**. Let **G** be an n-dimensional geometric lattice with $n \geqslant 4$. If, for each $W \in \mathbf{G}$ of dimension $n-4$, the elements $\geqslant W$ of **G** form a lattice embeddable as a large chunk of a 3-dimensional modular geometric lattice, then **G** can be embedded as a large chunk of an n-dimensional modular geometric lattice. This embedding is isometric (i.e., join- and dimension-preserving) and under suitable conditions provides the unique smallest isometric embedding of **G** into a modular geometric lattice. For more precise, technical statements, see Section 4.

It was originally hoped that our methods would yield generalizations of the Dembowski–Wagner Theorem [7; 6, p. 67] or the main result of [10]. In terms of lattices, these can be stated as follows. Let **G** be a finite geometric lattice of dimension $n \geqslant 2$ in which each $n-1$-space is not on at least two points, and in which all $n-1$-spaces have the same number of points and all $n-2$-spaces have the same number of points. The Dembowski–Wagner Theorem asserts that, if all points are on the same number of $n-1$-spaces, while any two $n-1$-spaces meet in an $n-2$-space, then **G** is a projective space. In [10] it is shown that **G** is a projective or affine space if $n \geqslant 4$ and any two $n-1$-spaces meet in 0 or in an $n-2$-space. These suggested that the following might be true: for sufficiently large n, an n-dimensional geometric lattice in which any two $n-1$-spaces meet in an $n-2$-space can be strongly embedded (in the sense of Section 4) in an n-dimensional modular geometric lattice. Unfortunately, this is false for all $n \geqslant 3$ when **G** is allowed to be infinite (see Section 5). We conjecture that the above statement is true when **G** is finite, but the proof of this would require entirely different methods from ours. All we can prove in this direction is a very special case of this (see Section 6) and the result on intersections of e- and f-spaces stated above.

In Section 3, we prove a preliminary embedding lemma which does not even require finite dimensionality. The situation is essentially the one mentioned in the first paragraph. The main difference is that, because of the use of this lemma in Section 4, we could not regard lines or planes as sets of points. Once one guesses that there is an embedding into a (generalized) projective space, the actual verification turns out to be a relatively straightforward application of the classical notion of adjoining ideal points and lines (see, e.g., [12]).

The main results are in Section 4. The complicated definitions given there are designed to make induction work. After induction is applied, we have to make sure that all the resulting modular lattices can be glued together coherently. The result of Section 3 can then be applied.

One unexpected bonus obtained from the embedding theorem is that, as a corollary, we deduce the existence of the well-known 6-dimensional projective representation of the Mathieu group M_{12} over $GF(3)$. This is found in Section 5. In that section examples are also given of geometric lattices not isometrically embeddable into any modular geometric lattice. Finally, in Section 6 we characterize some classical lattices in terms of a transitivity property of their automorphism groups.

We remark that the situations we will consider are similar to those of [4, pp. 148–149] and [16]. Embedding problems are not, however, studied there.

I am indebted to H. P. Young for pointing out that the main result of [10] was actually a theorem on lattices.

2. PRELIMINARIES

All lattices will have a 0 and 1. The elements of a lattice will sometimes be called its subspaces. We will frequently call the symbols $\vee$, $\wedge$, and $\leqslant$, join, intersection, and containment.

Let $\mathbf{G}$ be a geometric lattice (see [2, Chap. IV]). If $U \in \mathbf{G}$, all maximal chains from 0 to U have the same length $\dim U + 1$, where $\dim U$ is the dimension of U. We write $\dim \mathbf{G} = \dim 1$. The elements of dimension 0, 1, 2 are called points, lines, and planes. Each element $\neq 0$ is a join of points. If p is a point and $p \nleq U$, then $\dim p \vee U = \dim U + 1$. For all $U, V \in \mathbf{G}$, $\dim U + \dim V \geqslant \dim U \vee V + \dim U \wedge V$; the pair U, V is called a modular pair if equality holds, or, equivalently, if $(U \vee W) \wedge V = (U \wedge V) \vee W$ whenever $W \leqslant V$. The codimension of U is $\operatorname{codim} U = \dim \mathbf{G} - \dim U - 1$. Every element $\neq 1$ of $\mathbf{G}$ is the intersection of elements of codimension 1.

If $W \in \mathbf{G}$, $\mathbf{G}_W = [0, W] = \{X \in \mathbf{G} \mid X \leqslant W\}$ is a geometric lattice of dimension $\dim W$, and $\mathbf{G}^W = [W, 1] = \{X \in \mathbf{G} \mid X \geqslant W\}$ is a geometric lattice of dimension $\operatorname{codim} W$.

If $\mathbf{G}_1$ and $\mathbf{G}_2$ are geometric lattices, a join-monomorphism from $\mathbf{G}_1$ to $\mathbf{G}_2$ is an injective map $\varphi: \mathbf{G}_1 \to \mathbf{G}_2$ such that $(U \vee V)^\varphi = U^\varphi \vee V^\varphi$ for all $U, V \in \mathbf{G}_1$.

The product of two geometric lattices is defined on their product set by defining the operations componentwise.

A generalized projective space consists of a set S of points, together with certain distinguished subsets, called lines, such that the following axioms hold:

(PS1) Two distinct points are on a unique line;

(PS2) Each line has at least two points; and

(PS3) If a line intersects two sides of a triangle (not at their intersection), then it also intersects the third side.

This is a projective space if

(PS2′) Each line has at least three points.

A subspace of a generalized projective space is a subset of S closed under joins. The subspaces form a complete atomic complemented modular lattice, S being its set of atoms. Conversely, Birkhoff [1] showed that each modular geometric lattice **M** is isomorphic to the subspace lattice of some generalized projective space. Moreover, each such **M** is the product of a Boolean algebra and a finite number of simple modular geometric lattices, each coming from a projective space. The dimension of a generalized projective space is defined to be the dimension of the corresponding lattice.

LEMMA. *Let* **G** *be a geometric lattice of dimension* n, *and let* $1 \leqslant e \leqslant f \leqslant n-1$ *with* $e+f \geqslant n$. *Assume that every* e*-space* E *and* f*-space* F *of* **G** *satisfy*

$$\dim E \wedge F \neq e+f-n-1 \qquad \textit{if} \quad E \vee F = 1.$$

Then for any $e+f-n-1$*-space* W, $\mathbf{G}^W$ *is modular.*

Proof. We might as well pass to $\mathbf{G}^W$ and so assume that $W=0$. Thus, $e+f-n-1=-1$. We now proceed by induction on f. (PS1, 2) are obvious.

If $f=1$, then $e=f=1$, $n=2$, and any two lines have intersection $\neq 0$. Thus, (PS3) is clear.

Let $2 \leqslant f < n$. Fix an $n-1$-space H. Take any e-space E and $f-1$-space U with $E \vee U = H$. We claim that $\dim E \wedge U \neq e+(f-1)-(n-1)-1=-1$. For, there is an f-space F with $U < F \nless H$. Then $U = F \wedge H$, $E \vee F > H$, $E \vee F = 1$, and

$$\dim E \wedge U = \dim E \wedge F \wedge H = \dim E \wedge F \neq e+f-n-1.$$

By induction, $\mathbf{G}_H$ is a generalized projective space of dimension $n-1 \geqslant 2$. Since H is arbitrary, ($PS3$) is immediate. We must show **G** contains every subspace of the resulting generalized projective space.

It suffices to show $\mathbf{G}^p$ is modular for each point p. Let E and F be any e- and f-spaces with $p \nless E$, $p < F$, and $(E \vee p) \vee F = 1$. In $\mathbf{G}^p$, $\dim E \vee p = e$ and $\dim F = f-1$, where $e+(f-1)=\dim \mathbf{G}^p$. Suppose $(x \vee E) \wedge F = x$. Then $E \wedge F \leqslant E \wedge (x \vee E) \wedge F = E \wedge x = 0$, which is not the case by hypothesis. Thus, $(x \vee E) \wedge F > x$, so $\mathbf{G}^p$ inherits our hypothesis. Induction now completes the proof.

3. EMBEDDING LEMMA

In this section we will consider certain axioms concerning a poset **G**. We will frequently say *a is on b* when $a > b$ or $a < b$. The elements of **G** are assumed to be partitioned into four subsets: the points, lines, planes, and 3-spaces. If it were not for the application of the results of this section in the next one, we would simply regard the lines, planes, and 3-spaces as sets of points. However, we are forced to consider a more general setting. Our axioms are as follows.

(E1) For each point p, the poset $\mathbf{G}^p$ of lines, planes and 3-spaces on p is the set of all points, all lines, and some planes (respectively) of a generalized projective space of dimension $\geqslant 3$.

(E2) Two distinct points are on a unique line; no point is on any other point.

(E3) If two distinct planes are on (at least) two points, they are on a 3-space.

(E4) Each line, plane, and 3-space is on at least one point.

(E5) No element of $\mathbf{G}$ is on all points.

Obviously, (E1) is the crucial axiom. Note that $\mathbf{G}^p$ need not be finite-dimensional. Special cases of (E1) were considered in [4, 8, 10, 11, 14, 16].

EMBEDDING LEMMA. *Suppose* $\mathbf{G}$ *satisfies* (E1–5). *Then there is an order-monomorphism* φ *from* $\mathbf{G}$ *into the lattice* $\mathbf{M}$ *of all subspaces of a generalized projective space such that the following hold*:

(i) φ *maps points to points, lines to lines, planes to planes, and* 3-*spaces to* 3-*spaces.*

(ii) *For each point* p *of* $\mathbf{G}$, $(\mathbf{G}^p)^\varphi$ *contains all lines and planes of* $\mathbf{M}$ *containing* p^φ.

(iii) *If the generalized projective space* $\mathbf{G}^p$ *has finite dimension* $n-1$, *then* $\dim \mathbf{M} = n$.

Proof. Points will be denoted by a, b, c, p, q, r, x, y, lines by L, M, planes by E, F, and 3-spaces by T. The proof will be given in several steps.

(I) We begin with some elementary consequences of the axioms. Define $\vee$ and $\wedge$ as is usual in posets. Expressions such as $L \vee E$ and $L \wedge E$ are not always defined.

By (E2), we never have $p > q$. By (E4, 1), we never have $L > M$, $E > F$, or $T_1 > T_2$.

From (E1) we deduce the following. If $E \neq F$ and $E, F > L$, then $E \wedge F = L$. If $T_1 \neq T_2$ and $T_1, T_2 > E$, then $T_1 \wedge T_2 = E$. If $L \not< T$ and $L, T > p$, then $L \wedge T = p$. If $E \not< T$ and $E, T > L$, then $E \wedge T = L$. If $L \neq M$ and $L, M > p$, then $L \wedge M = p$; moreover $L \vee M$ exists and is a plane.

