2-TRANSITIVE DESIGNS *)

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INTRODUCTION

A great deal of work was done on 2-transitive groups during the last century and the beginning of this one. There has been a recent resurgence of interest in them for several reasons. First of all, many finite simple groups either have 2-transitive permutation representations or are closely related to groups that do. Also, recent work on finite simple groups has made the study of permutation groups more accessible. Finally, the close relationship between these groups and finite geometries has been recognized and has benefitted both group theory and geometry.

This survey will be concerned with designs having 2-transitive automorphism groups. A complete account of the relationship between designs and groups, as it was known in 1968, is contained in the beautiful book of DEMBOWSKI [40]. However, quite a lot has been done since then.

Since this is a combinatorics conference, I will try to minimize the group theory. However, the interplay between the groups and the designs they act on is fundamental to the subject: the fact that the automorphism group G of a design $\mathcal{D}$ permutes both the points and blocks of $\mathcal{D}$ suggests that these two actions should be played off against one another. Moreover, the manner in which designs occur in group-theoretic situations is a basic source for geometric problems and geometric theorems.

The difference between the study of 2-transitive designs and 2-transitive groups seems to be as follows. In the former case, one makes an assumption concerning the set stabilizer (or point-wise stabilizer) of a block: its transitivity properties, index in G, etc. In the latter case, one assumes structural properties of the stabilizer of one or more points. Just how fine a distinction this is can be seen from papers of O'NAN [128, 135], HARADA [63], ASCHBACHER [2,5], SHULT [149], KANTOR, O'NAN & SEITZ [107], and HERING, KANTOR & SEITZ [66], where designs are explicitly or

*) The preparation of this paper was supported in part by NSF Grant GP 37982X.

M. Hall, Jr. and J. H. van Lint (eds.), Combinatorics, 365-418.

implicitly obtained in the course of "purely" group-theoretic investigations.

So as not to give a false impression, it should be noted that the relationship between permutation groups, geometry and combinatorics has been known for a long time - see the books of BURNSIDE [18] and CARMICHAEL [32].

There are also important relationships between projective planes and groups. However, I will not discuss collineation groups of projective, affine, or inversive planes at all - that would require a survey paper of its own. Incidentally, most of the problems and methods considered here become meaningless or trivial in the case of such planes. I hope to demonstrate the richness of the geometric nature of a subject spawned in part by, but quite different from, projective planes.

The organization of this paper is as follows. Section 1 consists of little more than geometric and group-theoretic notation. Section 2 discusses the elementary, well-known construction of designs from 2-transitive groups.

In the remaining sections, G will be an automorphism group 2-transitive on the points of a design $\mathcal{D}$. One natural approach is to first try to find $\mathcal{D}$, and then find G. Unfortunately, even if $\mathcal{D}$ is known to be a projective or an affine space, it is still very difficult to determine G (see section 3). This fact is, in turn, undoubtedly partly to blame for the difficulties encountered in the situations described in sections 6-10.

Section 4 contains a brief discussion of the geometry of the Mathieu groups. These designs and groups will arise in later sections.

The subject matter of this survey properly begins in section 5. There, and in the remaining sections, a variety of possible restrictions on 2-transitive designs are discussed. In each case, classical projective or affine spaces satisfy the additional hypothesis and partly motivate its study. With the exception of section 5, the goal will be the determination of $\mathcal{D}$, not of G.

Section 5 is devoted almost exclusively to results of O'NAN. The main geometric application of his striking classification theorems is to the subgroup of G fixing all the blocks through a point of $\mathcal{D}$.

HALL [58] considered the case where the 2-transitive design $\mathcal{D}$ is a Steiner triple system. In section 6, a more general situation is studied: $\lambda = 1$, and the stabilizer of two points fixes all points on the line through them.

In section 7, it is assumed that the pointwise stabilizer of a block of $\mathcal{D}$ is transitive on the complement of the block. An added combinatorial bonus here is the relevance of geometric lattices.

However, the richest combinatorial structure occurs in section 8, where $\mathcal{D}$ is assumed to be a symmetric design. As the length of the section indicates, more work has been done in this case than in any other. There are also several applications, which are discussed in sections 8 and 9; these include difference sets (section 8), Hadamard matrices (section 9), symmetric 3-designs (section 9), the suborbit structure of permutation groups (section 8), and the reducibility of certain complex polynomials (section 8).

Section 9 briefly discusses symmetric 3-designs. Finally, section 10 contains a variety of miscellaneous topics. An appendix lists the known 2-transitive groups.

Throughout the paper -and especially in section 10- I have occasionally digressed slightly from the main topic. In most cases, geometric questions related to 2-transitive groups are raised, even if designs are not involved. In fact, it would be absurd to claim that the only relationship between combinatorics and 2-transitive groups is through designs. The best examples of this, which will not be described here, are the graph extension theorem of SHULT [147] and the growing theory of 2-graphs (SEIDEL [143]; HIGMAN [71]; TAYLOR [155,156]). Also, if G is 2- but not 3-transitive, G_x determines graphs on $S - \{x\}$ which have yet to be studied. Probably the most basic problem in the combinatorial approach to 2-transitive groups is to find ways to use groups, designs, and graphs simultaneously. Thus far, this problem has been considered briefly in only two papers: SIMS [150] and O'NAN [133].

I am indebted to the following people for comments which helped in the preparation of this survey: P. CAMERON, M. FRIED, N. ITO, W. KNAPP, H. NAGAO, P. NEUMANN, and R. NODA.

