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Finite Groups with a Split BN-pair of Rank 1. I*

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1. INTRODUCTION

A group is said to have a split BN-pair of rank 1 if it has a homomorphic image G having a (faithful) 2-transitive permutation representation on a set Ω such that, for $\alpha \in \Omega$, G_α has a normal subgroup Q regular on $\Omega - \alpha$. That is, Q is transitive on $\Omega - \alpha$, and no nontrivial element of Q fixes a point of $\Omega - \alpha$.

THEOREM 1.1. *Let G be a finite group 2-transitive on a set Ω. Suppose that, for $\alpha \in \Omega$, G_α has a normal subgroup Q regular on $\Omega - \alpha$. Then G has a normal subgroup M such that $M \leqslant G \leqslant$ Aut M and M acts on Ω as one of the following groups in its usual 2-transitive representation: a sharply 2-transitive group,* PSL(2, q), Sz(q), PSU(3, q), *or a group of Ree type.*

For $|\Omega|$ odd, this result has been proved by Shult [31]. The purpose of this paper is to prove Theorem 1.1 when $|\Omega|$ is even.

We remark that the groups listed in Theorem 1.1 all satisfy the hypotheses of the theorem. Also, sharply 2-transitive groups have been completely classified by Zassenhaus [44].

This theorem is one of a number of results of a similar nature. Zassenhaus groups are easily seen to satisfy the hypotheses of the theorem. The classification of Zassenhaus groups, due to Zassenhaus [43], Feit [10], Ito [20] and Suzuki [33], is implicitly required in the proof. Suzuki [34–36] has considered

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other special cases of Theorem 1.1. Further special cases are found in [15, 21, 23, 24 and 26]. We also note that recent results of Shult [30] and Kantor, O'Nan and Seitz [22] are similar to Theorem 1.1, and led to it.

The theorem can be viewed in a different manner. Tits [38] has classified all finite groups having a faithful irreducible BN-pair of rank ≥ 3. Theorem 1.1 extends this classification to finite groups having a split BN-pair of rank 1. Very recently, P. Fong and G. Seitz have used Theorem 1.1 in order to study finite groups having a BN-pair of rank 2.

We now indicate the approach used in the proof of Theorem 1.1 when $|\Omega|$ is even. The basic idea is to use induction in order to obtain the structure of the 2-Sylow subgroups of G. Once this has been accomplished, results of Alperin, Brauer and Gorenstein [1, 2] and Walter [39] can be applied.

The study of the 2-Sylow subgroups of G is based primarily on a study of the fusion of 2-elements of G. Standard fusion and transfer arguments are applied repeatedly. A useful tool is the fact that $G_{\alpha\beta}$ controls the fusion of those of its subsets which fix at least 3 points.

Another basic tool is the Brauer–Wielandt Theorem [41], which is applied to Klein groups in $G_{\alpha\beta}$ acting on Q. The structure of Q is studied only when it is clear that either Q is a p-group or some element of $G_{\alpha\beta}$ of prime order is fixed-point-free on Q; the Feit–Thompson Theorem [11] is never applied to Q. We also do not use Suzuki's method of generators and relations [33, 34, 35].

The structure of the paper is as follows. Sections 2 and 3 contain background material. In Section 4 we begin the proof of Theorem 1.1 by taking a counterexample of minimal order. Then $|\Omega|$ is even by Shult's result [31]. This section contains the fusion result mentioned above, together with an inductive lemma to be used throughout the proof.

By a result of Bender [4], we may assume that $G_{\alpha\beta}$ has even order. Let t be an involution in $G_{\alpha\beta}$. The action of $C(t)$ on the fixed points of t might be solvable, of unitary or Ree type, or contain PSL(2, q) in its usual representation. These possibilities are further divided as follows: the action is solvable of degree ≥ 4 (Section 5); t fixes just 2 points (Section 6); the action is of unitary or Ree type (Section 7); the action contains PSL(2, q) (Sections 8, 9). In the latter case, Section 9 considers the possibility that $C(t)$ has SL(2, q) as a normal subgroup. In Section 8, it is assumed that, for *any* involution t in $G_{\alpha\beta}$, the action of $C(t)$ on the fixed points of t contains PSL(2, q) for some odd prime power q depending on t, and that in each case $C(t)$ has PSL(2, q) as a normal subgroup. Within this framework, there are also a large number of subcases which must be considered.

Notation. Most of our notation is standard. All groups will be finite. If G is a group, $G^{\#} = G - \{1\}$, $G^{(1)}$ is the derived group of G, $\Phi(G)$ is the

Frattini subgroup, $O(G)$ is the largest normal subgroup of odd order, and Aut G is the automorphism group of G. If G is a p-group, $\Omega_1(G) = \langle x \in G \mid x^p = 1 \rangle$.

If $x \in G$ and $Y \subseteq G$ then $x^Y = \{x^y \mid y \in Y\}$. If $x, y \in G$ we write $x \sim y$, or $x \sim y$ in G, when x and y are conjugate in G, and we write $x \not\sim y$ otherwise.

If $X \subseteq H < G$, then X is weakly closed in H (with respect to G) if $g \in G$ and $X^g \subseteq H$ imply that $X = X^g$.

If p is a prime and m a positive integer, m_p will denote the p-part of m.

We use Wielandt's notation for permutation groups [42]. If G is a permutation group on Ω and $\alpha \in \Omega$, G_α is the stabilizer of α. If $\alpha \neq \beta \in \Omega$, then $G_{\alpha\beta}$ is the stabilizer of α and β, while $G_{\{\alpha,\beta\}}$ is the setwise stabilizer of $\{\alpha, \beta\}$. If $X \subseteq G$, $\Delta \subseteq \Omega$ and $\Delta^X = \Delta$, then X^Δ denotes the set of permutations induced by X on Δ. Our notation for the pointwise stabilizer of a subset of Ω will, however, differ from that of Wielandt (see Section 4). G is said to be semiregular on Ω if only $1 \in G$ fixes a point of Ω. G is regular on Ω if it is transitive and semiregular on Ω. We shall abuse this terminology slightly: if $t \in G$ is an involution, then t will be called a regular involution if $\langle t \rangle$ is semiregular on Ω.

We shall employ a useful but unusual convention concerning equality of certain types of groups. The following are typical examples. Let t be an involution in a permutation group G, Δ its set of fixed points, and $C_0(t)$ a subgroup of $C(t)$. Then, we write $C_0(t) = \mathrm{SL}(2, q)$ to mean that $C_0(t) \approx \mathrm{SL}(2, q)$ and that $C_0(t)^\Delta$ acts on Δ as $\mathrm{PSL}(2, q)$ in its usual 2-transitive permutation representation. Similarly, we write $C_0(t)^\Delta = \mathrm{PSU}(3, q)$ to mean that $C_0(t)^\Delta$ acts on Δ as $\mathrm{PSU}(3, q)$ in its usual 2-transitive permutation representation.

2. Background Lemmas

The Brauer–Wielandt Theorem is basic to our approach:

LEMMA 2.1 (Wielandt [41]). *Let $\langle t, u \rangle$ be a Klein group acting on a group X of odd order. Then*

(i) $X = C_X(t)C_X(u)C_X(tu)$; *and*

(ii) $|C_X(t)|\,|C_X(u)|\,|C_X(tu)| = |X|\,|C_X(\langle t, u\rangle)|^2$.

LEMMA 2.2. *Let S be a 2-Sylow subgroup of a group G. Suppose that $S_0 \lhd S$, where S/S_0 is abelian, and let $x \in S - S_0$. Assume that, for each $g \in G$ and each integer m, if $(x^m)^g \in S$ then $(x^m)^g \equiv x^m (\mathrm{mod}\ S_0)$. Then G has a normal subgroup G_0 such that $x \in G - G_0$ and G/G_0 is a 2-group.*

Proof. Compute the image of x under the transfer map $G \to S/S_0$.

LEMMA 2.3. *Let S be a 2-Sylow subgroup of a group G and let $S_0 \lhd S$ with S/S_0 cyclic. Suppose that x is an involution in $S - S_0$ conjugate to no* *element of* S_0. *Then* G *has a normal subgroup* G_0 *such that* $x \in G - G_0$ *and* G/G_0 *is a 2-group.*

This is clear from Lemma 2.2. Lemma 2.3 is essentially Thompson's transfer lemma.

LEMMA 2.4 (Burnside [8, p. 155]; [14, p. 203]). *If* S *is a 2-Sylow subgroup of a group* G, *then* $N(S)$ *controls fusion in* $Z(S)$.

LEMMA 2.5 (Burnside [8, p. 156]; [14, p. 46]). *If* S *is a 2-Sylow subgroup of a group* G, t *is an involution in* $Z(S)$, *and* $t \sim t_1 \in S - \langle t \rangle$, *then there is an elementary abelian subgroup* X *of* S *such that* $t \in X$ *and* $N(X)$ *has an element of odd order moving* t.

LEMMA 2.6. *Let* G *be 2-transitive on a set* Ω, *and let* $\alpha, \beta \in \Omega$, $\alpha \neq \beta$. *Suppose that* t *is an involution central in a 2-Sylow subgroup of* $G_{\alpha\beta}$ *and such that* $C(t)$ *is 2-transitive on the fixed points of* t. *If* S *is a 2-Sylow subgroup of* $C(t)$ *such that* $S_{\{\alpha,\beta\}}$ *is a 2-Sylow subgroup of* $C(t)_{\{\alpha,\beta\}}$, *then* S *contains a conjugate* $t' = (\alpha\beta) \cdots$ *of* t.

Proof. As $C(t)$ has an element interchanging α and β, $S_{\{\alpha,\beta\}}$ is a 2-Sylow subgroup of $G_{\{\alpha,\beta\}}$. Since G contains a conjugate $(\alpha\beta)$... of t, the lemma follows.

LEMMA 2.7 ([22, Lemma 3.4]). *Let* X *be a 2-group and* $Y \lhd X$, *where* $|X/Y| = k \geqslant 4$. *Let* A *be a subgroup of* Aut X *of odd order centralizing* Y *and transitive on* $(X/Y)^{\#}$. *Then either*

(i) *There is a unique* A-*invariant subgroup* X_1 *of* X *such that* $X = X_1 \times Y$; *or*

(ii) $k = 4$ *and there is a unique* A-*invariant subgroup* X_1 *of* X *such that* X_1 *is quaternion of order* 8, $X = X_1 Y$, $|X_1 \cap Y| = 2$ *and* $[X_1, Y] = 1$.

3. PSL(2, q), PSU(3, q), AND GROUPS OF REE TYPE

In this section we have compiled the properties of the groups of even degree characterized by Theorem 1.1 which will be required later.

LEMMA 3.1. *Set* $G = \mathrm{PSL}(2, q)$, *where* q *is odd. Let* $\bar{G}$ *be* $\mathrm{P\Gamma L}(2, q)$ *in its usual 2-transitive representation of degree* $q + 1$ *on a set* Ω.

(i) $\bar{G} = \mathrm{Aut}\, G$.

(ii) $\bar{G}/G$ *has an abelian 2-Sylow subgroup.*

(iii) *If q is not a square then* $|\bar{G}/G|_2 = 2$.

(iv) *If q is a square and $\alpha, \beta \in \Omega$, $\alpha \neq \beta$, then a 2-Sylow subgroup of $\bar{G}_{\alpha\beta}$ is metacyclic.*

(v) *Each involution in $\bar{G}$ — PGL(2, q) fixes $\sqrt{q} + 1$ points of Ω.*

(vi) *If q is a square then G is a subgroup of index 2 in precisely 3 subgroups of $\bar{G}$* : PGL(2, q), *$G\langle a\rangle$ with a an involution in $\bar{G}$* — PGL(2, q), *and* P$\hat{\text{G}}$L(2, q), *which acts on Ω as a Zassenhaus group.*

(vii) *If $q > 3$, the covering group of* PSL(2, q) *is* SL(2, q), unless $q = 9$. *The Schur multiplier of* PSL(2, 9) *has order* 6.

Proof. It is easy to check (i)–(iv). For (v) and (vi), see Fong and Wong [12, Section 1]. For (vii), see Schur [29].

LEMMA 3.2. *Let G be* PSU(3, q) *in its usual 2-transitive representation of degree $q^3 + 1$ on a set Ω, where q is odd. Let $\alpha, \beta \in \Omega$, $\alpha \neq \beta$.*

(i) *G_α has a normal subgroup Q of order q^3 regular on $\Omega - \alpha$.*

(ii) *$Z(Q) = \Phi(Q) = Q^{(1)}$ is elementary abelian of order q; $G_{\alpha\beta}$ is irreducible on $Q/Z(Q)$.*

(iii) *G has a single class of involutions.*

(iv) *If t is an involution in $G_{\alpha\beta}$, then $C_Q(t) = Z(Q)$ and $C(t) \triangleright C_0(t) =$* SL(2, q), *where $C(t)/C_0(t)$ is cyclic.*

(v) *A 2-Sylow subgroup S of G is quasidihedral if $q \equiv 1 (\text{mod } 4)$ and wreathed $Z_{2^a} \wr Z_2$ if $q \equiv 3 (\text{mod } 4)$.*

(vi) *Set $\bar{G}$ =* Aut G. *Then $\bar{G}$ is a permutation group on Ω.*

(vii) *$\bar{G} - G$ contains a single class of involutions of $\bar{G}$, each of which fixes $q + 1$ points.*

(viii) *If $a \in \bar{G}_{\alpha\beta} - G_{\alpha\beta}$ is an involution, then $C_G(a) =$* PGL(2, q) *and $C_Q(a) \cap Z(Q) = 1$.*

(ix) *A central extension of G by a 2-group splits.*

Proof. (i)–(vi) These are easy to verify.

(vii)–(viii) $\bar{G}/G$ has a cyclic 2-Sylow subgroup (Steinberg [32]). Let $a \in \bar{G}_{\alpha\beta} - G_{\alpha\beta}$ be induced by the involutory field automorphism of $GF(q^2)$. Then $C_G(a)$ is the full 3-dimensional orthogonal group over $GF(q)$, that is, $C_G(a) = \text{PGL}(2, q)$.

$\bar{G}$ acts on the projective plane PG(2, q^2). An involution $x \in \bar{G} - G$ is a collineation of this plane, and thus fixes $q^2 + 2$ or $q^2 + q + 1$ lines. If x fixes no points of Ω, then x fixes precisely $(q^3 + 1)/(q + 1)$ lines, each meeting Ω in $q + 1$ points, a contradiction.

Now assume that $x \in \bar{G}_{\alpha\beta}$. Let $\langle y \rangle$ be the 2-Sylow subgroup of $G_{\alpha\beta}$, so $|y| = (q^2 - 1)_2$. If $q \equiv 1 \pmod 4$, then $y^{-1}y^a = y^{q-1}$ is an involution. If $q \equiv 3 \pmod 4$, then $yy^a = y^{q+1}$ is an involution. In either case, a 2-Sylow subgroup of $\bar{G}_{\alpha\beta}$ has a single class of involutions not in $G_{\alpha\beta}$. This implies (vii) and (viii).

(ix) Let H be a central extension of G by a group $\langle z \rangle$ of order 2. Let t be an involution in $H - \langle z \rangle$ and set $L = C_H(t)$. Then $L/\langle z \rangle$ contains a characteristic subgroup $E/\langle z \rangle$ isomorphic to $\mathrm{SL}(2, q)$ such that L/E is cyclic. By Lemma 3.1 (vii) it follows that E has a characteristic subgroup E_0 such that $E = E_0 \times \langle z \rangle$.

A Sylow 2-subgroup S of $N_H(\langle t, z \rangle)$ is 2-Sylow in H. Set $S_0 = S \cap E_0$. Then $S \rhd S_0$, S/S_0 is abelian, and $z \notin S_0$. Now Lemma 2.2 implies that H has a normal subgroup H_0 of index 2, and $H = H_0 \times \langle z \rangle$.

We define groups of Ree type by means of the axioms of Ward [40]. Alternative characterizations are found in [15, 22, 28, 39].

LEMMA 3.3. *Let G be a group of Ree type, in its usual 2-transitive representation on a set Ω, $|\Omega| = q^3 + 1$, $q = 3^{2a+1}$, $a \geqslant 0$. Let $\alpha, \beta \in \Omega$, $\alpha \neq \beta$.*

(i) *A 2-Sylow subgroup S of G is elementary abelian of order* 8.

(ii) *$C(S) = S$ and $N(S)/C(S)$ acts on $S^{\#}$ as a Frobenius group of order* 21.

(iii) *G_α has a normal subgroup Q regular on $\Omega - \alpha$. If $q > 3$, then Q has class 3, $|Z(Q)| = q$, $Q^{(1)} = \Phi(Q)$, and $|\Phi(Q)| = q^2$.*

(iv) *$G_{\alpha\beta}$ is cyclic of order $q - 1$.*

(v) *An involution $t \in G_{\alpha\beta}$ fixes $q + 1$ points, and is the only element of $(G_{\alpha\beta})^{\#}$ fixing more than 2 points.*

(vi) $C(t) = \langle t \rangle \times \mathrm{PSL}(2, q)$.

(vii) *$C_Q(t) \cap Z(Q) = 1$, and if $q > 3$, then $C_Q(t)Z(Q) = \Phi(Q)$.*

(viii) *G is simple if $q > 3$, and if $q = 3$, then $G \approx \mathrm{P\Gamma L}(2, 8)$.*

(ix) Aut *G/G has odd order.*

(x) *A central extension of G by a 2-group splits.*

Proof. (i)–(viii) See Ward [40].

(ix) This has been checked for Ree groups by Ree [27]. The following proof for groups of Ree type is in the spirit of later sections. The notation is that of Section 4. We may assume that $q > 3$.

Aut G acts on Ω. Let $x \in \mathrm{Aut}\, G - G$, where $|x| = 2$ or 4 and $x^2 \in G$. We may assume that x centralizes the involution t in $G_{\alpha\beta}$. Let $\bar{G}$ be $G\langle x \rangle$, Δ the fixed points of t, and W the subgroup of $\bar{G}$ fixing each point of Δ. Then

$|W|$ is either 2 or 4. If $|W| = 4$ then $\bar{G}_{\alpha\beta}$ has a central 2-Sylow subgroup of order 4. If $|W| = 2$, then $(C_0(t)\langle x\rangle)^{\Delta} = \mathrm{PGL}(2, q)$ and again $\bar{G}_{\alpha\beta}$ has a central 2-Sylow subgroup of order 4.

Now $O(G_{\alpha\beta})$ is irreducible on $Q/\Phi(Q)$, $4 \nmid (|Q/\Phi(Q)| - 1)$, and t is fixed-point-free on $Q/\Phi(Q)$. Thus, a 2-Sylow subgroup of $G_{\alpha\beta}$ must be a Klein group, say $\langle t, u\rangle$. As $G_{\alpha\beta}$ is irreducible on $Q/\Phi(Q)$, it follows that u or tu is in $C(Q)$, and hence fixes each point of Ω, a contradiction.

(x) If X is a 2-Sylow subgroup of a central extension H of G by a group $\langle z\rangle$ of order 2, then $N_H(X)$ has a subgroup of order 7 transitive on $(X/\langle z\rangle)^{\#}$. By Lemma 2.7, X splits over $\langle z\rangle$, and consequently H splits over $\langle z\rangle$ [14, p. 246].