If $L \not< E$ and $L, E > p$, then $L \wedge E = p$. Suppose that, in addition,

$E > p \vee q > p$. Then $L \vee E$ exists and is a 3-space. For, $L \vee p \vee q$ and E are two planes on p and q, so (E3) applies.

If $p \nless L$, then $p \vee L$ exists and is a plane. For, by (E4), we can find $q < L$, and then $(p \vee q) \vee L$ is a plane (by (E1)).

Similarly, if $p \nless E$, then $p \vee E$ is a 3-space. For, by (E1, 4), E is on at least 2 concurrent lines, so (E3) can be applied.

(II) We first show that, if 4 lines are such that no 2 are on a common point, no 3 are coplanar, and 5 of the 6 pairs of lines are coplanar, then so is the sixth pair.

For, let the lines be L_1, L_2, L_3, L_4, where we must show L_3, L_4 are coplanar. Let $p_i < L_i$ (see (E4)). Then $p_i \vee L_j = L_i \vee L_j$ for $i \neq j$, $\{i, j\} \neq \{3, 4\}$. Thus, for i, j, k distinct, we have a 3-space $p_i \vee L_j \vee L_k = L_i \vee L_j \vee L_k$ (by (I)).

If $L_1 \vee L_3 \vee L_4 \neq L_2 \vee L_3 \vee L_4$, then, viewing from $\mathbf{G}^{p_4}$, we see two different planes containing the lines $p_4 \vee L_3$ and $p_3 \vee L_4$. Thus, L_3 and L_4 are coplanar in this case.

Suppose $L_1 \vee L_3 \vee L_4 = L_2 \vee L_3 \vee L_4$. By (E5), we can find $p_5 \nless L_1 \vee L_3 \vee L_4$. Then $p_5 \vee L_1 \vee L_2$ is a 3-space; in $\mathbf{G}^{p_5}$ it is a plane containing the distinct lines $p_5 \vee L_1$, $p_5 \vee L_2$. By (E1) we can find a line $L_5 = (p_5 \vee L_1) \wedge (p_5 \vee L_2)$. Here L_5 is on no point r of $L_1 \vee L_2 \vee L_3$; for otherwise, we can suppose $r \nless L_1$, and then $p_5 < p_5 \vee L_1 = L_5 \vee L_1 = r \vee L_1 < L_1 \vee L_2 \vee L_3$. By the preceding paragraph (applied to L_1, L_2, L_3, L_5 and L_1, L_2, L_4, L_5), L_3, L_5 and L_4, L_5 are coplanar pairs. Now use L_1, L_5, L_3, L_4 to get that L_3, L_4 are coplanar.

(III) Define an *ideal point* to be a family α of pairwise coplanar lines such that each point is on a unique member of α.

Let L, M be coplanar lines on no common point. We claim that there is a unique ideal point containing them.

Set $E = L \vee M$. For each $p \nless E$, $p \vee E$, is a 3-space; in $\mathbf{G}^p$, it is a plane containing the distinct lines $p \vee L$, $p \vee M$. We thus obtain a line $\alpha^p = (p \vee L) \wedge (p \vee M) \in \mathbf{G}$. Here L and α^p are on no common point, as otherwise $E = (\alpha^p \wedge L) \vee M \leqslant \alpha^p \vee M = p \vee M$. Also, if $q \nless \alpha^p$, E, then α^p and α^q are on no common point. By (II), any two α^p's are coplanar.

Fix $p \nless E$, $q \nless p \vee E$. For each $x < E$, as above we obtain a line $\alpha^x > x$, namely $\alpha^x = E \wedge (x \vee \alpha^p)$. Use L, α^p, α^q, α^x in (II) to get α^q, α^x coplanar. Now (II) shows that any two of our α^y's are coplanar. No two are on a common point. Both L and M occur as α^y's.

Thus, the set of α^y's is an ideal point containing L and M. It is unique as everything was unique at every stage of the construction. This proves our claim.

(IV) Let α be an ideal point. If $L \in \alpha$, we write $\alpha < L$ and say that α is on L. For each point p, let $p \vee \alpha = \alpha \vee p$ denote the unique line of α on p.

We write $\alpha < E$ if there is a line L with $\alpha < L < E$. Suppose $b < E$, $b \nless L$. Since L and $\alpha \vee b$ are coplanar, we must have $\alpha \vee b < E$. Thus, each line of α is on E or has no point on E.

We write $\alpha < T$ when $\alpha < E < T$ for some E. Let $\mathbf{G}_*$ consist of $\mathbf{G}$ and all ideal points. We have just partially ordered $\mathbf{G}_*$. A **-point* is defined to be a point or an ideal point.

For distinct *-points α, β, let $\alpha \vee_* \beta$ be the set of all *-points on every plane on α and β. There are at least two such planes. For, by (E2) and (III) there is at most one line of the form $p \vee \alpha = p \vee \beta$. By (E5) we can thus find distinct planes of the form $(q \vee \alpha) \vee (q \vee \beta)$, $(r \vee \alpha) \vee (r \vee \beta)$.

(V) Let α, β be distinct *-points and E_1, E_2 distinct planes on them. We claim that $\alpha \vee_* \beta$ is the set of all *-points common to E_1 and E_2. This amounts to saying that each *-point $\gamma \neq \alpha, \beta$ on E_1, E_2 is on each plane $E \neq E_1, E_2$ on α, β.

First, note that there is a point $x \nless E_1, E_2$ such that $x \vee E_1$ and $x \vee E_2$ are distinct 3-spaces. For, by (E5) there are points $y \nless E_1$ and $x \nless y \vee E_1$. We cannot have both $y \vee E_1 \geqslant y \vee E_2$ and $x \vee E_1 \geqslant x \vee E_2$, as then we would have $x < x \vee E_1 = E_1 \vee E_2 = y \vee E_1$.

Next note that $x \vee \alpha \ngtr x \vee \beta$. For otherwise, $x \vee \alpha = x \vee \beta$ and we may assume α is ideal. Let $p < E_1$, so $p \nless x \vee \alpha$ as $x \nless E_1$. Then $(p \vee \alpha) \vee (x \vee \alpha) > p, \beta$, so E_1, $(p \vee \alpha) \vee (x \vee \alpha) > p \vee \alpha$, $p \vee \beta$. By (II), $p \vee \alpha \ngtr p \vee \beta$. Thus, $E_1 = (p \vee \alpha) \vee (x \vee \alpha) > x$, which is not the case.

Set $E_3 = (x \vee \alpha) \vee (x \vee \beta)$. Since $x \vee E_1 > x \vee \alpha$, $x \vee \beta$, $x \vee \gamma$, we have $x \vee E_1 = E_3 \vee E_1$. Similarly, $x \vee E_2 = E_3 \vee E_2 > x \vee \gamma$. Thus, $E_3 = (x \vee E_1) \wedge (x \vee E_2) > x \vee \gamma > \gamma$.

Let $p < E$. By the preceding paragraph, we may assume that $p < E_1$, $p < E_2$, or $p \vee E_1 = p \vee E_2 = E_1 \vee E_2$. We may also assume that $p \neq \alpha$ and that p is on at most one of E_1, E_2, E_3; for if $p < E_i, E_j$ with $i \neq j$, then $E_i \wedge E = p \vee \alpha = E_i \wedge E_j \geqslant p \vee \gamma$, so $E > \gamma$.

If $p \nless E_1, E_3$, then $p \vee E_2 \leqslant p \vee E_1 = E_1 \vee E_2$, so $p \vee E_1 \neq p \vee E_3$, As for E_3, this implies that $E > \gamma$.

If $p < E_3$, then $p \nless E_1, E_2$, so $p \vee E_1 = p \vee E_2$. Let $\alpha < L < E_1$; then $p \vee \alpha \leqslant p \vee L < E_1 \vee E_2$. Similarly, $p \vee \gamma < E_1 \vee E_2$. Thus, $\gamma \leqslant p \vee \gamma \leqslant E_3 \wedge (E_1 \vee E_2) = p \vee \alpha < E$.

(VI) For each line (or plane) L (or E) of $\mathbf{G}$, let L^* (or E^*) be its set of *-points. Call a set of points a **-line* if it has the form $\alpha \vee_* \beta$ for distinct *-points α, β. According to (V), if $\alpha, \beta < E_1, E_2$ with $E_1 \neq E_2$, then $\alpha \vee_* \beta = E_1{}^* \cap E_2{}^*$.

Two distinct *-points α, β are on a unique *-line. For, let $\alpha \neq \gamma \in \alpha \vee_* \beta$, and let E_1, E_2 be as above. Then $\alpha, \gamma \in \alpha \vee_* \beta = E_1{}^* \cap E_2{}^*$ implies, by (V), that $\alpha \vee_* \gamma = E_1{}^* \cap E_2{}^* = \alpha \vee_* \beta$.

Note that L^* is a *-line for all lines L of **G**. (This is, of course, clear if L is on two points.) Let $p < L$, $q \nless L$. Then $q \vee L$ is a line of $\mathbf{G}^q$, and hence by (E1) is on at least 2 lines of $\mathbf{G}^q$ through q. By (III), L is on a *-point $\alpha \neq p$, so $L = p \vee \alpha$. By the definition in (V), $L^* = p \vee_* \alpha$.

Let $\mathbf{M}_*$ consist of the set of *-points and *-lines, ordered by set-inclusion.

(VII) We wish to show that $\mathbf{M}_*$ is a generalized projective space. The only axiom that needs to be checked is (PS3).

Let α, β, γ, δ, ϵ be distinct *-points with $\delta \in \alpha \vee_* \beta$, $\epsilon \in \gamma \vee_* \delta$, but $\gamma \notin \alpha \vee_* \beta$. We must show that $\alpha \vee_* \gamma$ and $\beta \vee_* \epsilon$ have a common *-point.

Take any point p. Denote by $\wedge_p$ the meet operation in the generalized projective space determined by $\mathbf{G}^p$. By (E5), we may assume that p, α, β, γ, δ, ϵ are not on a plane of **G**; by (E1) these then determine a plane T^p of the generalized projective space $\mathbf{G}^p$ (here T^p may not be in **G**). Consequently, $L^p = ((p \vee \alpha) \vee (p \vee \gamma)) \wedge_p (p \vee \beta) \vee (p \vee \epsilon))$ is a point of T^p. By (E1), L^p is a line of **G**, so $L^p = ((p \vee \alpha) \vee (p \vee \gamma)) \wedge ((p \vee \beta) \vee (p \wedge \epsilon))$.