1. BACKGROUND

A. Designs

A *design* $\mathcal{D}$ consists of a set S of points ("varieties" of wheat in the original statistical context), together with certain subsets called *blocks*, such that the following conditions hold for some integers v,b,k,r,λ: there are v points, b blocks, k points per block, r blocks per point, and λ blocks through any two distinct points. The following non-degeneracy conditions will also be assumed: $v \geq k+2 > 4$, and some k-subset of S is not a block. The *parameters* v, b, k, r, λ satisfy $vr = bk$ and $\lambda(v-1) = r(k-1)$. Also, $b \geq v$ (FISHER's inequality).

$\mathcal{D}$ is a *symmetric design* if $b = v$, or equivalently, if $r = k$. The parameters v, k, λ, and $n = k - \lambda$ then satisfy further restrictions (see DEMBOWSKI [40, § 2.1]), but these will not be needed. A *Hadamard design* is a symmetric design with $v = 2k+1$.

If x is a point of a design $\mathcal{D}$, then $\mathcal{D}_x$ denotes the set $S - \{x\}$ of points together with the sets $B - \{x\}$, where B is a block on x. $\mathcal{D}$ is called an *extension* of $\mathcal{D}_x$.

If B is a block of $\mathcal{D}$, then $\mathcal{D}_B$ denotes the set of points of B and the sets $B \cap C$, where C is any block other than B. This is again a design if $\mathcal{D}$ is symmetric.

The *complementary design* $\mathcal{D}'$ of $\mathcal{D}$ is the design having the same point set as $\mathcal{D}$, and whose blocks are the complements of those of $\mathcal{D}$.

A t-*design* is a design $\mathcal{D}$ such that each set of t points is in the same number $\lambda_t > 0$ of blocks. If $\lambda_t = 1$, $\mathcal{D}$ is also called a *Steiner system* $S(t,k,v)$.

A *line* of $\mathcal{D}$ consists of the intersection of all the blocks containing two given points. Two points are contained in a unique line. While lines of a design can usually have different sizes, they will automatically have the same size in this paper. Note that, when $\lambda = 1$, blocks are lines; in this case, I will use the more suggestive term line. Also, if $\lambda = 1$, a *subspace* of $\mathcal{D}$ is a set Δ of points such that, whenever x and y are distinct points of Δ, their line is contained in Δ.

An *automorphism* of $\mathcal{D}$ is a permutation of the points which also permutes the blocks. The automorphisms of $\mathcal{D}$ form a group Aut $\mathcal{D}$, the *automorphism group* of $\mathcal{D}$. The fact that Aut $\mathcal{D}$ permutes both the points and blocks is crucial.

If $\mathcal{D}$ is a symmetric design, the *dual* design $\tilde{\mathcal{D}}$ has the roles of points and blocks interchanged. $\tilde{\mathcal{D}}$ is symmetric, with the same parameters as $\mathcal{D}$. An *antiautomorphism* (or *correlation*) of $\mathcal{D}$ is an isomorphism $\theta: \mathcal{D} \to \tilde{\mathcal{D}}$. Then θ induces an isomorphism $\tilde{\mathcal{D}} \to \mathcal{D}$, also called θ, by acting on the points and blocks of $\tilde{\mathcal{D}}$ as θ does on the blocks of $\mathcal{D}$. θ is a *polarity* if $\theta^2 = 1$ (i.e. if $x \in y^\theta$ implies $y \in x^\theta$). If g is in Aut $\mathcal{D}$, so is $\theta^{-1}g\theta$. The group (Aut $\mathcal{D}$) $\langle\theta\rangle$ contains Aut $\mathcal{D}$ as a subgroup of index 2, and contains all antiautomorphisms of $\mathcal{D}$.

The following notation will be used for the classical geometries:
$PG_e(d,q)$, $1 \leq e \leq d-1$, denotes the design of points and e-spaces of $PG(d,q)$; and
$AG_e(d,q)$, $1 \leq e \leq d-1$, denotes the design of points and e-spaces of $AG(d,q)$.

As usual, $PG(2,q) = PG_1(2,q)$ and $AG(2,q) = AG_1(2,q)$. The automorphism group of $PG(d,q)$ is $P\Gamma L(d+1,q)$.

In section 7, *geometric lattices* will arise. These are (finite) lattices L such that each element is a join of points (i.e., atoms), and which satisfy the exchange condition: if x and y are points, and $X \in L$, then $x \not< X$ and $y < x \vee X$ imply $x < y \vee X$. Each $X \in L$ then has a dimension $\dim(X)$, where $\dim(0) = -1$, $\dim(X) = \dim(Y) - 1$ if $X < Y$ is maximal in Y, and $\dim(X \vee Y) + \dim(X \wedge Y) \leq \dim(X) + \dim(Y)$ (for all $X,Y \in L$). Moreover, *bases* of X can be introduced as sets of $\dim(X) + 1$ points of L, none of which is in the join of the rest. The usual replacement conditions then hold for bases.

B. Permutation groups

Let H be a group *inducing* a group of permutations on a finite set S of points. It is essential to allow the possibility that non-trivial elements of H induce the identity on S. $H(S)$ denotes the (normal) subgroup of H consisting of those $h \in H$ fixing every point of S, that is, the pointwise stabilizer of S. H^S denotes the group of permutations of S induced by H. Thus, $H^S \cong H/H(S)$.

x^h denotes the image of $x \in S$ under $h \in H$. X^h denotes the image of $X \subseteq S$ under $h \in H$: $X^h = \{x^h \mid x \in X\}$.