4. Beginning of Proof

Assume that G is a group of least order satisfying the hypotheses but not the conclusions of Theorem 1.1. Thus, G is 2-transitive on Ω, $|\Omega| = n$ is even, and G_α has a normal subgroup Q of odd order $n - 1$ regular on $\Omega - \alpha$.

Lemma 4.1.

(i) *G has no proper normal subgroup containing Q.*

(ii) *G has no normal subgroup of index 2.*

(iii) *G contains no odd permutations.*

(iv) *G has no regular normal subgroup.*

(v) *G has an involution fixing at least 4 points.*

(vi) *For each involution u, the number of fixed points of u is $\equiv n$* (mod 4).

Proof. (i) Let $G \rhd K \geqslant Q$. If K has a unique normal subgroup M as in Theorem 1, and clearly $C_G(M) = 1$, so that $G \leqslant \mathrm{Aut}\, M$ and G satisfies the conclusions of Theorem 1.1. If M is not unique, then K has a unique minimal normal subgroup L, and $M = LQ$ is a normal sharply 2-transitive subgroup of G.

(ii) As $|Q|$ is odd, such a subgroup would contain Q.

(iii) This is clear by (ii).

(iv) Let K be a regular normal subgroup of G. Then KQ is a sharply 2-transitive normal subgroup of G, contradicting (i).

(v) If $|G_\alpha|$ is odd, then G is solvable or contains a normal subgroup $\mathrm{PSL}(2, q)$, $q \equiv 3 \pmod 4$, containing Q (Bender [4]). If some involution fixes 2 points, but no involution fixes more than 2 points, then G has a normal subgroup PSL(2, q) containing Q, or G is A_6 in its usual representation (Hering [17]). None of these possibilities can occur.

(vi) By (iii), u is an even permutation, hence has an even number of 2-cycles.

An involution fixing no points of Ω will be called a *regular* involution.

Notation. Let α and β be distinct points of Ω.

Let X be any subset of G fixing at least two points. Define:

$$\begin{aligned} \Delta(X) &= \text{set of fixed points of } X; \\ N(X)^{\Delta(X)} &= \text{permutation group induced by } N(X) \text{ on } \Delta(X); \\ C_0(X) &= \langle C_{Q^g}(X) \mid \alpha^g \in \Delta(X) \rangle; \\ W_X &= \text{pointwise stabilizer of } \Delta(X) \text{ in } N(X). \end{aligned}$$

For an involution denoted t, we write $\Delta = \Delta(t)$ and $W = W_t$.

LEMMA 4.2. *If X is a subset of G fixing at least 3 points, then $C_0(X)^{\Delta(X)}$ is a 2-transitive group satisfying the hypotheses of Theorem 1.1 with $|\Delta(X)|$ even.*

Proof. We may assume that $\alpha \in \Delta(X)$. If $\beta, \gamma \in \Delta(X) - \alpha$, then $\gamma = \beta^h$, $h \in Q$. Let $x \in X$. Then $\beta^{hx} = \beta^h = \beta^{xh}$ implies that

$$[h^{-1}, x^{-1}] \in G_{\alpha\beta} \cap [Q, G_\alpha] \leqslant Q_\beta = 1.$$

Thus, $h \in C_Q(X)$, so that $C_Q(X)$ is transitive on $\Delta(X) - \alpha$. As α is any point of $\Delta(X)$ and $C_Q(X) \trianglelefteq N(X)_\alpha$, $C_0(X)^{\Delta(X)}$ is 2-transitive and satisfies the hypotheses of Theorem 1.1. Finally, $|\Delta(X)| = |C_Q(X)| + 1$ is even as $|Q|$ is odd.

LEMMA 4.3. *Let X and Y be subsets of $G_{\alpha\beta}$, each fixing at least 3 points. If X and Y are conjugate in G then they are conjugate in $G_{\alpha\beta}$.*

Proof. Let $Y = X^g$, $g \in G$. Then α, β, α^g, $\beta^g \in \Delta(Y)$. Let $\alpha^{gh} = \alpha$, $\beta^{gh} = \beta$, where $h \in C_0(Y)$ (Lemma 4.2). Then $gh \in G_{\alpha\beta}$, and $X^{gh} = Y^h = Y$.

LEMMA 4.4. *Let X be a subset of $G^{\#}$ fixing at least 3 points. Then $[C_0(X), W_X] = 1$, and one of the following holds.*

(i) $|\Delta(X)| = 2^a$, $|C_0(X)^{\Delta(X)}| = 2^a(2^a - 1)$, *and $C_0(X)$ is a sharply 2-transitive group.*

(ii) $|\Delta(X)| = q + 1$ *and* $C_0(X) = \mathrm{PSL}(2, q)$ *for some odd prime power q.*

(iii) $|\Delta(X)| = q + 1$ *and* $C_0(X) = \mathrm{SL}(2, q)$ *for some odd prime power q.*

(iv) $|\Delta(X)| = q^3 + 1$ *and $C_0(X)$ is a central extension of* PSU(3, q) *by a group of odd order, where q is an odd prime power.*

(v) $|\Delta(X)| = q^3 + 1$ *and* $C_0(X)$ *is a central extension of a group of Ree type by a group of odd order, where* $q = 3^{2a+1}$.

Proof. Let $X \subseteq G_\alpha$. Then $[C_Q(X), W_X] \leqslant Q \cap W_X = 1$. As $N(X)$ normalizes W_X, we have $[C_0(X), W_X] = 1$. By Lemma 4.2, the minimality of $|G|$, and the definition of $C_0(X)$, it follows that $C_0(X)^{\Delta(X)}$ is solvable of order $2^a(2^a - 1)$, $C_0(X)^{\Delta(X)} = \mathrm{PSL}(2, q)$, $C_0(X)^{\Delta(X)} = \mathrm{PSU}(3, q)$, or $C_0(X)^{\Delta(X)}$ is of Ree type.

Clearly, $C_0(X)^{\Delta(X)} \approx C_0(X)/C_0(X) \cap W_X$ and $C_0(X) \cap W_X \leqslant Z(C_0(X))$. From the definition of $C_0(X)$ it follows that $C_0(X)$ has no normal subgroup of index 2.

If $C_0(X)^{\Delta(X)}$ is unitary or of Ree type, then (iv) or (v) holds by Lemmas 3.2(ix) and 3.3(x). Suppose that $C_0(X)^{\Delta(X)} = \mathrm{PSL}(2, q)$ with $q > 3$ and q odd. In this case we have $C_Q(X) \leqslant C_0(X)^{(1)}$, so that $C_0(X) = C_0(X)^{(1)}$. Thus, if neither (ii) nor (iii) holds, then $C_0(X)$ is a homomorphic image of the covering group of PSL(2, 9) (Lemma 3.1(vii)). However, in this case, if P is a 3-Sylow subgroup of $C_0(X)_\alpha$, then $P = C_Q(X) \times (P \cap W_X)$, so that a result of Gaschütz [14, p. 246] implies that $C_0(X)$ splits over $P \cap W_X$, a contradiction.

Finally, suppose that $C_0(X)^{\Delta(X)}$ is solvable of order $2^a(2^a - 1)$. Then $C_0(X)$ has a normal 2-Sylow subgroup R such that $R^{\Delta(X)}$ is regular. It follows that $C_0(X) = R \cdot C_Q(X)$. Using Lemma 2.7 and the fact that $C_0(X)$ has no normal subgroup of index 2, we have $|R \cap W_X| \leqslant 2$. Consequently, (i), (ii), or (iii) holds.

LEMMA 4.5. *If* $\langle t, u\rangle$ *is a Klein group in* $G_{\alpha\beta}$ *with* $t \sim u \sim tu$, $|\Delta(\langle t, u\rangle)| = 2$, *and* $C_0(t)^{\Delta(t)} = \mathrm{PSL}(2, q)$, *then* Q *is elementary abelian and* $n = q^3 + 1$.

Proof. $t \sim u \sim tu$ in $G_{\alpha\beta}$ (Lemma 4.3). As $\langle t, u\rangle$ acts on $Z(Q)$, one and hence all involutions in $\langle t, u\rangle$ centralize elements of $Z(Q)^\#$. However, $C_0(t)_{\alpha\beta}$ is irreducible on $C_Q(t)$. Thus, $C_Q(t) \leqslant Z(Q)$, and it follows that $Q \leqslant Z(Q)$. Also, $n = q^3 + 1$ follows from Lemma 2.1.

LEMMA 4.6. *Let* t *be an involution in* $G_{\alpha\beta}$ *such that* $C(t)_{\alpha\beta}$ *contains a 2-Sylow subgroup of* $G_{\alpha\beta}$. *Then* $C(t)$ *contains a 2-Sylow subgroup of* G *provided that either*

(i) $n_2 \leqslant |\Delta|_2$, *or*

(ii) Q *is a* p*-group of order* p^a, $|C_Q(t)| = p^b$, *and either* b *is odd or* a *is even.*

Moreover, in either case $n_2 = |\Delta|_2$.

Proof. (i) $|G|_2 = n_2|G_{\alpha\beta}|_2 \leqslant |\Delta|_2 |C(t)_{\alpha\beta}|_2 = |C(t)|_2$. Moreover, $n_2 = |\Delta|_2$.

(ii) If a is even then $n_2 = 2 \leqslant |\Delta|_2$. If a is odd and b is odd then $n_2 = (p+1)_2 = (p^b+1)_2$.

5. The Solvable Case

THEOREM 5.1. *Let t be an involution in $G_{\alpha\beta}$ such that $C(t)^\Delta$ is solvable and $|\Delta| > 2$. Then*

(1) $|\Delta| = 4$; *and*
(2) *if $G_{\alpha\beta}$ contains no Klein group, then $C_0(t) = \mathrm{SL}(2,3)$.*

Proof. Suppose the theorem is false. Let $|\Delta| = k \geqslant 4$. If $k = 4$, we are assuming that $G_{\alpha\beta}$ contains no Klein group and $C_0(t) = A_4$.

If there is an involution $z \in G_{\alpha\beta}$ fixing just 2 points, then $z \in Z(G_{\alpha\beta})$. Also, $z \neq t$ and z^Δ fixes just 2 points. Then $|\Delta| = 4$ and $G_{\alpha\beta}$ has a Klein group, a contradiction. Thus, there is no such involution z.

Write $k = 2^f, f \geqslant 2$.

LEMMA 5.2. *$C_0(t)$ has a normal 2-Sylow subgroup T of order k. If W contains no Klein group then $T \times \langle t\rangle$ is the unique elementary abelian subgroup of $C(t)$ of order $2k$. If t^g is in $T \times \langle t\rangle$, then g is in $N(T\langle t\rangle)$.*

Proof. The first assertion follows from Lemma 4.4 and our conditions on t. Suppose that W contains no Klein group, and let S be a 2-Sylow subgroup of $C(t)$. If $k = 4$, then $G_{\alpha\beta}$ contains no Klein group, and the second assertion is clear.

Suppose that $k > 4$. If $S - T(S \cap W)$ contains no involution, the uniqueness of $T\langle t\rangle$ is again clear. Let $u \in S - T(S \cap W)$ be an involution. Since $C(t)^\Delta$ is solvable, it is a subgroup of the group of 1-dimensional affine semilinear mappings on $GF(k)$ (Huppert [19]). By hypothesis, $u^\Delta \neq 1$, so that u^Δ acts as a field automorphism. Thus, u^Δ fixes $\sqrt{k}$ points, that is, $|C_T(u)| = \sqrt{k}$. As $T(S \cap W) = T \times (S \cap W)$, the second assertion follows.

If $t^g \in T \times \langle t\rangle$, then $(T\langle t\rangle)^{g^{-1}} \leqslant C(t)$. By the uniqueness of $T\langle t\rangle$, we have $g \in N(T\langle t\rangle)$.

LEMMA 5.3. *Suppose that $k > 4$, W contains no Klein group, and $T\langle t\rangle - \langle t\rangle$ contains conjugates of t. Then:*

(i) *$T\langle t\rangle$ contains k conjugates of t, namely, the elements of Tt;*
(ii) *$N(T\langle t\rangle)$ is transitive on Tt; and*
(iii) *$\langle t\rangle$ is a 2-Sylow subgroup of W.*

Proof. (i) and (ii). If $T\langle t\rangle$ does not contain k conjugates of t it contains $2k-1$ such conjugates and, by Lemma 5.2, $N(T\langle t\rangle)$ is transitive on $(T\langle t\rangle)^{\#}$. Then $H = N(T\langle t\rangle)/C(T\langle t\rangle)$ is a linear group acting on $(T\langle t\rangle)^{\#}$ as a primitive group of degree $2k-1$ with subdegrees $1, k-1, k-1$. $|H|$ is odd (Wielandt [42, p. 8, Ex. 3.13]), so that H is solvable (Feit–Thompson [11]). Let M be a normal subgroup of H regular on $(T\langle t\rangle)^{\#}$. Then M is fixed-point-free on $T\langle t\rangle$, so that M is cyclic and $H/M \leqslant \mathrm{Aut}\, GF(2k)$. Now $2^f - 1 = |C_Q(t)|$ divides $f+1$, whereas $f > 2$.

Thus, $T\langle t\rangle$ has k conjugates of t. Let $t^g \in T$, $g \in G$. By Lemma 5.2, $g \in N(T\langle t\rangle)$. However, $T^g \neq T$ and $|T \cup T^g| > k+1$, whereas $T\langle t\rangle$ has only k conjugates of t. This contradiction implies that Tt is the set of conjugates of t in $T\langle t\rangle$.

(iii) A 2-Sylow subgroup of $C(T\langle t\rangle)$ has the form $T \times Y$ with Y 2-Sylow in W. Let X be a 2-Sylow subgroup of $N(T\langle t\rangle)$ normalizing $T \times Y$. Since $|Tt| = k = 2^f$, X is transitive on Tt. If $|Y| > 2$, then t is the only involution which is a square in $T \times Y$, so that $X \leqslant C(t)$, a contradiction.

For purposes of Lemmas 5.4 and 5.5, we recall that we are assuming that, if $G_{\alpha\beta}$ contains a Klein group, then $k > 4$. We also make the following observation, which will be used frequently in Sections 5, 6 and 7. If $\langle u, v\rangle$ is a Klein group in $G_{\alpha\beta}$, and $u^{\Delta(v)} \neq 1$, then either $C(v)^{\Delta(v)}$ is solvable, or the action of $u^{\Delta(v)}$ has been described in Section 3.

LEMMA 5.4. *$G_{\alpha\beta}$ contains no involution u such that $C(u)^{\Delta(u)}$ is nonsolvable.*

Proof. Suppose that $C(u)^{\Delta(u)}$ is nonsolvable. We first assume that $u \in C(t)$. Since $C_0(\langle t, u\rangle) \leqslant C_0(t) \cap C_0(u)$, $|\Delta(\langle t, u\rangle)| = 4$ and $|\Delta| = 16$. By Lemma 4.1(vi), $|\Delta(u)| = 28$ and $|\Delta(tu)| = 4$, 16 or 28. Also, W contains no Klein group, as otherwise there is an involution $v \in W$ such that $t^{\Delta(v)} \neq 1$ (Lemma 2.1), and then we must have $|\Delta(v)| = 16^2$ (see Section 3), whereas $(16^2 - 1) \nmid |Q|$ (Lemma 2.1).

$C_0(u)^{\Delta(u)}$ is a Ree group. For otherwise, it is unitary. By Lemma 4.4, $C_0(t) \approx C_0(t)^{\Delta}$, so that $A_4 = (C_0(t) \cap C(u))^{\Delta} \approx (C_0(t) \cap C(u))^{\Delta(u)}$. Then $t^{\Delta(u)}$ is a field automorphism (Lemma 3.2). If v is the involution in $C_0(u)_{\alpha\beta}$, then $\langle t, u, v\rangle$ is elementary abelian of order 8, and $\langle t, u, v\rangle \cap W = \langle t\rangle$. Thus, $\langle u, v\rangle^{\Delta}$ is a Klein group in $C(t)^{\Delta}_{\alpha\beta}$, which is not possible.

Let S be a 2-Sylow subgroup of $C(u)$ containing $C_T(u)\langle t\rangle$. Then $E = S \cap C_0(u) \geqslant S \cap C_0(\langle t, u\rangle) = C_T(u)$. By Section 3, there is no involution $x \in W_u$ such that $u^{\Delta(x)} \neq 1$ and $C(\langle u, x\rangle)^{\Delta(x)}$ is a Ree group. Thus, W_u contains no Klein group, so that $\Omega_1(S) = E\langle u\rangle$. Note that $S_{\alpha\beta} = \langle t\rangle(S \cap W_u)$ is 2-Sylow in $C(u)_{\alpha\beta}$.

If u is weakly closed in S then $S_{\alpha\beta}$ is a 2-Sylow subgroup of $G_{\alpha\beta}$, contradicting Lemma 2.6. As in Lemma 5.3(i) it follows that u is conjugate to all elements of Eu. Then t must be conjugate to all elements of $(E\langle u\rangle)^{\#} - Eu = E^{\#} \geqslant C_T(u)$, contradicting Lemma 5.3(i).

Thus, $u \notin C(t)$. Let v be an involution in $C(t)_{\alpha\beta} \cap C(u)_{\alpha\beta}$. Since $v \in C(t)_{\alpha\beta}$, $C(v)^{\Delta(v)}$ and $C(tv)^{\Delta(tv)}$ must be solvable. If $|\Delta(v)| \geqslant 8$, we can replace t by v in the above argument. Thus, $|\Delta(v)| = 4$. As $|\Delta(\langle t, v\rangle)| \neq 2$, we have $|\Delta(\langle t, v\rangle)| = 4$ and $|\Delta| = |\Delta(tv)| = 16$. Then $|Q| = 15^2 \cdot 3/9 = 3|C_Q(u)|\,|C_Q(uv)|/|C_Q(\langle u, v\rangle)|^2$ (Lemma 2.1). If $|C_Q(\langle u, v\rangle)| = 3$ then $C_Q(u)$ is a 3-group and $5^2 \nmid |Q|$. If $C_Q(\langle u, v\rangle) = 1$ then $5^2 = |C_Q(u)|\,|C_Q(uv)|$, so that $|C_Q(u)| = 5$, contradicting Lemma 4.1(vi).

LEMMA 5.5. *$G_{\alpha\beta}$ contains no Klein group.*

Proof. Let $\langle x, y\rangle$ be a Klein group in $G_{\alpha\beta}$ containing t such that $C(x)_{\alpha\beta}$ contains a 2-Sylow subgroup of $G_{\alpha\beta}$. Set $\ell = |\Delta(\langle x, y\rangle)|$. Since $|\Delta| > 4$, $\ell > 2$. By Lemma 5.4, x, y and xy fix ℓ or ℓ^2 points. By Lemma 2.1, at most one of these fixes ℓ points, and $n - 1 = (\ell^2 - 1)^2(\ell^i - 1)/(\ell - 1)^2 = (\ell + 1)^2(\ell^i - 1)$, $i = 1$ or 2. If $i = 1$, $n \not\equiv 0 \pmod{2\ell}$. In either case, $n \not\equiv 0 \pmod{\ell^2}$. Thus, by Lemma 4.6, $|\Delta(x)| \neq \ell^2$, so that $|\Delta(x)| = \ell$ and $C(x)$ contains a 2-Sylow subgroup of G.