By (E3,5), there is a point q for which $p \vee q$ is not on T^p. Then q, α, β, γ, δ, ϵ are not on a plane of **G**, as otherwise p, q, α, β, γ, δ, ϵ would be on a 3-space of **G** by (I). We can thus define L^q as above: $L^q = ((q \vee \alpha) \vee (q \vee \gamma)) \wedge ((q \vee \beta) \vee (q \vee \epsilon))$.

By (I), there are 3-spaces $T_1 = p \vee ((q \vee \alpha) \vee (q \vee \gamma))$ and $q \vee ((p \vee \alpha) \vee (p \vee \gamma))$ of **G**. Both are on $p \vee (q \vee \alpha) = (p \vee \alpha) \vee (q \vee \alpha) = q \vee (p \vee \alpha)$ and $p \vee (q \vee \gamma) = q \vee (p \vee \gamma)$ by (IV). Thus, they are equal and are on $p \vee L^q$ and $q \vee L^p$. Similarly, $T_2 = p \vee ((q \vee \beta) \vee (q \vee \epsilon))$ is a 3-space of **G** on $p \vee L^q$ and $q \vee L^p$. Here $T_1 \neq T_2$ by our choice of q. Thus, $T_1 \wedge T_2 = p \vee L^q = q \vee L^q$. By (III), there is a *-point σ on L^p and L^q.

By (V),

$$\sigma \in ((p \vee \alpha) \vee (p \vee \gamma))^* \cap ((q \vee \alpha) \vee (q \vee \gamma))^* = \alpha \vee_* \gamma.$$

Similarly, $\sigma \in \beta \vee_* \epsilon$. This proves (PS3). (Remark: For a beautiful proof of (PS3) in a similar context, see [12, pp. 52–53], where, however, more points than we have available are used, but only 3 dimensions are required.)

(VIII) Finally, let $\mathbf{M}$ denote the lattice of all subspaces of the generalized projective space $\mathbf{M}_*$. Each point of $\mathbf{G}$ may be regarded as a point of $\mathbf{M}$. Each line L of $\mathbf{G}$ determines a unique line L^* of $\mathbf{M}$. Let E be a plane of $\mathbf{G}$. Let α and β be distinct *-points of E. Then, by definition, $\alpha \vee_* \beta \subset E^*$. Thus, $E^* \in \mathbf{M}$. By (VII), E^* is a plane of $\mathbf{M}$.

Similarly, for each 3-space T of $\mathbf{G}$, let T^* be the set of all *-points on some plane on T. Take distinct *-points $\alpha, \beta \in T^*$. By (E4), we can find $p < T$. Then $(p \vee \alpha) \vee (p \vee \beta)$ is a line or plane of $\mathbf{G}$. Thus, by definition, $\alpha \vee_* \beta \subset T^*$, and $T^* \in \mathbf{M}$.

We thus obtain an order-preserving injection $\varphi: \mathbf{G} \to \mathbf{M}$, defined by $p \to p$, $L \to L^*$, $E \to E^*$, $T \to T^*$. By (VI), $(\mathbf{G}^p)^\varphi$ contains all L^*, E^* with $p < L, E$, so $(\mathbf{G}^p)^\varphi$ contains all lines and planes of $\mathbf{M}$ through p^φ. Each point and line of $\mathbf{M}$ is in $\mathbf{M}_*$, and hence is the intersection of planes of $\mathbf{G}^\varphi$. This completes the proof of the lemma.

4. Main Results

We begin with some definitions.

DEFINITION 1. An *isometry* from a geometric lattice $\mathbf{G}_1$ to a geometric lattice $\mathbf{G}_2$ is a join-monomorphism $\theta: \mathbf{G}_1 \to \mathbf{G}_2$ which preserves dimension. $\mathbf{G}_1$ is said to be isometrically embedded into $\mathbf{G}_2$.

Clearly, a join-monomorphism θ is an isometry if and only if $\dim 1^\theta = \dim 1$. A basic property of isometries is that, if U, V is a modular pair in $\mathbf{G}_1$, then $(U \wedge V)^\theta = U^\theta \wedge V^\theta$. For, $(U \wedge V)^\theta \leqslant U^\theta \wedge V^\theta$ and

$$\begin{aligned}\dim(U \wedge V)^\theta &= \dim U + \dim V - \dim U \vee V \\ &= \dim U^\theta + \dim V^\theta - \dim U^\theta \vee V^\theta \geqslant \dim U^\theta \wedge V^\theta.\end{aligned}$$

In particular, if $\mathbf{G}_1$ is modular, then θ is also a lattice monomorphism.

DEFINITION 2. Let $\mathbf{G}$ be a geometric lattice and $\mathbf{M}$ a modular geometric lattice. $\mathbf{G}$ is *strongly embedded* in $\mathbf{M}$ (via θ) if there is an isometry $\theta: \mathbf{G} \to \mathbf{M}$ such that, for all $W \in \mathbf{G}^\theta$,

(SE1) $\dim \mathbf{G} = \dim \mathbf{M} = n \geqslant 2$;

(SE2) If $\dim W \leqslant n - 4$, every element $> W$ of $\mathbf{M}$ of dimension $\leqslant \dim W + 2$ is the intersection (in $\mathbf{M}$) of those members of $\mathbf{G}^\theta$ containing it; and

(SE3) If $\dim W = n - 3$, $(\mathbf{G}^\theta)^W$ is either $\mathbf{M}^W$ or is obtained from $\mathbf{M}^W$ by removing a line and all its points.

Here, (SE3) is intended to allow $(\mathbf{G}^\theta)^W$ to be a (possibly degenerate) projective plane or an affine plane.

DEFINITION 3. For a geometric lattice $\mathbf{G}$ of dimension $n \geqslant 2$, let $\mathfrak{P}(\mathbf{G})$ denote the set of all proper subsets Σ of $\mathbf{G}$ satisfying the following conditions:

(P1) $U \in \Sigma$, $U \leqslant V \in \mathbf{G}$ imply $V \in \Sigma$.

(P2) If $U, V \in \Sigma$ is a modular pair, then $U \wedge V \in \Sigma$.

(P3) For each $W \in \mathbf{G}$ of dimension $\leqslant n-3$, $\Sigma^W = \{U \in \Sigma \mid W \leqslant U\}$ is either all of $\mathbf{G}^W$ or is maximal in $\mathbf{G}^W$ with respect to satisfying (P1,2).

Define

$$\mathfrak{L}(\mathbf{G}) = \{\Sigma \mid \{1\} \subset \Sigma \subset \mathbf{G} \text{ and } \Sigma \text{ is the intersection of two members of } \mathfrak{P}(\mathbf{G})\}.$$

We remark that, in (P3), conditions (P1,2) are automatically inherited by Σ^W. This is clear for (P1). If $U, V \in \Sigma^W$, then $U \wedge V \geqslant W$, so if $\dim U + \dim V = \dim U \vee V + \dim U \wedge V$ in $\mathbf{G}^W$, then the same is true in $\mathbf{G}$, and hence (P2) holds for Σ^W.

The following result is the key to induction.

UNIQUENESS LEMMA. *Let $\mathbf{G}$ be a geometric lattice strongly embedded in a modular geometric lattice $\mathbf{M}$ via θ. Set $\mathbf{G}' = \mathbf{G}^\theta$, and define operations on $\mathbf{G}'$ in the natural way so it becomes a lattice isomorphic to $\mathbf{G}$.*

(UL1) *If $U \in \mathbf{G}'$, then U has the same dimensions in $\mathbf{G}'$ and in $\mathbf{M}$.*

(UL2) *Joins are the same in $\mathbf{G}'$ and $\mathbf{M}$. If $U, V \in \mathbf{G}'$ is a modular pair, then $U \wedge V$ is the same in $\mathbf{G}'$ and $\mathbf{M}$. $\mathbf{G}'$ is strongly embedded in $\mathbf{M}$ (via the inclusion map).*

(UL3) *If $W \in \mathbf{G}'$ has dimension $\leqslant n-3$, then $\mathbf{G}'^W$ is strongly embedded in $\mathbf{M}^W$ (via the inclusion map).*

(UL4) *Let $\Sigma \subset \mathbf{G}$. Then $\Sigma \in \mathfrak{P}(\mathbf{G})$ if and only if Σ^θ is the set of all elements of $\mathbf{G}$ containing a point σ of $\mathbf{M}$. Moreover, $\sigma = \wedge \Sigma^\theta$ (the intersection being taken in $\mathbf{M}$).*

(UL5) *Let $\{1\} \subset \Sigma \subset \mathbf{G}$. Then $\Sigma \in \mathfrak{L}(\mathbf{G})$ if and only if Σ^θ is the set of all elements of $\mathbf{G}$ containing a line λ of M. Moreover, $\lambda = \wedge \Sigma^\theta$, and every line of $\mathbf{M}$ arises in this way except possibly when $n = 2$.*

(UL6) *If $\mathbf{G}$ can be strongly embedded in $\mathbf{M}_1$, then $\mathbf{M}$ and $\mathbf{M}_1$ are isomorphic. Every nontrivial automorphism of $\mathbf{G}$ induces a nontrivial automorphism of $\mathbf{M}$.*

(UL7) *In* (SE3) *assume that* $(\mathbf{G}^\theta)^W$ *always equals* $\mathbf{M}^W$. *Let* φ *be an isometry from* $\mathbf{G}$ *into a modular geometric lattice* $\mathbf{N}$. *Then there is a unique isometry* $\varphi^*: \mathbf{M} \to \mathbf{N}$ *such that* $\theta\varphi^* = \varphi$.

Proof. Let $\vee$ and $\wedge$ denote the operations in $\mathbf{M}$.

(UL1) θ is an isometry.

(UL2) This is clear from the definitions.

(UL3) Clearly $\mathbf{G}^W \subseteq \mathbf{M}^W$, where $\mathbf{M}^W$ is modular. By (UL1), $\mathbf{G}'^W$ and $\mathbf{M}^W$ have dimension $n - 1 - \dim W$, and the elements of $\mathbf{G}'^W$ have the same dimension in $\mathbf{G}'^W$ and $\mathbf{M}^W$.

Take $U \in \mathbf{G}'^W$ of codimension $\geqslant 3$. Each element of $\mathbf{M}^W$ containing U as a subspace of codimension $\leqslant 2$ is also an element of $\mathbf{M}$ containing U as a subspace of codimension $\leqslant 2$. Also, $(\mathbf{G}'^W)^U = \mathbf{G}'^U$. Thus, (SE2) for $\mathbf{G}'^W$, $\mathbf{M}^W$ follows from (SE2) for $\mathbf{G}'$, $\mathbf{M}$. (SE3) is checked in the same way.