$H_X = \{h \in H \mid X^h = X\}$ is the (set) *stabilizer* of X in H. Clearly, H_X contains the *pointwise stabilizer* $H(X)$ of X, and H_X induces the permutation group $H_X^X \cong H_X/H(X)$ on X. It is convenient to abbreviate $H_{\{x\}} = H_x$. If, say, $X,Y \subseteq S$ then $H_{XY} = H_X \cap H_Y$.

x^H denotes the *orbit* of x under H: $x^H = \{x^h \mid h \in H\}$. The orbits of H partition S.

H is *transitive* if $x^H = S$ for some (and hence each) x. Clearly, H is transitive on each of its orbits. H is *regular* if it is transitive and

$H_x = 1$ for some (and hence each) x. H is *primitive* on S if H is transitive on S and H_x is a maximal subgroup of H. H is *t-transitive* on S if it acts transitively on the ordered t-subsets of S. In this case, H_x is (t-1)-transitive on S - {x}. H is *sharply t-transitive* if it is regular on the set of ordered t-subsets of S; for $t \geq 2$, all such H have been determined (ZASSENHAUS [171,172]; JORDAN [89, pp.345-361]; HALL [57, pp.72-73]).

The *rank* of a transitive group H is the number of orbits of H_x. Thus, having rank 2 is the same as being 2-transitive. An *involution* in H is an element of order 2.

C. Preliminary lemmas

(1) ORBIT THEOREM. *If* $G \leq \mathrm{Aut}\,\mathcal{D}$, *then* G *has at least as many block-orbits as point-orbits. If* $\mathcal{D}$ *is symmetric, these numbers are the same* (see DEMBOWSKI [40, p.78]).

(2) *If* $\mathcal{D}$ *is a symmetric design, then each* $g \in \mathrm{Aut}\,\mathcal{D}$ *fixes the same number of points and blocks* (see DEMBOWSKI [40, p.81]).

(3) *If* $\mathcal{D}$ *is symmetric and* $1 \neq g \in \mathrm{Aut}\,\mathcal{D}$, *then* g *fixes at most* $\frac{1}{2}v$ *points* (FEIT [44]). As noted in KANTOR [102], FEIT's proof shows that, if g fixes exactly $\frac{1}{2}v$ points, then g is an involution and $v = 4n$.

(4) *Let* H *act as a permutation group on* S. *Let* $K \leq H$. *Then the normalizer* $N_H(K)$ *of* K *is contained in the set-stabilizer* $H_{\Omega(K)}$ *of the set* $\Omega(K)$ *of fixed points of* K. *Moreover, if* $g \in G$ *then* $\Omega(K^g) = \Omega(K)^g$.

(5) ORBIT LENGTH. *If* $X = x^H$ *is an orbit of* H *on* S, *then* $|X| = |H:H_x|$ *is the index in* H *of the stabilizer of* x.

(6) *Suppose* H *is as in* 1B *and let* X *and* Y *be orbits of* H. *If* d *is the g.c.d of* $|X|$ *and* $|Y|$, *and* $x \in X$, *then each orbit of* H_x *on* Y *has size divisible by* $|Y|/d$. *In particular, if* $d = 1$ *then* G_x *is transitive on* Y.

2. CONSTRUCTIONS

A. Basic construction

(1) *Let* G *be* 2*-transitive on the finite set* S. *Let* B *be any* k*-subset of* S, *and assume that* G *is not transitive on (unordered)* k*-subsets of* S. *Then the distinct sets* B^g, $g \in G$, *are the blocks of a design* $\mathcal{D} = \mathcal{D}(G,S,B)$.

PROOF. Each B^g has $|B| = k$ points. If $x^g = y$, then g sets up a 1-1-correspondence between the blocks on x and those on y; this provides us with r. The same proof yields λ. □

(2) $G \leq \mathrm{Aut}\,\mathcal{D}(G,S,B)$, *and* G *is transitive on blocks. Hence,* $\mathcal{D}(G,S,B)$ *has* $b = |G:G_B|$ *blocks* (see 1C(5)). *In particular,* $\mathcal{D}(G,S,B)$ *is symmetric if and only if* $|G:G_B| = v$.

Of course, the trouble with this construction is that B, and hence $\mathcal{D}$, may be totally unrelated to the action or structure of G. It is necessary to choose B carefully if $\mathcal{D}$ is to provide information about G. This is what will be done in later sections. One can, for example, assume that B is the set of fixed points of G_{xy}, $x \neq y$, or that G(B) is transitive on S - B.

In almost every case of interest, B is an orbit of some subgroup of G, so that G_B is transitive on B. Note that, *if* $\lambda = 1$, *then necessarily* G_B *is* 2*-transitive on* B.

PROOF. If $x,y,x',y' \in B$, $x \neq y$ and $x' \neq y'$, then any $g \in G$ such that $x^g = x'$ and $y^g = y'$ must fix B. □

If G is t-transitive, then $\mathcal{D}(G,S,B)$ is clearly a t-design.

B. When is $\lambda = 1$?

Suppose $\mathcal{D} = \mathcal{D}(G,S,B)$, where B is the set of fixed points of some subset W of G_{xy} (where $x \neq y$). In this situation (as in the general one) it is natural to ask when $\lambda = 1$. The simplest answer is due to WITT [169]:

(1) $\lambda = 1$ *if* $W^g \subseteq G_{xy}$ *and* $g \in G$ *imply* $W^g = W$.