Let T^* be the 2-Sylow subgroup of $C_0(y)$. $C(x)$ contains a conjugate X of $T^* \times \langle y\rangle$. Then $|X^{\Delta(x)}| \leqslant \ell$, so that $|X \cap W_x| \geqslant 2\ell^2/\ell \geqslant 8$. Choose $v \in (X \cap W_x)^{\#}$ such that $|\Delta(v)|$ is maximal. Then $|(X \cap W_x)^{\Delta(v)}| \geqslant 4$. However, $C(v)^{\Delta(v)}$ is solvable by Lemma 5.4, so that this is impossible.

LEMMA 5.6. (i) $n = k^2$.
 (ii) *$T^{\#}$ consists of $k - 1$ regular involutions.*
 (iii) *Tt consists of k conjugates of t, permuted transitively by $N(T\langle t\rangle)$.*
 (iv) $W = \langle t\rangle$.

Proof. By Lemmas 2.6, 5.2, 5.3(iii), and 5.5, or their proofs if $k = 4$, $\langle t\rangle$ is a 2-Sylow subgroup of W, $T\langle t\rangle$ contains all involutions in $C(t)$, and either (ii) and (iii) hold or $k = 4$ and $(T\langle t\rangle)^{\#}$ consists of 7 conjugates of t.

We first show that either (i), (ii), and (iii) hold or $n = 28$ and G has a single class of involutions. If $\gamma \notin \Delta$, then t normalizes $G_{\gamma\gamma t}$ and hence centralizes some involution $t_1 \in G_{\gamma\gamma t}$. Then $t_1 \in T\langle t\rangle$. By Lemma 5.5, no 2 involutions in $T\langle t\rangle$ have common fixed points. Thus, the conjugates of t lying in $T\langle t\rangle$ determine a partition of Ω into subsets of k elements. It follows that either $n = k \cdot k$ or $k = 4$ and $n = 7 \cdot 4 = 28$. In the latter case, G has one class of involutions.

Since $C(T\langle t\rangle) = T \times W_{t_1}$ for each conjugate t_1 of t inside $T\langle t\rangle$, it also follows that $O(W) = 1$. Thus, $W = \langle t\rangle$.

It remains to show that $n \neq 28$. If $n = 28$, then $k = 4$, all involutions in G are conjugate, and $|G_{\alpha\beta}|_2 = 2$ or 4. Let $M = O(G_{\alpha\beta})$ and let $t \sim t' = (\alpha, \beta) \cdots$ with $t' \in C(t)$ (Lemma 2.6). Then $M = C_M(t)C_M(t')C_M(tt')$. As $W = \langle t\rangle$,

$C_M(t) = 1$. If $C_M(t') > 1$, then, since $k = 4$ and $W_{t'} = \langle t' \rangle$, $C_M(t') \sim C_Q(t)$, which is impossible. Thus, $C_M(t') = 1$, and similarly $C_M(tt') = 1$. Consequently, $|G_{\alpha\beta}| = 2$ or 4. If $G_{\alpha\beta} = \langle t \rangle$, a result of Ito [21] yields a contradiction. If $|G_{\alpha\beta}| = 4$, then no involution in T is a square in $TG_{\alpha\beta}$. However, if $t_1 \in T \cap Z(TG_{\alpha\beta})^{\#}$, then t_1 and t are conjugate in $N(TG_{\alpha\beta})$ (Lemma 2.4), and this is a contradiction.

We now complete the proof of Theorem 5.1 (compare Harada [16]). Since $W = \langle t \rangle$ and $C(t)^{\Delta}_{\alpha\beta}$ is cyclic, $C(t)_{\alpha\beta}$ is cyclic (Lemma 5.5).

Set $\mathscr{I} = Tt$, and regard $N(T\langle t \rangle)$ as a permutation group on $\mathscr{I}$. By Lemma 5.6(iii), $N(T\langle t \rangle)^{\mathscr{I}}$ is transitive. Set $A = C_Q(t)$. Then

$$A \leqslant N(T\langle t \rangle) \cap C(t),$$

and $A^{\mathscr{I}}$ is regular on $\mathscr{I} - \{t\}$. Thus, $N(T\langle t \rangle)^{\mathscr{I}}$ satisfies the hypotheses of Theorem 1.1. By Lemma 5.6(iv), $T\langle t \rangle = C(\mathscr{I})$. Also, $N(T\langle t \rangle) \cap C(t)$ acts on $\mathscr{I}$ as $C(t)_\alpha$ acts on T, that is, as $C(t)_\alpha$ acts on Δ.

We claim that $N(T\langle t \rangle)^{\mathscr{I}}$ is solvable. This is clear if $|\mathscr{I}| = k = 4$. Let $|\mathscr{I}| = k = 2^f > 4$. If $N(T\langle t \rangle)^{\mathscr{I}}$ is not solvable, the minimality of G implies that $N(T\langle t \rangle)^{\mathscr{I}}$ contains $\mathrm{PSL}(2, 2^f - 1)$, and then $|(N(T\langle t \rangle) \cap C(t))^{\mathscr{I}}| \geqslant (2^f - 1)(2^f - 2)/2$. On the other hand, $|(N(T\langle t \rangle) \cap C(t))^{\mathscr{I}}| = |C(t)^{\Delta}_{\alpha}| \leqslant (2^f - 1)f$. This is a contradiction unless $f = 3$. If $f = 3$ and $(N(T\langle t \rangle))^{\mathscr{I}}$ contains $\mathrm{PSL}(2, 7)$, then $C(t)_{\alpha\beta}$ contains an element g of order 3 inverted by an element of $N(T\langle t \rangle)$. Moreover, in this case, $n = 64$ and $n - 1 = 7 \cdot 3^2$. Thus, $C_Q(g) > 1$ and $|\Delta(g)| > 2$. By Lemma 4.3, g is inverted in $G_{\alpha\beta}$, whereas g is centralized by a 2-Sylow subgroup $\langle t \rangle$ of $G_{\alpha\beta}$. This is a contradiction.

Thus, $N(T\langle t \rangle)$ has a normal subgroup R containing $C(\mathscr{I}) = T\langle t \rangle$ such that $R^{\mathscr{I}}$ is regular. Clearly, $|R| = 2k^2$ and A is regular on $(R/T\langle t \rangle)^{\#}$. By Lemma 5.6(ii), T is a minimal normal subgroup of RA, so that $T \leqslant Z(R)$.

Suppose that $k = 4$ and R/T is quaternion of order 8. Then $T\langle t \rangle/T = Z(R/T)$, so that $x^2 = t^* \in Tt$ for some $x \in R$. Then $C(t^*) \geqslant \langle T, x \rangle$, where $T \leqslant Z(R) \leqslant C(x)$, so that $x \in C(T\langle t \rangle)$, contradicting Lemma 5.6(iv).

By Lemma 2.7, $R/T = T_1/T \times T\langle t \rangle/T$, where $C(t)_\alpha$ normalizes $T_1 = [R, C(t)_\alpha]$. Then $T \leqslant Z(T_1)$, and A is regular on $(T_1/T)^{\#}$.

Let S be a 2-Sylow subgroup of $N(T\langle t \rangle)$ containing both R and a 2-Sylow subgroup of $C(t)_{\alpha\beta}$. Then $|S^{\mathscr{I}}| = k|S^{\mathscr{I}}_{\alpha\beta}|$, and by Lemma 5.6(i) we have $|S| = k^2|C(t)_{\alpha\beta}|_2 = n_2|G_{\alpha\beta}|_2 = |G|_2$. Thus, S is a 2-Sylow subgroup of G. Clearly, $S = T_1 S_{\alpha\beta} \rhd T_1$, where $t \in S_{\alpha\beta}$, $S_{\alpha\beta}$ is cyclic, and $T_1 \cap S_{\alpha\beta} = 1$.

By Lemma 2.3, $t \sim t_1 \in T_1$. Then $t_1 \notin T$ (Lemma 5.6(iii)). Since A is transitive on $(T_1/T)^{\#}$ and $T \leqslant Z(T_1)$, each coset $\neq T$ of T in T_1 consists of k involutions. Thus, T_1 is elementary abelian of order k^2. However, $t \sim t_1$ and $C(t)$ contains no elementary abelian subgroup of order $> 2k$. This contradiction proves Theorem 5.1.

6. 2-Involutions

In this section we consider the possibility that G contains 2-involutions, that is, involutions fixing exactly two points.

Theorem 6.1. (i) *G contains no 2-involutions.*

(ii) *If t is a nonregular involution such that $C_0(t)^{\Delta} = \mathrm{PSL}(2, q)$, then $n > q^2 + 1$.*

Proof. Suppose that $G_{\alpha\beta}$ contains a 2-involution z. Then z inverts every element in Q and $z \in Z(G_{\alpha\beta})$. By Lemma 4.1(v) there exists an involution t in $G_{\alpha\beta}$ which has more than 2 fixed points. We consider the Klein group $\langle t, z\rangle$. Since $z \in C(t)$ fixes just 2 points of Δ, we have $C_0(t)^{\Delta} = \mathrm{PSL}(2, q)$ for some odd prime power q. Clearly, z is the only 2-involution in $G_{\alpha\beta}$. Hence, $|\Delta(tz)| > 2$ and $C_0(tz)^{\Delta(tz)} = \mathrm{PSL}(2, q')$ for some q'. By Lemma 2.1, $n - 1 = qq'$. If, say, $q \geqslant q'$, we have $n \leqslant q^2 + 1$.

It thus suffices to prove (ii). Suppose that $t \in G_{\alpha\beta}$ is an involution such that $|\Delta| > 2$ and $C_0(t)^{\Delta} = \mathrm{PSL}(2, q)$, where $n \leqslant q^2 + 1$. Then $|\Omega - \Delta| \leqslant q^2 - q$. Let γ be an arbitrary point in $\Omega - \Delta$, and set $\gamma' = \gamma^t$.

Lemma 6.2. $C_0(t) = \mathrm{PSL}(2, q)$.

Proof. Otherwise, $C_0(t) \approx \mathrm{SL}(2, q)$ by Lemma 4.4. Let u be the unique involution in $C_0(t)$. Then $\Delta(u) \supseteq \Delta$. Hence, $|\Omega - \Delta(u)| \leqslant q^2 - q$. If $\gamma \in \Omega - \Delta(u)$, then $|C_0(t)_\gamma| \geqslant q(q^2 - 1)/(q^2 - q) = q + 1$. On the other hand, $C_0(t)_\gamma$ has odd order since the unique involution u of $C_0(t)$ does not fix γ. Also, $(q, |C_0(t)_\gamma|) = 1$ since Q is regular on $\Omega - \alpha$. However, $\mathrm{SL}(2, q)$ has no such subgroup (Dickson [9, pp. 285–286]), a contradiction.

Lemma 6.3. (i) $n = q^2 + 1$.

(ii) $q \equiv 1 \pmod 4$.

(iii) *$C_0(t)_{\{\gamma,\gamma'\}}$ is a dihedral group of order $q + 1$ which is self-normalizing in $C_0(t)$.*

(iv) *$C_0(t)$ acts transitively on the set of nontrivial orbits of $\langle t\rangle$.*

Proof. Let $X = C_0(t)_{\{\gamma,\gamma'\}}$. As above, $|X| \geqslant q + 1$ and $(q, |X|) = 1$. We thus have one of the following situations (Dickson [9, pp. 285–286]):

(a) X is a dihedral group of order $q + 1$;

(b) $X \approx A_4$;

(c) $X \approx S_4$ and $q \equiv \pm 1 \pmod 8$; or

(d) $X \approx A_5$ and $q \equiv \pm 1 \pmod{10}$.

If (a) holds then $n = q^2 + 1$ and $C_0(t)$ is transitive on the orbits of $\langle t\rangle$ on $\Omega - \Delta$. Clearly (iii) holds, and (ii) follows from Lemma 4.1(vi).

Suppose that (b), (c) or (d) holds. As $|X : C_0(t)_{\gamma\gamma'}| \leqslant 2$, $C_0(t)_{\gamma\gamma'}$ contains a subgroup isomorphic to A_4. Then there exists a Klein group $\langle v_1, v_2\rangle \leqslant G_{\alpha\beta}$ such that $v_1 \sim v_2 \sim v_1v_2$. Thus,

$$(*) \qquad |C_Q(\langle v_1, v_2\rangle)|^2 |Q| = |C_Q(v_1)|^3$$

by Lemma 2.1. Assume now that q is a prime. We have $q \mid |Q|$, so that by (*) $q \mid |C_Q(v_1)|$ and $q \mid |C_Q\langle v_1, v_2\rangle|$ since $|Q| \leqslant q^2$. Then $|C_Q(v_1)| \geqslant |C_Q(\langle v_1, v_2\rangle)|^2 \geqslant q^2 \geqslant n - 1$, a contradiction.

Thus, q is not a prime. For each of the cases (b), (c) and (d), $3 \mid |X|$ and hence $3 \nmid q$. Also, q is an odd prime power and $q \leqslant |X| - 1$. This implies that $q = 49$ and that we have case (d) for any choice of γ in $\Omega - \Delta$. Hence, $\frac{1}{2}q(q^2 - 1)/60$ divides

$$(|Q| - q)/2 = q((|Q|/q) - 1)/2, \quad \text{and} \quad 40 \mid ((|Q|/q) - 1) \leqslant q - 1 = 48.$$

Therefore, we have $|Q| = 49 \cdot 41$. By (*), $41 \cdot 7 \mid |C_Q\langle v_1, v_2\rangle|$ and $|\Delta(v_1)| > n$, a contradiction.

LEMMA 6.4. *All involutions in $C_0(t)$ are 2-involutions.*

Proof. As $|C_0(t)_{\alpha\beta}| = (q - 1)/2$ and $q \equiv 1 \pmod 4$, there is an involution $u \in C_0(t)_{\alpha\beta}$. If u is a 2-involution the lemma is clear. Suppose that u fixes some point $\gamma \in \Omega - \Delta$. Then u fixes $\gamma' = \gamma^t$. Hence, by Lemma 6.3, $C_0(t)_\gamma = C_0(t)_{\gamma\gamma'} = C_0(t)_{\{\gamma,\gamma'\}}$. This group is a dihedral group of order $q + 1$ and contains $(q + 1)/2$ conjugates of u. The total number of conjugates of u in $C_0(t)$ is $q(q + 1)/2$. Counting in two ways the pairs (u, γ) with u an involution in $C_0(t)$ and $\gamma \notin \Delta$ a fixed point of u, we find that

$$\tfrac{1}{2}q(q + 1)|\Delta(u) - (\Delta \cap \Delta(u))| = |\Omega - \Delta|(q + 1)/2 = q(q - 1)(q + 1)/2.$$

Then $|\Delta(u)| = q + 1$, and tu is a 2-involution by Lemmas 6.3(i) and 2.1.

A 2-involution $z' \in G_{\gamma\gamma'}$ centralizes t and fixes no points of Δ. Let $H = \langle z'\rangle C_0(t)$. Then H is PGL(2, q) and $H_{\gamma\gamma'}$ is a dihedral group of order $2(q + 1)$. The product of z' with an involution in $C_0(t)_{\gamma\gamma'}$ is an involution in $H_{\gamma\gamma'} - C_0(t)_{\gamma\gamma'}$ conjugate in H to z'. This is a contradiction since $G_{\gamma\gamma'}$ contains only one 2-involution.

LEMMA 6.5. (i) *$G_{\alpha\beta}$ contains a unique 2-involution z.*

(ii) *$C_0(t)_\gamma$ is cyclic of order $(q + 1)/2$.*

(iii) *If u is an involution in G, then either $|\Delta(u)| = 2$ or $|\Delta(u)| = q + 1$ and $C_0(u) =$ PSL(2, q).*

Proof. (i) is obvious. By Lemma 6.4, $|C_0(t)_\gamma|$ is odd, so that (ii) follows from Lemma 6.3(iii). Let u be an arbitrary involution in G. If u has no fixed points, then u is an odd permutation. By Lemma 4.1(iii), u has at least 2 fixed points. Assume that $u \in G_{\alpha\beta}$ and $|\Delta(u)| > 2$. Then $C_0(u)^{\Delta(u)} = \mathrm{PSL}(2, q')$ for some q'. The argument at the beginning of the proof of Theorem 6.1, together with Lemmas 6.2 and 6.3(i) for u or zu, shows that $q' = q$ and $C_0(u) = \mathrm{PSL}(2, q)$.

LEMMA 6.6. (i) $W = \langle t \rangle$.

(ii) *If u is an involution different from t in $C(t)$, then* $|\Delta \cap \Delta(u)| \leq 2$.

Proof (Hering [18]). (i) By Lemma 6.3, $C_0(t)$ acts transitively on the set of nontrivial orbits of $\langle t \rangle$ and, for each of these orbits $\{\gamma, \gamma'\}$, $C_0(t)_{\{\gamma,\gamma'\}}$ is self-normalizing in $C_0(t)$. Hence, $C_0(t)_{\{\gamma,\gamma'\}}$ fixes only one nontrivial orbit of $\langle t \rangle$. As W centralizes $C_0(t)$, W must fix each orbit $\{\gamma, \gamma'\}$, so that W is an elementary abelian 2-group. If W contains an involution $u \neq t$, then $|C_Q(u)| \geq |C_Q(\langle t, u \rangle)|^2 = q^2 = n - 1$, a contradiction.

(ii) Let $|\Delta \cap \Delta(u)| > 2$. Then $|\Delta \cap \Delta(u)| = |\Delta(\langle t, u \rangle)| = \sqrt{q} + 1$ since $u \notin W$ by (i). Let $\gamma \in \Delta(u) - (\Delta \cap \Delta(u))$. By Lemma 6.5(ii), $|C_0(\langle t, u \rangle)_\gamma|$ divides $|C_0(t)_\gamma| = (q + 1)/2$. On the other hand, $|C_0(\langle t, u \rangle)_\gamma|$ divides $|C_0(u)_\gamma|$, where $|C_0(u)_\gamma| = q(q - 1)/2$ by Lemma 6.5(iii). It follows that $C_0(\langle t, u \rangle)_\gamma = 1$, so that

$$q - \sqrt{q} = |\Delta(u) - (\Delta \cap \Delta(u))| \geq |C_0(\langle t, u \rangle)| = \sqrt{q}(q - 1)/2,$$

a contradiction.

LEMMA 6.7. *Let $C_1(t)$ be the subgroup of $C(t)$ generated by the 2-involutions in $C(t)$. Then*

(i) $C_1(t) = \mathrm{PGL}(2, q)$; *and*

(ii) *All involutions in $C_1(t)$ are 2-involutions.*

Proof. Since $t \in G_{\{\gamma,\gamma'\}}$, t commutes with the unique 2-involution z' in $G_{\gamma\gamma'}$. This involution fixes no points of Δ, so that $\langle z' \rangle C_0(t) = \mathrm{PGL}(2, q)$. Hence, the number of involutions in $\langle z' \rangle C_0(t)$ is $q(q + 1)/2 + q(q - 1)/2$.

