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ORIGINAL PAPER

Strongly Regular Graphs Satisfying the 4-Vertex Condition

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Abstract

We survey the area of strongly regular graphs satisfying the 4-vertex condition and find several new families. We describe a switching operation on collinearity graphs of polar spaces that produces cospectral graphs. The obtained graphs satisfy the 4-vertex condition if the original graph belongs to a symplectic polar space.

Keywords 4-vertex condition · Strongly regular graph · Sympletic polar space

Mathematics Subject Classification 05E30

1 Introduction

In this note we look at graphs with high combinatorial regularity, where this regularity is not an obvious consequence of properties of their group of automorphisms.

A graph Γ is said to satisfy the *t-vertex condition* if, for all triples (T, x_0, y_0) consisting of a t-vertex graph T together with two distinct distinguished vertices x_0, y_0 of T, and all pairs of distinct vertices x, y of Γ, the number of isomorphic copies of T in Γ, where the isomorphism maps x_0 to x and y_0 to y, does not depend on the choice of the pair x, y but only on whether x, y are adjacent or nonadjacent.

This concept was introduced by Hestenes and Higman [14] (who refer to the unpublished Sims [32]) in order to study rank 3 graphs. Clearly, a rank 3 graph satisfies the t-vertex condition for all t. If the graph Γ satisfies the t-vertex condition, where Γ has v vertices and $3 \leq t \leq v$, then Γ also satisfies the $(t-1)$-vertex condition. A graph satisfies the 3-vertex condition if and only if it is strongly regular (or complete

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or edgeless). It satisfies the v-vertex condition if and only if it is rank 3. Thus, we get a hierarchy of conditions of increasing strength between strongly regular and rank 3.

The present paper will focus almost exclusively on the case $t = 4$. A simple criterion for the 4-vertex condition is given in Proposition 2.1. Previously not many graphs were known that satisfy the 4-vertex condition without being rank 3. Here we survey the known examples and give several new constructions. One of our constructions proceeds by switching symplectic graphs (see Sect. 7). As a consequence we find

Theorem 1.1 *For $v \geq 4$ there are at least $\lfloor v^{1/6} \rfloor!$ strongly regular graphs of order at most v satisfying the 4-vertex condition.*

It follows that among all non-isomorphic strongly regular graphs of order at most v that satisfy the 4-vertex condition the fraction that is determined by their spectrum goes to 0 when v goes to infinity.

2 The 4-Vertex Condition

A graph of order v is called *strongly regular* with parameters (v, k, λ, μ) if it is neither complete nor edgeless, each vertex has degree k, any two adjacent vertices have exactly λ common neighbors, and any two non-adjacent vertices have exactly μ common neighbors.

A graph with vertex set V has *rank r* if its automorphism group is transitive on V and has exactly r orbits on $V \times V$. Rank 3 graphs are strongly regular.

If x is a vertex of the graph Γ, then the *local graph* $\Gamma(x)$ of Γ at x is the induced subgraph in Γ on the neighborhood of x. We say that Γ is *locally* P when all local graphs of Γ have property P. If Γ is strongly regular, then its *1st subconstituent* (at a vertex x) is the local graph at x, while its *2nd subconstituent* (at x) is the induced subgraph on the non-neighborhood of x. If xy is an edge (resp. nonedge) in Γ, then the subgraph induced on $\Gamma(x) \cap \Gamma(y)$ is called a λ-graph (resp. μ-graph).

See [4] for further information about strongly regular graphs.

Details on the parameters of graphs satisfying the 4-vertex condition are given in [14]. In particular, we have the following simple criterion for the 4-vertex condition:

Proposition 2.1 (Sims [32]) *A strongly regular graph Γ with parameters (v, k, λ, μ) satisfies the 4-vertex condition, with parameters (α, β), if and only if the number of edges in $\Gamma(x) \cap \Gamma(y)$ is α (resp. β) whenever the vertices x, y are adjacent (resp. nonadjacent). In this case, $k\left(\binom{\lambda}{2} - \alpha\right) = \beta(v - k - 1)$.*

The equality here follows by counting 4-cliques minus an edge.

It immediately follows that the collinearity graph of a generalized quadrangle (cf. [28]) or partial quadrangle (cf. [7]) satisfies the 4-vertex condition (with $\alpha = \binom{\lambda}{2}$ and $\beta = 0$). The same holds for a graph Γ with $\lambda \leq 1$.

If Γ is locally strongly regular, say with local parameters (v', k', λ', μ') (where clearly $v' = k$ and $k' = \lambda$), then $\Gamma(x) \cap \Gamma(y)$ has valency λ' (resp. μ') when $x \sim y$ (resp. $x \nsim y$) so that Γ satisfies the 4-vertex condition with $\alpha = \lambda\lambda'/2$ and $\beta = \mu\mu'/2$.

2.1 A Few Rank 4 Examples

Below we give a small table with the parameters of some edge-transitive rank 4 graphs satisfying the 4-vertex condition. Except for the example with group $HJ.2$ due to Reichard [30], these do not seem to have been noticed in print.

v	k	λ	μ	λ′	μ′	α	β	Group	Name	Refs.
144	55	22	20	–	9	87	90	$M_{12}.2$		
280	36	8	4	–	2	1	4	HJ.2		[30]
300	104	28	40	–	8	78	160	$PGO_5(5)$	$NO_5^-(5)$	Sect. 6
325	144	68	60	–	30	1153	900	$PGO_5(5)$	$NO_5^+(5)$	Sect. 6
512	196	60	84	14	20	420	840	$2^9.\Gamma L_3(8)$	Dual hyperoval	Sect. 4
729	112	1	20	0	0	0	0	$3^6.2.L_3(4).2$	Games graph	[3]
1120	729	468	486	297	306	69498	74358	$PSp_6(3).2$	disj. t.i. planes	Sect. 5
1849	462	131	110	–	–	2980	1845	$43^2{:}(42\times D_{22})$	power diff. set	Sect. 3.6

The numbers λ', μ' give the valency of the λ-and μ-graphs in case these are regular (and then $\alpha = \lambda\lambda'/2$ and $\beta = \mu\mu'/2$).

The examples on 144 and 729 vertices also satisfy the 5-vertex condition.

2.2 Strongly Regular Graphs with Strongly Regular Subconstituents

As we saw, graphs that are locally strongly regular satisfy the 4-vertex condition. Sometimes it follows that also the 2nd subconstituents must be strongly regular.

Lemma 2.2 *Suppose that a strongly regular graph with parameters* $(v, k, \lambda, \mu) = (4t^2, 2t^2-\varepsilon t, t^2-\varepsilon t, t^2-\varepsilon t)$ *(where* $\varepsilon = \pm 1$*) has first subconstituents that are strongly regular with parameters* $(v', k', \lambda', \mu') = \left(2t^2-\varepsilon t, t^2-\varepsilon t, \frac{1}{2}t(t-\varepsilon), t(\frac{1}{2}t-\varepsilon)\right)$*. Then its second subconstituents are strongly regular with parameters* $(v'', k'', \lambda'', \mu'') = \left(2t^2+\varepsilon t-1, t^2, \frac{1}{2}t(t-\varepsilon), \frac{1}{2}t^2\right)$.

More generally, the spectrum of the 2nd subconstituent at any vertex of a strongly regular graph follows from that of the 1st subconstituent—see [8], Theorem 5.1.

Call the three parameter sets in the above lemma $A(\varepsilon t)$, $B(\varepsilon t)$, and $C(\varepsilon t)$, respectively. They occur again in Sect. 3.3. The parameter sets $A(t)$ and $A(-t)$ are known as (*negative*) *Latin square parameters* $LS_t(2t)$ (resp. $NL_t(2t)$). The complementary graphs have parameters $LS_{t+1}(2t)$ (resp. $NL_{t-1}(2t)$).

Cameron et al. [8] studied the situation of a primitive strongly regular graph such that, for some vertex, both subconstituents are strongly regular, and found that such a graph either has a vanishing Krein parameter q_{11}^1 or q_{22}^2, or has Latin square or negative Latin square parameters. They conjectured that every non-grid example of the latter has parameters as in the above lemma or has a complement with these parameters.

3 Survey of the Known Examples and Results

3.1 Complements

A graph satisfies the t-vertex condition if and only if its complement does.

3.2 Generalized Quadrangles

Higman [13] observed that the collinearity graphs of generalized quadrangles satisfy the 4-vertex condition (and there are many examples that are not rank 3, cf. [23]).