PROOF. If $x,y \in B^g$, then W^g fixes x and y by 1C(4). That is, $W^g \subseteq G_{xy}$, so $W^g = W$. Thus, $B^g = B$. □

This result has been used in a variety of circumstances. For example, if G_{xy} is cyclic, it applies to every subgroup $W \neq 1$ of G_{xy} fixing more than two points; this was very useful in the determination of all such groups (KANTOR, O'NAN & SEITZ [107]). The designs and groups that arise here are very interesting. Assume that G does not have a regular normal subgroup and that G_{xy} has such a subgroup W. Then $v = q^3 + 1$, $r = q^2$, and $k = q + 1$ for some prime power q. (In the terminology of DEMBOWSKI [40, p.104], these are the parameters of a *unital*.) There are just two possibilities. One is that G is PSU(3,q) or PGU(3,q), and the design consists of the absolute points and non-absolute lines of a unitary polarity of $PG(2,q^2)$. (See O'NAN [128] for a detailed study of this design.) In the other case, $q = 3^{2e+1}$ for some $e \geq 0$, and G is a group of Ree type (see WARD [163] and KANTOR, O'NAN & SEITZ [107] for some properties of G and the design); the case q = 3 will arise again in section 6, where the design is called $\mathcal{D}(4)$.

There is, of course, an obvious t-design analogue of WITT's result.

There are some other interesting conditions which imply $\lambda = 1$. The most striking one is due to O'NAN [130]:

(2) *Suppose* B *is the set of fixed points of* $W \leq G_{xy}$. *Assume that no element of* $G_{xy} - W$ *is conjugate in* G_x *to an element of* W. *Then* $\lambda = 1$.

It is worthwhile to compare this with 2B(1). The main hypothesis there concerns conjugates of W, while in 2B(2) it concerns conjugates of elements of W. On the other hand, 2B(1) considers *all* conjugates, while 2B(2) only considers conjugates in G_x.

The proof of 2B(2) is elementary, but not straightforward. The main application is as follows:

(3) *Suppose* N *is a normal subgroup of* G_x, $y \neq x$, $N_y \neq 1$, *and* N_y *fixes more than two points. Then* 2B(2) *applies to* $W = N_y$ (O'NAN [130]).

PROOF. Suppose $g \in G_x$. Then $W^g \cap G_{xy} \leq N^g \cap G_{xy} = N_y = W$. □

Note that N fixes every block through x. Both 2B(2) and 2B(3) are crucial in the proofs of the theorems in section 5.

(4) *Suppose* $1 \neq G_{xy} < K < G_x$ *and* $B = \{x\} \cup y^K$. *Then*

(i) $\mathcal{D}(G,S,B)$ *has* $\lambda = 1$ *if for any three points* x,y,z, G_x *has an element interchanging* x *and* y (O'NAN, unpublished; ATKINSON [6]; a t-design version has been found by NEUMANN [122]); *and*

(ii) *if* $|y^K| \leq 3$ *then* G *acts on a design with* $\lambda = 1$ *as a 2-transitive automorphism group* (O'NAN, unpublished; ATKINSON [6]).

A few comments are needed concerning 2B(4). If $\lambda = 1$ for a given $\mathcal{D}(G,S,B)$, let $x,y \in B$, $x \neq y$, and set $K = G_{xB}$. Then $G_{xy} < K < G_x$. This makes the hypothesis of 2B(4) seem more reasonable.

One example of 2B(4i) is provided by the following unpublished result of SHULT (applied to $H = G_x$ acting on $X = S - \{x\}$). Suppose H is transitive on X, and some involution $t \in H$ fixes exactly one point. Then, if $y,z \in X$, $y \neq z$, there is a conjugate of t interchanging y and z.

3. COLLINEATION GROUPS

A. Projective spaces

Let G be a collineation group of PG(d,q) which is 2-transitive on points. The only known examples are: $G \geq PSL(d+1,q)$, the group of projective collineations of determinant 1; and the peculiar but fascinating example $G \cong A_7$ acting on PG(3,2).

It seems unlikely that other examples exist, but this has been verified in only a few cases. WAGNER [162] proved this for $d \leq 4$, D.G. HIGMAN (unpublished) for d = 5,6, and KANTOR [102] for d = 7,8, or when $d = s^\alpha$ for a prime divisor s of q-1. The same conclusion holds if some non-trivial element of G fixes a (d-2)-space pointwise (WAGNER [162], HIGMAN [68], KANTOR [102]).

Here are two interesting properties of G.

(1) *If* E *is a plane, then* G_E^E *contains* PSL(3,q) (WAGNER [162]).

(2) *If* H *is a hyperplane, then* G_H *is 2-transitive on* S - H (KANTOR [93]).

Additional (but technical) properties of G are found in KANTOR [102].

(3) Since the example $G \cong A_7$ will arise in sections 6-8, it is perhaps worthwhile to discuss it in some detail. By one of the flukes of nature, $A_8 \cong PSL(4,2)$ (see 4A(2) for a proof). Thus, PG(3,2) does indeed have a collineation group $G \cong A_7$. Thus, A_8 can be regarded as acting on the 8 cosets of G, or on the 15·14/2 2-sets of points of PG(3,2). By 1C(6), G is transitive on these 2-sets. It follows that G is indeed 2-transitive.

By 1C(5), if $x \neq y$ then $|G_{xy}| = 12$. Take a point z not on the line L through x and y. It is easy to see that G cannot contain any non-trivial elation (= transvection), so $G_{xyz} = 1$. Again by 1C(5), G_{xy} must be transitive (and hence even regular) on the 12 points not in L.

It is now not difficult to prove $G_x \cong PSL(3,2)$ and $G_{xy} \cong A_4$.