If u is an arbitrary 2-involution in $C(t)$, then t leaves invariant $\Delta(u)$. Hence, by Lemma 6.5(i) the number of 2-involutions in $C(t)$ is not greater than the number of subsets of cardinality 2 of Ω which are invariant under t. Obviously, this number is $q(q + 1)/2 + q(q - 1)/2$. Hence, $C_1(t) = \langle z' \rangle C_0(t)$.

For the rest of this section let t' be an involution in $G_{\{\alpha,\beta\}} - G_{\alpha\beta}$ which is conjugate to t. Furthermore, let $C_1(t')$ be the subgroup of $C(t')$ generated by all 2-involutions and $H = C_1(t')_{\alpha\beta}$.

LEMMA 6.8. (i) *H is a cyclic group of order $q + 1$ containing z.*

(ii) *H is semiregular on $\Omega - \{\alpha, \beta\}$.*

(iii) *$C_1(t')_{\{\alpha,\beta\}}$ is a dihedral group of order* $2(q + 1)$.

Proof. By Lemma 6.5(ii), $C_0(t')$ is transitive on $\Omega - \Delta(t')$. Hence $|C_1(t')_\alpha| = q + 1$. Also, $C_0(t')_\alpha$ is a cyclic group of order $(q + 1)/2$. Since $q \equiv 1 \pmod 4$ and $z \in C_1(t')_\alpha$, we get $C_1(t')_\alpha = \langle z \rangle \times C_0(t')_\alpha$. Here $C_1(t')_\alpha = C_1(t')_{\alpha\alpha t'} = H$, so that we have (i).

Let $h \in H$ be an element of prime order. If $|h| = 2$, then $h = z$ and h fixes only α and β. Let $|h| > 2$. Then $h \in C_0(t')$. Here $\langle h \rangle$ is the only subgroup of $C_0(t')_{\{\alpha,\beta\}}$ of its order. By Lemma 6.3(iv), $C_0(t')$ acts on the nontrivial orbits of $\langle t' \rangle$ as it does on the conjugates of $\langle h \rangle$. Thus, h again fixes only α and β. As H is regular on $\Delta(t')$, this implies (ii).

Finally, (iii) follows from (i) together with Lemma 6.3(iii).

LEMMA 6.9. *H contains an r-Sylow subgroup R for some prime r such that*

(i) *R acts irreducibly on Q;*

(ii) *$C(R)_{\alpha\beta} \leqslant C(H)_{\alpha\beta}$;*

(iii) *$N(R)_{\alpha\beta}$ is isomorphic to a subgroup of the group of 1-dimensional semilinear transformations over $GF(q^2)$; and*

(iv) *R is an r-Sylow subgroup of G.*

Proof. Let $q = p^s$ with p a prime. As $q \equiv 1 \pmod 4$, there is a prime r such that $r \mid (q^2 - 1)$ and $r \nmid (p^i - 1)$ for $1 \leqslant i < 2s$ (see Birkhoff and Vandiver [5, Theorem V]). Let R be an r-Sylow subgroup of G. Then R has at least 2 fixed points, because $r \nmid q^2(q^2 + 1)$. Let $R \leqslant G_{\alpha\beta}$. Because of the property $r \nmid (p^i - 1)$ for $1 \leqslant i < 2s$, we have $C(x) \cap Q = 1$ for $x \in R^\#$. Hence, $|R| \mid (q^2 - 1)$ and therefore $|R| \mid (q + 1)$, so that we can assume that $R \leqslant H$. Then Q is elementary abelian, R acts irreducibly on Q, and (ii) and (iii) follow from a lemma of Huppert [19, Hilffsatz 2].

LEMMA 6.10. *$|G_{\{\alpha,\beta\}} : N(R)_{\{\alpha,\beta\}}|$ is odd.*

Proof. Suppose that this index is even. Then the involution t', which centralizes R, must normalize a second conjugate of R. Thus, there exists an element $g \in G_{\{\alpha,\beta\}}$ such that $t' \in N(R^g)$ and $R^g \neq R$.

Suppose that $R^g \leqslant C(t')$. Then $R^g \leqslant H$ since $H = C_1(t')_{\alpha\beta} \lhd C(t')_{\{\alpha,\beta\}}$ and H contains an r-Sylow subgroup of G by Lemma 6.9(iv). However, this is impossible as H is cyclic and we assumed that $R^g \neq R$.

Therefore, $R^g \nleqslant C(t')$, and t' inverts every element in R^g. By Lemma 6.8(iii) there exists a 2-involution $y \in C_1(t'^g)_{\{\alpha,\beta\}} - C_1(t'^g)_{\alpha\beta}$ which inverts every element in H^g. Then $yt' \in C(R^g)_{\alpha\beta}$, and by Lemma 6.9(ii), $yt' \in C(H^g)_{\alpha\beta}$. Hence, t' acts on H^g in the same way as y does, and $D = \langle t', H^g \rangle$ is a dihedral group of order $2(q + 1)$. As $z = z^g \in H^g$ and $q \equiv 1 \pmod 4$, $\langle z, t' \rangle$ is a 2-Sylow subgroup of D. Also, $z \in C_1(t')$ and $t' \notin C_1(t')$, so that $zt' \in C(t') - C_1(t')$. By Lemmas 6.7 and 6.5(iii), zt' fixes $q + 1$ points.

Thus, all elements in $D - H^g$ fix $q + 1$ points. Furthermore, each of them interchanges α and β since $D_{\alpha\beta} = H^g$. Hence, $\Delta(x) \cap \{\alpha, \beta\} = \phi$ for all $x \in D - H^g$. Let x_1 and x_2 be involutions in $D - H^g$ and consider $\Delta(x_1) \cap \Delta(x_2)$. Clearly $x_1x_2 \in H^g$. If $\gamma \in \Omega - \{\alpha, \beta\}$, then $(H^g)_\gamma = 1$ by Lemma 6.8(ii). Thus, $\Delta(x_1) \cap \Delta(x_2) = \phi$ if $x_1 \neq x_2$. This implies that

$$\Big| \bigcup_{x \in D - H^g} \Delta(x) \Big| = (q + 1)^2 > q^2 + 1 = n,$$

which is a contradiction.

We can now complete the proof of Theorem 6.1. Let $T = D \times \langle t\rangle$ be a 2-Sylow subgroup of $(C_1(t) \times \langle t\rangle)_{\{\alpha,\beta\}}$, where D is a dihedral group and a 2-Sylow subgroup of $C_1(t)$ (Lemma 6.7). Then $T \leqslant G_{\{\alpha,\beta\}}$ and, by Lemma 6.10, we may assume that T is contained in a 2-Sylow subgroup S of $G_{\{\alpha,\beta\}}$ such that $S \leqslant N(R)_{\{\alpha,\beta\}}$. Since $n = q^2 + 1 \equiv 2 \pmod 4$, S is a 2-Sylow subgroup of G.

As $q \equiv 1 \pmod 4$ we can write $D = \langle e, z'\rangle$, where $|e| = (q - 1)_2$, $\Omega_1(\langle e\rangle) = \langle z\rangle$, z' is a 2-involution, $\langle e, z'\rangle$ is a dihedral group, $T = \langle e, z'\rangle \times \langle t\rangle$, $T_{\alpha\beta} = \langle e\rangle \times \langle t\rangle$, and $\langle e, z'\rangle$ is generated by the 2-involutions of T.

Since R is cyclic of odd order, $T/C_T(R)$ is cyclic. If x is any 2-involution in $C(R)$ then $\Delta(x) \subseteq \Delta(R) = \{\alpha, \beta\}$ and $x = z$. Hence, $\langle e, z'\rangle \cap C(R) = \langle e\rangle$.

Since $T/\langle e\rangle$ is a Klein group, $C_T(R)$ must be a subgroup of index 2 in T containing $\langle e\rangle$. On the other hand, $C_S(R)_{\alpha\beta}$ is cyclic by Lemma 6.9. This implies that $|C_S(R)_{\alpha\beta}| \leqslant |e| = (q - 1)_2$, since G contains no odd permutations. Therefore, $C_S(R)_{\alpha\beta} = \langle e\rangle$, and $C_S(R) = C_T(R)$. Since $S \cap C(R)$ and $S \cap G_{\alpha\beta}$ are normal subgroups of S,

$$\langle e\rangle = C_S(R)_{\alpha\beta} = S \cap C(R) \cap G_{\alpha\beta} \lhd S.$$

By Lemma 6.9, $N(R)_{\alpha\beta}/C(R)_{\alpha\beta}$ is cyclic. Hence, $S_{\alpha\beta}/\langle e\rangle$ is cyclic. Also, $C_T(R) \lhd S$. Then $S/\langle e\rangle = S_{\alpha\beta}C_T(R)/\langle e\rangle$ is abelian with 2 generators, so that $\Omega_1(S) = T$. Therefore, $\langle e, z'\rangle$ is the subgroup of S generated by all 2-involutions, and $\langle e, z'\rangle \lhd S$. Also, $S/\langle e, z'\rangle$ is cyclic and $u \in S - \langle e, z'\rangle$. This contradicts Lemma 2.3 and proves Theorem 6.1.

7. The Unitary and Ree Cases

By Lemma 4.4 and Theorems 5.1 and 6.1, for each involution $u \in G_{\alpha\beta}$, $C_0(u)^{\Delta(u)}$ is PSL(2, q), PSU(3, q), or of Ree type. In this section, we show that the second and third possibilities do not occur, and that Klein groups fix just two points.

Lemmas 3.1, 3.2, and 3.3 will be used very frequently throughout this section.

THEOREM 7.1. *$G_{\alpha\beta}$ contains no Klein group fixing more than 2 points.*

Proof. We begin with two lemmas.

LEMMA 7.2. *For each involution $t \in G_{\alpha\beta}$, W_t contains no Klein group.*

Proof. Let $\langle t, u\rangle$ be a Klein group in W_t. In view of Section 3, $\{|C_Q(t)|, |C_Q(u)|, |C_Q(tu)|\} = \{q, q^2, q^2\}, \{q, q^2, q^3\}$ or $\{q, q^3, q^3\}$, where $q = |C_Q(\langle t, u\rangle)|$. By Lemma 2.1, $n - 1 = q^3, q^4$ or q^5, respectively. By Theorem 6.1, we must have $|C_Q(t)| = q$, $|C_Q(u)| = |C_Q(tu)| = q^3$, and $C_0(u)^{\Delta(u)}$ and $C_0(tu)^{\Delta(tu)}$ are unitary or of Ree type. Thus, by Section 3, neither W_u nor W_{tu} contains a Klein group.

Both $C_0(u) \cap W_u$ and $C_0(tu) \cap W_{tu}$ have odd order (Lemma 4.4). If $t^{\Delta(u)} \in C_0(u)^{\Delta(u)}$, there is a conjugate t' of t in $C(u)$ such that $\langle t, t'\rangle$ is a Klein group. Suppose that $t^{\Delta(u)} \notin C_0(u)^{\Delta(u)}$, so that $C_0(u)^{\Delta(u)}$ is unitary. By Lemma 3.2, $C(t^{\Delta(u)}) \cap C_0(u)^{\Delta(u)}$ does not contain a 2-Sylow subgroup of $C_0(u)^{\Delta(u)}$. Thus, there is an involution t' conjugate to t under $C_0(u)$ such that $\langle t, t'\rangle$ is a Klein group. In either case, $\langle t, t', u\rangle$ is elementary abelian of order 8 and $t \sim t'$.

However, t', $t'^{\Delta(u)}$, and $t'^{\Delta(tu)}$ fix $q + 1$ points (Lemmas 3.2 and 3.3). Thus, $\Delta(t') \subseteq \Delta(u) \cap \Delta(tu) = \Delta$. Then $\Delta(t') = \Delta$, contradicting Lemma 2.1.

We mention one immediate consequence of Lemma 7.2: for each non-regular involution t such that $C_0(t) \approx \mathrm{SL}(2, q)$, $\langle t\rangle = Z(C_0(t))$.

Let $\langle t, u\rangle$ be a Klein group in $G_{\alpha\beta}$ fixing more than 2 points.

LEMMA 7.3. *We may assume that $C(t)_{\alpha\beta}$ contains a 2-Sylow subgroup of $G_{\alpha\beta}$.*

Proof. Let T be a 2-Sylow subgroup of $G_{\alpha\beta}$ containing $\langle t, u\rangle$, and suppose that $v \in \Omega_1(Z(T))^{\#}$, $v \notin \langle t, u\rangle$. If Theorem 7.1 is known for Klein groups in $G_{\alpha\beta}$ containing v, then $\langle t, u\rangle \cap W_v = 1$ and $\langle t, u\rangle^{\Delta(v)}$ contains an involution acting as a field automorphism (Section 3), hence fixing more than 2 points of $\Delta(v)$, a contradiction.

Let S be a 2-Sylow subgroup of $C(t)$ such that $\langle t, u\rangle \leqslant S_{\alpha\beta}$ and $S_{\{\alpha,\beta\}}$ is 2-Sylow in $C(t)_{\{\alpha,\beta\}}$. Set $q = |C_Q(\langle t, u\rangle)|$. Then $|C_Q(t)|$, $|C_Q(u)|$, and $|C_Q(tu)|$ are among the numbers q^2, q^3 since $C_0(\langle t, u\rangle)^{\Delta(\langle t,u\rangle)} = \mathrm{PSL}(2, q)$ (see Section 3). Consequently, $\{|C_Q(t)|, |C_Q(u)|, |C_Q(tu)|\} = \{q^2, q^2, q^2\}$, $\{q^2, q^2, q^3\}$, $\{q^2, q^3, q^3\}$, or $\{q^3, q^3, q^3\}$. By Lemma 2.1, $|Q| = q^4, q^5, q^6$ or q^7, respectively. Theorem 6.1 eliminates the first possibility.

Case 1. $\{q^2, q^2, q^3\}$.

Here $q \equiv 1 \pmod 4$ (Lemma 4.1(vi)). Let $\langle t, u\rangle = \langle a, b\rangle$ with $|C_Q(a)| = |C_Q(b)| = q^2$ and $C_0(ab)^{\Delta(ab)} = \mathrm{PSU}(3, q)$ (Lemma 3.3). If $C_0(\langle a, b\rangle)$ is $\mathrm{SL}(2, q)$ then, since $C_0(\langle a, b\rangle) \leqslant C_0(a) \cap C_0(b)$, both $C_0(a)$ and $C_0(b)$ are $\mathrm{SL}(2, q^2)$ with involutions a and b respectively. Then $a = b$ is the involution in $C_0(\langle a, b\rangle)$, a contradiction.

If $C_0(\langle a, b\rangle)$ is $\mathrm{PSL}(2, q)$, let v be the unique involution in $C_0(ab)_{\alpha\beta}$. Then $C_0(\langle ab, v\rangle) = \mathrm{SL}(2, q)$, $|C_Q(v)|$ and $|C_Q(abv)|$ are q, q^2 or q^3, and

$$q^5 = |Q| = q^3|C_Q(v)|\,|C_Q(abv)|/q^2.$$

By Lemma 7.2, $|C_Q(v)| = |C_Q(abv)| = q^2$, and the argument of the preceding paragraph, applied to $\langle ab, v\rangle$, yields a contradiction.

Case 2. $\{q^2, q^3, q^3\}$.

Once again, $q \equiv 1 \pmod 4$ (Lemma 4.1(vi)), so that S is a 2-Sylow subgroup of G (Lemma 4.6). Suppose that $C_0(t)^\Delta = \mathrm{PSL}(2, q^2)$, so that u^Δ is a field automorphism. Then S has a normal subgroup S_1 such that $u \notin S_1$ and all involutions in $S - S_1$ act on Δ as field automorphisms, and such that S/S_1 is cyclic (Lemma 3.1). Then $u \sim u' \in S_1$ (Lemma 2.3), where u'^Δ fixes 0 or 2 points. Since $t^{\Delta(u')} \in C(u')^{\Delta(u')}$, this is impossible by Lemma 3.2.

Thus, $C_0(t)^\Delta = \mathrm{PSU}(3, q)$. Clearly, $S \trianglerighteq (S \cap C_0(t)) \times (S \cap W)$ with $S \cap C_0(t)$ quasidihedral and $S \cap W$ cyclic or generalized quaternion. If $t \sim t' = (\alpha\beta) \cdots \in S$ (Lemma 2.6), then t' fixes $q + 1$ points of Δ (Lemma 3.2). Thus, there is a Klein group $\langle t, t_1\rangle$ in $S_{\alpha\beta}$ with $t \sim t_1$. Consequently, there is an elementary abelian subgroup X of $S_{\alpha\beta}$ containing t such that $N(X)_{\alpha\beta}$ has an element g of odd order moving t (Lemma 2.5). X contains no Klein group $\langle t, t_2\rangle$ with $t \sim t_2 \sim tt_2$, as otherwise $|Q| = (q^3)^3/q^2$. Thus, $|X| > 4$. On the other hand, $X^\Delta \leqslant C(t)^\Delta_{\alpha\beta}$ implies that $|X| \leqslant 8$ (by Lemma 7.2).

Thus, $|X| = 8$. If $|g| = 7$, we could find a Klein group $\langle t, t_2\rangle$ in X of the above type. Thus, $|g| = 3$, so that X contains a Klein group $\langle v, v'\rangle$ with $v \sim v' \sim vv'$. $|C_Q(v)| = q^2$, as $|C_Q(\langle t, v\rangle)| = q$ and $|Q| \neq (q^3)^3/q^2$. Now the proof of Lemma 4.5 shows that Q is abelian, whereas $C_Q(t)$ is nonabelian.

Case 3. $\{q^3, q^3, q^3\}$.

Once again, S is a 2-Sylow subgroup of G (Lemma 4.6). We have $S \trianglerighteq E \times F$ with $E = S \cap C_0(t)$, $F = S \cap W$, E quasi dihedral, wreathed, or elementary abelian of order 8, and F cyclic or generalized quaternion. By Lemmas 3.2 and 3.3 and the preceding cases, all involutions fix $q^3 + 1$ points.

If $C_0(t)^\Delta$ is of Ree type, then $S = E \times F$ (Lemma 3.3). Clearly, $\Omega_1(S) =$

$E \times \langle t \rangle \leqslant Z(S)$ and $C(t) \cap N(S)$ permutes $\Omega_1(S)^{\#}$ with orbits of lengths 1, 7, 7. Thus, $N(S)$ is transitive on $\Omega_1(S)^{\#}$. It follows that $N(S)/C(S)$ acts on $\Omega_1(S)$ as a subgroup of $\mathrm{GL}(4, 2) \approx A_8$ of order $15 \cdot 7 \cdot 3$, which is impossible.