(UL4) Since $\mathfrak{P}(\mathbf{G})^\theta = \mathfrak{P}(\mathbf{G}')$, we need only consider subsets Σ of $\mathbf{G}'$. For $\sigma \in \mathbf{M}$, write $[\sigma] = \{X \in \mathbf{G}' \mid \sigma \leqslant X\}$.

We proceed by induction on n. Let $n = 2$. Plug $W = 0$ into (SE3) to get that $\mathbf{G}'$ is $\mathbf{M}$ or one of the natural affine subplanes of $\mathbf{M}$. Then $\Sigma \in \mathfrak{P}(\mathbf{G}')$ if and only if $\Sigma = \mathbf{G}'^p$ for some point p of $\mathbf{G}'$ or Σ is a parallel class. Thus, (UL4) is clear here.

Let $n \geqslant 3$. Take a point σ of $\mathbf{M}$. We claim that $[\sigma] \in \mathfrak{P}(\mathbf{G}')$. (P1) is clear, while (P2) follows from (UL2). We must prove (P3). Take any point $p \neq \sigma$ of $\mathbf{G}'$. Then $[\sigma]^p$ consists of all elements of $\mathbf{G}'^p$ containing the point $p \vee \sigma$ of $\mathbf{M}^p$. By induction, $[\sigma]^p \in \mathfrak{P}(\mathbf{G}'^p)$. Comparison of (P3) for $\mathbf{G}'^p$ and $\mathbf{G}'$ shows that we only need to check that $[\sigma]$ is maximal in $\mathbf{G}'$ with respect to (P1,2).

Suppose $[\sigma] \subset \Sigma \subseteq \mathbf{G}'$, where Σ satisfies (P1,2). Let $U \in \Sigma - [\sigma]$. There is a point $p \leqslant U$ of $\mathbf{G}'$. Then Σ^p contains $[\sigma]^p$ and U, while satisfying (P1,2). By induction, $[\sigma]^p$ is maximal in $\mathbf{G}'^p$, so $\Sigma^p = \mathbf{G}'^p$. Take any point $q \not< p \vee \sigma$ of $\mathbf{G}'$. Then $p \vee q \in \mathbf{G}'^p = \Sigma^p$, but $p \vee q \notin [\sigma]$. As above, $\Sigma^q = \mathbf{G}'^q$. Now $p, q \in \Sigma$, so $0 \in \Sigma$ by (P2), and hence $\Sigma = \mathbf{G}'$. This proves the maximality of $[\sigma]$.

Conversely, let $\Sigma \in \mathfrak{P}(\mathbf{G}')$. If Σ contains a point p of $\mathbf{G}'$, then $\Sigma \supseteq [p]$ by (P1), and then $\Sigma = [p]$ by what has just been proved. We may thus assume that Σ contains no point p of $\mathbf{G}'$. For each p, Σ^p is then a proper subset of $\mathbf{G}'^p$ satisfying (P1–3). By induction, $\Sigma^p = [\lambda^p]$ for some point $\lambda^p > p$ of $\mathbf{M}^p$. Here, λ^p is a line of $\mathbf{M}$.

Let $q \not< \lambda^p$ be any point of $\mathbf{G}'$. Define λ^q as above. Then $\Sigma^q = [\lambda^q]$ and $p \not< \lambda^q$. Both $p \vee \lambda^q$ and $q \vee \lambda^p$ are planes of $\mathbf{M}$ containing $p \vee q$. We claim that λ^p and λ^q are coplanar. Suppose that $p \vee \lambda^q$ is the intersection (in $\mathbf{M}$)

of those $U \in \mathbf{G}'$ containing it. Then $p \vee \lambda^q \leqslant U$ implies that $U \in \Sigma^q \subseteq \Sigma$ and hence that $U \in \Sigma^p$, so $\lambda^p \leqslant U$. Thus, $\lambda^p \leqslant p \vee \lambda^q$ in this case. Similarly, if $q \vee \lambda^p = \wedge\, [q \vee \lambda^p]$, then $\lambda^q \leqslant p \vee \lambda^p$. By (SE2, 3), the only other alternative is: $n - 1 = 2$ and $p \vee \lambda^q$ and $q \vee \lambda^p$ are both the unique exceptional line of $\mathbf{G}'^p$ which can occur in (SE3). Thus, λ^p and λ^q are coplanar in any case.

Define $\sigma = \lambda^p \wedge \lambda^q$. It suffices to show that $\Sigma \supseteq [\sigma]$. Since $n \geqslant 3$, we can find a point r of $\mathbf{G}'$ with $r \nless \lambda^p \vee \lambda^q$. As before, $\lambda^r \wedge \lambda^p \neq 0 \neq \lambda^r \wedge \lambda^q$. Since $\lambda^r \nless \lambda^p \vee \lambda^q$, it follows that $\lambda^r > \lambda^p \wedge \lambda^q = \sigma$.

Consequently, $\sigma < \lambda^x$ for each point x of $\mathbf{G}$. However, whenever $\sigma < X \in \mathbf{G}$, we can find such an $x \leqslant X$, and then $\lambda^x = \sigma \vee x \leqslant X$ implies that $X \in \Sigma^x \subseteq \Sigma$.

(UL5) Let λ be a line of $\mathbf{M}$. Then $\lambda = \sigma \vee \tau$ for points σ, τ of $\mathbf{M}$, so $[\lambda] = [\sigma] \cap [\tau]$. If $\lambda = \wedge\, [\lambda]$, then $[\lambda] \in \mathfrak{L}(\mathbf{G})$ by (UL4). By (SE2, 3), $\lambda \neq \wedge\, [\lambda]$ implies that $n = 2$ and $[\lambda] = \{1\}$.

Conversely, if $\Sigma \in \mathfrak{L}(\mathbf{G})$, then, by (UL4), $\Sigma = [\sigma] \cap [\tau] = [\sigma \vee \tau]$.

(UL6) If $n = 2$, $\mathbf{G}'$ is $\mathbf{M}$ or a natural affine subplane of $\mathbf{M}$, and (UL6) is clear. When $n > 2$, just use (UL4, 5).

(UL7) The proof is very similar to those of (UL4, 5). We may assume that $\mathbf{G} \subseteq \mathbf{N}$ and φ is the inclusion map. For $\sigma \in \mathbf{N}$, let $\langle\sigma\rangle = \{X \in \mathbf{G} \mid \sigma \leqslant X\}$. We make the inductive hypothesis that, for each Σ in $\mathfrak{P}(\mathbf{G})$ (or $\mathfrak{L}(\mathbf{G})$), $\Sigma = \langle\sigma\rangle$ for a point (or line) σ of $\mathbf{N}$, and σ is the intersection in $\mathbf{N}$ of Σ. When $n = 2$, $\mathbf{G}' = \mathbf{M}$ and this is clear. Suppose $n \geqslant 3$.

Let $\Sigma \in \mathfrak{P}(\mathbf{G})$. We may assume that Σ contains no point p of $\mathbf{G}$. Since $\Sigma^p \subset \mathbf{G}^p \subseteq \mathbf{N}^p$, it follows by induction that $\Sigma^p = \langle\lambda^p\rangle$ for a line $\lambda^p > p$ of $\mathbf{N}$, and λ^p is the intersection of Σ^p. Take any point $q \nless \lambda^p$ of $\mathbf{G}$. Then $q \vee \lambda^p$ is a line of $\mathbf{N}^p$, and hence by induction is the intersection in $\mathbf{N}$ of those $U \in \mathbf{G}$ containing it. Since $U \in \Sigma^q$, $U \geqslant \lambda^q$. Thus, $q \vee \lambda^p \geqslant \lambda^q$. As in the proof of (UL4), $\Sigma \supseteq \langle\sigma\rangle$ for a point σ of $\mathbf{N}$. However, $\Sigma^p = \langle p \vee \sigma\rangle$ for each p. Thus, $\Sigma = \langle\sigma\rangle$. Moreover, it is clear that σ is the intersection of Σ.

Next, let $\Sigma \in \mathfrak{L}(\mathbf{G})$. Since $\Sigma = \Sigma_1 \cap \Sigma_2$ with $\Sigma_1, \Sigma_2 \in \mathfrak{P}(\mathbf{G})$, $\Sigma = \langle\sigma\rangle \cap \langle\tau\rangle = \langle\sigma \vee \tau\rangle$ with σ and τ distinct points of $\mathbf{N}$. Take $p \nleqslant \sigma \vee \tau$ and $q \nless p \vee \sigma \vee \tau$. Then $\sigma \vee \tau = (p \vee \sigma \vee \tau) \wedge (q \vee \sigma \vee \tau)$. Here, $p \vee \sigma \vee \tau$ and $q \vee \sigma \vee \tau$ are the intersections in $\mathbf{N}$ of elements of $\mathbf{G}$. Hence, so is $\sigma \vee \tau$. This completes the induction.

It follows from (UL4, 5) that φ induces a map φ^* from the points and lines of $\mathbf{M}$ to those of $\mathbf{N}$, namely, α^{φ^*} is the intersection in $\mathbf{N}$ of $[\alpha]^{\theta^{-1}\varphi}$ for each point or line α of $\mathbf{M}$. Here, φ^* is injective and preserves inclusion.

There is thus a unique extension of φ^* to an isometry $\varphi^*: \mathbf{M} \to \mathbf{N}$. Since $\theta\varphi^*$ and φ agree on the points of $\mathbf{G}$ while preserving joins, $\theta\varphi^* = \varphi$. The uniqueness of φ^* is easy to check.

The restriction made in (UL7) is probably essential. Without it, (UL7) is false when $n = 2$ (for example, $\mathbf{G} = AG(2, 3)$ is isometrically embeddable in $\mathbf{N} = PG(2, 4)$), and we suspect higher dimensional counterexamples also exist.

We are now ready to state the dimension theorem.

THEOREM 1. *Let $\mathbf{G}$ be an n-dimensional geometric lattice. Let $1 \leqslant e$, $f \leqslant n - 1$ with $e + f \geqslant n$ but not $e = f = n - 1$. Suppose that no e-space and f-space with join 1 have their intersection of dimension $e + f - n - 1$. Then $\mathbf{G}$ can be strongly embedded in an n-dimensional modular geometric lattice.*

Proof. Take any $e + f - n - 1$-space W. By Section 2, $\mathbf{G}^W$ is modular. On the other hand, we have $n - 4 \geqslant e + f - n - 1 \geqslant -1$. Thus, for each $n - 4$-space U, $\mathbf{G}^U$ is 3-dimensional and modular. Thus, the theorem is a special case of the following embedding theorem.