More generally the 4-vertex condition holds for partial quadrangles. For example, the Hill graph with parameters $(v, k, \lambda, \mu) = (4096, 234, 2, 14)$ (derived from the cap constructed in [15]) has a rank 10 group and satisfies the 4-vertex condition with $\alpha = 1, \beta = 0$.

Reichard [31] showed that the collinearity graphs of generalized quadrangles satisfy the 5-vertex condition, and that the collinearity graphs of generalized quadrangles $\mathrm{GQ}(s, s^2)$ satisfy the 7-vertex condition.

More generally the 5-vertex condition holds for partial quadrangles.

3.3 Binary Vector Spaces with a Quadratic Form

The first non-rank-3 graph satisfying the 5-vertex condition was constructed by A. V. Ivanov [21]: a strongly regular graph Γ_0 whose subconstituents Γ_1, Γ_2 satisfy the 4-vertex condition. The parameters are as follows.

	v	k	λ	μ	α	β	$\|G\|$	Remarks
Γ_0	256	120	56	56	784	672	$2^{20} \cdot 3^2 \cdot 5 \cdot 7$	Rank 4: $1 + 120 + 120 + 15$
Γ_1	120	56	28	24	216	144	$2^{12} \cdot 3^2 \cdot 5 \cdot 7$	Rank 4: $1 + 56 + 56 + 7$
Γ_2	135	64	28	32	168	192	$2^{12} \cdot 3^2 \cdot 5 \cdot 7$	Intransitive: $120 + 15$

In [6] an infinite family of graphs $\Gamma^{(m)}$ $(m \geq 1)$ is constructed by taking as vertex set $\mathbb{F}_2^{2m}$, where vectors are adjacent when the line joining them meets the hyperplane at infinity in a fixed hyperbolic quadric minus a maximal t.i. subspace. The graphs $\Gamma^{(m)}$ have parameters $A(2^{m-1})$ (see Sect. 2.2). They have a rank 4 group (for $m \geq 4$) and satisfy the 4-vertex condition.

The local graphs $\Delta^{(m)}$ are strongly regular with parameters $B(2^{m-1})$. They have a rank 4 group (for $m \geq 4$) and satisfy the 4-vertex condition.

By Lemma 2.2 also the 2nd subconstituents $\mathrm{E}^{(m)}$ are strongly regular, with parameters $C(2^{m-1})$.

We checked by computer that the graph $\Gamma^{(4)}$ is isomorphic to the above Γ_0.

In [30] it is shown that the graphs $\Gamma^{(m)}$ satisfy the 5-vertex condition.

In [29] it is shown that the graphs $\Gamma^{(m)}$ are triplewise 5-regular, a.k.a. (3,5)-regular, where (s, t)-regularity is the analog of the t-vertex condition where s instead of two vertices are distinguished. It follows that the 2nd subconstituents $\mathrm{E}^{(m)}$ of the graphs $\Gamma^{(m)}$ also satisfy the 4-vertex condition.

In [22], two infinite families of graphs are constructed. One is the above $\Gamma^{(m)}$. The second family has members $\Sigma^{(m)}$ with vertex set $\mathbb{F}_2^{2m}$, where vectors are adjacent when the line joining them hits the hyperplane at infinity either in a fixed elliptic quadric minus a maximal t.i. subspace S or in $S^{\perp} \backslash S$. The graphs $\Sigma^{(m)}$ have parameters $A(-2^{m-1})$, have rank 5 (for $m \geq 5$), and satisfy the 4-vertex condition.

Let $\Gamma(V, X)$ be the graph on a vector space V where two vectors are adjacent precisely when the joining line hits the subset X of the hyperplane PV at infinity. Since $\Gamma(V, X)$ is strongly regular if and only if X is a 2-character set ([11]), that is, if and only if $|X \cap H|$ takes only two distinct values when H runs through the hyperplanes of PV, the set $(Q \setminus S) \cup (S^{\perp} \backslash S)$ is a 2-character set when Q is an elliptic quadric, and S a maximal t.i. subspace.

Let V be a vector space over $\mathbb{F}_2$. Then the local graph of $\Gamma(V, X)$ is the collinearity graph of the partial linear space with point set X and whose lines are the projective lines (of size 3) contained in X.

The local graphs $\mathrm{T}^{(m)}$ are strongly regular with parameters $B(-2^{m-1})$. They are intransitive (for $m \geq 5$).

It follows from Lemma 2.2 that also the 2nd subconstituents $\Upsilon^{(m)}$ are strongly regular, with parameters $C(-2^{m-1})$. There is a tower of graphs here: If Υ is the 2nd subconstituent of $\Sigma^{(m)}$ at a vertex x, and $s \in S$, then the local graph of Υ at its vertex $x + s$ is isomorphic to $\Sigma^{(m-1)}$. (For a proof, see Appendix A.)

In [22] it is conjectured that the graphs $\Sigma^{(m)}$ satisfy the 5-vertex condition, and that the graphs $\mathrm{T}^{(m)}$ and $\Upsilon^{(m)}$ satisfy the 4-vertex condition. The former was proved in [30]. The latter is proved in Appendix A. In [29] it is announced that $\Sigma^{(m)}$ is even $(3, 5)$-regular, but we are not aware of a proof in print.

3.4 Block Graphs of Steiner Triple Systems

Higman [13] investigated for which v-point Steiner triple systems the block graph satisfies the 4-vertex condition. He found that either the system is a projective space $\mathrm{PG}(m, 2)$ or v is one of 9, 13, 25. In [25] the cases 13 and 25 are ruled out, so that the only other example is the affine plane $\mathrm{AG}(2, 3)$. The examples are rank 3.

3.5 Smallest Example

In [26] it is shown that the smallest non-rank-3 strongly regular graphs satisfying the 4-vertex condition have $v = 36$ vertices. There are three examples. All have $(v, k, \lambda, \mu) = (36, 14, 4, 6)$ and $\alpha = 0$, $\beta = 4$.

3.6 Cyclotomic Examples

Given (q, e, J), where $e \,|\, (q-1)/2$ and J is a set of nonnegative integers, and a fixed primitive element η of $\mathbb{F}_q$, consider the cyclotomic graph with vertex set $\mathbb{F}_q$, where two elements are adjacent when their difference is in $D = \{\eta^{ie+j} \mid 0 \le i < (q-1)/e,\ \ j \in J\}$. In some cases this yields a strongly regular graph that satisfies the 4-vertex condition. We give a few examples. The examples on 11^2 and 23^2 vertices are due to Klin and Pech [27].

q	p^f	e	J	η	α	β	rk
1849	43^2	4	$\{0\}$	Any	2980	1845	4
146689	383^2	4	$\{0\}$	Any	11353825	10662960	4
121	11^2	6	$\{0, 1, 2\}$	Any	200	206	5
625	5^4	6	$\{0, 1, 2\}$	Any	5913	6022	5
5041	71^2	6	$\{0, 1, 2\}$	Any	395641	396270	5
529	23^2	8	$\{0, 1, 2, 3\}$	$\eta^2 = \eta + 4$	4215	4300	5

In all cases $q = p^f$ where p is semiprimitive mod e (that is, $e \mid (p^i + 1)$ for some i), so that the parameters of the strongly regular graph can be found in [4, Thm. 7.3.2].

4 Graphs From Hyperovals

In [17], Huang et al. constructed various families of graphs. The complement of one of them can be described as follows ([2]). For $q = 2^m$, take $\mathbb{F}_q^3$ as the vertex set of Γ. Let π be the plane at infinity of $\mathbb{F}_q^3$. Let H^* be a dual hyperoval of π (that is, a set of $q+2$ lines, no three on a point). The plane π is partitioned into two parts, $\frac{1}{2}(q+1)(q+2)$ points on two lines of H^* and $\frac{1}{2}q(q-1)$ exterior points on no line of H^*. Two vertices of Γ are adjacent when the line joining them hits π in one of the exterior points. Then Γ is strongly regular and has parameters

$$(v, k, \lambda, \mu) = \big(q^3, \tfrac{1}{2}q(q-1)^2, \tfrac{1}{4}q(q-2)(q-3), \tfrac{1}{4}q(q-1)(q-2)\big).$$

Its local graphs are strongly regular with parameters

$$\big(\tfrac{1}{2}q(q-1)^2, \tfrac{1}{4}q(q-2)(q-3), \tfrac{1}{8}q(q^2-9q+22), \tfrac{1}{8}q(q-3)(q-4)\big).$$

Hence, as noted in Sect. 2, Γ satisfies the 4-vertex condition. If $m = 3$, then Γ has rank 4.