If $C_0(t)^{\Delta}$ is unitary, then by using a different Klein group if necessary we may assume that $C_0(\langle t, u\rangle) = \mathrm{SL}(2, q)$ (Lemma 3.2). Then $C_Q(\langle t, u\rangle) = Z(C_Q(t)) = Z(C_Q(u)) = Z(C_Q(tu))$, and it follows that $Z(Q) = Z(C_Q(t))$. If $t \sim t_1 \in C(t)_{\alpha\beta}$, then t and t_1 are conjugate in $G_{\alpha\beta}$, so that $Z(C_Q(t)) = Z(Q) = Z(C_Q(t_1))$. By Lemma 3.2(viii), it follows that $S - EF$ contains no conjugate of t. If $t \sim t' \in S - \{t\}$, then $t' \in EF$. As $C_0(t)$ has one class of involutions, we may assume that $t' \in \Omega_1(Z(EF)) \leqslant Z(S)$. By Lemma 2.4, all involutions in $\Omega_1(Z(EF))$ are conjugate. By Lemma 2.3, $\Omega_1(S) \leqslant EF$. Thus, $\Omega_1(S) = \Omega_1(E) \times \langle t \rangle$. However, t is not a square in $\Omega_1(S)$, whereas a central involution in E is a square in $\Omega_1(E) \leqslant \Omega_1(S)$. This contradicts the fact that $N(S)$ is transitive on $\Omega_1(Z(S))^{\#}$.

This completes the proof of Theorem 7.1.

COROLLARY 7.4. *For each nonregular involution t, $C_0(t)$ is* $\mathrm{PSL}(2, q)$ *or* $\mathrm{SL}(2, q)$ *for some q.*

Proof. Theorems 5.1, 6.1 and 7.1.

COROLLARY 7.5. (i) *If t is an involution in $G_{\alpha\beta}$, then $C(t)^{\Delta}_{\alpha\beta} \neq 1$.*

(ii) *If t is an involution weakly closed in a 2-Sylow subgroup of $G_{\alpha\beta}$, and if $C_0(t) = A_4$, then a 2-Sylow subgroup of $G_{\alpha\beta}$ is a Klein group.*

Proof. (i) Otherwise, by Corollary 7.4 and Theorems 5.1 and 7.1, $C_0(t) = \mathrm{SL}(2, 3)$ and $G_{\alpha\beta}$ contains no Klein group. Let S be a 2-Sylow subgroup of $C(t)$. Then, S is a 2-Sylow subgroup of G as $S = EF$ with $E = S \cap C_0(t)$ quaternion of order 8 and $F = S \cap W$ a cyclic or generalized quaternion group. By Lemma 2.6, S contains a conjugate $t^g \neq t$ of t. Since $t^g = ef$ with $e \in E, f \in F$ and $|e| = |f| = 4$, we have $e \in C(t^g)$ but $e^2 \notin \langle t^g \rangle$. However, S contains no element $e^{g^{-1}}$ whose square is not in $\langle t \rangle$, a contradiction.

(ii) By Theorem 7.1 and part (i), a 2-Sylow subgroup S of $C(t)$ has the form $S = TF$, where $T < C_0(t)$ is a Klein group, $|F : F \cap W| = 2$, and $F \cap W$ is cyclic or generalized quaternion. $S - \{t\}$ contains an involution $t' \sim t$ (Lemma 2.6). If $t' \notin T \times (F \cap W)$, then t' fixes 2 points of Δ, which we may assume to be α and β. However, this contradicts the fact that t is weakly closed in a 2-Sylow subgroup of $G_{\alpha\beta}$ (Lemma 4.3). Thus, $t' \in T \times (F \cap W)$ and we may assume that $t' \in Z(S)$. $T \times \langle t \rangle$ is the only subgroup of $C(t)$ that is elementary abelian of order 8 and contains 4 conjugates of t. Thus, $T \times \langle t \rangle$ is weakly closed in $C(t)$ and $T \times (F \cap W) = C_S(T \times \langle t \rangle)$ is also weakly closed in S. Therefore, the fusion of the conjugates of t in $T \times \langle t \rangle$ is controlled by $N(T \times (F \cap W))$. If $|F \cap W| > 2$, then t is the only square in $T \times (F \cap W)$, which is a contradiction. Thus, $|F \cap W| = 2$ and F is a Klein group by Theorem 5.1 (ii).

COROLLARY 7.6. *$G_{\alpha\beta}$ contains no elementary abelian subgroup of order* 8.

Proof. Let X be such a subgroup and $t \in X$. By Theorem 7.1, $X \cap W = \langle t \rangle$ and X^{Δ} contains no field automorphisms, contradicting Lemma 3.1.

THEOREM 7.7. *G is simple.*

Proof. Let $1 \neq K \trianglelefteq G$. If $Q \leqslant K$ then $K = G$ by Lemma 4.1(i). Let $Q \nleqslant K$. As $G = KG_{\alpha} \trianglerighteq KQ$, $G = KQ$. Let t be an involution in $G_{\alpha\beta}$. Then $t \in K$ as $|Q|$ is odd.

Since $[t, Q] \leqslant K \cap Q$, $Q = C_Q(t)(K \cap Q)$, so that $G = KQ = KC_Q(t)$. Let $C_0(t)^{\Delta} = \mathrm{PSL}(2, q)$ with $q = p^e$, p prime. Then G/K is an abelian p-group, and $C_Q(t) \cap G^{(1)} = 1$ as $C(t)_{\alpha\beta}$ is irreducible on $C_Q(t)$. Thus, $[C_Q(t), C(t)_{\alpha\beta}] \leqslant C_Q(t) \cap G^{(1)} = 1$, contradicting Corollary 7.5(i).

THEOREM 7.8. *Suppose that a 2-Sylow subgroup S of G is not dihedral. Then S contains a proper elementary abelian subgroup of order* 8.

Proof. If G has no elementary abelian subgroup of order 8, then, by a result of Alperin [2, Proposition 1], S is (a) the 2-Sylow subgroup of PSU(3, 4), (b) quasidihedral, or (c) wreathed $Z_{2^a} \wr Z_2$.

In (a), $\Omega_1(S) = Z(S)$ is a Klein group. If $t \in Z(S)^{\#}$, then

$$S \trianglerighteq (S \cap C_0(t))(S \cap W).$$

However, S has no normal quaternion subgroup.

Thus, S has the form (b) or (c), or is elementary abelian of order 8. G is not isomorphic to M_{11} [7]. Consequently, for some prime p and $e \geqslant 1$, G is isomorphic to $\mathrm{PSU}(3, p^e)$, $\mathrm{PSL}(3, p^e)$ or a group of Ree type and $p = 3$ (Theorem 7.7, Alperin, Brauer and Gorenstein [1, 2], and Walter [39]).

In view of the known structure of $C(t)$, t an involution, we have $p \mid |Q|$. A p-Sylow subgroup P of G thus fixes just one point, say α, and then $N(P)$ fixes α. If $N(P)$ is maximal in G, then G is $\mathrm{PSU}(3, p^e)$ or of Ree type in its usual 2-transitive representation, which is assumed to be false. Similarly, G is not isomorphic to $\mathrm{PSL}(3, p^e)$.

We remark that the possibility that S is dihedral will not arise in Sections 8 and 9.

8. THE PSL CASE

For each involution $u \in G_{\alpha\beta}$, we have $C_0(u) = \mathrm{PSL}(2, q)$ or $\mathrm{SL}(2, q)$ for some q (Corollary 7.4). In this section, we assume that *each* such group $C_0(u)$ has the form $\mathrm{PSL}(2, q)$; in Theorem 8.9 we will show that this situation does not occur.

Let t be any involution central in a 2-Sylow subgroup of $G_{\alpha\beta}$. Let S be a 2-Sylow subgroup of $C(t)$ such that $S_{\{\alpha,\beta\}}$ is a 2-Sylow subgroup of $C(t)_{\{\alpha,\beta\}}$.

LEMMA 8.1. *Let $C_0(t) = \mathrm{PSL}(2, q)$.*

(i) *$D = C_0(t) \cap S$ is a dihedral group.*

(ii) *$C = W \cap S$ is a cyclic or generalized quaternion group.*

(iii) *$D \times C \trianglelefteq S$.*

(iv) *There is an involution $r \in D \cap Z(S)$.*

(v) *If $v \in S - DC$ is a nonregular involution, then $D\langle v\rangle$ is dihedral and $C\langle v\rangle$ is dihedral or quasidihedral.*

(vi) *$C_0(t)_{\alpha\beta}$ is fixed-point-free on Q if $q \equiv 3 \pmod 4$.*

(vii) *Q is nilpotent if $q \equiv 3 \pmod 4$.*

Proof. (i), (iii), and (iv) are clear. (ii) follows from Theorem 7.1.

If $v \in S - DC$ is a nonregular involution, then $(C_0(t)\langle v\rangle)^{\Delta} = \mathrm{PGL}(2, q)$, so that $(D\langle v\rangle)^{\Delta} \approx D\langle v\rangle$ is dihedral. $C_C(v)$ acts faithfully on $\Delta(v)$ (Theorem 7.1). If $q \equiv 3 \pmod 4$, then $|\Delta \cap \Delta(v)| = 2$, while if $q \equiv 1 \pmod 4$, then $|\Delta \cap \Delta(v)| = 0$. It follows from Lemma 4.1 (vi) that $t^{\Delta(v)}$ is in $C(v)^{\Delta(v)} - C_0(v)^{\Delta(v)}$, and by Theorem 7.1 and Lemma 3.1 $C_C(v) = \langle t\rangle$. Thus, (v) holds.

Let $q \equiv 3 \pmod 4$. Then $C_0(t)_{\alpha\beta}$ is cyclic of odd order $(q-1)/2$. Also, $C_0(t)_{\alpha\beta}$ centralizes $S_{\alpha\beta}^{\Delta}$ and W, so that $C_0(t)_{\alpha\beta}$ centralizes $S_{\alpha\beta}$. Suppose that $1 \neq x \in C_0(t)_{\alpha\beta}$ and $|\Delta(x)| \geq 3$. As x is inverted in $C_0(t)$, it is inverted in $G_{\alpha\beta}$ (Lemma 4.3). Since $C(x)_{\alpha\beta}$ contains a 2-Sylow subgroup $S_{\alpha\beta}$ of $G_{\alpha\beta}$, this is impossible. This proves (vi).

If $(q-1)/2 > 1$, then (vii) follows from a theorem of Thompson [37]. If $(q-1)/2 = 1$, then $C_0(t) = A_4$. By Theorem 5.1, $S_{\alpha\beta}$ contains a Klein group $\langle t, u\rangle$. If $t \sim u$ then $t \sim u$ in $G_{\alpha\beta}$. By Corollary 7.6 and Lemma 2.5, we may assume that $t \sim u \sim tu$, so that $|Q| = 3^3$ (Lemma 2.1) and Q is nilpotent.

We may thus suppose that t is weakly closed in $S_{\alpha\beta}$. Let $t \sim t' = (\alpha\beta) \cdots \in C(t)$ (Lemma 2.6). Then $t' \in C_0(t) \times W$. It follows that $C(t)$ contains 4 or 7 conjugates of t. If $\gamma \in \Omega - \Delta$, then t normalizes $G_{\gamma\gamma^t}$. Since t is weakly closed in $S_{\alpha\beta}$, $G_{\gamma\gamma^t}$ contains an odd number of conjugates of t. Then t centralizes some involution $t_1 \sim t$, $t_1 \in G_{\gamma\gamma^t}$. Since no 2 conjugates of t lying in $C_0(t)W$ fix common points, the 4 or 7 conjugates of t inside $C_0(t)W$ determine a partition of Ω into sets of 4 points. Thus, $n = 28$ or 16, $|Q| = 27$ or 15, and Q is nilpotent, as claimed.

We note that Lemma 8.1(i)–(v) holds for any involution u in $G_{\alpha\beta}$, where S is then taken to be a 2-Sylow subgroup of $C(u)$ such that $S_{\{\alpha,\beta\}}$ is 2-Sylow in $C(u)_{\{\alpha,\beta\}}$.

THEOREM 8.2. *If v is an involution in $G_{\alpha\beta}$ and $C_0(v) = \mathrm{PSL}(2, q)$, then $G_{\alpha\beta}$ contains a Klein group.*

Proof. Suppose that $G_{\alpha\beta}$ contains no Klein group. Then $C(t)_{\alpha\beta} \sim C(v)_{\alpha\beta}$ contains a 2-Sylow subgroup $C_1 \geqslant C$ of $G_{\alpha\beta}$. Also, $|C_0(t)_{\alpha\beta}|$ is odd, so that $q \equiv 3 \pmod 4$. Clearly, $S = DC_1$, $|C_1 : C| \leqslant 2$, and $\Omega_1(S) \leqslant DC$. Let r be as in Lemma 8.1(iv).

LEMMA 8.3. (i) *$t \sim r$ or rt.*

(ii) *We may assume that $r = (\alpha\beta) \cdots$.*

(iii) *C_1 is cyclic.*

Proof. Let $t \sim t' = (\alpha\beta) \cdots \in S$ (Lemma 2.6). Then $t' \in DC$. As $C_0(t)$ has one class of involutions, $t' \sim r$ or rt under $C_0(t)$. Choosing D suitably, we may assume that $t' = r$ or rt. Then $C_1 \leqslant C(t')$, so that $q \equiv 3 \pmod 4$ implies that C_1 is cyclic.

LEMMA 8.4. *S is not a 2-Sylow subgroup of G, and $C(t)$ contains regular involutions.*

Proof. Clearly, the first statement implies the second. Suppose that S is a 2-Sylow subgroup of G. By Lemma 8.3, $S^{(1)} \leqslant D$ and $t \sim r$ or rt, where $\langle r, t\rangle \leqslant Z(S)$.

If D is not a Klein group, then $\langle r, t\rangle = \Omega_1(Z(S))$ and $r \in S^{(1)}$. By Lemma 2.4, $N(S)$ is transitive on $\langle r, t\rangle^{\#}$, whereas $t \notin S^{(1)}$.

Thus, D is a Klein group. If $C_1 > C$ then we may assume that $r^{\Delta} \in Z(S^{\Delta}) \cap (S^{\Delta})^{(1)}$. As above, Lemma 2.4 yields a contradiction. Thus, $C_1 = C$, S is abelian, and $\Omega_1(S) = D \times \langle t\rangle$. As $t \sim t' = r$ or rt and $|C^{\Delta(t')}| = |C|$, S is elementary abelian of order 8, contradicting Theorem 7.8.

LEMMA 8.5. *$t \in Z(G_{\alpha\beta})$.*

Proof. By Lemmas 4.6 and 8.4, $n \equiv 0 \pmod 4$ and Q is not a p-group. By Lemma 8.1(vi), $Q = P \times L$ with $q \mid |P|$, $(|P|, |L|) = 1$, and $L \neq 1$.

Suppose that $t \notin Z(G_{\alpha\beta})$ and let $X = C(L)_{\alpha\beta}$. Then $tX \in Z(G_{\alpha\beta}/X)$ and $[t, X] \neq 1$. Clearly, $|\Delta(X)| \geqslant |L| + 1$. By Lemma 4.3, $\langle r, t\rangle$ acts on $\Delta(X)$. Also, $|C(X)_{\alpha\beta}|$ is odd as t^X is the set of involutions in $G_{\alpha\beta}$. Thus, (i) $C_0(X) = \mathrm{PSL}(2, \ell)$ for some $\ell \equiv 3 \pmod 4$, or (ii) $C_0(X)^{\Delta(X)}$ is solvable.

(i) Suppose that $C_0(X) = \mathrm{PSL}(2, \ell)$. As $S_{\alpha\beta} \approx S_{\alpha\beta}^{\Delta(X)}$, $S_{\alpha\beta} = \langle t\rangle$. Since $(C_0(X)\langle t\rangle)^{\Delta(X)} = \mathrm{PGL}(2, \ell)$, $C_0(X)_{\alpha\beta}\langle t\rangle$ is cyclic. The proof of Lemma 8.1(vi) shows that both $C_0(t)_{\alpha\beta}$ and $C_0(X)_{\alpha\beta}$ are fixed-point-free on Q. Then $C_0(X)_{\alpha\beta} \leqslant C(t)_{\alpha\beta}$ implies that $\frac{1}{2}(\ell - 1) \mid (q - 1)$. Also, as $C_0(t)_{\alpha\beta}$ is fixed-point-free on L, $\frac{1}{2}(q - 1) \mid (|L| - 1) = \ell - 1$. However, $\frac{1}{2}(q - 1)$ and $\frac{1}{2}(\ell - 1)$ are odd, so that $q - 1 = \ell - 1$, a contradiction.

(ii) Thus, $C_0(X)^{\Delta(X)}$ is solvable. Since $C_0(\langle t, X\rangle) \leqslant C_0(X) \cap C_0(t)$, $|\Delta(X)| = 4$ or 16. Consequently, $|L| = 3$ or 5. Since $C_0(t)_{\alpha\beta}$ is fixed-point-free on L and since $|C_0(t)_{\alpha\beta}|$ is odd, it follows that $C_0(t) = A_4$, and this contradicts Theorem 5.1.

LEMMA 8.6. $n = 1 + q(q^2 + 1)/2$.

Proof [22, Lemmas 4.3 and D.1]. If $x = (\alpha, \beta) \cdots$ is an involution then $x \in C(t)$ (Lemma 8.5) and x^{Δ} is regular. There is a conjugate $t_1 \in C_0(t)\langle t\rangle$ of t such that $t_1^{\Delta} = x^{\Delta}$. Now $xt_1 \in W \leqslant C(C_0(t)\langle t\rangle) \leqslant C(t_1)$, so that $(xt_1)^2 = 1$ and $xt_1 \in \langle t\rangle$. There are thus $2 \cdot (q - 1)/2$ involutions $(\alpha, \beta) \cdots$. By Lemma 8.4 there are regular involutions in $C(t)$. It follows that there are $(q - 1)/2$ conjugates of t interchanging α and β. On the other hand, t has $(n - 1)/q$ conjugates in G_α and $n(n - 1)/(q + 1)q$ conjugates in G. Thus,

$$n(n - 1)/(q + 1)q = (n - 1)/q + (n - 1)(q - 1)/2,$$

which implies that $n = 1 + q(q^2 + 1)/2$.

LEMMA 8.7. *$C_0(t)_{\alpha\beta}C_1W$ is cyclic.*

Proof. Let $x \in (C_1W)^{\#}$ have prime order and fix a point not in Δ. Then $|x|$ is odd and $x \in W$. Thus, $\Delta \subset \Delta(x)$ and, by Lemmas 8.5 and 8.6, $C(x)^{\Delta(x)}$ is solvable. As $C_0(t) < C_0(x)$, $|\Delta(x)| = 16$ and $C_0(t) = A_4$, contradicting Theorem 5.1.

Thus, if $t \sim t' = r$ or rt, then $(C_1W)^{\Delta(t')}$ is semiregular. It follows that C_1W is cyclic of order dividing $q + 1$. Also, $C_0(t)_{\alpha\beta}C_1W/W$ is cyclic and $W \leqslant Z(C_0(t)_{\alpha\beta}C_1W)$, so that $C_0(t)_{\alpha\beta}C_1W$ is abelian. As $|C_0(t)_{\alpha\beta}| = (q - 1)/2$, $C_0(t)_{\alpha\beta}C_1W$ is cyclic.