THEOREM 2. *Let $\mathbf{G}$ be a geometric lattice of dimension $n \geqslant 3$. Suppose that, for each $n - 4$-space U, $\mathbf{G}^U$ can be strongly embedded in a 3-dimensional modular geometric lattice. Then $\mathbf{G}$ can be strongly embedded in an n-dimensional modular geometric lattice.*

Proof. We use induction on n. There is nothing to prove if $n = 3$. Suppose $n \geqslant 4$. Let p and q denote distinct points of $\mathbf{G}$.

(I) $\mathbf{G}^p$ satisfies the hypothesis of the theorem. For, $\dim \mathbf{G}^p = n - 1 \geqslant 3$, and if $U \in \mathbf{G}^p$ has codimension 3 in $\mathbf{G}^p$, it has codimension 3 in $\mathbf{G}$, while $(\mathbf{G}^p)^U = \mathbf{G}^U$.

By induction, we obtain a modular geometric lattice $\mathbf{M}(\mathbf{G}^p)$ such that $\mathbf{G}^p$ is strongly embedded in $\mathbf{M}(\mathbf{G}^p)$. We may assume that $\mathbf{G}^p \subseteq \mathbf{M}(\mathbf{G}^p)$ and that the inclusion map yields the strong embedding.

If $\sigma \in \mathbf{M}(\mathbf{G}^p)$, set $[\sigma] = \{X \in \mathbf{G}^p \mid \sigma \leqslant X\}$.

(II) Let $\mathbf{G}_\natural$ consist of the sets $[p] = \mathbf{G}^p$ and the members of $\mathfrak{P}(\mathbf{G}^p)$, $\mathfrak{L}(\mathbf{G}^p)$, and $\mathfrak{L}(\mathbf{G}^{p \vee q})$, for all p, q. By (UL3, 4, 5), these are the subsets of $\mathbf{G}$ corresponding to the points of $\mathbf{G}$ and $\mathbf{M}(\mathbf{G}^p)$, the lines of $\mathbf{M}(\mathbf{G}^p)$, and most of the lines of $\mathbf{M}(\mathbf{G}^{p \vee q})$ (all of them if $n > 4$).

Partially order $\mathbf{G}_\natural$ by the *opposite* of set inclusion.

We claim that $[q] < \Sigma \in \mathbf{G}_\natural$ implies that Σ is in $\mathfrak{P}(\mathbf{G}^q)$, $\mathfrak{L}(\mathbf{G}^q)$, or $\mathfrak{L}(\mathbf{G}^{q \vee r})$ for some r.

Suppose first that $\Sigma \in \mathfrak{P}(\mathbf{G}^p)$, $p \neq q$. Then $\Sigma \subsetneq [p] \cap [q] = [p \vee q]$. By (UL4), $\Sigma = [p \vee q] \in \mathfrak{P}(\mathbf{G}^q)$.

Next let $[q] < \Sigma \in \mathfrak{L}(\mathbf{G}^p)$. Then, by (UL5), $\Sigma = [p \vee q] \cap \Sigma_1$ with $\Sigma_1 \in \mathfrak{P}(\mathbf{G}^p)$. From (P3) and (UL4) we see that $\Sigma = \Sigma_1^{p \vee q} \in \mathfrak{P}((\mathbf{G}^p)^{p \vee q}) = \mathfrak{P}((\mathbf{G}^q)^{p \vee q})$. By (UL3), $(\mathbf{G}^q)^{p \vee q}$ is strongly embedded in $\mathbf{M}(\mathbf{G}^q)^{p \vee q}$. Thus, by (UL4), $\Sigma = [\sigma]$ for some point σ of $\mathbf{M}(\mathbf{G}^q)^{p \vee q}$, that is, for some line $\sigma > p \vee q$ of $\mathbf{M}(\mathbf{G}^q)$. By (UL5), $\Sigma \in \mathfrak{L}(\mathbf{G}^q)$.

Finally, suppose $[q] < \Sigma \in \mathfrak{L}(\mathbf{G}^{p \vee r})$ with p, q, r distinct. If $q < p \vee r$ and, say, $p \neq q$, then $p \vee r = p \vee q$ and $\Sigma \in \mathfrak{L}(\mathbf{G}^{p \vee q})$. We may thus assume that $q \not< p \vee r$. $(\mathbf{G}^p)^{p \vee r}$ is strongly embedded in $\mathbf{M}(\mathbf{G}^p)^{p \vee r}$. By (UL5) and (SE3), $\Sigma = [\sigma]$ for some line σ of $\mathbf{M}(\mathbf{G}^p)^{p \vee r}$, that is, for some plane σ of $\mathbf{M}(\mathbf{G})^p$. Since $\Sigma \subsetneq [p] \cap [q]$, $\sigma > p \vee q$, so σ is a line of $\mathbf{M}(\mathbf{G}^p)^{p \vee q}$. Again by (UL3, 5), $\Sigma \in \mathfrak{L}((\mathbf{G}^p)^{p \vee q}) = \mathfrak{L}(\mathbf{G}^{p \vee q})$.

This proves our claim. Note that the same arguments show that $\mathfrak{P}(\mathbf{G}^{p \vee q}) \subseteq \mathfrak{L}(\mathbf{G}^p)$.

(III) Define $\mathbf{G}_{\#}$ as follows. If $n > 4$, $\mathbf{G}_{\#} = \mathbf{G}_{\natural}$. If $n = 4$, $\mathbf{G}_{\#}$ is $\mathbf{G}_{\natural}$ together with a new symbol $T(L)$ for each line L such that $\mathbf{G}^L$ is an affine plane (cf. (SE3)). Here, $T(L)$ will play the role of the line at infinity in $\mathbf{G}^L$.

Order $\mathbf{G}_{\#}$ as follows. Order agrees on $\mathbf{G}_{\natural}$ and $\mathbf{G}_{\#}$. We never have $T(L) < g$ for any $g \in \mathbf{G}_{\#}$. For $\Sigma \in \mathbf{G}_{\#}$, write $\Sigma < T(L)$ if and only if one of the following holds:

(i) $\Sigma = \mathbf{G}^p$ with $p < L$;

(ii) $\Sigma = \mathbf{G}^L$;

(iii) $[L] \neq \Sigma \in \mathfrak{P}(\mathbf{G}^p)$, where $p < L$ and $\Sigma \cap \mathbf{G}^L$ is a parallel class of lines of $\mathbf{G}^L$;

(iv) $\Sigma \in \mathfrak{L}(\mathbf{G}^p)$, where $p < L$, $\Sigma \subset \mathbf{G}^L$, and Σ contains no plane of $\mathbf{G}$; or

(v) $\Sigma \in \mathfrak{L}(\mathbf{G}^p)$, where $p < L$, and $\Sigma \cap \mathbf{G}^L = \varnothing$.

Note that this is all well-defined. This is clear for (i) and (ii). In (iii), p is the only point of L with $\Sigma \in \mathfrak{L}(\mathbf{G}^p)$; in (v), p is also unique. If Σ is as in (iv), then $\Sigma \in \mathfrak{L}(\mathbf{G}^q)$ for each $q < L$, but the remaining requirements are unaltered.

Let $\Sigma \in (\mathbf{G}_{\natural})^p$, $p < L$. In (II) we saw that $\Sigma = [\sigma]$, where σ is p or a point, line, or plane of $\mathbf{M}(\mathbf{G}^p)$. By (UL5), there is at most one plane $> L$ of $\mathbf{M}(\mathbf{G}^p)$ not of the form $[\alpha]$ with $\alpha \in \mathbf{M}(\mathbf{G}^p)$. If this plane is identified with $T(L)$, the conditions (i–v) are precisely what are needed to ensure that $\sigma < T(L)$ in $\mathbf{M}(\mathbf{G}^p)$. Thus, $(\mathbf{G}_{\#})^p$ is indeed partially ordered. Moreover, suppose $\sigma < T(L)$ in $\mathbf{M}(\mathbf{G}^p)$, and let $r \not< L$. Then $r \vee L$ is a point of the affine plane $\mathbf{G}^L = (\mathbf{G}^p)^L$, so $p \vee L \not< T(L)$ in $\mathbf{M}(\mathbf{G}^p)$. Thus, $p \vee r \leqslant \sigma < T(L)$ is impossible. Consequently, $\Sigma \not\subseteq \mathbf{G}^r$ for any point $r \not< L$.

We can now prove that $\mathbf{G}_\#$ is a poset. For suppose $\Sigma_1 < \Sigma < T(L)$ with $\Sigma_1, \Sigma \in \mathbf{G}_\natural$. Let $\Sigma_1 \subseteq \mathbf{G}^r$ for a point r of $\mathbf{G}$. Since $\Sigma_1 \supset \Sigma$, it follows that $\Sigma \subseteq \mathbf{G}^r$, so $r < L$. Then $\Sigma_1 < T(L)$ since $(\mathbf{G}_\#)^r$ is partially ordered.

(IV) $\mathbf{G}_\#$ is a poset. Its points, lines, and planes are defined to be those of its elements which, respectively, have the form $[p]$ or are in $\mathfrak{P}(\mathbf{G}^p)$ or $\mathfrak{L}(\mathbf{G}^p)$ for some p. Its 3-spaces will be the elements of $\mathfrak{L}(\mathbf{G}^{p \vee q})$ for all $p \neq q$, and also the symbols $T(L)$.

We now check that $\mathbf{G}_\#$ satisfies (E1–5) of Section 2. Note that, by (II, III), $(\mathbf{G}_\#)^p$ consists of all elements of $\mathfrak{P}(\mathbf{G}^p)$, $\mathfrak{L}(\mathbf{G}^p)$, and $\mathfrak{L}(\mathbf{G}^{p\vee q})$, $p \neq q$, together with those of the $T(L)$ having $p < L$.

(E1) By (UL4, 5), $\mathfrak{P}(\mathbf{G}^p)$ and $\mathfrak{L}(\mathbf{G}^p)$ correspond to all the points and lines of the modular geometric lattice $\mathbf{M}(\mathbf{G}^p)$. Consider $\Sigma \in \mathfrak{L}(\mathbf{G}^{p\vee q})$. By (II), $\Sigma = [\sigma]$ for a line σ of $\mathbf{M}(\mathbf{G}^p)^{p\vee q}$; thus, σ is a plane of $\mathbf{M}(\mathbf{G}^p)$. Similarly, if $p < L$ and $T(L)$ exists, then, by (III), $T(L)$ is essentially a plane of $\mathbf{M}(\mathbf{G}^p)$. We saw in (III) that our ordering of $(\mathbf{G}_\#)^p$ corresponds correctly to that of $\mathbf{M}(\mathbf{G}^p)$. Finally, $\dim \mathbf{M}(\mathbf{G}^p) = n - 1 \geqslant 3$. Thus, (E1) holds.