B. PERIN's results

What happens if some kind of additional transitivity is assumed in 3A? This question was posed and almost completely answered by PERIN [139]:

Suppose G *is transitive on the triangles of points of* PG(d,q). *Then* $G \geq PSL(d+1,q)$, *except perhaps if* $q = 2$ *and* d *is odd.* (In 3A(3), the collineation group A_7 of PG(3,2) was shown to be transitive on triangles.) *If* G *is transitive on tetrahedra, then* $G \geq PSL(d+1,q)$.

The proof is ingenious and surprisingly easy. It depends solely on elementary number theory and elementary group theory.

PERIN's results are certainly the strongest and most useful ones concerning 2-transitive collineation groups of finite projective spaces. They arise several times in later sections of this survey. They have also been useful elsewhere: they were involved in one of the first proofs used for the determination of the 2-transitive permutation representations of the groups PSL(n,q). This in turn led CURTIS, KANTOR & SEITZ [36] to the determination of the 2-transitive representations of all the finite Chevalley groups.

C. Affine spaces

(1) Now let G be a collineation group of AG(d,q) 2-transitive on points. Here, the question is whether G must contain the translation group V of the space. The only known counterexample is $G \cong PSL(3,2) \cong PSL(2,7)$ acting on AG(3,2).

Suppose, for a given d and q, G must contain V. Then by 3A(2), each 2-transitive collineation group of PG(d,q) must contain PSL(d+1,q). The only d and q for which it is known that $G > V$ must hold is $d = s^{\alpha}$ for a prime divisor s of q-1 (KANTOR [102]).

It is important to note that there are many 2-transitive groups $G > V$. The classification of these groups is equivalent to the classification of finite groups of semilinear transformations transitive on non-zero vectors.

(2) If G is also transitive on triangles, it can be shown that $G > V$, except perhaps if $q = 2$ and d is odd. If G is transitive on tetrahedra, then $G > V$.

D. Generalizations

PERIN [139] studied the more general situation in which G is a collineation group of PG(d,q) transitive on e-spaces. When combined with KANTOR [100], the result is that G is 2-transitive on points if $2 \leq e \leq d-2$, except for groups of order $31 \cdot 5$ line-transitive on PG(4,2); if $3 \leq e \leq d-3$, then $G \geq PSL(d+1,q)$ except perhaps if $q = 2$ and d is odd; if $4 \leq e \leq d-4$, then $G \geq PSL(d+1,q)$.

Suppose next that G is a collineation group of AG(d,q) transitive on e-spaces, where $1 \leq e \leq d-1$. It is then easy to see (by 1C(6)) that G_x is transitive on the e-spaces through x. If $2 \leq e \leq d-2$, this essentially reduces the problem to the one of the preceding paragraph.

4. THE MATHIEU GROUPS

The Mathieu groups will appear several times in the remainder of this survey. The following brief description of these groups and some of their properties is based primarily on WITT [169,170] and LÜNEBURG [111].

A. M_{22}, M_{23}, and M_{24}

(1) There are unique Steiner systems $\mathcal{W}_{22} = S(3,6,22)$, $\mathcal{W}_{23} = S(4,7,23)$, and $\mathcal{W}_{24} = S(5,8,24)$, discovered by WITT [169]. If x is a point of $\mathcal{W}_v$, then $(\mathcal{W}_v)_x = \mathcal{W}_{v-1}$ for v = 24, 23, and $(\mathcal{W}_{22})_x = PG(2,4)$.

Aut $\mathcal{W}_v$ is (v-19)-transitive on points and transitive on blocks. Write $M_{24} = \text{Aut}\,\mathcal{W}_{24}$ and $M_{23} = \text{Aut}\,\mathcal{W}_{23}$. If x and y are in $\mathcal{W}_{24}$, $x \neq y$, then $(M_{24})_{\{x,y\}} = \text{Aut}\,\mathcal{W}_{22}$ contains $M_{22} = (M_{24})_{xy}$ as a subgroup of index 2. The three groups M_{24}, M_{23} and M_{22} are simple groups, the "large" Mathieu groups.

If x,y and z are distinct points of $\mathcal{W}_{24}$, then $(M_{24})_x = M_{23}$, $(M_{24})_{xy} = M_{22}$, $(M_{24})_{\{x,y\}} = \text{Aut}\,\mathcal{W}_{22} = \text{Aut}\,M_{22}$, $(M_{24})_{xyz} = PSL(3,4)$, and $(M_{24})_{\{x,y,z\}} = P\Gamma L(3,4)$. Suppose B is a block of $\mathcal{W}_{24}$. Then $(M_{24})_B^B \cong A_8$ and $(M_{24})(B)$ is regular on S-B; here, $(M_{24})(B)$ induces an elation group of $(\mathcal{W}_{24})_{xyz} = PG(2,4)$ if $x,y,z \in B$.

(2) M_{24} provides an easy proof that $A_8 \cong PSL(4,2)$. Namely, consider $G = (M_{24})_B$. By 4A(1), $G^B \cong A_8$. But G(B) is elementary abelian of order

16, and is normal in G. It follows readily that A_8 is isomorphic to a subgroup of the automorphism group $PSL(4,2)$ of $G(B)$. Since $|A_8| = |PSL(4,2)|$, this proves $A_8 \cong PSL(4,2)$.

(3) Any two blocks B and C of $\mathcal{W}_{24}$ meet in 0, 2 or 4 points. M_{24} is transitive on the ordered pairs of blocks whose intersections have a fixed size. Also, if $|B \cap C| = 4$, then the symmetric difference $B + C$ is a block.

Any two blocks of $\mathcal{W}_{23}$ meet in 1 or 3 points. M_{23} is again transitive on the ordered pairs of blocks meeting in 1 or in 3 blocks.

Any two blocks of $\mathcal{W}_{22}$ meet in 0 or 2 points. M_{22} is transitive as above.