We can now complete the proof of Theorem 8.2. By Lemma 8.7 and [22, Theorem 1.1 or Lemma D.5], $G_{\alpha\beta} > C_0(t)_{\alpha\beta}C_1W$. That is, $C(t)^{\Delta}$ must have odd field automorphisms. Let $q = q'^b$ with b an odd prime.

By Lemmas 8.1 and 8.6, $Q = C_Q(t) \times L$ with $|L| = (q^2 + 1)/2$, and $C_0(t)_{\alpha\beta}\langle t\rangle$ is fixed-point-free on L. If L has a proper nontrivial characteristic subgroup L_1 then we have $|L_1| \geqslant q - 1$ and $|L/L_1| \geqslant q - 1$, whereas $|L| = (q^2 + 1)/2$. Thus, L is an ℓ-group for some prime ℓ.

We have $q'^{2b} + 1 = q^2 + 1 = 2\ell^a$ for some a. Then $q'^2 + 1$ is an even divisor of $2\ell^a$, so that $q'^2 + 1 = 2\ell^{a'}$, $a' < a$. Now

$$2\ell^a = (2\ell^{a'} - 1)^b + 1 > \ell^{a'b},$$

so that $a \geqslant a'b \geqslant 3a'$. Then

$$0 = 2\ell^{a'}(-1)^{b-1}\binom{b}{b-1} + (2\ell^{a'})^2(-1)^{b-2}\binom{b}{b-2} \pmod{\ell^{3a'}},$$

or $0 = 2b - 4\ell^{a'}b(b-1)/2 \pmod{\ell^{2a'}}$. Thus, $0 = 2b \pmod{\ell^{a'}}$, $b = \ell^{a'}$ (as b is prime), and finally $0 = 2b - 4\ell^{a'}b(b-1)/2 = 2\ell^{a'} \pmod{\ell^{2a'}}$.

This contradiction proves Theorem 8.2.

THEOREM 8.9. *For some involution $u \in G_{\alpha\beta}$, $C_0(u) = \mathrm{SL}(2, q)$ for some q.*

Proof. Assume that $C_0(u)$ has the form $\mathrm{PSL}(2, q)$ for each nonregular involution u. By Theorem 8.2, $G_{\alpha\beta}$ contains a Klein group. Let t be an involution central in a 2-Sylow subgroup of $G_{\alpha\beta}$. We use the notation of Lemma 8.1. Let $\langle t, u\rangle$ be a Klein group in $S_{\alpha\beta}$.

LEMMA 8.10 (Bender [4, Lemma 3.8]). *Let $r = (\alpha\beta)\cdots$ be an involution, and let $a, b \in Q$ satisfy $ab = ba$ and $(ar)^3 = (br)^3 = 1$. Then $b = a$ or a^{-1}.*

Proof. Assume that $b \neq a$. Set $e = (a^{-1}b)^r \notin G_\alpha$. Then

$$arara = brbrb = r,$$
$$a = ra^{-1}brbrba^{-1}r = ra^{-1}br \cdot b \cdot ra^{-1}br$$

(as $ab = ba$), so that $a = ebe$. Set $f = b^{-1}(ba)^{1/2}$. Then $f \in Q$ and

$$fbf = b^{-1}(ba)^{1/2} \cdot b \cdot b^{-1}(ba)^{1/2} = b^{-1}(ba) = a.$$

Now $ebe = a = fbf$,

$$(f^{-1}e)^b = b^{-1}f^{-1}eb = fe^{-1} = ((f^{-1}e)^{-1})^{f^{-1}},$$

and hence $(f^{-1}e)^{bf} = (f^{-1}e)^{-1}$. However, $bf \in Q$ has odd order, so that $(bf)^2 \in C(f^{-1}e)$ implies that $bf \in C(f^{-1}e)$. As $f \in Q$ and $e \notin G_\alpha$, $f^{-1}e \notin G_\alpha$. Then $bf \in Q$ fixes both α and $\alpha^{f^{-1}e}$, so that $bf = 1$ and $a = fbf = b^{-1}$.

LEMMA 8.11. *Suppose that u is a nonregular involution. Let S_1 be a 2-subgroup of $C(u)$ and let v be a nonregular involution in $C(u)$. Assume:*

(a) $S_1 \cap C_0(u) = D_1$ *is dihedral*;
(b) $S_1 \cap W_u = C_1$ *is cyclic*;
(c) $S_1 = (D_1C_1)\langle v\rangle$;
(d) $v^{\Delta(u)} \notin C_0(u)^{\Delta(u)}$; *and*
(e) $|D_1| \leqslant |C_1|$.

Then $N(S_1) \leqslant C(u)$.

Proof. As in Lemma 8.1, $D_1\langle v\rangle$ is dihedral and $C_1\langle v\rangle$ is dihedral or quasidihedral. Let $D_1\langle v\rangle = \langle e, v\rangle$ and $C_1 = \langle f\rangle$, where $|e| = |D_1| \leqslant |f|$. As $S_1 = (D_1 \times C_1)\langle v\rangle$, $S_1^{(1)} = \langle e^2, f^2\rangle$. Thus, $\Omega_1(S_1^{(1)}) = \langle r, u\rangle$, where $\langle r\rangle = Z(D_1\langle v\rangle)$.

We claim that u is the only involution in $\langle r, u\rangle$ contained in a normal cyclic subgroup of S_1 of order $|f|$. Clearly, $C_1 \lhd S_1$. Let $h \in S$ and suppose that $|h| = |f|$, $\langle h\rangle \lhd S_1$, and $u \notin \langle h\rangle$. As $|f| > |e^2|$, $D_1 \times C_1$ has exponent $|f|$ and $h \notin D_1C_1$. Also, $[f, h] \in \langle f\rangle \cap \langle h\rangle = 1$. However, h acts on $\langle f\rangle$ as v does, and $C_1\langle v\rangle$ is dihedral or quasidihedral. This is a contradiction as $|C_1| \geqslant |D_1| \geqslant 4$.

Thus, $N(S_1) \leqslant C(u)$.

LEMMA 8.12. *If $t \in Z(S_{\alpha\beta})$ is suitably chosen, then S is a 2-Sylow subgroup of G.*

Proof. Otherwise, for each involution $t \in Z(S_{\alpha\beta})$, $C(t)$ does not contain a 2-Sylow subgroup of G.

By Lemmas 2.1, 2.5, 4.1(vi), 4.5, and 4.6, and Corollary 7.6, we have $n \equiv 0 \pmod 4$, $|\Delta| \equiv |\Delta(u)| \equiv |\Delta(tu)| \equiv 0 \pmod 4$, and $t \not\sim u$, tu. We thus have $S = (D \times C)\langle u\rangle$, and all conjugates of t are in DC. By Lemma 8.1, $D\langle u\rangle$ is dihedral, say $\langle e, u\rangle$ with $|e| = |D|$, and $C\langle u\rangle$ is dihedral or quasidihedral, say $\langle f, u\rangle$ with $|f| = |C|$. Choose r as in Lemma 8.1(iv). We have $S^{(1)} = \langle e^2, f^2\rangle$ and $\langle r, t\rangle = \Omega_1(Z(S) \cap S^{(1)})$.

By hypothesis, $N(S)$ moves t to $t' = r$ or rt. As $C \leqslant C(t')$, C is cyclic. By Lemma 8.11, $|e^2| > |f^2|$. Thus, $N(S) \leqslant C(r)$ and $t' = rt$.

Clearly, $C(r)$ has a 2-Sylow subgroup $R > S$. We claim that r is a regular involution. For otherwise, $C_0(r) = \mathrm{PSL}(2, m)$, $m \equiv 3 \pmod 4$ (Lemma 4.1(vi)). Then $R \trianglerighteq (R \cap C_0(r)) \times (R \cap W_r)$, where $R \cap W_r$ is cyclic or generalized quaternion. As $\Delta \cap \Delta(r) = \phi$, $t \in (R \cap C_0(r))(R \cap W_r)$. It follows that t is conjugate in $C(r)$ to an involution in $Z(R)$, which is not the case. Thus, r is regular.

As $|Q| = n - 1 \equiv 3 \pmod 4$, Q is not a p-group (Lemma 4.6). By Lemma 2.1, we may assume that $C_0(u) = \mathrm{PSL}(2, \ell)$ with $(q, \ell) = 1$. By Lemma 4.1(vi), $\ell \equiv 3 \pmod 4$. As $r \in C(u)$, $r \in C_0(u)W_u$.

If $u \in Z(S_{\alpha\beta})$ we can repeat our previous argument and find a regular involution $r' \in C_0(u)$ such that $u \sim ur'$. Since $C_0(u)$ has a single class of involutions, it follows that $r \in C_0(u)$.

If $u \notin Z(S_{\alpha\beta})$ let $S_1 = (D_1 \times C_1)\langle t\rangle$ be a 2-Sylow subgroup of $C(u)$, with $D_1\langle t\rangle$ dihedral and $C_1\langle t\rangle$ dihedral or quasidihedral. If S_1 is a 2-Sylow subgroup of G, then some conjugate u_1 of u centralizes $S_{\alpha\beta} = C\langle u\rangle$. We may then assume that $u_1 \in C_S(C\langle u\rangle) \leqslant DC$, whereas u is not conjugate to any element of $\langle r, t\rangle$. Thus, $N(S_1)$ moves u to some other element u' of $Z(S_1)$.

Then $C_1 \leqslant C(u')$ implies that C_1 is cyclic. By Lemma 8.11, $|D_1| > |C_1|$. As before, an involution in $Z(S_1) \leqslant D_1C_1$ centralized by a 2-Sylow subgroup of $N(S_1)$ must be in D_1. Since $\Omega_1(Z(S_1)) = \Omega_1(Z(D_1\langle t\rangle)) \times \langle u\rangle$ contains u', and since $r \in C_0(u)W_u$, some conjugate of r is in D_1. Thus, we again find that $r \in C_0(u)$.

We may assume that $r = (\alpha\beta) \cdots$. Since $r \in C_0(t) \cap C_0(u)$, $C_0(t) = \mathrm{PSL}(2, q)$, and $C_0(u) = \mathrm{PSL}(2, \ell)$, we can find elements $a \in C_Q(t)$ and $b \in C_Q(u)$ such that $(ar)^3 = 1 = (br)^3$. However, $(|a|, |b|) = 1$ and Q is nilpotent (Lemma 8.1(vii)), contradicting Lemma 8.10.

The proof of Theorem 8.9 now splits into four cases.

Case 1. $q \equiv 3 \pmod 4$ and $C(t)_{\alpha\beta} - \{t\}$ contains no conjugate of t.

Here $S = (D \times C)\langle u\rangle$. By Lemma 2.3, $u \sim r$ or rt. If $t \sim t' = (\alpha\beta) \cdots \in S$ (Lemma 2.6) then $t' \in DC$, and we also have $t' \sim r$ or rt. Since $\Omega_1(Z(S)) = \langle r, t\rangle$, all involutions in $\langle r, t\rangle$ must be conjugate (Lemma 2.4). Then $u \sim t$, which is not the case.

Case 2. $q \equiv 3 \pmod 4$ and there is a Klein group $\langle t, u\rangle$ in $G_{\alpha\beta}$ with $t \sim u$.

By Corollary 7.6 and Lemma 2.5 we may assume that $t \sim u \sim tu$. Once again, $\Omega_1(Z(S)) = \langle r, t\rangle$. Suppose that two of r, t, rt are conjugate. Then all are conjugate in $N(S)$ (Lemma 2.4). As $C \leqslant C(r)$, C is cyclic, say $C = \langle f\rangle$. By Lemma 8.1, $D\langle u\rangle$ is dihedral, say $D = \langle e, u\rangle$ with $|e| = |D|$, and $C\langle u\rangle = \langle f, u\rangle$ is dihedral or quasidihedral. Thus, $S^{(1)} = \langle e^2, f^2\rangle$, where $N(S)$ is transitive on $\Omega_1(S^{(1)})^{\#}$, so that $|e^2| = |f^2|$, contradicting Lemma 8.11.

Thus, r, t and rt are nonconjugate. As $u \sim t$ and $C\langle u\rangle$ is dihedral or quasidihedral, $S_{\alpha\beta} = C\langle u\rangle$ has at most one class of involutions $\nsim t$. Since $S_{\alpha\beta}$ is a 2-Sylow subgroup of $G_{\alpha\beta}$, $s = r$ or rt is a regular involution. In particular, no conjugate of s is in $S - DC$.

Let $g \in G$ be such that $u^g = t$ and $\langle t, s, u\rangle^g \leqslant S$. Then $s^g \in DC$ and $u^g = t$ imply that $s^g t \nsim s^g, t$ and hence $s^g t \sim st$. Also, $su \sim s^g u^g = s^g t \sim st \nsim t \sim u$.

If r is a regular involution, take $s = r$. As $D\langle u\rangle$ is dihedral of order $\geqslant 8$, $u \sim ru = su$, a contradiction.

Thus, $s = rt$, and $su \sim st$ states that $r(tu) \sim r$. However, from the dihedral group $D\langle tu\rangle$ we find that $tu \sim r(tu)$. Then $t \sim tu \sim r(tu) \sim r$, a contradiction.

Case 3. $q \equiv 1 \pmod 4$, and $C(t)_{\alpha\beta} - \{t\}$ contains no conjugate of t.

As usual, $\langle r, t\rangle \leqslant Z(S)$, where now $r \in G_{\alpha\beta}$. Thus, none of r, t and rt are conjugate, and we have $t \sim t' = (\alpha\beta) \cdots \in S - DC$ (by Lemma 2.6). Let

$g \in G$ be such that $t'^g = t$ and $\langle r, t, t'\rangle^g \leqslant S$. Then $t^g \in S - DC$, $|\langle r^g, t^g\rangle \cap DC| = 2$, and hence r^g or $(rt)^g \in DC$.

If $r^g \in DC$ then $r^g t \not\sim r, t$ implies that $r^g t \sim rt$. However, as $D\langle t'\rangle$ is dihedral, $t \sim t' \sim rt' \sim r^g t'^g = r^g t \sim rt$, a contradiction.

Thus, $(rt)^g \in DC$. As above, $t' \sim rt'$. Now $S \geqslant DC\langle r^g\rangle$, where $C\langle r^g\rangle$ is dihedral or quasidihedral (Lemma 8.1). If $|C| > 2$, then $r^g \sim r^g t = (rt')^g \sim t'^g$, a contradiction. Thus, $C = \langle t\rangle$ and $\Omega_1(S) = D\langle t'\rangle \times \langle t\rangle$.

Clearly $\Omega_1(S_{\alpha\beta}) = \langle r, t\rangle$. It follows that $C(r)$ contains a 2-Sylow subgroup S of G, and $C(r)_{\alpha\beta}$ contains a 2-Sylow subgroup $S_{\alpha\beta}$ of $G_{\alpha\beta}$, but $C(r)_{\alpha\beta} - \{r\}$ contains no conjugate of r. Replacing t by r in the preceding argument, we find that r, like t, is not a square of an element of $\Omega_1(S)$. Since r is certainly a square in $D\langle t'\rangle$, this is a contradiction.

Case 4. $q \equiv 1 \pmod 4$ and $C(t)_{\alpha\beta} - \{t\}$ contains a conjugate of t.

By Corollary 7.6, $\Omega_1(S_{\alpha\beta}) = \langle r, t\rangle$, so that t, r and rt are conjugate in $N(S_{\alpha\beta})_{\alpha\beta}$ and hence in $N(S)$ (Lemma 2.4). Also, $D \times C \trianglelefteq S$ with S/DC abelian (Lemma 3.1). If $S^{(1)} \leqslant D$, then $S^{(1)} = 1$.

Suppose that $S - DC$ contains no involutions. Then $\Omega_1(S) = D \times \langle t\rangle$. However, $r \sim t$ in $N(\Omega_1(S))$, so we have $|D| = 4$. Then $q \not\equiv 1 \pmod 8$, so that q is not a square and $|S : DC| \leqslant 2$. Since $S = DS_{\alpha\beta}$, r is not a square in S. As $r \sim t$ in $N(S)$, we have $|C| = 2$ and $S = D \times \langle t\rangle$ is elementary abelian of order 8. Although this already contradicts Theorem 7.8, we wish to point out the simple reason why this is impossible. Clearly, $N(S)/C(S)$ is a Frobenius group of order 21. By Lemma 4.5, $n = q^3 + 1$. As $C(S) = S \times O(W) = S \times O(W_u) = S \times O(W_{tu})$, $O(W)$ fixes Ω pointwise, so that $C(S) = S$. Since $N(S)$ is transitive on the Klein groups in S, there is an element $g \in N(S) \cap N(\langle t, u\rangle)$ such that $\langle t, u\rangle\langle g\rangle \approx A_4$. As g normalizes $\langle t, u\rangle$, $g \in G_{\alpha\beta}$. Since $\langle t, u, g\rangle$ acts on Q and $C_Q(\langle t, u\rangle) = 1$, we have $C_Q(g) \neq 1$. Now $C_0(g)$ is not $\mathrm{SL}(2, \ell)$ for some ℓ, since G contains no quaternion subgroup. Thus, $C_0(g)$ contains a Klein group $\langle v, v'\rangle$. This group is conjugate to $\langle t, u\rangle$; hence $\langle v, v'\rangle$ fixes 2 points, say γ and δ. Then g must fix γ and δ. However, $\langle v, v'\rangle \leqslant C_0(g)$, so that $\langle v, v'\rangle$ cannot fix points of $\Delta(g)$, a contradiction.

Consequently, there is an involution $v \in S - DC$. By Theorem 7.1 and Lemma 3.1, $\Omega_1(S) \leqslant DC\langle v\rangle$ and S/DC is abelian. Since $r \sim t \sim rt$, $v \sim t$ by Lemma 2.2.

By Lemma 8.1, $D\langle v\rangle = \langle e, v\rangle$ and $C\langle v\rangle = \langle f_1, v\rangle$ with $|D| = |e|$ and $|C| = |f_1|$. Then $\Omega_1(S) \geqslant \langle e, f_1^2, v\rangle$. Since $\Omega_1(S)/D \leqslant \Omega_1(DC\langle v\rangle/D) \approx \Omega_1(\langle f_1, v\rangle)$, we have $\Omega_1(S) = \langle e, f, v\rangle$ with $f = f_1$ or f_1^2. If C is cyclic, then $f \in C$, while if C is generalized quaternion, then once again $f = f_1^2 \in C$. Thus, $\Omega_1(S) = (\langle e^2, ev\rangle\langle f\rangle)\langle v\rangle$ with $\langle e^2, ev\rangle$ and $\langle f, v\rangle$ dihedral groups and $v \notin \langle e^2, ev\rangle\langle f\rangle$. Now $\Omega_1(S)^{(1)} = \langle e^2, f^2\rangle$ and $N(\Omega_1(S))$ is transitive on

$\langle r, t\rangle^{\#}$, so that $|\langle e^2, ev\rangle| = |f|$. Applying Lemma 8.11 to $\Omega_1(S)$ thus yields a contradiction.

This completes the proof of Theorem 8.9.

9. The SL Case

In view of the preceding sections, the proof of Theorem 1.1 will be completed once we have proved.