5 Disjoint t.i. Planes in Symplectic 6-Space

Let V be a 6-dimensional vector space over $\mathbb{F}_q$, provided with a nondegenerate symplectic form. Let Γ be the graph with as vertices the totally isotropic planes, adjacent when disjoint.

Proposition 5.1 *The graph Γ is strongly regular, with parameters $v = (q^3+1)(q^2+1)(q+1)$, $k = q^6$, $\lambda = q^2(q^3-1)(q-1)$, $\mu = (q-1)q^5$. If q is even, then Γ is rank*

3, *otherwise rank* 4. *Its local graph* Δ *is strongly regular with parameters* $v' = k$, $k' = \lambda$, $\lambda' = \mu' - q^2(q-2)$ *and* $\mu' = q^2(q-1)(q^3-q^2-1)$. *It follows that* Γ *satisfies the* 4*-vertex condition.*

For convenience, we give the parameters of $\bar{\Delta}$, the complement of Δ:
$\bar{v} = q^6$, $\bar{k} = (q^2+1)(q^3-1)$, $\bar{\lambda} = q^4+q^3-q^2-2$, $\bar{\mu} = q^4+q^2$.

Proof The dual polar graph Σ belonging to $\mathrm{Sp}_6(q)$ is distance-regular of diameter 3 and has eigenvalue -1. It follows that its distance-3 graph Γ is strongly regular (see [5], Prop. 4.2.17). More generally, the distance 1-or-2 graph of the symplectic dual polar space $\mathrm{Sp}_{2m}(q)$ is distance-regular (cf. [5], Prop. 9.4.10). For $m = 3$ it is the complement of Γ.

For any vertex x, the subgraph induced by Σ on $\Sigma_3(x)$ is isomorphic to the symmetric bilinear forms graph on $\mathbb{F}_q^3$ (see [5], Prop. 9.5.10). If q is odd, then distance j ($j = 0, 1, 2, 3$) in $\Sigma_3(x)$ corresponds to $\mathrm{rk}(f-g) = j$ in the symmetric bilinear forms graph and hence to distance $\lfloor (j+1)/2 \rfloor$ in the quadratic forms graph (see [5], Sect. 9.6). It follows that Δ is the complement of the quadratic forms graph, and has parameters as claimed.

If q is even, then Γ is rank 3 (by triality, it is the complement of the $O_8^+(q)$ polar graph), and Δ is the complement of the rank 3 graph $VO_6^+(q)$, with parameters as claimed. □

6 Nonsingular Points Joined by a Tangent

Let V be a vector space of dimension $2m+1$ over $\mathbb{F}_q$ with q odd, and let Q be a nondegenerate quadratic form on V. We also use Q as the symbol for the set of singular projective points.

The projective space PV has $(q^{2m+1}-1)/(q-1)$ points, $(q^{2m}-1)/(q-1)$ singular, and q^{2m} nonsingular. The nonsingular points come in two types: there are $\frac{1}{2}q^m(q^m+\varepsilon)$ points of type ε (where $\varepsilon = \pm 1$), with $\varepsilon = +1$ (resp. -1) for points x for which $x^\perp$, the hyperplane of points orthogonal to x, is hyperbolic (resp. elliptic).

Consider the graph $NO^\varepsilon_{2m+1}(q)$ that has as vertex set the set of nonsingular points of type ε, where two points are adjacent when the joining line is a tangent.

Proposition 6.1 (Wilbrink [34], cf. [3]) *Let* $m \geq 2$. *The graph* $NO^\varepsilon_{2m+1}(q)$ *is strongly regular with parameters* $v = \frac{1}{2}q^m(q^m+\varepsilon)$, $k = (q^{m-1}+\varepsilon)(q^m-\varepsilon)$, $\lambda = 2(q^{2m-2}-1)+\varepsilon q^{m-1}(q-1)$, $\mu = 2q^{m-1}(q^{m-1}+\varepsilon)$.

For $m = 1$, $\varepsilon = -1$ the graph is edgeless. For $m = 1$, $\varepsilon = 1$ we have the triangular graph $T(q+1)$. Wilbrink also handled the case of even q. We give an explicit proof here; for a different and more general proof see [1].

Proof The neighbors of a vertex x lie on the tangents joining x with a singular point of $x^\perp$, and $x^\perp$ has $(q^{m-1}+\varepsilon)(q^m-\varepsilon)/(q-1)$ singular points. This gives the value of k.

A common neighbor z of two adjacent vertices x, y lies on the line xy (and there are $q-2$ choices) or on some other tangent T on x. In the latter case the plane $\langle x, y, z\rangle$

meets Q in a conic or double line. If it is a conic, then z is uniquely determined on T by the fact that yz is the tangent on y other than xy. If it is a double line, then each nonsingular point of $T \setminus \{x\}$ is suitable. Let p be the singular point on xy. Then $\{p, x\}^\perp/\langle p\rangle$ is a nondegenerate $(2m-2)$-space of type ε, and has $a = (q^{m-2}+\varepsilon)(q^{m-1}-\varepsilon)/(q-1)$ singular points. It follows that xy is in a planes that hit Q in a double line, and in q^{2m-2} planes that hit Q in a conic. Consequently, $\lambda = q-2+q^{2m-2}+(q-1)qa$, as desired.

A common neighbor z of two nonadjacent vertices x, y determines a nondegenerate plane $\pi = \langle x, y, z\rangle$ in which xz and yz are tangents, so that x, y, z are exterior points. Now x, y are on two tangents each, and π contains 4 common neighbors of x, y. If Q is a quadratic form on a $(2m+1)$-space, then a point p is exterior if and only if $(-1)^m \det(Q)\, Q(p)$ is a nonzero square. In order to have p exterior in π but a ε-point in V, the $(2m-2)$-space $\pi^\perp$ must be an ε-subspace of the $(2m-1)$-space $\{x, y\}^\perp$. Since there are $b = \frac{1}{2}q^{m-1}(q^{m-1}+\varepsilon)$ such ε-subspaces, we find $\mu = 4b$, as desired. □

The automorphism group $\mathrm{P\Gamma O}_{2m+1}(q)$ of the graph contains $\mathrm{PGO}_{2m+1}(q)$. The latter has $(q+3)/2$ orbits on pairs of vertices [1]. Hence, the graph has rank $(q+3)/2$ if q is prime.

For $m=2$, $\varepsilon=-1$, this is the collinearity graph of a semi-partial geometry found by Metz. Its lines have size $s+1=q$ and there are $t+1=q^2+1$ lines on each point. Each point outside a line has either 0 or $\alpha=2$ neighbors on the line. See Debroey [9], voorbeeld 1.1.3d, and Debroey–Thas [10], example 1.4d, and Hirschfeld–Thas [16], p. 268, and Brouwer-van Lint [3], Sect. 7A, and Brouwer-Van Maldeghem Sect. 8.7, example (ix).

For $m=2$, $\varepsilon=+1$ this is the collinearity graph of a geometry with $t+1=(q+1)^2$ lines of size $s+1=q$ on each point, such that each point outside a line has 0, 2, or q neighbors on the line ([3], Sect. 7B).

We shall prove that these graphs satisfy the 4-vertex condition. First a lemma.