(4) If B_0 is a block of $\mathcal{W}_{22}$, there are 16 points outside B_0 and 16 blocks missing B_0. These form a symmetric design with parameters $v = 16$, $k = 6$, $\lambda = 2$ and full automorphism group $(\mathrm{Aut}\, M_{22})_{B_0} \cong S_6 \cdot V \cong Sp(4,2) \cdot V$, where $V = M_{22}(B_0)$ is an elementary abelian group of order 16. This design is $S^{-1}(4)$ in the notation of 8B(4).

(5) The remarks in 4A(4) can be interpreted in $\mathcal{W}_{24}$ as follows (CAMERON [24]). Fix a block B^* of $\mathcal{W}_{24}$ and set $S^* = S - B^*$. If $x,y \in B^*$, $x \neq y$, let S_{xy} be the set of all blocks B such that $B \cap B^* = \{x,y\}$. By 4A(4), $|S^*| = |S_{xy}| = 16$, and S^* and S_{xy} yield a symmetric design.

Let $z \in B^* - \{x,y\}$. Then S_{xy} and S_{xz} also determine a symmetric (16,6,2)-design: call $B \in S_{xy}$ and $C \in S_{xz}$ *incident* if $|B \cap C \cap S^*| = 1$. All the resulting symmetric designs are isomorphic (they are $S^{-1}(4)$ in the notation of 8B(4)).

B. M_{11} and M_{12}

(1) There are unique Steiner systems $\mathcal{W}_{11} = S(4,5,11)$ and $\mathcal{W}_{12} = S(5,6,11)$. If x, y and z are three points of $\mathcal{W}_{12}$, then $(\mathcal{W}_{12})_x = \mathcal{W}_{11}$, $(\mathcal{W}_{11})_{xy}$ is the miquelian inversive plane of order 3, and $(\mathcal{W}_{11})_{xyz} = AG(2,3)$. Write $M_v = \mathrm{Aut}\, \mathcal{W}_v$, $v = 11,12$. Then M_v is sharply $(v-7)$-transitive on the points of $\mathcal{W}_v$, and is transitive on blocks. M_{11} and M_{12} are both simple; $(M_{12})_{\{x,y\}}^B = P\Gamma L(2,9)$.

$G_B \cong G_B^B \cong S_6$ if B is a block of $\mathcal{W}_{12}$. Also, S-B is another block, and $G_{\{B,S-B\}} \cong \mathrm{Aut}\, S_6$.

(2) $\mathcal{W}_{12}$ is obtained from $\mathcal{W}_{24}$ as follows. Let B and C be blocks of the latter design such that $|B \cap C| = 2$. Then their symmetric difference B+C has size 12, and $(M_{24})_{B+C}$ is just M_{12}. Note that $|B \cap (B+C)| = 6$; the blocks of $\mathcal{W}_{12}$ are precisely the intersections of size 6 of B+C with blocks of $\mathcal{W}_{24}$.

If $x,y,z \in B+C$, then $(\mathcal{W}_{12})_{xyz} = AG(2,3)$ is embedded in $(\mathcal{W}_{24})_{xyz} = PG(2,4)$ as the unital preserved by $P\Gamma U(3,2) = (\mathcal{W}_{12})_{\{x,y,z\}}$. The latter group is precisely the full collineation group of $AG(2,3)$.

Moreover, the complement of B+C again has the form B_1+C_1 (where $|B_1 \cap C_1| = 2$), and $(M_{24})_{\{B+C,B_1+C_1\}} = \mathrm{Aut}\, M_{12}$ contains M_{12} as a subgroup of index 2.

(3) In the notation of 4B(2), fix a point $p \notin B+C$. Then $M_{11} = (M_{24})_{B+C,p}$ is 3-transitive on B+C (as well as on $(B_1+C_1) - \{p\}$). The $\mathcal{W}_{12}$ determined by B+C has exactly 22 blocks through p. Together with the points of $\mathcal{W}_{12}$ (i.e., B+C), these form a 3-design $\mathcal{D}$ with $v = 12$ and $k = 6$. If $x \in B+C$, then $\mathcal{D}_x$ is a symmetric (11,5,2)-design. $\mathrm{Aut}\, \mathcal{D} = M_{11}$, and $\mathrm{Aut}\, \mathcal{D}_x = (M_{11})_x \cong PSL(2,11)$. The designs $\mathcal{D}_x$ and $\mathcal{D}$ will reappear in sections 8 and 9.

C. Applications and characterizations

(1) The Mathieu groups are intimately linked to the sporadic simple groups of CONWAY [34,35], HIGMAN & SIMS [72], MCLAUGHLIN [119] and FISCHER [48]. For descriptions of these groups, see the above papers, LÜNEBURG [111], and SEIDEL [143].

Several characterizations of Mathieu groups will appear in sections 7 and 9. The following characterizations do not, however, fit into the framework of those sections.

(2) Let $\mathcal{D}$ be a Steiner system $S(t,k,v)$. Suppose $G \leq \mathrm{Aut}\, \mathcal{D}$ is transitive on the ordered (t+1)-tuples of points not contained in a block, and also on the ordered (t+2)-tuples of points no t+1 of which are contained in a block. Then $\mathcal{D}$ is $PG(2,q)$, $AG_2(d,2)$, $\mathcal{W}_{22}$, $\mathcal{W}_{23}$, or $\mathcal{W}_{24}$ (TITS [158]).