(E2) Consider distinct points $[p]$, $[q]$ of $\mathbf{G}_\#$. By (II), any line of $\mathbf{G}_\natural$ on both of them is in $\mathfrak{P}(\mathbf{G}^p)$ and hence is unique; $[p] \ngtr [q]$ by definition.

(E3) Let Σ_1, Σ_2 be distinct planes of $\mathbf{G}_\#$ on the distinct points $[p]$, $[q]$. By (II), $\Sigma_1 = [\sigma_1]$ and $\Sigma_2 = [\sigma_2]$ for distinct lines σ_1, σ_2 of $\mathbf{M}(\mathbf{G}^p)$. Since $\sigma_1, \sigma_2 > p \vee q$, there is a plane $\tau > \sigma_1, \sigma_2$ of $\mathbf{M}(\mathbf{G}^p)$. Since $\tau > p \vee q$, τ is a line of $\mathbf{M}(\mathbf{G}^p)^{p\vee q}$. By (UL3, 5) and (III), $\tau \in \mathfrak{L}(\mathbf{G}^{p\vee q})$ or $\tau = T(p \vee q)$.

(E4) This is clear from the definitions.

(E5) Suppose $g \in \mathbf{G}_\#$ is on all points of $\mathbf{G}_\#$. Since $\mathbf{G}$ contains an independent set of 5 points, $g \in (\mathbf{G}_\#)^p$ is on each of an independent set of 4 points of the generalized projective space $(\mathbf{G}_\#)^p$. However, in $(\mathbf{G}_\#)^p$, g has dimension $\leqslant 2$ by (E1), which is a contradiction.

(V) By the Embedding Lemma of Section 2, there is a complemented modular lattice $\mathbf{M}$ and a poset monomorphism $\varphi: \mathbf{G}_\# \to \mathbf{M}$ which satisfy: $\dim \mathbf{M} = (n - 1) + 1$; points, lines, planes, and 3-spaces are mapped to the same kinds of elements of $\mathbf{M}$; and for each $[p]$, every line and plane of $\mathbf{M}$ containing $[p]^\varphi = p^*$ is in $(\mathbf{G}_\#{}^p)^\varphi$. Thus, $(\mathbf{G}_\#{}^p)^\varphi$ and $\mathbf{M}^{p^*}$ contain precisely the same lines and planes of $\mathbf{M}$.

Define an isometry $\theta(p): \mathbf{M}(\mathbf{G}^p) \to \mathbf{M}^{p^*}$ as follows. Let $\sigma, \tau \in \mathbf{M}(\mathbf{G}^p)$. If σ is p or a point or line of $\mathbf{M}(\mathbf{G}^p)$, let $\sigma^{\theta(p)} = [\sigma]^\varphi$. Extend this inductively to all of $\mathbf{M}(\mathbf{G}^p)$ by setting $(\sigma \vee \tau)^{\theta(p)} = \sigma^{\theta(p)} \vee \tau^{\theta(p)}$.

Let $U \in \mathbf{G}^p$. Write $U = (p \vee p_1) \vee \cdots \vee (p \vee p_k)$ with each p_i a point of $\mathbf{G}$. Then

$$\begin{aligned} U^{\theta(p)} &= (p \vee p_1)^{\theta(p)} \vee \cdots \vee (p \vee p_k)^{\theta(p)} \\ &= [p \vee p_1]^{\varphi} \vee \cdots \vee [p \vee p_k]^{\varphi} \\ &= (p^* \vee p_1^*) \vee \cdots \vee (p^* \vee p_k^*) \\ &= p^* \vee p_1^* \vee \cdots \vee p_k^*. \end{aligned} \tag{*}$$

Consequently, if $p \neq q < U$, then we can assume $p_1 = q$ and get $U^{\theta(p)} = U^{\theta(q)}$.

We thus obtain a map $\theta\colon \mathbf{G} \to \mathbf{M}$ by defining $0^\theta = 0$ and $U^\theta = U^{\theta(p)}$ whenever $p \leqslant U \in \mathbf{G}$.

(VI) Finally, we will show that $\mathbf{G}$ is strongly embedded in $\mathbf{M}$ via θ.

For $U, V \in \mathbf{G}$, $(U \vee V)^\theta = U^\theta \vee V^\theta$ follows from $0^\theta = 0$ and (*). Also, θ is injective. For, if $U^\theta = V^\theta$, we may assume $U, V \neq 0$. Let $p \leqslant U$. Then $U^{\theta(p)} = U^\theta = (U \vee V)^\theta = (U \vee V)^{\theta(p)}$, so $U = U \vee V$. Similarly, $V = U \vee V$. Thus, θ is a join-monomorphism. Clearly, θ preserves dimension.

(SE1) In (V) we noted that $\dim \mathbf{M} = n = \dim \mathbf{G}$.

(SE2, 3) Let $W \in \mathbf{G}$, $S \in \mathbf{M}$, $W^\theta < S$, $\dim W \leqslant n - 3$, and $\dim S \leqslant \dim W + 2$. Note that, since $n \geqslant 4$, W is not 0 if $\dim W = n - 3$, as is the case in (SE3).

First suppose $W \neq 0$. Let $p \leqslant W$. Then $W^{\theta(p)} = W^\theta < S$, so $W < S^{\theta(p)^{-1}}$ with $W \in \mathbf{G}^p$, $S^{\theta(p)^{-1}} \in \mathbf{M}(\mathbf{G}^p)$, and $\dim S^{\theta(p)^{-1}} \leqslant \dim W + 2$. If $\dim W = n - 3$, then (SE3) for $\mathbf{G}^p$, $\mathbf{M}(\mathbf{G}^p)$ implies that $S^{\theta(p)^{-1}} \in \mathbf{G}^p$, except possibly for those $S^{\theta(p)^{-1}}$ corresponding to a certain line of $\mathbf{M}(\mathbf{G}^p)^W$ and all its points. Then $S = S^{\theta(p)^{-1}\theta} \in \mathbf{G}^\theta$, except possibly for those S corresponding to a certain line of $(\mathbf{M}(\mathbf{G}^p)^W)^\theta = (\mathbf{M}^{p^\theta})^{W^\theta} = \mathbf{M}^{W^\theta}$ and all its points. This proves (SE3) for $\mathbf{G}$, $\mathbf{M}$. Next, we can write $S^{\theta(p)^{-1}} = V_1 \wedge \cdots \wedge V_k$ with $V_i \in \mathbf{G}^p$ (by (SE2) for $\mathbf{G}^p$, $\mathbf{M}(\mathbf{G}^p)$; here $\wedge$ denotes the operation in $\mathbf{M}(\mathbf{G}^p)$). Then $S = V_1^{\theta(p)} \wedge \cdots \wedge V_k^{\theta(p)} = V_1^\theta \wedge \cdots \wedge V_k^\theta$, so (SE2) holds when $W \neq 0$.

Finally, let $W = 0$. Then (SE2) requires that every point and line of $\mathbf{M}$ be the intersection of elements of $\mathbf{G}^\theta$. We only need to check this for a line λ of $\mathbf{M}$. By the Embedding Lemma, λ is the intersection of planes π of $(\mathbf{G}_{\neq})^\varphi$. Here, $\pi^{\varphi^{-1}} = [\sigma]$ for some line σ of some $\mathbf{M}(\mathbf{G}^p)$. Then $\pi = \sigma^{\theta(p)}$. By (SE2) for $\mathbf{G}^p$, $\mathbf{M}(\mathbf{G}^p)$, σ is the intersection of elements of $\mathbf{G}^p$. Hence, π is the intersection of elements of $\mathbf{G}^\theta$. Consequently, so is λ.

This completes the proof of the theorem.

5. Some Examples and Remarks

Each of the following examples satisfies conditions very similar to those of Theorems 1 or 2. In all but Example 4, there is a geometric lattice which cannot be isometrically embedded into any modular geometric lattice. All but Example 2 are simple (i.e., indecomposable) lattices.

Example 1. $\mathbf{G} = \mathbf{W}_{22}, \mathbf{W}_{23}, \mathbf{W}_{24}$. These Witt spaces [17, 18] have dimensions 3, 4, and 5. If codim $U = 2$, then $\mathbf{G}^U = [U, 1]$ is $PG(2, 4)$ while $\mathbf{G}_U = [0, U]$ is distributive. There is no isometry θ from $\mathbf{G}$ into a modular geometric lattice $\mathbf{M}$.

Proof. Suppose $\theta: \mathbf{M} \to \mathbf{G}$ exists. We may assume that $\mathbf{G} = \mathbf{W}_{22}$, $\mathbf{G} \subset \mathbf{M}$, and θ is the inclusion map. Let $\wedge$ denote the operation in $\mathbf{M}$. Fix a line L of $\mathbf{G}$. There are lines L_1, L_2, K of $\mathbf{G}$ such that L, L_1, L_2 are coplanar and no two have a common point of $\mathbf{G}$, while $L \vee K$, $L_1 \vee K$ and $L_2 \vee K$ are planes $\neq L \vee L_1$. Each of the 6 points of $\mathbf{G}$ on $L \vee L_1$ is on one of L, L_1, L_2. Since dim $L \vee L_1 \vee L_2 \vee K = 3$, we obtain a point σ of $\mathbf{M}$ by

$$\sigma = L \wedge L_1 = L \wedge (L_1 \vee K) = L \wedge K.$$

By symmetry, $\sigma = L \wedge L_2$. Each of the planes of $\mathbf{G}$ other than $L \vee L_1$ which contains L has such a K. Thus, there are $22/2$ lines $\sigma \vee x$, $x \in \mathbf{G}$. (Note that σ is equally well determined by L or K.) Each such line is on 5 planes of $\mathbf{G}$, while each plane of $\mathbf{G}$ on σ (such as $L \vee L_1 \vee L_2$) contains 3 such lines. Thus, σ is on $11 \cdot 5/3$ planes of $\mathbf{G}$, which is ridiculous.

Remark 1. *Let* $\mathbf{G}$ *be a finite n-dimensional geometric lattice. Suppose* $-1 \leqslant k \leqslant n - 3$, *and for all k-spaces* U, $\mathbf{G}^U$ *is a projective space and* $\mathbf{G}_U$ *is distributive. Then* $\mathbf{G}$ *is a projective space, an affine space over* $GF(2)$, *a Witt space* $\mathbf{W}_{22}$, $\mathbf{W}_{23}$, *or* $\mathbf{W}_{24}$, *or* $\mathbf{G}$ *is the lattice associated in the natural way with a* 3-(112, 12, 1) *design* (in which case $\mathbf{G}^p$ would, dubiously, be a projective plane of order 10 for each point p). This is essentially [6, pp. 76–77], [18], and [10, p. 70] or [8], translated in terms of lattices; the proof is an easy counting argument. This result has no infinite analogue—see Example 5.