Lemma 6.2 *Let S be a solid such that $Q\big|_S$ is nondegenerate. Let x, y, z be distinct nonsingular points of the same type ε such that $\langle z, x\rangle$ and $\langle z, y\rangle$ are tangents and $\langle x, y\rangle$ is nondegenerate. Put $\pi = \langle x, y, z\rangle$. Then there are either* 0 *or* 2 *nonsingular points $w \in S\setminus\pi$ of type ε such that $\langle x, w\rangle$, $\langle y, w\rangle$, and $\langle z, w\rangle$ are tangents. For x, y, z given, the number of w only depends on the type of S. It equals* 2 *if and only if the nonzero number $2\Big(\frac{B(z,z)B(x,y)}{B(x,z)B(y,z)}-1\Big)\det(Q\big|_S)$ is a square.*

Proof Replace x by $\frac{B(z,z)}{B(x,z)}x$ and y by $\frac{B(z,z)}{B(y,z)}y$. Then $B(x,z)=B(z,z)=B(y,z)$. Put $x_0=x-z$, $y_0=y-z$, $w_0=w-z$, then $B(x_0,z)=B(y_0,z)=B(w_0,z)=0$. Since the lines $\langle z, x\rangle$, $\langle z, y\rangle$, and $\langle z, w\rangle$ are tangents, the points x_0, y_0, z_0 are singular, that is, $Q(x_0)=Q(y_0)=Q(w_0)=0$. The line $\langle x, w\rangle$ is a tangent, so $Q(x+tw)=0$ has a unique solution t. Now

$$\begin{aligned} Q(x+tw) &= Q(z+x_0+t(z+w_0)) = Q((1+t)z+x_0+tw_0) \\ &= (1+t)^2Q(z)+Q(x_0+tw_0) = (1+t)^2Q(z)+tB(x_0,w_0). \end{aligned}$$

It follows that $\left(2 + \frac{B(x_0,w_0)}{Q(z)}\right)^2 = 4$, that is $\frac{B(x_0,w_0)}{Q(z)} \in \{0, -4\}$.

As $Q\big|_S$ is nondegenerate, $z^\perp \cap S$ is a nondegenerate plane. If $B(x_0, w_0) = 0$, then $\langle x_0, w_0\rangle$ is a totally singular line in this plane, impossible. Hence, $B(x_0, w_0) = -4Q(z)$. Similarly, $B(y_0, w_0) = -4Q(z)$.

In the plane $z^\perp \cap S$, let u be the point of intersection of the tangents through the points x_0 and y_0 and write $w_0 = ax_0 + by_0 + cu$. Then $B(x_0, u) = B(y_0, u) = 0$ and $-4Q(z) = B(x_0, w_0) = B(x_0, ax_0 + by_0 + cu) = bB(x_0, y_0)$. Similarly, $-4Q(z) = B(y_0, w_0) = aB(x_0, y_0)$, so that $a = b = \frac{-4Q(z)}{B(x_0,y_0)}$, independent of w. Also,

$$0 = Q(w_0) = Q(ax_0 + by_0 + cu) = abB(x_0, y_0) + c^2 Q(u) = \frac{16Q(z)^2}{B(x_0, y_0)} + c^2 Q(u).$$

If $-B(x_0, y_0)Q(u)$ is a square, then we have two solutions for c (so also w_0 and, therefore, w) and otherwise none. Since u is an exterior point in the plane $\sigma = z^\perp \cap S$, the number $-Q(u) \det Q\big|_\sigma$ is a square. Also, $\det Q\big|_S = Q(z) \det Q\big|_\sigma$ and $B(x, y) = B(x_0, y_0) + B(z, z)$. □

Proposition 6.3 *The graph $NO^\varepsilon_{2m+1}(q)$ satisfies the 4-vertex condition.*

Proof By Proposition 2.1 it suffices to check for $x \neq y$ that the number of edges in $\Gamma(x) \cap \Gamma(y)$ does not depend on the choice of the points x, y, but only on whether x, y are adjacent or not.

Since Aut Γ is edge-transitive, we only need to check $\Gamma(x) \cap \Gamma(y)$ for $x \nsim y$.

Claim: this subgraph $\Gamma(x) \cap \Gamma(y)$ is regular of valency $4q^{2m-3} + 3\varepsilon q^{m-1} - 4\varepsilon q^{m-2} - 1$. In other words, this is the value of μ in the local graph (which is regular, but not strongly regular).

If $x \sim z \sim y$, $x \nsim y$, then $\pi = \langle x, y, z\rangle$ is a nondegenerate plane in which the common neighbors of x, y form a 4-cycle, so that x, y, z have two common neighbors in π, say a and b.

The plane π lies in $(q^{2m-3} - \varepsilon q^{m-2})/2$ solids of type $O^-(4, q)$, equally many solids of type $O^+(4, q)$, and $(q^{m-2} + \varepsilon)(q^{m-1} - \varepsilon)/(q - 1)$ degenerate solids.

If S is a degenerate solid through π with apex p, we see that $w \in S \setminus \pi$ is in $\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)$ if and only if gets projected from p onto an element of $\{a, b, z\}$ in π. Hence, $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z) \cap S \setminus \pi| = 3(q - 1)$. Hence, the total number of choices for w equals $3(q^{m-2} + \varepsilon)(q^{m-1} - \varepsilon)$.

Now let S be a nondegenerate solid on π, and let $p = S \cap \pi^\perp$. By Lemma 6.2, the number of w in S is 0 or 2, depending on the determinant of Q restricted to S. Since $\pi^\perp$ contains equally many points p with $Q(p)$ a square as with $Q(p)$ a non-square, the total number of choices for w equals the number of choices for p which is $q^{2m-3} - \varepsilon q^{m-2}$.

So the induced subgraph on $\Gamma(x) \cap \Gamma(y)$ has valency $2 + 3(q^{m-2} + \varepsilon)(q^{m-1} - \varepsilon) + (q^{2m-3} - \varepsilon q^{m-2}) = 4q^{2m-3} + 3\varepsilon q^{m-1} - 4\varepsilon q^{m-2} - 1$. □

7 Polar Switching

A *polar space* is a partial linear space such that for each line L any point outside L is collinear to either all or precisely one of the points of L. A *singular subspace* is a line-closed set of points, any two of which are collinear. The polar space is called *nondegenerate* when no point is collinear to all points. Finite nondegenerate polar spaces are the sets of totally isotropic (t.i.) or totally singular (t.s.) points and lines in a vector space over a finite field provided with a suitable symplectic, quadratic or hermitian form. The *rank* of the polar space is the (vector space) dimension of its maximal singular subspaces.

Let **P** be a nondegenerate polar space of rank $d \geq 3$ in a vector space V over $\mathbb{F}_q$. Its collinearity graph Γ_0 is strongly regular and satisfies the 4-vertex condition (since it is rank 3). We shall construct cospectral graphs that satisfy the 4-vertex condition (but are not rank 3) by a switching construction. Let $x^\perp$ be the set of points collinear with x (including x itself).

Suppose U is a maximal singular subspace of **P** (i.e., a maximal clique in Γ_0), and let H_1, H_2 be two hyperplanes of U. We can redefine adjacency and make the points x with $x^\perp \cap U = H_1$ or H_2 adjacent to the points in H_2 or H_1, respectively, and leave all other adjacencies unchanged. This is an example of WQH-switching (Wang, Qiu & Hu [33], cf. [20]) and yields a graph cospectral with Γ_0. One can repeat this interchange of hyperplanes and get arbitrary permutations of all hyperplanes. We generalize this, even allowing different designs on U.

7.1 Construction

Let P be the point set of **P**, and let the subset U be (the set of points of) a totally isotropic d-space. Let **D** be a symmetric design with the same parameters as the symmetric design of points and hyperplanes of $\mathrm{PG}(d-1, q)$, so its parameters are 2-$\left(\frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1}, \frac{q^{d-2}-1}{q-1}\right)$. Let φ be a bijection from the set $\mathcal{H}$ of hyperplanes of U to the blocks of **D**. We assume that the points of U are also the points of **D**.

Following ideas in [12, 24] we define a graph Γ_φ on the vertex set of Γ_0 as follows:

1. Vertices in U are pairwise adjacent.
2. Distinct vertices $x, y \notin U$ are adjacent if $x \in y^\perp$.
3. Vertices $x \in U$, $y \notin U$ are adjacent if $x \in (y^\perp \cap U)^\varphi$.

Clearly, $\Gamma_\varphi = \Gamma_0$ if we take the hyperplanes of U for the blocks of **D** and φ as the identity.

Theorem 7.1 *The graph Γ_φ is strongly regular with the same parameters as the classical graph Γ_0.*

Proof Let x and y be any two vertices. We show that the number of common neighbors z of x, y in Γ_φ does not depend on φ (but depends on whether x, y are equal, adjacent or nonadjacent in Γ_φ).

If $x, y \in U$, then any $z \in U$ is a common neighbor. The number of $z \in P \setminus U$ such that $x, y \in (z^\perp \cap U)^\varphi$ does not depend on φ: each hyperplane H of U such that $x, y \in H^\varphi$ contributes $|H^\perp \setminus U|$ such z.

Suppose that $x, y \notin U$. Then we are counting the z in $(x^\perp \cap U)^\varphi \cap (y^\perp \cap U)^\varphi$, and also the z in $(x^\perp \cap y^\perp) \setminus U$. The numbers of such z does not depend on φ.