THEOREM 9.1. *For each involution $t \in G_{\alpha\beta}$, $C_0(t)$ is not* $\mathrm{SL}(2, q)$.

Proof. Assume that $C_0(t) = \mathrm{SL}(2, q)$ for some involution $t \in G_{\alpha\beta}$. We begin by introducing some of the notation to be used in Section 9.

LEMMA 9.2. *Let S be a 2-Sylow subgroup of $C(t)$ such that $S_{\{\alpha,\beta\}}$ is a 2-Sylow subgroup of $C(t)_{\{\alpha,\beta\}}$.*

(i) *$E = C_0(t) \cap S$ is a generalized quaternion group of order $(q^2 - 1)_2 = 4k$, where k is a power of* 2.

(ii) *$F = W \cap S$ is cyclic or generalized quaternion of order* ≥ 4.

(iii) *$E \lhd S, F \lhd S, E \cap F = \langle t\rangle$ and $[E, F] = 1$.*

(iv) *E and F have cyclic subgroups $\langle e_1\rangle$ and $\langle f_1\rangle$, respectively, which are normal in S, such that $|e_1| = \frac{1}{2}(q^2 - 1)_2 = 2k$ and $|F : \langle f_1\rangle| = 1$ if F is cyclic or* 2 *if F is generalized quaternion.*

(v) *S is a 2-Sylow subgroup of G.*

Proof. As E is a 2-Sylow subgroup of $C_0(t)$, we have (i). By Theorem 7.1, F is cyclic or generalized quaternion.

By Lemma 3.1(ii), $S^{(1)} \leq EF$. Thus, $N(S)$ normalizes $\Omega_1(Z(S) \cap S^{(1)}) = \langle t\rangle$, and (v) follows.

If $|F| = 2$ and $t \sim t' = (\alpha, \beta) \cdots \in S$ (Lemma 2.6), then $C_E(t') = \langle t\rangle$ and $\langle t'\rangle E$ is quasidihedral since t is not a square in $C(t')$ (Lemma 3.1 and Theorem 7.1). Also, $\Omega_1(S) \leq \langle t'\rangle E$ (Lemma 3.1). This contradicts Theorem 7.8, and proves (ii).

LEMMA 9.3. *Let $t \sim u_0 \in EF - \langle t\rangle$.*

(i) *There is an element $e_2 \in S - EF$ such that $e_1 \sim e_2$, $\langle u\rangle = \Omega_1(\langle e_2\rangle) \leq EF$ and $t \neq u \sim t$.*

(ii) *If F is a generalized quaternion group, then*

$$e_2^{\,4} \in \mathrm{PGL}(2, q) - \mathrm{PSL}(2, q).$$

(iii) *If F is cyclic and $S - EF$ contains an involution, then*

$$e_2^\Delta \in \mathrm{PGL}(2, q) - \mathrm{PSL}(2, q).$$

(iv) *If the hypotheses of* (ii) *or* (iii) *hold then $e_2^2 = rs$ with $r \in E$, $s \in F$ and $|f_1| \geqslant |e_1| = |e_2| = |s| = 2k$.*

Proof. (i) As $C_0(t)$ has just one class of elements of order 4, we may assume that $u_0 = xy$ with $x \in \langle e_1 \rangle$, $y \in F$ and $|x| = |y| = 4$. Then $e_1 \in C(u_0)$, $u_0 \notin \langle e_1 \rangle$, so that S contains an element e_2 such that $e_2 \sim e_1$ and $t \notin \langle e_2 \rangle$. Since $|e_2| \geqslant 4$ and no involution in $S - EF$ is a square (Lemma 3.1 and Theorem 7.1), $\langle u \rangle = \Omega_1(\langle e_2 \rangle) \leqslant EF$. Since $e_2^k \notin F$, $e_2 \notin EF$.

(ii) We can find $a \in E$ and $b \in F$ such that $|ab| = 2$, $x^a = x^{-1}$, $y^b = y^{-1}$ and $e_1^a = e_1^{-1}$. Then $\langle e_1, ab \rangle$ is a dihedral group centralizing u_0. Thus, S contains a dihedral group $\langle e_2, g \rangle$ with $|g| = 2$. Here g or $e_2 g$ is not in EF. By Lemma 3.1 and Theorem 7.1, g or $e_2 g$ is in EF and $(C_0(t)\langle e_2 \rangle)^\Delta = \mathrm{PGL}(2, q)$.

(iii) Let v be an involution in $S - EF$. Since $(E\langle v \rangle)^\Delta$ is dihedral, $\langle e_1, v \rangle$ is dihedral. Also, v is a nonregular involution, as otherwise $0 \equiv n \equiv q + 1 \pmod 4$ by Lemma 4.1(iii), whereas $v^\Delta \notin \mathrm{PSL}(2, q)$ is a regular involution. Then $C_F(v) = \langle t \rangle$, as otherwise t is a square in $C(v)$, contradicting Theorem 7.1, Lemma 3.1 and the fact that $|\Delta| \equiv |\Delta(v)| \pmod 4$ (Lemma 4.1(vi)). It follows that v inverts the subgroup $\langle y \rangle$ of order 4 in the cyclic group F. As in (ii), from the dihedral group $\langle e_1, v \rangle \leqslant C(u_0)$, we obtain a dihedral group $\langle e_2, g \rangle$ in S, and (iii) follows as above.

(iv) By (ii) and (iii), $e_2^2 \in EF$. Then $e_2^2 = rs$, where $r \in E$, $s \in F$ and $|r| = |s|$ as $u \neq t$. Since $|e_2^2| = |e_1|/2$, (iv) follows.

LEMMA 9.4. $S > EF$.

Proof. This follows from Lemmas 2.6 and 9.3(i).

LEMMA 9.5. *Let v be an involution in $S - EF$.*

(i) *v is a nonregular involution.*

(ii) *$E\langle v \rangle$ is quasidihedral, and $F\langle v \rangle$ is dihedral or quasidihedral.*

(iii) *$E\langle v \rangle = \langle e, v \rangle$, where $|e| = |E| = 4k$ and $e^v = e^{-1}t$.*

(iv) *If $a \in E$ and $b \in F$ have order* 8, *$b^v = b^{-1}$, and $v \sim t$, then $a^2 b^2 \sim t$.*

Proof. (i) If $q \equiv 3 \pmod 4$ this is clear. If $q \equiv 1 \pmod 4$ this follows from Lemma 4.1 (vi).

(ii) By Lemma 4.1(vi), $|\Delta| \equiv |\Delta(v)| \pmod 4$. Thus, $t^{\Delta(v)}$ is not a square, so that $C_E(v) = C_F(v) = \langle t \rangle$.

(iii) This follows from (ii).

(iv) Since $a \in \langle e^2 \rangle$, $a^v = a^{-1}$. Then $(ab)^v = (ab)^{-1}$, so that $\langle ab, v \rangle$ is dihedral of order 8. It follows that $v \sim uv$, where $u = a^2b^2$.

If $t \sim v' \in S - EF$, then $v' = vrs$, $r \in E$, $s \in F$. Since $b^{v'} = (b^{-1})^{rs} = b$ or b^{-1} and $F\langle v' \rangle$ is dihedral or quasidihedral, $b^{v'} = b^{-1}$.

As $\langle t, u, v \rangle \leqslant C(v)$, we can find $g \in G$ such that $v^g = t$ and $\langle t, u, v \rangle^g \leqslant S$. Now $u^g = (vvu)^g = tv'$, where $v' = (vu)^g \sim v^g \sim t$. If $v' \in S - EF$, then, using the dihedral group $\langle v', b \rangle$, we find that $v' \sim tv'$. Thus, $u \sim u^g = tv' \sim v' \sim t$. If $v' \in EF$, then there is a dihedral group $\langle r_1, v' \rangle$ with $r_1 \in E - \langle t \rangle$ and once again $v' \sim tv'$.

LEMMA 9.6. *F is generalized quaternion.*

Proof. Assume that F is cyclic. If there are no involutions in $S - EF$, then S has no elementary abelian subgroup of order 8, contradicting Theorem 7.8. Let v be an involution in $S - EF$. Then $S = EF\langle v \rangle\langle a \rangle$, where a^Δ is a field automorphism. By Theorem 7.1, $F\langle a \rangle$ is cyclic or generalized quaternion.

Since $C_0(t)$ has one class of elements of order 4, all involutions in $EF - \langle t \rangle$ are conjugate in $C(t)$. $\Omega_1(S) \leqslant EF\langle v \rangle$ (Theorem 7.1) and S/EF is abelian (Lemma 3.1). By Lemma 9.5(i), v is nonregular.

Define e_1, f_1 and e as in Lemmas 9.2(iv) and 9.5(iii). Then, $\langle e_1 \rangle = \langle e^2 \rangle$ and $e \in EF\langle v \rangle - EF$.

Suppose that all involutions in EF are conjugate to t. Let e_2 be as in Lemma 9.3. We have $f_1^v \in f_1^{-1}\langle t \rangle$, $e_2^\Delta \in \mathrm{PGL}(2, q) - \mathrm{PSL}(2, q)$, $e_2 \in EF\langle v \rangle - EF$, and $|e_2| > 2$. Then $e_2 = xyv$, $x \in E$, $y \in F$, so that

$$e_2^2 = xyvxyv \in xvy^{-1}\langle t \rangle xyv = x(\langle t \rangle x)^v \subseteq E,$$

and $t \in \langle e_2 \rangle$, a contradiction.

Thus, the involutions in $EF - \langle t \rangle$ are not conjugate to t, and we may assume that $v \sim t$ (Lemma 2.6). Since e^Δ is an odd permutation, whereas f_1^Δ is even, we have $|f_1| = |f_1^{\Omega-\Delta}| \leqslant \frac{1}{2}|\Omega - \Delta|_2 = \frac{1}{2}|e| = |e_1|$. By Lemma 9.5(iv), either $|f_1| = 4$ or $\langle v, f_1 \rangle$ is quasidihedral of order 16.

We claim that $S = EF\langle v \rangle$. For suppose that $a^\Delta \neq 1$, and set $S_0 = EF\langle v \rangle\langle a^2 \rangle$. By Lemma 2.2, there is an integer m and a $g \in G$ such that $(a^m)^g \in S$, but $(a^m)^g \not\equiv a^m \pmod{S_0}$. If $t^g \in EF$ then $t^g = t$, $(a^m)^g = a^m[a^m, g]$, and $[a^m, g]^\Delta \in (C(t)^\Delta)^{(1)} = C_0(t)^\Delta$, so that $[a^m, g] \in C_0(t)W \cap S = EF \leqslant S_0$, a contradiction. If $t^g \notin EF$ then t^g is not a square in S, so that $(a^m)^g t^g$ and $a^m t$ are in $EF\langle v \rangle \leqslant S_0$, a contradiction, proving our claim.

If $|F| = 4$ then $S = (E\langle v \rangle)\langle u \rangle$ for an involution $u \in EF - \langle t \rangle$. All involutions in the quasidihedral group $E\langle v \rangle$ are conjugate to t. As $u \not\sim t$, this contradicts Lemma 2.3.

Thus, $\langle v, f_1\rangle$ is quasidihedral of order 16. Set $S_1 = E\langle f_1^2\rangle\langle v\rangle = \langle e, v, f_1^2\rangle$. A conjugate of t lying in $S - S_1$ would have the form $e^i v f_1^j$ with j odd. If i is odd then $e^i v \in E \leqslant C(F)$ and $|e^i v f_1^j| = |f_1^j| = 8$. If i is even then

$$e^i v f_1^j e^i v f_1^j = e^i (f_1^j)^v (e^i)^v f_1^j = e^i f_1^{-j} t e^{-i} f_1^j = t.$$

Thus, S_1 contains all conjugates of t lying in S.

By Lemma 2.2, there is an integer m and a $g \in G$ such that $(f_1^m)^g \in S$ but $(f_1^m)^g \not\equiv f_1^m \pmod{S_1}$. We have seen that $f_1^m \neq t$. Thus, $|f_1^m| \geqslant 4$ and $(f_1^{2m})^g \in EF$. It follows that $t^g = t$, $\langle f_1^m\rangle^g = W \cap S = \langle f_1^m\rangle$, and hence $(f_1^m)^g \equiv f_1^m \pmod{\langle f_1^2\rangle}$, a contradiction.

LEMMA 9.7. *$S_{\alpha\beta}$ is a 2-Sylow subgroup of $G_{\alpha\beta}$.*

Proof. By Lemma 9.6, no involution in $EF - \langle t\rangle$ centralizes F. By Lemma 9.5(ii), the same is true of each involution in $S - EF$.

LEMMA 9.8. (i) *There is an involution in $S - EF$.*
(ii) *$G_{\alpha\beta}$ contains a Klein group.*

Proof. If $q \equiv 1 \pmod 4$, (ii) follows from Lemma 9.6. If $q \equiv 3 \pmod 4$, (i) and (ii) are equivalent. Assume that $S - EF$ contains no involution.

By Lemma 2.6, $EF - \{t\}$ contains a conjugate of t. By Lemmas 9.6 and 9.3(ii), $S - EF$ contains an element b such that b^Δ is an involution in $\mathrm{PGL}(2, q) - \mathrm{PSL}(2, q)$. Then $b^2 \in F^\#$ and $F\langle b\rangle$ is a generalized quaternion group (Lemma 9.6). Thus, $F\langle b\rangle = \langle f, b\rangle$ with $f^b = f^{-1}$ and $|f| \geqslant 8$. We may now assume that $|b| = 4$.

Since $b^\Delta = f^\Delta$ is an involution in $\mathrm{PGL}(2, q) - \mathrm{PSL}(2, q)$, it is an odd permutation. Then $b^{\Omega-\Delta}$ and $f^{\Omega-\Delta}$ are also odd permutations. However, $t \in \langle b\rangle \cap \langle f\rangle$, so that $\langle b\rangle^{\Omega-\Delta}$ and $\langle f\rangle^{\Omega-\Delta}$ are semiregular and have different orders, a contradiction.

LEMMA 9.9. *$S = EF\langle v\rangle$, where E and F are generalized quaternion groups of order $4k$, $E \cap F = \langle t\rangle$, $[E, F] = 1$, $E \lhd S$, $F \lhd S$, v is an involution in $S - EF$, and $E\langle v\rangle$ and $F\langle v\rangle$ are quasidihedral groups.*

Proof. By Lemmas 9.6, 9.8 and 9.5, $S = EF\langle v\rangle\langle a\rangle$ with E and F generalized quaternion, v a nonregular involution in $S - EF$, $E\langle v\rangle$ and $F\langle v\rangle$ quasidihedral, and a^Δ a field automorphism. Also, $(C_0(t)\langle v\rangle)^\Delta = \mathrm{PGL}(2, q)$.

Let $E\langle v\rangle = \langle e, v\rangle$ and $F\langle v\rangle = \langle f, v\rangle$ with $|E| = |e|$ and $|F| = |f|$. Then e^Δ and $f^\Delta = v^\Delta$ are odd permutations and $\langle e\rangle^{\Omega-\Delta}$ and $\langle f\rangle^{\Omega-\Delta}$ are semiregular. Thus, $4k = |e| = |f| = |\Omega - \Delta|_2$.

It remains to show that $S = EF\langle v\rangle$. Suppose that $a^{\Delta} \neq 1$. By Theorem 7.1 and Lemma 3.1, $F\langle a\rangle$ is a generalized quaternion group. Then we may assume that $|a| = 4$ and $F\langle a\rangle = \langle g, a\rangle$ with $|g| = |F| = |f| = |\Omega - \Delta|_2 = 4k = (q^2 - 1)_2 \geq 8$. Since $\langle g\rangle^{\Omega-\Delta}$ is semiregular, it follows that $g^{\Omega-\Delta}$ is an odd permutation. Then $g^{\Delta} = a^{\Delta}$ is also odd, so that $a^{\Omega-\Delta}$ is odd. However, $\langle g\rangle^{\Omega-\Delta}$ and $\langle a\rangle^{\Omega-\Delta}$ are semiregular and $|g^{\Omega-\Delta}| \geq 8 > |a^{\Omega-\Delta}|$, so that this is impossible. This proves Lemma 9.9.

LEMMA 9.10. *If $k = 2$ then all involutions in G are conjugate.*

Proof. Here $|S| = 64$. Define e by Lemma 9.5(iii), and f_1 by Lemma 9.2(iv). Then $e^v = e^{-1}t$, $f_1^e = f_1^v = f_1^{-1}$, and $e^{f_1} = ef_1^2 = te$. Also, $C_S(e) = \langle e\rangle$. Thus, a result of Brauer and Fong [6] implies that either all involutions in G are conjugate, or $G \approx M_{12}$. As the latter possibility does not occur [7], the lemma follows.

In unpublished research, P. Fong has studied simple groups G whose 2-Sylow subgroups have the structure described in Lemma 9.9 with $k > 2$. His main result is that all involutions in G are conjugate. We only require a special case of this result.

In Lemmas 9.11–9.18 *we assume that $k > 2$.* These lemmas are due to Fong.

We use the following notation: $E\langle v\rangle = \langle e, v\rangle$, $F\langle v\rangle = \langle f, v\rangle$, $|e| = |f| = |E| = |F| = 4k$, $u = e^k f^k$, and $m = vu$. If $S - EF$ contains a conjugate of t, we also assume that $v \sim t$.

LEMMA 9.11. *$S = \langle e, f, m\rangle$, where $e^{2k} = f^{2k} = t$, $m^2 = 1$, $[e^2, f^2] = 1$, $f^e = e^{-2}f^{-1}$, $(ef)^2 = (e^{-1}f)^2 = 1$, $e^m = e^{-1}$, and $f^m = f^{-1}$.*

Proof. $m^2 = e^k f^k (e^k f^k)^v = 1$, $(f^k)^e = (f^k)^v = f^{-k}$, $e^{f^k} = et$, and

$$e^m = e^{vu} = (e^{-1}t)^{e^k f^k} = e^{-1}.$$

Similarly, $f^m = f^{-1}$. As $|ev| = |fv| = 4$, $ev \in E$, and $fv \in F$, we have $ef^{-1} = ev(fv)^{-1} = evfvt = fvevt = fv(ev)^{-1} = fe^{-1}$. Thus, $|e^{-1}f| = 2$, and similarly, $|ef| = 2$. Also, $f^e = e^{-1}fe = e^{-1}e^{-1}f^{-1}$.

LEMMA 9.12. *$S^{(1)} = \langle e^2, f^2\rangle$, and $\Omega_1(S^{(1)})^{\#} = \langle t, u\rangle^{\#}$ consists of the involutions in S which are squares in S.*

Proof. As $S = \langle e, f, m\rangle$, $\langle e, m\rangle^{(1)} = \langle e^2\rangle$, $\langle f, m\rangle^{(1)} = \langle f^2\rangle$, and $f^e f^{-1} = e^{-2}f^{-2}$, $S^{(1)} = \langle e^2, f^2\rangle$. Also e, f and m are involutions (mod $S^{(1)}$). Thus, each involution in S which is a square must be in $S^{(1)}$.

LEMMA 9.13. *Set* $V = S^{(1)}\langle emf\rangle$.