(3) Let $\mathcal{D}$ be a Steiner system $S(t,2t-2,v)$. Assume that, whenever B and C are distinct blocks and $|B \cap C| = t-1$, necessarily B+C is a block. Then $\mathcal{D}$ is $AG_2(d,2)$ or $\mathcal{W}_{24}$. This striking result is due to CAMERON [29]. Actually, CAMERON proves a stronger theorem characterizing $PG_1(d,2)$ and $\mathcal{W}_{23}$.

(4) I know of no satisfactory characterizations of the designs $\mathcal{W}_{11}$ or $\mathcal{W}_{12}$ in terms of the action of M_{11} or M_{12} on them. These 2-transitive designs do not seem to fit into any known general design setting as do the other three WITT designs. (There is, however, a lattice-theoretic setting; see KANTOR [103].)

5. NORMAL SUBGROUPS OF G_x

A. Situation

G is 2-transitive, and N is a non-trivial normal subgroup of G_x. Of course, I have in mind placing some restrictions on G_x of a geometric nature. Nevertheless, many purely group-theoretic results have been proved recently which are very useful to geometry. Here are two of these: if N is regular on $S - \{x\}$, then G is of known type (HERING, KANTOR & SEITZ [66], SHULT [146]); if $|S|$ is odd, $|N|$ is even, and all involutions in N fix only x, then G is either known or has a regular normal elementary abelian subgroup (ASCHBACHER [3]).

B. O'NAN's results

The best work presently being done on 2-transitive groups is due to O'NAN. Some of his general results are described in 5B and then applied in 5C.

(1) *Suppose* N *is abelian and not semiregular* (*i.e.*, $N_y \neq 1$ *for some* $y \in S-\{x\}$). *Then* $P\Gamma L(n,q) \geq G \geq PSL(n,q)$ *for some* $n \geq 3$ *and* q (O'NAN [130]).

(2) *Suppose* $N \cap N^g = N$ *or* 1 *for all* $g \in G$, *and* N *is not semiregular. Then* $P\Gamma L(n,q) \geq G \geq PSL(n,q)$ *for some* $n \geq 3$ *and* q (O'NAN [132]).

(3) *Suppose* N *is cyclic. Then either* G *has a regular normal subgroup, or* $G \geq PSL(2,q)$ *or* $PSU(3,q)$ *for a prime* q, *or* G *is* $P\Gamma L(2,8)$ (O'NAN, unpublished; ASCHBACHER [4]).

(4) *If* N *is abelian, and* $|N|$ *and* $|\Omega|$ *are odd, then* G *has a regular normal subgroup* (O'NAN [135]).

Further beautiful results are found in O'NAN [133]. While these are not strictly geometric, he finds very ingenious ways to use designs and graphs in his arguments.

O'NAN [134] considered the 3-transitive analogue of the above situation. He classified those 3-transitive groups G such that G_{xy} has a non-trivial abelian normal subgroup.

C. Applications

O'NAN's applications of his results are also basic for his proofs. Let $\mathcal{D}$ be a design and suppose $G \leq \text{Aut}\,\mathcal{D}$ is 2-transitive on points. Let N be the group of $g \in G_x$ fixing all blocks on x. Then N is normal in G_x.

Clearly, N is a very natural geometric subject. It corresponds to groups of central collineations of projective spaces, and dilatation groups of affine ones.

By 5B(2), $\mathcal{D}$ is a projective space if $N_y \neq 1$ for some $y \neq x$. By 5B(3,4), $\mathcal{D}$ is severely restricted if N is cyclic or if N is abelian and $|N|$ and v are both odd. The same is true if $|N|$ is even but each involution in N fixes only x (ASCHBACHER [3]). However, the case N abelian, $|N|$ odd, and v even has not yet been settled.

A slightly different application of 5B(2) is found in 8E(1).

There are, of course, analogous applications to t-designs with $t > 2$.

6. G_{xy} FIXES k POINTS

A. Situation

G is 2-transitive on S. If $x \neq y$, G_{xy} fixes precisely k points, where $2 < k < v$.

Let L be the set of fixed points of G_{xy}. By WITT's result 2B(1) (with $W = G_{xy}$), $\{L^g \mid g \in G\}$ yields a design $\mathcal{D}$ with $\lambda = 1$. Moreover, G_L^L is sharply 2-transitive on L, from which it follows that k is a prime power.

Possibly the main property of $\mathcal{D}$ and G is that the set of fixed points of any subgroup of G is a subspace of $\mathcal{D}$ (defined in 1A). In spite of all the subspaces of $\mathcal{D}$ this usually guarantees, it is very hard to get solid information about $\mathcal{D}$.

B. Known examples of $\mathcal{D}$

(1) $AG_1(d,k)$.

(2) $PG_1(d,2)$.

(3) A unique design $\mathcal{D}(4)$ with $v = 28$, $k = 4$. In this case, necessarily $G \cong P\Gamma L(2,8)$.

Note that, even if $\mathcal{D}$ is $AG_1(d,k)$ or $PG_1(d,2)$, in view of section 3 it is still very difficult to determine G. This fact is undoubtedly one of the major obstacles to the study of $\mathcal{D}$ itself. Note also that $G \cong A_7$ occurs here for $PG_1(3,2)$, in which case $G_{xy} \cong A_4$ is regular on S-L (cf. 3A(3)).

C. Classification theorems

The study of the present situation was initiated by HALL [58] in the case $k = 3$. His and all subsequent results have depended on 2-subgroups of G.

(1) *Suppose* $k = 3$ *and some line is the set of fixed points of an involution. Then* $\mathcal{D}$ *is* $AG_1(d,3)$, $PG(2,2)$, *or* $PG_1(3,2)$. (M. HALL [58] combined with J. HALL [55] or TEIRLINCK [157].)