The remainder of the proof concerns the case $x \in U$, $y \notin U$. If $z \in U$ then the requirements are $z \neq x$ and $z \in (y^\perp \cap U)^\varphi$. The number of such z does not depend on φ.

So we need to count the $z \notin U$. First set $I := y^\perp \cap U$, so $Y := \langle y, I \rangle$ is totally isotropic. If $z \in Y$ then $I^\varphi = (z^\perp \cap U)^\varphi$, and x, z are adjacent if and only if x, y are adjacent. The number of such z is independent of φ.

It remains to count the z in $y^\perp \setminus Y$ such that $x \in (z^\perp \cap U)^\varphi$; here $z^\perp \cap U \neq I$ as $z \notin Y$. Let $H \neq I$ be a hyperplane of U such that $x \in H^\varphi$. The number of H does not depend on φ (note that $x \in I^\varphi$ if and only if x, y are adjacent in Γ_φ). We show that the number of z in $y^\perp \setminus Y$ with $z^\perp \cap U = H$ does not depend on φ or H. Using bars to project $(H \cap I)^\perp$ into the nondegenerate rank 2 polar space $(H \cap I)^\perp/(H \cap I)$, we see totally isotropic lines $\bar{U}$ and $\bar{Y}$ meeting at a point $\bar{I}$, and a nondegenerate 2-space $\langle \bar{y}, \bar{H} \rangle$; the number of $\bar{z}$ in $\langle \bar{y}, \bar{H} \rangle^\perp \setminus \bar{I}$ does not depend on φ or H, so neither does the number of required z. □

7.2 Isomorphisms

7.2.1 Emptying Bijections φ

Call a vertex $e \in U$ *emptying* for φ if $\bigcap\{H \mid H \in \mathcal{H},\ e \in H^\varphi\} = \emptyset$. Call φ *emptying* if the subspace U is spanned by emptying vertices.

Call a vertex $f \in U$ *dually emptying* for φ if $\bigcap\{H^\varphi \mid f \in H \in \mathcal{H}\} = \emptyset$. Call φ *dually emptying* if the subspace U is spanned by dually emptying vertices.

If a is not emptying, then $\bigcap\{H \mid H \in \mathcal{H},\ a \in H^\varphi\} = \{b\}$ for some vertex b. If b is not dually emptying, then $\bigcap\{H^\varphi \mid b \in H \in \mathcal{H}\} = \{a\}$ for some vertex a. This establishes a 1-1 correspondence between not emptying vertices a and not dually emptying vertices b.

Proposition 7.2 *If a permutation φ of $\mathcal{H}$ is not dually emptying, then it is in* $\mathrm{P\Gamma L}(U)$.

Proof Let E denote the set of emptying vertices of U, and put $A = U \setminus E$. Let F denote the set of dually emptying vertices of U, and put $B = U \setminus F$. Let $\psi\colon B \to A$ be the 1-1 correspondence found above. We show that if L is a line in U with $|L \cap B| \geq q$, then $L \subseteq B$ and L^ψ is a line.

Indeed, let $b, b' \in L \cap B$ and set $M = \langle b^\psi, b'^\psi \rangle$. Then $L \subseteq H$ is equivalent to $M \subseteq H^\varphi$ so that $(L \cap B)^\psi = M \cap A$. If all points of L are in B with a single exception w, then all points of M are in A with a single exception v, and all hyperplanes H with $w \in H$ satisfy $v \in H^\varphi$ (since every line meets every hyperplane), and $v = w^\psi$, that is, w was no exception.

If φ is not dually emptying, then there exists a hyperplane H such that $U \setminus H \subseteq B$. By the above this implies $B = U$ and ψ is in $\mathrm{P\Gamma L}(U)$ and induces φ on the set $\mathcal{H}$. □

7.2.2 Large Cliques

We use the presence of maximal cliques of various sizes to study the structure of the graphs Γ_φ when φ is a permutation.

Abbreviate the size $\frac{q^i-1}{q-1}$ of an i-space with m_i, so that maximal singular subspaces have size m_d. Since m_d is the Delsarte-Hoffman upper bound for the size of cliques in Γ_φ, each vertex outside a clique of this size is adjacent to precisely m_{d-1} vertices inside, cf. [4, Proposition 1.1.7].

Lemma 7.3 *Let $d \geq 3$.*

(i) If $M \neq U$ is a maximal singular subspace of $\mathbf{P}$, then $C = (M \setminus U) \cup \bigcap\{H^\varphi \mid M \cap U \subseteq H \in \mathcal{H}\}$ is a maximal clique in Γ_φ of size at least $q^{d-2}(q+1)$ (and $C \setminus U = M \setminus U$).
(ii) If $C \neq U$ is a maximal clique in Γ_φ of size at least $q^{d-2}(q+1)$, then $M = \langle C \setminus U\rangle$ is a maximal singular subspace of $\mathbf{P}$. If, moreover, $|C| = m_d$, then $M \setminus U = C \setminus U$.

Proof (i) Let M be a maximal singular subspace other than U. Then $C = (M \setminus U) \cup \bigcap\{H^\varphi \mid M \cap U \subseteq H \in \mathcal{H}\}$ is the largest clique in Γ_φ containing $M \setminus U$. (Indeed, the set of hyperplanes of U of the form $m^\perp \cap U$ where $m \in M \setminus U$ equals the set of hyperplanes containing $M \cap U$, so C is a clique. No further point outside $U \cup C$ can be adjacent to all of C, since $|M \setminus U| > m_{d-1}$.) If $\dim M \cap U = d-1$, then $|C| = |M| = m_d$. If $\dim M \cap U \leq d-2$, then $|C| \geq |M \setminus U| \geq m_d - m_{d-2} = q^{d-2}(q+1)$.

(ii) Let $C \neq U$ be a maximal clique of size at least $q^{d-2}(q+1)$. If $|C \setminus U| \leq m_{d-1}$, then $|C \cap U| \geq q^{d-2}(q+1) - m_{d-1} > m_{d-2}$. The set $C \cap U$ is the intersection of sets H^φ, each of size m_{d-1}, and any two distinct such sets meet in m_{d-2} points. It follows that no two different H occur, that is, $H = c^\perp \cap U$ is independent of the choice of $c \in C \setminus U$. Now C is contained in, and hence equals, $H^\varphi \cup (C \setminus U)$, and $|C \setminus U| = m_d - m_{d-1} > m_{d-1}$, a contradiction.

If S is a clique in Γ_0, then also $\langle S\rangle$ is a clique in Γ_0. In particular, $\langle C \setminus U\rangle$ is a singular subspace. It is maximal since $|\langle C \setminus U\rangle| > m_{d-1}$.

If $|C| = m_d$, then each vertex outside C is adjacent to precisely m_{d-1} vertices inside. Hence no point outside $C \cup U$ can be adjacent to all of $C \setminus U$. □

Lemma 7.4 *If the permutation φ is dually emptying, then U is uniquely determined within the graph Γ_φ.*

Proof The subspace U is a clique of size m_d in Γ_φ, with the two properties

(i) in the subgraph induced on its complement $P \setminus U$ all maximal cliques N have size $m_d - m_i$ (where $m_i = |\langle N\rangle \cap U|$) for some i, $0 \leq i \leq d-1$, and
(ii) the number of maximal cliques of size m_d disjoint from U equals the number of maximal singular subspaces disjoint from any given one.

Let $E \neq U$ be a clique of Γ_φ of size m_d with the same two properties. First we use (i) to see that $E \cap U$ must be a hyperplane in U.

Since E is a maximal clique, and φ is a permutation, $E \cap U$ is an intersection of hyperplanes and hence a subspace of U. By hypothesis, we can find a dually emptying point f of U not in E. If $g \in f^\perp \cap (E \setminus U)$ (g will exist unless $f^\perp \cap E = U \cap E$) and M is a maximal singular subspace containing f and g, and meeting U in $\{f\}$, then $C = M \setminus \{f\}$ is a maximal clique in Γ_φ of size $m_d - 1$. And $N = C \setminus E$ is a maximal clique in $P \setminus E$ of size $m_d - m_i - 1$ in case $|M \cap E| = m_i$. (Note that $C \setminus U = M \setminus U$.)