Example 2. If $\mathbf{G}$ is any geometric lattice containing $\mathbf{W}_{22}$ as an interval, then there is no isometry from $\mathbf{G}$ into a modular geometric lattice. In particular, this is the case if $\mathbf{G}$ is the product of $\mathbf{W}_{22}$ and a modular geometric lattice of dimension $\geqslant 3$. For such a $\mathbf{G}$, $\mathbf{G}^U$ is modular for all $U \in \mathbf{G}$ of codimension 2 and for all but one element U of codimension 3.

EXAMPLE 3. Let $\mathbf{G}$ be the 3-dimensional lattice consisting of the empty set, the points, the pairs of points, and the circles of a finite inversive plane [6, Chap. 6]. For each point p, $\mathbf{G}^p$ is an affine plane of the same order n. One of the main results on finite inversive planes is Dembowski's theorem [6, p. 268], which implies that when n is even, $\mathbf{G}$ can be strongly embedded into $PG(3, n)$. One of the central problems in the area is whether or not this is also true when n is odd. In the infinite case, examples are known in which some of the affine planes $\mathbf{G}^p$ are non-desarguesian [9], and it is easy to check that $\mathbf{G}$ is not isometrically embeddable into a modular geometric lattice. (Of course, infinite examples of geometric lattices not isometrically embeddable into any modular geometric lattice are easy to come by—just consider the 2-dimensional lattice whose elements $\neq 0, 1$ are the points and lines of an infinite-dimensional projective space.)

EXAMPLE 4. $\mathbf{G} = \mathbf{W}_{11}, \mathbf{W}_{12}$. These Witt spaces [17, 18] are geometric lattices of dimensions 4 and 5, and if codim $U = 2$, then $\mathbf{G}^U$ is $AG(2, 3)$ and $\mathbf{G}_U$ is distributive. If codim $W = 3$, $\mathbf{G}^W$ is an inversive plane of order 3. Hence, $\mathbf{G}^W$ can be strongly embedded in $PG(3, 3)$ [6, p. 273; 18]. By Theorem 2, $\mathbf{G}$ can be strongly embedded in a modular lattice $\mathbf{M}$; we may assume that $\mathbf{G} \subset \mathbf{M}$ and that the inclusion map induces the embedding. By (UL3, 6), $\mathbf{M}^W = PG(3, 3)$. Here, W has just one or two points. Consequently, since W is arbitrary, we must have $\mathbf{M} = PG(5, 3)$. This yields the well-known description of $\mathbf{W}_{12}$ inside $PG(5, 3)$ [3]. Moreover, by (UL6), M_{12} extends from $\mathbf{W}_{12}$ to a subgroup of $PSL(6, 3)$; this is the standard 6-dimensional projective representation of M_{12}.

The difference between the behavior of $\mathbf{W}_{24}$ and $\mathbf{W}_{12}$ is due to the fact that, in the standard representation of $\mathbf{W}_{24}$ inside $PG(11, 2)$ [15], its points, lines, planes, 3-spaces, and 4-spaces are represented by points, lines, planes, 3-spaces, and 6-spaces of $PG(11, 2)$.

Remark 2. *Let* $\mathbf{G}$ *be a finite n-dimensional geometric lattice. Suppose* $0 \leqslant k \leqslant n-3$, *and for all k-spaces U,* $\mathbf{G}^U$ *is an affine space and* $\mathbf{G}_U$ *is distributive. Then* $\mathbf{G}$ *is a Boolean algebra, an inverse plane, a Witt space* $\mathbf{W}_{11}$ *or* $\mathbf{W}_{12}$, *or* $\dim \mathbf{G} = 3$ *and* $\mathbf{G}^U$ *is an affine plane of order* 13 *for each line U.* Here, $\mathbf{G}$ is a Boolean algebra if each $\mathbf{G}^U$ is $AG(2, 2)$. We must have $\dim \mathbf{G}^U = 2$, for if $\dim \mathbf{G}^U \geqslant 3$, we may assume that U is a point and $\dim \mathbf{G} = 4$, and an elementary counting argument yields a contradiction [6, p. 77]. If $\dim \mathbf{G} = 3$, $\mathbf{G}$ is an inversive plane. Suppose $\dim \mathbf{G} = 4$ and, for each line U, $\mathbf{G}^U$ is an affine plane. Then each $\mathbf{G}^U$ has the same order q, and there are $(q^2+2)(q^2+1)\, q^2(q^2-1)/(q+2)(q+1)\, q(q-1)$ 3-spaces. By the result of Dembowski [6, p. 268] mentioned earlier, if q is even, it is a power of 2. Consequently, $q = 2, 3, 4, 8$, or 13. The case $q = 2$ has already been handled, while $q = 3$ leads to $\mathbf{W}_{11}$ [18]. If $q = 4$ or 8, Dembowski's theorem implies that, for each point p, $\mathbf{G}^p$ can be strongly embedded in $PG(3, q)$. By Theorem 2 and (UL3, 6), $\mathbf{G}$ can be strongly embedded in $PG(4, q)$. We may assume that $\mathbf{G} \subset PG(4, q)$. Since planes of $\mathbf{G}$ have just 3 points, skew lines of $\mathbf{G}$ are skew in $PG(4, q)$. Thus, the total number of points of $PG(4, q)$ on all lines of $\mathbf{G}$ is

$$(q^2 + 2) + \binom{q^2+2}{2}(q-1) > (q^5 - 1)/(q - 1),$$

which is impossible. This only leaves the possibility $q = 13$. (The preceding argument shows that when $q = 13$, some $\mathbf{G}^p$ would have to be a non-miquelian inversive plane of order 13; it seems unlikely that such an inversive plane exists.) Finally, when $\dim \mathbf{G} \geqslant 5$ and $\mathbf{G}^U$ is not $AG(2, 2)$, we can count the number of $n - 1$-spaces and use [18] to find that $\mathbf{G}$ is $\mathbf{W}_{12}$. Again, there is no infinite analogue of this result. (We note that the case $n = q = 4$, $k = 2$, of Remark 2 is stated in Witt [18, Satz 6].)

EXAMPLE 5. For each integer $n \geqslant 3$, there is an n-dimensional geometric lattice $\mathbf{G}$ such that any two $n - 1$-spaces meet in an $n - 2$-space while $\mathbf{G}$ is not isometrically embeddable into a modular geometric lattice.

Proof. We will imitate the usual construction of free projective planes. Define a sequence $\{\mathbf{G}_k\}$ of "partial geometries" of points and blocks inductively as follows (compare [5]).

$\mathbf{G}_0$ is any set of cardinality $\geqslant n + 2$. Its elements are called points, and it has no blocks. If k is odd, and if N is any set of n points of $\mathbf{G}_{k-1}$ not on a block of $\mathbf{G}_{k-1}$, adjoin a block whose only points are those in N; $\mathbf{G}_k$ is $\mathbf{G}_{k-1}$ together with all the new blocks. If $k > 0$ is even, and if B_1, B_2 are distinct blocks of $\mathbf{G}_{k-1}$ having $| B_1 \cap B_2 | < n - 1$ common points, adjoin $n - 1 - | B_1 \cap B_2 |$ new points which are on no block $\neq B_1, B_2$; $\mathbf{G}_k$ is $\mathbf{G}_{k-1}$ together with all these points. Set $\mathbf{G}_\infty = \bigcup_k \mathbf{G}_k$. Then any n points of $\mathbf{G}_\infty$ are on a unique block, and any two blocks have exactly $n - 1$ common points. $\mathbf{G}_\infty$ determines a geometric lattice $\mathbf{G}$, whose elements are 1, the blocks, and the sets of at most $n - 1$ points. For $X \neq 0, 1$ in $\mathbf{G}$, define degree $(X) = \min\{k \mid \text{all points of } X \text{ are in } \mathbf{G}_k\}$ if $\dim X \leqslant n - 2$ and degree $(X) = \min\{k \mid X \in \mathbf{G}_k\}$ if $\dim X = n - 1$.

Let W be any $n - 3$-space. Then $\mathbf{G}^W$ is a projective plane. The restriction of the degree function grades $\mathbf{G}^W$ in the sense that each point (line) is on at most two lines (points) of smaller degree. The usual proof [13, p. 28] now shows that $\mathbf{G}^W$ contains no closed configurations.

If there were an isometry from $\mathbf{G}$ into a modular geometric lattice, then there would be an isomorphism from $\mathbf{G}^W$ into a desarguesian projective plane. This is impossible as $\mathbf{G}^W$ has no desargues configurations.

This example shows that Theorem 1 is false for $e = f = n - 1$. We conjecture, however, that it is true in the finite case. This is tied up with some interesting questions concerning finite projective planes.

There are obviously several variations possible in the above construction. For example, lines of **G** could have been arranged to have various numbers of points instead of always having two; some intersections of $n - 1$-spaces could have been allowed to have dimension $\geqslant -1$ but $\leqslant n - 4$.

6. Automorphism Groups

On page 80 of [2], Birkhoff observes that the classical geometric lattices —projective and affine spaces—are highly symmetric: their automorphism groups are highly transitive. The following result characterizes these lattices in terms of just such a property.

Theorem 3. *Let* **G** *be a finite geometric lattice of dimension* $n \geqslant 3$. *Let* Γ *be its automorphism group. Suppose that* Γ *is transitive on* (i) *the ordered pairs of distinct* $n - 1$*-spaces having nonzero intersection, and* (ii) *the ordered pairs consisting of an* $n - 1$*-space and a point not on it. Then* **G** *is a Boolean algebra, a projective space, an affine space, or* $\mathbf{W}_{22}$.

Proof. (I) Let **G** be a counterexample with the fewest number of points.

Γ is transitive on $n - 1$-spaces and on $n - 2$-spaces. Any two $n - 1$-spaces meet in 0 or an $n - 2$-space.

It is easy to see that, since **G** is not a Boolean algebra, some and hence every block does not contain at least two points. Consequently, by [10], $n = 3$ and, for each point x, $\mathbf{G}^x$ is a projective plane of order q not depending on x.

Let v be the number of points, k the number of points per plane, and h the number of points per line. In [8] it is shown (using more efficiently the exact same information as in [10, Section 4]) that these parameters must satisfy either (A) $q = h^2$, $k = h(h^2 - h + 1)$, or (B) $q = h^3 + h$, $k = h^2(h^2 - h + 1)$. In either case, $v - k = q^2(h - 1)$.