(i) $V = \langle e^2\rangle \times \langle emf\rangle \lhd S$, *where* $|e^2| = |emf|$.
(ii) m *inverts* V.
(iii) V *is weakly closed in* S.
(iv) $N(V)$ *controls fusion in* V.

Proof. (i) By Lemma 9.12, $|S : V| = 4$. We have $(e^2)^{emf} = (e^{-2})^f = (e^{-2})^v = e^2$. Similarly, $(f^2)^{emf} = f^2$ so that V is abelian. Also, $emfemf = ef^{-1}e^{-1}f = eeff$ (Lemma 9.11). Thus, $V \geqslant \langle e^2\rangle \times \langle emf\rangle \geqslant \langle e^2, emf, f^2\rangle = V$ and $|emf| = 2|e^2f^2| = |e^2|$.

(ii) $e^m = e^{-1}$ and $(emf)^m = e^{-1}mf^{-1} = e^{-1}fm = f^{-1}em = f^{-1}me^{-1} = (emf)^{-1}$.

(iii) If $V \neq V^g \leqslant S$, $g \in G$, then $u \in V^g$ (Lemma 9.12), so that $V^g \leqslant C_S(u) = V\langle m\rangle$. Then $VV^g = C_S(u)$, and $V \cap V^g \leqslant Z(C_S(u))$. However, m inverts $V \cap V^g$ and $|V \cap V^g| = \frac{1}{2}|V| > 4$, a contradiction.

(iv) This is immediate by (iii).

LEMMA 9.14. *There is a 3-element* $b \in N(V)$ *such that* $\langle b\rangle$ *is transitive on* $\langle t, u\rangle^{\#}$.

Proof. As $u \sim ut$, it suffices to show that $t \sim u$ (Lemma 9.13(iv)). If t is not weakly closed in EF, this follows from Lemma 9.3. We may thus assume that $t \sim v$ (Lemma 2.6).

Let $v^g = t$ and $\langle v, u, t\rangle^g \leqslant S$, where $g \in G$. Since $|\langle t^g, u^g\rangle \cap EF| = 2$, we may assume that $u^g \notin EF$. Also $(e^2f^2)^v = e^{-2}f^{-2}$, so that $v \sim vu \sim (vu)^g = tu^g$. However, $u^g \in S - EF$ inverts e^2 (Lemma 9.5), so that $u^g \sim tu^g \sim v \sim t$.

LEMMA 9.15. *Each involution in* S *is conjugate in* S *to one of*: t, u, ef, ef^{-1}, mf, em, *or* m.

Proof. We need only consider involutions in $S - S^{(1)}$. Suppose that e^if^j is an involution, with i and j integers. If i is even and j is odd, then

$$e^if^{j-1}fe^if^{j-1}f = e^if^{j-1}e^{-i}ff^{j-1}f = f^{2j} \neq 1.$$

Thus, by symmetry i and j are both odd. By Lemma 9.11, $f^{e^2} = e^{-2}fe^2 = e^{-2}e^{-2}f$. Then $(e^if)^{e^2} = e^{i-4}f$, so that $ef \sim e^if$ if $i \equiv 1 \pmod 4$. Also, $(e^if^j)^{f^2} = f^{-2}e^if^{j+2} = e^if^{j+4}$. Thus, $ef \sim e^if^j$ if $i \equiv j \equiv 1 \pmod 4$. As $(e^if^j)^m = e^{-i}f^{-j}$, $ef \sim e^if^j$ if $i \equiv j \pmod 4$. Replacing f by f^{-1}, we have $ef^{-1} \sim e^if^j$ if $i \equiv -j \pmod 4$.

By Lemma 9.11, $|S : \langle e, f\rangle| = 2$. An involution not yet considered must then have the form e^imf^j. Note that $(e^imf^{2j})^e = e^ie^{-1}mef^{-2j} = e^{i-2}mf^{-2j}$.

If i is odd and $1 = e^imf^{2j}e^imf^{2j} = e^if^{-2j}e^{-i}f^{2j} = f^{4j}$, then $e^imf^{2j} = e^im$ or e^imt. As $(e^im)^u = e^imt$ and $(e^im)^e = e^{i-2}m$, $em \sim e^im \sim e^imt$ for i odd. Similarly, $mf \sim mf^j \sim tmf^j$ for j odd.

As $(e^{2i}mf^{2j})^e = e^{2i-2}mf^{-2j}$ and $(e^{2i}mf^{2j})^f = e^{-2i}mf^{2j+2}$, $m \sim e^{2i}mf^{2j}$ for all i and j.

Finally, suppose that i and j are odd. Then $e^imf^je^imf^j = e^ime^{-i}f^{-j}mf^j = e^{2i}f^{2j} \neq 1$. This proves the lemma.

LEMMA 9.16. *Either $m \sim ef$ or $m \sim ef^{-1}$.*

Proof. Set $U = C_S(u) = V\langle m\rangle = \langle e^2, f^2, ef, m\rangle$. By Lemma 9.14 there is a 2-Sylow subgroup S_1 of $C(u)$ such that $S_1 > U$. As $|S : U| = 2$, $S^a = S_1$ and $t^a = u$ for some $a \in N(U)$. Clearly, $|N(U)/C(U)U|_2 = 2$. If we let $L/C(U)U = O(N(U)/C(U)U)$, where $L \geqslant C(U)U$, then $N(U) = SL$. We may thus assume that $a \in L$ and $a \notin C(u)$. By Lemma 9.13(iii), $\langle a\rangle$ is transitive on $\langle t, u\rangle^{\#}$.

Thus, there is a 3-element $d \in N(U)$ such that $\langle d\rangle$ is transitive on $\langle t, u\rangle^{\#}$. By Lemma 9.15, each involution in $U - V$ is conjugate in S to m, ef, or ef^{-1}, where m^S, $(ef)^S$, and $(ef^{-1})^S \subseteq U - V$. If $m \not\sim ef$ and $m \not\sim ef^{-1}$, then $(m^S)^d = m^S$. However, $|m^S| = |S : C_S(m)| = |S : \langle t, u, m\rangle| = (4k)^2/8 = 2k^2 \equiv 2 \pmod 3$. Thus, we can find distinct elements $m_1, m_2 \in m^S \cap C(d)$. Then d centralizes the element $m_1m_2 \neq 1$ of V, whereas d centralizes no involution in V.

LEMMA 9.17. *If $mf \sim t$ then $e^{-k}mf \sim e^{-k}u$.*

Proof. Let S_1 be a 2-Sylow subgroup of $C(u)$ containing $C_S(u)$. As $mf \sim t \sim u$, there is a $g \in G$ such that $(mf)^g = u$ and $C_S(mf)^g \leqslant S_1$. We have $t \in \langle e^2\rangle \leqslant C_S(mf)$ since $(e^2)^{mf} = (e^{-2})^f = e^2$. By Lemma 9.12, applied to S_1, $t^g = t$ or tu. If $t^g = t$ then mf and u are conjugate in $C(t)$. If $t^g = tu$ then, for a suitable b in Lemma 9.14, $t^{gb} = t$ and $(mf)^{gb} = u^b = tu$, and once again mf and u are conjugate in $C(t)$.

Let $(mf)^{g'} = u$, $g' \in C(t)$. We may assume that $C_S(mf)^{g'} \leqslant C_S(u) = V\langle m\rangle$. Then $\langle e^2\rangle^{g'} \leqslant V\langle m\rangle$ and $(e^k)^{g'} \in V$. Thus, $(e^k)^{g'} \in C_0(t) \cap V = \langle e^2\rangle$, so that $(e^k)^{g'} = e^{\pm k}$. Replacing g' by $g'm$ if necessary, we have $(e^k)^{g'} = e^k$ and $(mf)^{g'} = u$. The result follows.

LEMMA 9.18. (i) *If $ef \sim t$, then $muf \sim f^k$ or $emu \sim f^k$.*

(ii) *If $ef^{-1} \sim t$, then $muf \sim f^k$ or $emu \sim f^k$.*

(iii) *If $ef \not\sim t \not\sim ef^{-1}$, then $e^{-k}mf \sim e^{-k}u$.*

Proof. (i) Let $(ef)^g = t$ and $C_S(ef)^g \leqslant S$, where $g \in G$. As $(ef)^{mf} = (e^{-1}f^{-1})^f = f^{-1}e^{-1} = ef$, $(e^kf^k)^{ef} = (e^kf^{-k})^f = e^{-k}f^{-k} = u$ and $u^{mf} = (e^kf^k)^f = e^{-k}f^k = ut$, we have $t \in \langle mf, u\rangle^{(1)} \leqslant C_S(ef)^{(1)}$. Then $t \neq t^g \in S^{(1)}$, so that $t^g = u$ or ut. We may assume that $t^g = u$.

Also, $muf = u \cdot mf \in C_S(ef)$, and $emu = ef \cdot mf \cdot u \in C_S(ef)$, so that $(muf)^g$ and $(emu)^g$ are in

$$C_S(ef)^g \cap C(t)^g \leqslant S \cap C(t^g) = C_S(u) = V\langle m\rangle = (\langle e^2\rangle \times \langle emf\rangle)\langle m\rangle.$$

Here $mufmuf = uf^{-1}uf = t$, and $(emu)^2 = t$. As m inverts V (Lemma 9.13(ii)) and $(emf)^2 = e^2f^2$, $(muf)^g$ and $(emu)^g$ are in $\langle e^k\rangle \times \langle e^{k/2}f^{k/2}\rangle$. In $N(V)$ these are conjugate to elements with square t. Thus, $muf \sim e^{\pm k}$ or $f^{\pm k}$, and $emu \sim e^{\pm k}$ or $f^{\pm k}$. However, $e^k \not\sim f^k$ as these are not conjugate in $C(t)$, and $muf \not\sim emu$ as otherwise $(muf)^g = ((emu)^g)^{\pm 1}$ and $g \in C(t)$. Thus either muf or $emf \sim f^k \sim f^{-k}$.

(ii) As $(muf)^m = muf^{-1}$ we can replace f by f^{-1} in the above argument.

(iii) Suppose first that $mf \sim t$, and set $S_0 = S^{(1)}\langle e, mf\rangle$. Then $S = S_0\langle em\rangle$, and each involution in S_0 is conjugate in S to t, u or mf (Lemma 9.15). By Lemmas 9.14 and 2.3, $em \sim t$. Similarly, if $em \sim t$ then $mf \sim t$ and (iii) holds by Lemma 9.17.

Suppose now that $em \not\sim t \not\sim mf$. Set $S_1 = S^{(1)}\langle ef, m\rangle$, so that $S = S_1\langle mf\rangle$. By Lemma 2.3, mf is conjugate to an involution in $S_1 - S^{(1)}$, hence to ef, ef^{-1}, or m (Lemma 9.15). By Lemma 9.16, $mf \sim ef$ or ef^{-1}.

Using Lemma 9.11, we find that $C_S(ef) = \langle ef, t, u, mf\rangle$ and $C_S(ef^{-1}) = \langle ef^{-1}, t, u, mf^{-1}\rangle$ have order 16, while $C_S(m) = \langle m, t, u\rangle$ has order 8. Also, $|C_S(mf)| \geqslant |\langle mf, em, e^2\rangle| = 2 \cdot 2 \cdot 2k > 16$ and $|C_S(em)| = |C_S(mf)|$. Thus, since $mf \not\sim t \sim u$, $C_S(mf)$ is a 2-Sylow subgroup of $C(mf)$.

Let $ef^{\pm 1} \sim mf$. Then $(ef^{\pm 1})^g = mf$ and $C_S(ef^{\pm 1})^g \leqslant S$ for some $g \in G$. Then $\langle t, u\rangle^g \leqslant S$. However, $t \sim u \not\sim ef, ef^{-1}, em, mf, m$, so that $\langle t, u\rangle^g = \langle t, u\rangle$ (Lemma 9.15). Then $mf = (ef^{\pm 1})^g \in C(\langle t, u\rangle)^g = C(\langle t, u\rangle)$, which is not the case.

From now on we again allow the possibility that $k = 2$.

LEMMA 9.19. (i) *All involutions in G are conjugate.*

(ii) *All elements of order* 4 *in* W *are conjugate in* $N(W) = C(t)$.

Proof. Recall that, since $\langle t\rangle = C_0(t) \cap W \leqslant Z(W)$, $N(W) = C(t)$.

We first show that (i) and (ii) are equivalent. Suppose that (ii) holds. By Lemma 9.10, we may assume that $k > 2$. Since $t \sim u$, (i) follows from Lemma 2.3 and the fact that all elements of order 4 in $C_0(t)$ are conjugate. Now assume that (i) holds. Let y_1 and y_2 be elements of order 4 in W. Then e^ky_1 and e^ky_2 are involutions, so that $(e^ky_1)^g = e^ky_2$ for some $g \in G$. Since $e^2 \in C(e^ky_1) \cap C(e^ky_2)$, t is a square in $C(e^ky_1)$ and $C(e^ky_2)$. By Lemma 9.12 (which holds even if $k = 2$) we may assume that $\langle e^ky_1, t\rangle^g = \langle e^ky_2, t\rangle$. Since $t \sim e^ky_2t$ in $C(e^ky_2)$, we may now assume that $t^g = t$. Then $g \in C(t) = N(W)$ and $(e^k)^ge^{-k} = (y_1^{-1})^gy_2 \in C_0(t) \cap W = \langle t\rangle$. It follows that $y_1 \sim y_2$ in $C(t)$, so that (ii) holds.

In particular, by Lemma 9.10 both (i) and (ii) hold if $k = 2$.

Assume that $k > 2$. The quasidihedral group $F\langle v\rangle$ has 2 classes of elements of order 4. Recall that $m = vu$. As $emu = ev \in E$ and $f^k \in F$ both have square t, they are not conjugate in G. By Lemma 9.18, either $vf = muf \sim f^k$ or $vf^{1-k} = f^k umf = e^{-k}mf \sim e^{-k}u = f^k$. Thus, all elements of order 4 in F are conjugate in G, hence in $N(W)$.

We now complete the proof of Theorem 9.1. By Lemmas 9.8(ii), 9.19 and 4.5, Q is elementary abelian of order q^3.

Since $C(t)^{\Delta}$ is 3-transitive, $C(t) = C_0(t)\langle v\rangle X$, where $X \geq W$ and X^{Δ} fixes more than 2 points. Let $x \in X^{\#}$. We claim that $\Delta(x) \subseteq \Delta$. Suppose that $\Delta(x) \not\subseteq \Delta$. Clearly, $C_0(t) \cap C(x)$ is $\mathrm{SL}(2, q')$, where q is a power of q'. As $C_0(\langle t, x\rangle) \leq C_0(t) \cap C(x)$, $C_0(\langle t, x\rangle) = \mathrm{SL}(2, q')$. As Q is abelian, $C_0(x) = \mathrm{SL}(2, q'')$, where $q'' > q'$ is a power of q'. The involution in $C_0(x) \cap W_x$ must be the involution in $C_0(\langle t, x\rangle) \cap W_{\langle t,x\rangle}$, whereas $t \notin W_x$, a contradiction.

Thus, X is semiregular on $\Omega - \Delta$. It follows that X is fixed-point-free on $[Q, t]$.

A comparison of Lemma 9.19(ii) with the structure of Frobenius complements (see Passman [25]) shows that $X/O(X) \approx \mathrm{SL}(2, \ell)$ with $\ell = 3$ or 5. If $\ell = 5$, $X \approx \mathrm{SL}(2, 5) \times O(X)$. If $\ell = 3$, it is easily seen that X has a normal subgroup X_1 such that X_1 is the direct product of a quaternion group and a group of odd order. Thus, X has a normal subgroup X^* such that $X^* = L \times K$, $(|L|, |K|) = 1$, and either $X = X^*$ and $L \approx \mathrm{SL}(2, 5)$, or $|X : X^*| = 3$, L is quaternion of order 8, and $X/K \approx \mathrm{SL}(2, 3)$.

There is an element of order 4 in $C_0(t) \cap C(K)$. Also, K centralizes an element of order 4 in $L \leq W$. Thus, K centralizes an involution $t^g \neq t$ in $C_0(t)W$, where $g \in G$ (Lemma 9.19(i)). Also, $t \in C_0(t^g)W_{t^g}$.

We claim that $K = 1$. If this is not so, let $M \leq K$ have odd prime order. Then $M \leq C(t^g) = C_0(t^g)\langle v^g\rangle X^g$ and $M \leq C(t)$. It follows that $M \leq C_0(t^g)K^g$. As $t \in C_0(t^g)W_{t^g}$, there is an element $d \in L^g$ such that $|d| = 4$ and $t^d = t^g t$. Since d centralizes $C_0(t^g)K^g$, $M^d = M$ and $\Delta(M) = \Delta(M^d) \subseteq \Delta(t) \cap \Delta(t^g t)$. Then $\langle t, t^g t\rangle$ is a Klein group fixing at least $|\Delta(M)|$ points of Ω. As $M^{\Delta} \leq X^{\Delta}$ fixes more than 2 points, this contradicts Theorem 7.1.

Thus, $C(t) = C_0(t)X\langle v\rangle$, where $C_0(t)$ is $\mathrm{SL}(2, q)$ and $X \approx \mathrm{SL}(2, \ell)$ for $\ell = 3$ or 5. We claim that X centralizes $C_0(t)$. If $\ell = 5$, then $X \leq W$, and this is clear. If $\ell = 3$, then either $X \leq W$ or $W = X^* = X \cap W$ is quaternion of order 8 and X^{Δ} is generated by a field automorphism of order 3. In the latter case, since $W\langle v\rangle$ is quasidihedral $C(t)/C_0(t)$ is isomorphic to the group S_4. However, $C(t)/C_0(t)W$ is abelian of order 6, a contradiction. Thus, $[C_0(t), X] = 1$.

By a result of Fong and Wong ([12], Main Theorem or (3H) and (3J)),

q is a power of ℓ. However, L is fixed-point-free on the group $[Q, t]$ of order q^2. This contradiction proves Theorem 9.1, and completes the proof of Theorem 1.1.

10. Corollaries

We now note some easy consequences of Theorem 1.1.

Corollary 10.1. *Let G be a 2-primitive group in which the stabilizer of a point is solvable. Then* $\mathrm{PSL}(2, q) \leqslant G \leqslant \mathrm{P\Gamma L}(2, q)$ *for some q.*

Results of this type are in Passman [26].

Corollary 10.2. *Let G be a 3-transitive group on a set Ω in which the stabilizer of 3 points is cyclic. Then* $\mathrm{PSL}(2, q) \leqslant G \leqslant \mathrm{P\Gamma L}(2, q)$ *for some q.*

Proof. Let $\alpha \in \Omega$. If $G_\alpha^{\Omega-\alpha}$ has a regular normal subgroup, we can apply Theorem 1.1. If $G_\alpha^{\Omega-\alpha}$ has no regular normal subgroup, then by [22], $G_\alpha^{\Omega-\alpha}$ is $\mathrm{PSL}(2, q)$, $\mathrm{PGL}(2, q)$, $\mathrm{Sz}(q)$, $\mathrm{PSU}(3, q)$, $\mathrm{PGU}(3, q)$ or of Ree type, in its usual 2-transitive representation. The corollary now follows from a result of Suzuki [36].

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