Why is N maximal? No point can be added since $|N| > m_{d-1}$, unless $q = 2$ and $|N| = |M \cap E| = m_{d-1}$. In that case, no point outside U can be added since $\langle N \rangle = M$. And no point inside U can be added since N determines all hyperplanes on f, and f is dually emptying.

Since $M \cap E \neq \emptyset$, we have $1 \leq i \leq d-1$, and $m_d - m_i - 1$ is not of the form $m_d - m_h$, violating (i). Therefore, $f^{\perp} \cap E = U \cap E$, so that $H = \langle E \setminus U \rangle \cap U$ and $H^{\varphi} = E \cap U$ are hyperplanes.

Now we use (ii) to arrive at a contradiction.

We claim that if a maximal clique F of size m_d is disjoint from E, then $\langle F \setminus U \rangle$ is disjoint from $\langle E \setminus U \rangle$. Suppose not. Since $\langle E \setminus U \rangle \setminus U = E \setminus U$ and $\langle F \setminus U \rangle \setminus U = F \setminus U$ by Lemma 7.3(ii), a common vertex must lie in U. If $\langle F \setminus U \rangle$ meets U in m_e vertices with $e \geq 2$, then F meets U in a subspace of dimension e, but that would meet H^{φ}, impossible. So, $\langle F \setminus U \rangle$ meets U in a singleton $\{f\}$ on the hyperplane H. As F has size m_d, f is not dually emptying, so $\bigcap\{H^{\varphi} \mid f \in H\} = \{f'\}$ for some point f'. Now $f' \in E \cap F$, a contradiction. This shows our claim.

By the claim and Lemma 7.3, we have an injection from the set of maximal cliques of size m_d disjoint from E into the set of maximal singular subspaces disjoint from $\langle E \setminus U \rangle$. Since E satisfies (ii), both sets have the same size, so the injection is also a surjection.

On the other hand, since φ is dually emptying, there is a dually emptying point o in $U \setminus H$. This o lies in a maximal singular subspace O disjoint from $\langle E \setminus U \rangle$, and this O is not in the image of the surjection. Contradiction. □

Lemma 7.5 *Let* $\mathbf{P}$ *be a nondegenerate polar space with point set* P*, and* U *a maximal totally isotropic subspace. Let* $h\colon P \setminus U \to P \setminus U$ *be a bijection preserving collinearity. Then* h *can be uniquely extended to an automorphism* h' *of* $\mathbf{P}$.

Proof Indeed, we can extend h as follows. For $u \in U$, let R be a maximal t.i. subspace with $U \cap R = \{u\}$. Then $R \setminus \{u\}$ is a subspace of $\mathbf{L}$ of size $|U| - 1$ and is mapped by h to a similar subspace S. In $\mathbf{P}$ this subspace is contained in a unique maximal t.i. subspace $\langle S \rangle$ $(= S^{\perp})$ and we can define $h'(u) = v$ when $\langle S \rangle \setminus S = \{v\}$.

This is well-defined: if R' is a maximal t.i. subspace with $U \cap R' = \{u\}$ and R, R' meet in codimension 1, and h maps $R' \setminus \{u\}$ to S', then $\langle S \cap S' \rangle = (S \cap S') \cup \{v\}$. Since the graph on such subspaces R, adjacent when they meet in codimension 1, is connected, v is well-defined.

This preserves orthogonality: if $u \in x^{\perp}$, then there is a maximal t.i. subspace R containing u, x with $R \cap U = \{u\}$. Now $h(u) = v$ lies in the t.i. subspace $\langle h(R \setminus \{u\}) \rangle$ which also contains $h(x)$. □

Proposition 7.6 *Let* $\mathbf{P}$ *be a nondegenerate polar space and* U *a maximal t.i. subspace. Let* φ *and* χ *be permutations of* $\mathcal{H}$ *such that* Γ_{φ} *is isomorphic to* Γ_{χ}*. Then* φ *and* χ *are in the same* $\mathrm{P\Gamma L}(U)$*-double coset in Sym*$(\mathcal{H})$.

Proof If $\varphi \in \mathrm{P\Gamma L}(U)$, then Γ_{φ} is isomorphic to Γ_0 and its group of automorphisms is transitive on the set of maximal singular subspaces. If $\varphi \notin \mathrm{P\Gamma L}(U)$, then according to Lemma 7.4 and Proposition 7.2 the maximal singular subspace U can be recognized in Γ_{φ}, and hence Γ_{φ} is not isomorphic to Γ_0. Since by assumption Γ_{φ} and Γ_{χ} are isomorphic, either both or neither are isomorphic to Γ_0. In the former case both φ and χ are in $\mathrm{P\Gamma L}(U)$ and the claim holds. Assume in the following that φ and χ are not in $\mathrm{P\Gamma L}(U)$.

We have the set P, the point set of $\mathbf{P}$, with three structures defined on it. The polar space structure $\mathbf{P}$, with relation $\perp$, and the two graph structures Γ_φ and Γ_χ. We translate what it means for Γ_φ and Γ_χ to be isomorphic in terms of the polar space.

Let $g : \Gamma_\varphi \to \Gamma_\chi$ be an isomorphism. By Lemma 7.4, it sends U to itself.

The number of common neighbors of a triple of points in U equals $\lambda - 1$ for collinear triples and is smaller for noncollinear triples. It follows that g preserves projective lines in U, and hence induces a permutation $\bar{g}$ of $\mathcal{H}$ that is in $\mathrm{P\Gamma L}(U)$.

Let h denote the restriction of g to $P \setminus U$. Then h preserves collinearity (since we have $\{x, y, z\}^\perp \cap (P \setminus U) = \{x, y\}^\perp \cap (P \setminus U)$ for a triple of pairwise adjacent points x, y, z of $P \setminus U$ if and only if x, y, z are collinear). By Lemma 7.5, h can be uniquely extended to an automorphism h' of $\mathbf{P}$.

Let $\bar{h}$ be the permutation of $\mathcal{H}$ induced by h'. Then $\bar{h} \in \mathrm{P\Gamma L}(U)$.

For $x \in U$ and $y \notin U$, if x and y are adjacent in Γ_φ, then x^g and y^g are adjacent in Γ_χ. This says that $x \in (y^\perp \cap U)^\varphi$ implies that $x^g \in (y^{g\perp} \cap U)^\chi$: g maps the points of any hyperplane of U to the points of another hyperplane. Then $(y^\perp \cap U)^{\varphi g} = (y^{g\perp} \cap U)^\chi = (y^{h\perp} \cap U)^\chi = (y^\perp \cap U)^{\bar{h}\chi}$, so that $\varphi\bar{g} = \bar{h}\chi$. □

Theorem 7.7 *Let $d \geq 3$. There are at least $q^{d-2}!$ pairwise nonisomorphic strongly regular graphs having the same parameters as the collinearity graph Γ_0 of the polar space $\mathbf{P}$.*

Proof Let $q = p^e$, where p is prime. Then $|\mathrm{P\Gamma L}(U)| < eq^{d^2}$. In view of Proposition 7.6, we have obtained at least $m_d!/|\mathrm{P\Gamma L}(U)|^2 > q^{d-2}!$ pairwise nonisomorphic strongly regular graphs unless $(d, q) = (3, 2)$. For $(d, q) = (3, 2)$, we have four $\mathrm{P\Gamma L}(U)$-double cosets in $\mathrm{Sym}(\mathcal{H})$. □

Similar estimates would follow if one generalized Lemma 7.4 to show that U is uniquely determined in $\mathbf{P}$ for arbitrary designs $\mathbf{D}$ (that is, for φ that are not permutations). The blocks of $\mathbf{D}$ are then found as $\{\Gamma_\varphi(x) \cap U \mid x \in P \setminus U\}$. In [24, Corollary 3.2] it is shown that for $d \geq 4$ there are at least $q^{d-2}!$ choices for $\mathbf{D}$. Hence, one would obtain the same estimate as in Theorem 7.7 for $d \geq 4$.

7.3 Switched Symplectic Graphs with 4-Vertex Condition

We show that in the symplectic case the graphs Γ_φ satisfy the 4-vertex condition. Let $\mathbf{P}$ be $\mathrm{Sp}_{2d}(q)$, and let V be a $2d$-dimensional vector space over $\mathbb{F}_q$, provided with a nondegenerate symplectic form.