If (A) holds, then $h \neq 2$, as otherwise **G** would be $\mathbf{W}_{22}$ by [18].

(II) We next boost the given amount of transitivity. Fix a plane E, and let Γ_E be its (global) stabilizer. By hypothesis, Γ_E is transitive on the points $\nless E$, the lines $< E$, and the planes meeting E in a line. The points and lines of E form a design. By [6, p. 78], Γ_E is transitive on the points $< E$. Also, Γ is transitive on the points of **G**.

Consequently, Γ_x is transitive on the planes $> x$ and on those $\not> x$. Again by [6, p. 78], Γ_x is transitive on the points $\neq x$; that is, Γ is 2-transitive on points.

The stabilizer Γ_L of a line L is 2-transitive on the h points $< L$ and on the $q + 1$ planes $> L$. In cases (A) and (B), $(h, q + 1) = 1$. It follows that, if $x < L$, Γ_{xL} is transitive on the planes $> L$. Thus, Γ_x is transitive on the incident point-line pairs of the projective plane $\mathbf{G}^x$.

(III) By (II), $|\Gamma_x|$ is divisible by the number $k(v - 1)q$ of planes $\not> x$. Thus, if $y \neq x$ then $h^2 - h + 1 \mid |\Gamma_{xy}|$.

Also by (II), Γ_E is transitive on the $(q + 1)k/h$ lines $L < E$ and on the $q^2(h - 1)$ points $y \not< E$. Then $q^2(h - 1) \mid \Gamma_{yE} : \Gamma_{yLE} \mid = ((q + 1)k/h) \mid \Gamma_{LE} : \Gamma_{yLE} \mid$. Now assume (A). Then

$$(q^2(h - 1), (q + 1)k/h) = (h^4(h - 1), (h^2 + 1)(h^2 - h + 1))$$
$$= (2, h - 1),$$

so $(q + 1)k/h(2, h - 1)$ divides $|\Gamma_{yE} : \Omega_{yLE}|$.

Thus, Γ_{yE} is either transitive on the lines $< E$ or has exactly two orbits of such lines of the same length. Correspondingly, Γ_{LE} is either transitive on the points $\not< E$ or has two orbits of such points, each of length $h^4(h - 1)/2$.

(IV) Consider case (B). By (II), Γ_x is transitive on the points of the plane $\mathbf{G}^x$. By [6, p. 177], $q \neq 10$, so $h > 2$.

Suppose Γ_x has an element inducing an involutory collineation of $\mathbf{G}^x$. Since $q = h^3 + h$ is not a square, this must be a perspectivity [6, p. 172], and then $\mathbf{G}^x$ is desarguesian [6, pp. 193, 197], whereas q is not a prime power.

(V) We now eliminate case (B) by showing that some element of Γ_x induces an involution on $\mathbf{G}^x$, contrary to (IV). At the same time we will show that, in (III), Γ_{yE} was in fact transitive on the lines $< E$.

Since $h > 2$, there is a prime $s \mid h^2 - h + 1$, $s \neq 3$. By (III), Γ_{xy} has a Sylow s-subgroup $\Sigma \neq 1$. Let $\mathbf{G}_*$ consist of 0 and those elements of $\mathbf{G}$ fixed by Σ which contain a point fixed by Σ. Clearly $\mathbf{G}_*$ is a sublattice of $\mathbf{G}$.

By a standard result on permutation groups, the normalizer $N(\Sigma)$ of Σ in Γ is 2-transitive on the points of $\mathbf{G}_*$ [17]. If L is a line in $\mathbf{G}_*$, $s \nmid h(h - 1)(h - 2)$ implies that L contains at least 3 points of $\mathbf{G}_*$. Since $s \nmid (q^2 + 1)q^2$, L is contained in at least 2 planes in $\mathbf{G}_*$, and since $s \nmid k - h$ each such plane contains a triangle from $\mathbf{G}_*$. Thus, $\mathbf{G}_*$ is a 3-dimensional geometric lattice. Moreover, if E and F are planes of $\mathbf{G}_*$ such that $E \wedge F = L$ is a line, then a Sylow s-subgroup of Γ_{EF} fixes at least 2 points of L and so is conjugate in Γ_{EF} to Σ. As above, $N(\Sigma)_L$ is 2-transitive on the planes $> L$ of $\mathbf{G}_*$. Similarly, as $s \nmid v - k - 1$, Σ is a Sylow s-subgroup of the stabilizer of a nonincident point-plane pair from $\mathbf{G}_*$, so $N(\Sigma)$ is transitive on such pairs.

Since $\mathbf{G}$ was a minimal counterexample, $\mathbf{G}_*$ must be a projective space $PG(3, m)$ or an affine space $AG(3, m)$. In either case, $\mathbf{G}_*^x$ is a subplane of $\mathbf{G}^x$. Since a line of $\mathbf{G}_*^x$ has $(q + 1) - (m + 1)$ points of $\mathbf{G}^x$ not in $\mathbf{G}_*^x$, $s \mid q - m$. Then $s \nmid h - m$, so $\mathbf{G}_*$ must be $PG(3, m)$.

By [6, p. 39], some element of $N(\Sigma)_x$ induces an involution on $\mathbf{G}_*^x$, while for any line L and plane E of $\mathbf{G}_*$ with $L < E$, $N(\Sigma)_{LE}$ is transitive on the points $\nless E$ of $\mathbf{G}_*$. By (IV), the first statement eliminates (B). The second implies that Γ_{LE} is transitive on the points $\nless E$ of $\mathbf{G}$. For, if Γ_{LE} were intransitive, by (III) it would have two orbits of points $\nless E$ of length $h^4(h - 1)/2$ not divisible by s; then $\mathbf{G}_*$ would have a point in each of these orbits, and $N(\Sigma)_{LE}$ could not be transitive on the points $\nless E$ of $\mathbf{G}_*$.

(VI) Only case (A) remains. By (III) and (V), if $y \nless E$ then Γ_{yE} is transitive on the lines $< E$. By [6, p. 78], Γ_{yE} is transitive on the points $x < E$. Thus, Γ_{xE} is transitive on the lines $M > x$ with $M \nless E$. By [6, p. 214], $\mathbf{G}^x$ is desarguesian. In particular, $q = h^2 = q_0^{2e}$ with q_0 a prime and e an integer.

By (III), $h^2 - h + 1 \mid | \Gamma_{xy} |$. If $\gamma \in \Gamma_{xy}$ has order s^i for some i, then, as in (V), γ induces a planar collineation γ^x of $\mathbf{G}^x$, and $| \gamma^x | \mid 2e$. Since $(h^2 - h + 1)/(h + 1, 3)$ does not divide $2e$, we cannot have $| \gamma^x | = s^i$ for each such s^i. Thus, we can choose our prime $s \neq 3$ dividing $h^2 - h + 1$ such that some $\gamma \in \Gamma_{xy}$ of order s induces the identity on $\mathbf{G}^x$. As in (V), γ fixes a point $\neq x$ of each line $> x$. Then γ fixes a plane $F \ngtr x$, and hence fixes each point z of F. Now γ induces a perspectivity of $\mathbf{G}^z$, but $s \nmid q^2(q - 1)$, so γ induces the identity on $\mathbf{G}^z$. Since each point $\nless F$ of $\mathbf{G}$ is the intersection of two lines meeting F in a point, γ fixes each point of $\mathbf{G}$, which is ridiculous.

This contradiction completes the proof of the theorem.

It seems very likely that condition (ii) of Theorem 3 is superfluous. However, we have only been able to show this in case (B). Of course, one could ask what happens when the automorphism group of a finite geometric lattice $\mathbf{G}$ is 2-transitive on the $n - 1$-spaces. But here the answer is easy: $\mathbf{G}$ must be a Boolean algebra or a projective space. In fact, by [10], this conclusion requires no assumption concerning automorphism groups, only that all $n - 1$-spaces have the same number of points, all $n - 2$-spaces have the same number of points, and any two $n - 1$-spaces have intersection an $n - 2$-space.

Note Added in proof. R. Wille has called my attention to his paper On incidence geometries of grade n, in *Atti Conv. Geom. Comb. Appl.* (1971), 421–426 (Perugia). If his main result (Theorem 6) is specialized to the geometric lattice case, it becomes our Theorem 1. His results do not, however, contain our Theorem 2 (or even Remark 2 of Section 5).

References

1. G. Birkhoff, Combinatorial relations in projective geometries, *Ann. of Math.* **36** (1935), 743–748.
2. G. Birkhoff, "Lattice Theory," 3rd ed., AMS Colloq. Publ. **25** (1966).
3. H. S. M. Coxeter, Twelve points in $PG(5, 3)$ with 95040 self-transformations. *Proc. Roy. Soc.* (*A*) **247** (1958), 279–293.
4. H. H. Crapo and G.-C. Rota, Geometric lattices, *in* J. C. Abbott, "Trends in Lattice Theory," Van Nostrand, New York, 1970.
5. P. Dembowski, Homomorphismen von λ-Ebenen, *Arch. Math.* **10** (1959), 46–50.
6. P. Dembowski, "Finite Geometries," Springer, Berlin-Heidelberg-New York, 1968.
7. P. Dembowski and A. Wagner, Some characterizations of finite projective spaces, *Arch. Math.* **11** (1960), 465–469.
8. J. Doyen and X. Hubaut, Finite regular locally projective spaces, *Math. Z.* **119** (1971), 83–88.
9. G. Ewald, Beispiel einer Möbiusebene mit nichtisomorphen affinen Unterebenen, *Arch. Math.* **11** (1960), 146–150.
10. W. M. Kantor, Characterizations of finite projective and affine spaces, *Can. J. Math.* **21** (1968), 64–75.
11. W. M. Kantor, Jordan groups, *J. Algebra* **12** (1969), 471–493.
12. M. Pasch and M. Dehn, "Vorlesungen über neuere Geometrie," 2nd ed., Springer, Berlin, 1926.
13. G. Pickert, "Projektive Ebenen," Springer, Berlin-Göttingen-Heidelberg, 1955.
14. J. Tits, Sur les systèmes de Steiner associés aux trois "grands" groupes de Mathieu, *Rendic. Mat.* **23** (1964), 166–184.
15. J. A. Todd, On representations of the Mathieu groups as collineation groups, *J. London Math. Soc.* **34** (1959), 406–416.
16. R. Wille, Verbandstheoretische Charakterisierung n-stufiger Geometrien, *Arch. Math.* **18** (1967), 465–468.
17. E. Witt, Die 5-fach transitiven Gruppen von Mathieu, *Abh. Hamburg* **12** (1938), 256–264.
18. E. Witt, Über Steinersche Systeme, *Abh. Hamburg* **12** (1938), 265–275.