The parameters of Γ_0 are $v = (q^{2d} - 1)/(q - 1)$, $k = q(q^{2d-2} - 1)/(q - 1)$, $v - k - 1 = q^{2d-1}$, $\lambda = q^2(q^{2d-4} - 1)/(q - 1) + q - 1$, $\mu = (q^{2d-2} - 1)/(q - 1)$ and $\binom{\lambda}{2} - \alpha = \frac{1}{2}q^{2d-1}(q^{2d-4} - 1)/(q - 1)$, $\beta = \frac{1}{2}q(q^{2d-2} - 1)(q^{2d-4} - 1)/(q - 1)^2$, and those of Γ_φ will turn out to be the same.

Proposition 7.8 *The graph Γ_φ satisfies the 4-vertex condition.*

Proof Let x, y be two vertices of Γ_φ. We show that the number of edges in $\Gamma_\varphi(x) \cap \Gamma_\varphi(y)$ is independent of φ, and only depends on whether x, y are adjacent or nonadjacent. Since Γ_0 satisfies the 4-vertex condition, Γ_φ does too.

Count edges ab in $\Gamma_\varphi(x) \cap \Gamma_\varphi(y)$. The vertices x, y, a, b are pairwise adjacent, except that x and y need not be adjacent. We distinguish several cases depending on which of x, y, a, b are in U. Each of the separate counts will be independent of φ. If $x \notin U$ then let $X = x^\perp \cap U$. If $y \notin U$ then let $Y = y^\perp \cap U$.

Case $x, y, a, b \notin U$. In this case adjacencies and counts do not involve φ.

Case $a, b \in U$. Here a, b must be chosen distinct from x, y in case $x, y \in U$, or distinct from x and in Y^φ in case $x \in U$, $y \notin U$ (and the count depends on whether $x \sim y$), or in $X^\varphi \cap Y^\varphi$ in case $x, y \notin U$ (and the count depends on whether $X = Y$). In all cases the count is independent of φ.

Case $x, y, a \in U$, $b \notin U$. For each hyperplane H such that $x, y \in H^\varphi$ we count the $b \in H^\perp \setminus U$ and the $a \in H^\varphi$ distinct from x, y.

Case $x, y \in U$, $a, b \notin U$. For any two hyperplanes H, H' of U with $x, y \in H^\varphi \cap H'^\varphi$ count adjacent a, b with $a \in H^\perp \setminus U$ and $b \in H'^\perp \setminus U$. (The counts will depend on whether $H = H'$, but not on φ.)

Case $x, a \in U$, $y, b \notin U$. For each hyperplane H with $x \in H^\varphi$, count the $a \in H^\varphi \cap Y^\varphi$ distinct from x, and $b \in H^\perp \setminus U$ adjacent to y. (Here $H = Y$ occurs when $x \sim y$. The counts for $H \neq Y$ do not depend on H.)

Case $x \in U$, $y, a, b \notin U$. For any two hyperplanes H, H' with $x \in H^\varphi \cap H'^\varphi$, count edges ab with $a \in H^\perp$ and $b \in H'^\perp$ in $y^\perp \setminus (U \cup \{y\})$. (Here $H = Y$ or $H' = Y$ occur when $x \sim y$. The counts for $H, H' \neq Y$ do not depend on the hyperplanes chosen but only on whether $H = Y$ or $H' = Y$ or $H = H'$.)

Finally the least trivial case.

Case $a \in U$, $x, y, b \notin U$. Count a, H, b with $a \in X^\varphi \cap Y^\varphi$ and H a hyperplane of U on a and $b \in \langle x, y, H\rangle^\perp \setminus (U \cup \{x, y\})$. The count for a depends on whether $X = Y$, that for b depends on whether $H = X$ or $H = Y$ or $H \supseteq X \cap Y$, but does not otherwise depend on the choice of H.

Since all counts were independent of φ, this proves our proposition. □

By Theorem 7.7, this shows that there are many strongly regular graphs which satisfy the 4-vertex condition. But we still have to show the simplified version of this statement given in the introduction as Theorem 1.1.

Proof of Theorem 1.1. Note that here v refers to a nonnegative integer as in Theorem 1.1 and no longer is the number of vertices in Γ_φ.

Apply Theorem 7.7 for $d = 3$ to find at least $q!$ strongly regular graphs satisfying the 4-vertex condition on $\tilde{v}$ vertices, for $\tilde{v} = \frac{q^6-1}{q-1}$. Given v, there is a prime q between $v^{1/6}$ and $2v^{1/6}$ by Bertrand's postulate. Now $\tilde{v} < 2q^5 < 64v^{5/6} < v$ for $v > 2^{36}$. Checking the prime powers q for $7 \leq q \leq 64$ one sees that there is a q with $\tilde{v} \leq v \leq q^6$ for $v \geq 19608$. One easily verifies the assertion for $v < 19608$ using rank 3 graphs. □

Further graphs with the same parameters satisfy the 4-vertex condition. Additional examples can be obtained by repeated WQH-switching, see §7.4 and [20], and there are more examples among the graphs constructed in [18]. We have not tried (much) to determine precisely which graphs in [18] do satisfy the 4-vertex condition. Similarly, we do not know when WQH-switching preserves the 4-vertex condition.

7.4 Small Examples

7.4.1 Examples on 63 Vertices

In [19] a large number of strongly regular graphs are found by applying GM-switching to the $Sp_6(2)$ polar graph. Among these are 280 non-rank-3 strongly regular graphs with $(v, k, \lambda, \mu) = (63, 30, 13, 15)$ satisfying the 4-vertex condition. All have $\alpha = 30$ and $\beta = 45$. Three of these are among the Γ_φ constructed above.

We list for each occurring group size the number of examples found.

$\|G\|$	4	8	16	32	48	64	96	128	192	256	384	512	768	1344	1536	4608
#	3	16	76	62	1	60	2	30	5	12	3	3	2	1	3	1

None of these examples has a transitive group. We list the orbit lengths in the seven cases with fewer than six orbits.

$\|G\|$	768	768	1344	1536	1536 (twice)	4608
orbits	3+12+48	1+6+24+32	7+56	1+6+24+32	3+4+8+48	3+12+48

7.4.2 Permutations of Hyperplanes

Let **P** be $Sp_{2d}(q)$, and let φ be a permutation of the set $\mathcal{H}$ of hyperplanes of U. For $(d, q) = (3, 2), (3, 3), (4, 2)$, the number of double cosets of $P\Gamma L(d, q)$ in $\mathrm{Sym}(\mathcal{H})$ is 4, 252, and 3374, respectively, and these are the numbers of non-isomorphic graphs Γ_φ. In each case, exactly one has rank 3. None of the others has a transitive group (since U can be recognized). The pointwise stabiliser of U in $\mathrm{Aut}(\Gamma_0)$ has size $N = q^{\binom{d+1}{2}}(q-1)$ and is always contained in $\mathrm{Aut}(\Gamma_\varphi)$. Hence, N divides $|\mathrm{Aut}(\Gamma_\varphi)|$.

Case $(d, q) = (3, 3)$. Here $N = 1458$. We list the group sizes for the 251 graphs Γ_φ other than Γ_0.

$\|G\|/N$	1	2	3	4	6	8	12	16	18	24	39	54	72	144
#	172	26	29	6	3	2	2	2	1	1	3	1	2	1

We list the orbit lengths in the five cases with fewer than six orbits.

$\|G\|/N$	39 (thrice)	72	144
orbits	13+351	1+12+108+243	1+12+108+243

Case $(d, q) = (4, 2)$. Here $N = 1024$. We list the group sizes for the 3373 graphs Γ_φ other than Γ_0.

$\lvert G\rvert/N$	1	2	3	4	5	6	7	8	12	16
#	3148	85	40	24	4	10	6	26	1	4
$\lvert G\rvert/N$	18	21	24	32	56	60	96	192	288	1344
#	1	2	11	2	2	1	2	2	1	1

We list the orbit lengths in the eight cases with fewer than six orbits.

$\lvert G\rvert/N$	12	18	24	56 (twice)
orbits	3+12+48+192	6+9+96+144	3+12+48+192	1+14+112+128

$\lvert G\rvert/N$	60	288	1344
orbits	15+240	3+12+48+192	7+8+16+224

7.4.3 Other Polar Spaces

We made the same exhaustive investigation of all permutations φ for the other choices of **P** in the cases $(d, q) \in \{(3, 2), (3, 3), (4, 2)\}$. The only non-rank-3 examples satisfying the 4-vertex condition occur for $O_7(3)$. Here we obtain 252 graphs in total, of which one is rank 3, and three more satisfy the 4-vertex condition. They all have two orbits (of sizes 13+351) and an automorphism group of size 56862. All other graphs Γ_φ obtained from $O_7(3)$ have more than two orbits.

One might wonder whether a graph Γ_φ from $O_{2d+1}(q)$ satisfies the 4-vertex condition if and only if it has at most two orbits. And whether a non-rank-3 graph Γ_φ can only satisfy the 4-vertex condition if **P** is $Sp_{2d}(q)$ or $O_{2d+1}(q)$.

7.4.4 Other Designs

There are four 2-(15, 7, 3) designs **D** other than that of the hyperplanes of PG(3, 2). We investigated the case where $(d, q) = (4, 2)$ and **P** is $Sp_2(8)$, so that the resulting examples satisfy the 4-vertex condition. We generated several hundred thousand graphs Γ_φ for each of these designs. None of these graphs occurs for two different designs. We believe our enumeration to be complete.

$\lvert \mathrm{Aut}(\mathbf{D})\rvert$	Point orbits	Block orbits	# Γ_φ
576	3+12	3+12	113519
168	7+8	1+14	340730
168	1+14	7+8	328078
96	1+6+8	1+6+8	677460

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Appendix: Details on Ivanov's Graphs

In Sect. 3.3 we discussed the graphs $\Gamma^{(m)}$ from [6] and $\Sigma^{(m)}$ from [22]. Here we give some more detail on the latter.

For $m \geq 2$, consider $V = \mathbb{F}_2^{2m}$ provided with the elliptic quadratic form $q(x) = x_1^2 + x_2^2 + x_1x_2 + x_3x_4 + \ldots + x_{2m-1}x_{2m}$. Identify the set of projective points (1-spaces) in V with $V^* = V \setminus \{0\}$. Let $Q = \{x \in V^* \mid q(x) = 0\}$ and let S be the maximal t.s. subspace given by $S = \{x \in V^* \mid x_1 = x_2 = 0 \text{ and } x_{2i-1} = 0\ (2 \leq i \leq m)\}$. Then $S^\perp = \{x \in V^* \mid x_{2i-1} = 0\ (2 \leq i \leq m)\}$. The graph $\Sigma^{(m)}$ has V as vertex set, where two distinct vertices v, w are adjacent when $v - w \in (Q \cup S^\perp) \setminus S$. Let $\mathrm{T}^{(m)}$ and $\Upsilon^{(m)}$ be the induced subgraphs on the neighbors (nonneighbors) of the vertex 0. Put $R = V^* \setminus (Q \cup S^\perp)$.

Proposition *(i) For $m \leq 4$, the graphs $\Sigma^{(m)}$ are rank 3, and are isomorphic to the complement of $VO_{2m}^-(2)$.*

(ii) For $m \geq 5$, the automorphism group of $\mathrm{T}^{(m)}$ has two vertex orbits $S^\perp \setminus S$ and $Q \setminus S$, of sizes $3 \cdot 2^{m-1}$ and $2^{2m-1} - 2^m$, respectively. For $2 \leq m \leq 4$, the group is rank 3, and the graph is the complement of $NO_{2m}^-(2)$.

(iii) For $m \geq 5$, the automorphism group of $\Upsilon^{(m)}$ has two vertex orbits S and R of sizes $2^{m-1} - 1$ and $2^{2m-1} - 2^m$, respectively. For $3 \leq m \leq 4$, the group is rank 3, and the graph is the complement of $O_{2m}^-(2)$.

(iv) The λ- and μ-graphs in $\Upsilon^{(m)}$ and the μ-graphs in $\mathrm{T}^{(m)}$ are all regular of valency $2^{m-2}(2^{m-2}+1)$. In particular, $\Upsilon^{(m)}$ satisfies the 4-vertex condition.

(v) The λ-graphs in $\mathrm{T}^{(m)}$ have vertices of valencies in $0, 2^{2m-4} - 2^m, 2^{2m-4}, 2^{2m-3} - 2^m$. Edges not in a line contained in Q have λ-graphs with a single isolated vertex and $\lambda - 1$ vertices of valency 2^{2m-4}. For edges in a line contained in Q the λ-graphs have a single vertex with valency $2^{2m-3} - 2^m$, and $2^{m-3} - 1$ vertices with valency $2^{2m-4} - 2^m$, and the remaining $2^{2m-3} + 2^{m-3}$ vertices have valency 2^{2m-4}. In particular, $\mathrm{T}^{(m)}$ satisfies the 4-vertex condition, with $\alpha = 2^{2m-5}(2^{2m-3} + 2^{m-2} - 1)$ and $\beta = \frac{1}{2}\mu\mu' = 2^{2m-4}(2^{m-2}+1)^2$.

(vi) The local graph of $\Upsilon^{(m)}$ at a vertex $s \in S$ is isomorphic to $\Sigma^{(m-1)}$.

Proof (i)–(iii) This is clear, and can also be found in [22]. (iv)–(v) (the part about $\mathrm{T}^{(m)}$):

Let $(v, w) = q(v+w) - q(v) - q(w)$ be the symmetric bilinear form belonging to q. Let $X = (Q \cup S^\perp) \setminus S$. Then $\mathrm{T}^{(m)}$ is the graph with vertex set X, where two vertices x, y are adjacent when the projective line $\{x, y, x+y\}$ they span is contained in X. If at least one of x, y is in $S^\perp \setminus S$, then this is equivalent to $(x, y) = 1$. If both are in $Q \setminus S$, then this is equivalent to $((x, y) = 0 \text{ and } x + y \notin S)$ or $((x, y) = 1 \text{ and } x + y \in S^\perp \setminus S)$.

Let x, y, z be pairwise adjacent vertices. The valency c of z in the λ-graph $\lambda(x, y)$ is the number of common neighbors of x, y, z. Distinguish several cases.

If $z = x + y$, then if $x, y, z \in Q$ we find $c = |\{x, y\}^\perp \cap (Q \setminus S)| - 3 = 2^{2m-3} - 2^m$. If $z = x + y$ and at least one of x, y, z lies in $S^\perp$, then $c = 0$.

Now let $z \neq x + y$. The claims are true for $m \leq 4$. Let $m \geq 5$ and use induction on m. Choose coordinates so that x, y, z have final coordinates 00 and let x', y', z' be these points without the final two coordinates. If they have c' common neighbors w' in $\mathrm{T}^{(m-1)}$, then we find $2c'$ common neighbors $w = (w', 0, *)$. Moreover (since x, y, z are linearly independent), we find 2^{2m-5} common neighbors $(w', 1, q'(w'))$ in Q, where w' runs through all vectors with the desired inner products with x', y', z'. Altogether $c = 2c' + 2^{2m-5}$, as claimed.

For the μ-graphs the argument is similar and simpler: by the definition of adjacency three dependent vertices are pairwise adjacent, so that the case $z = x + y$ does not occur here.

(iv) (the part about $\Upsilon^{(m)}$): Let $Y = V^* \setminus X$. Then $\Upsilon^{(m)}$ is the graph with vertex set Y, where two vertices x, y are adjacent when the projective line $\{x, y, x + y\}$ they span is not contained in Y. The same argument as before yields the valencies of the λ- and μ-graphs.

(vi) Consider the graph $\Sigma^{(m)}$. The nonneighbors z of 0 that are neighbors of s are the vertices of the form $z = s + b$ with $z \in S \cup R$ and $b \in (Q \cup S^\perp) \setminus S$. It follows that $s + z \in Q \setminus s^\perp$. Let $s = (0 \dots 01)$, then $Q \setminus s^\perp$ can be identified with $W = \mathbb{F}_2^{2m-2}$ via $w \to i(w) = (w, 1, \bar{q}(w))$ for $w \in \mathbb{F}_2^{2m-2}$ and $\bar{q}(w)$ determined by $q(i(w)) = 0$. The local graph of Υ at s can be identified with the graph with vertices w, where w, w' are adjacent when the line joining $i(w), i(w')$ has third point $(w+w', 0, *) \in (Q \cup S^\perp) \setminus S$, that is, the line joining w, w' has third point $w'' = w + w'$ satisfying $w'' \notin T$ and ($\bar{q}(w'') = 0$ or $w'' \in T^\perp$) where $T = \{w \in W \mid w_1 = w_2 = w_3 = w_5 = \dots = w_{2m-3} = 0\}$. But this is $\Sigma^{(m-1)}$. □

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