(2) *Suppose some involution fixes just one point. Then* G *has a regular normal elementary abelian* p*-subgroup, where* p *is an odd prime and* $p|k$. (This is an easy consequence of GLAUBERMAN [53] and FEIT & THOMPSON [46]. The case $k = 3$ is in HALL [58], and is very elementary.)

The best result known is due to HARADA [63]:

(3) *Assume that all involutions fix at most* k *points. Then one of the following holds:*
   (i) $\mathcal{D}$ *is* $AG(2,k)$, $PG(2,2)$ *or* $PG_1(3,2)$;
   (ii) $\mathcal{D}$ *is* $AG_1(3,k)$ *with* k *odd; or*
   (iii) $\mathcal{D}$ *is an affine translation plane of odd order* k.
   (Actually, this is slightly different from HARADA's original formulation; see the Appendix of KANTOR [105].)

The only known non-desarguesian examples of (iii) have order $k = 9$. Results of HUPPERT [77] imply that there is a unique such example with G solvable. If G is non-solvable, results contained in CZERWINSKI [37] and HERING [65] show that the "exceptional" nearfield plane of order 9 is the only example possible; unfortunately, as of the writing of the present survey, these results had not quite been completely proved.

(4) *Assume that* G *is transitive on non-incident point-line pairs. Then* $\mathcal{D}$ *is* $AG_1(d,k)$ *or* $PG_1(d,2)$. (HALL and BRUCK for $k = 3$; see HALL [60] or DEMBOWSKI [40, pp.100-101]; KANTOR [99] in general. Other special cases are due to ITO [84] and OSBORN [136]. A variation on this theme is found in BUEKENHOUT [14].)

(5) *If some non-trivial element of* G_x *fixes all lines through* x, *then either* $\mathcal{D}$ *is* $PG_1(d,2)$ *or* $\mathcal{D}(4)$, *or* G *has a regular normal elementary abelian subgroup.*

PROOF. 5B(2) or 5B(3) applies to a non-trivial normal subgroup of G_x minimal with respect to fixing all lines through x. □

It is easy to see that 6C(5) contains 6C(2); however, 6C(2) is the far more useful result.

D. Subplanes

(1) In [58], HALL showed that, when $k = 3$, $\mathcal{D}$ has a subspace $PG(2,2)$ or $AG(2,3)$. Because of the 2-transitivity of G, $\mathcal{D}$ has many such concrete subplanes. What is lacking is a way to tie these subplanes together into a projective or affine space.

KANTOR [105] proved the following awkward result, which both generalizes HALL's result and implies 6C(4). $\mathcal{D}$ *must have a subspace such that either*
   (i) $|\Delta| = k^i$, $i \geq 2$, *and* G_Δ^Δ *is 2-transitive, has a regular normal subgroup, and has no involution fixing more than one point;*
   (ii) $k = 3$ *and* Δ *is* $PG(2,2)$;
   (iii) Δ *is an affine translation plane, and* G_Δ^Δ *contains the translation group and is flag-transitive or has exactly two flag-orbits; or*
   (iv) k *is a power of* 2, *and* Δ *is the design* $\mathcal{D}(k)$ *obtained from the dual of the complement of a completed conic in* $PG(2,2k)$.

Note that, if k is prime, Δ is $AG_1(i,k)$ in (i) and $AG(2,k)$ in (iii), so $\mathcal{D}$ must have $AG(2,k)$ as a subplane if $k > 3$ is prime. As in 6C(2), it is very likely that CZERWINSKI [37] and

HERING [65] will imply that the only non-desarguesian planes which might arise in (iii) are the exceptional nearfield plane of order 9 or a semifield plane of odd order.

Once again, a method is needed for tying all these subplanes together.

(2) In this context, it is natural to recall the standard methods of gluing planes together to form projective or affine spaces: the axioms of VEBLEN & YOUNG [161], and the theorem of BUEKENHOUT [11]. Groups are not needed for these (nor even finiteness).

Let $\mathcal{D}$ be a design with $\lambda = 1$. If each triangle is contained in a subspace which is a projective plane, then $\mathcal{D}$ consists of the points and lines of a projective space (VEBLEN & YOUNG [161]).

If each triangle of $\mathcal{D}$ is contained in an affine plane of order > 3, then $\mathcal{D}$ consists of the points and lines of an affine space (BUEKENHOUT [11]). This is false if $k = 3$ (see HALL [58]). But here, if Aut $\mathcal{D}$ is primitive on points (e.g., if Aut $\mathcal{D}$ is 2-transitive), then $\mathcal{D}$ is an affine space. (This is contained in FISCHER [47]; it is also an easy consequence of HALL [58] and GLAUBERMAN [53]).

J. HALL [55] and TEIRLINCK [157] have also handled the case where each triangle of $\mathcal{D}$ is in a projective or affine plane (a situation which arises in proving 6C(1)).

There are further interesting geometric questions of this sort that can be asked, with or without a group present; see BUEKENHOUT & DEHERDER [17].

E. Higher transitivity

It is natural to modify the situation under consideration as follows: G is t-transitive on S, and the stabilizer of t points fixes exactly k points, where $2 < t < k < v$. This time, the design $\mathcal{D}$ is a Steiner system $S(t,k,v)$. If B is a block, G_B^B is sharply t-transitive.

(1) Suppose that $t = 3$. The only known examples of $\mathcal{D}$ are: (i) $AG_2(d,2)$, and (ii) if $PGL(2,q^i)$, $i \geq 2$, is regarded as acting on $GF(q^i) \cup \{\infty\}$, the blocks of $\mathcal{D}$ are the sets $(GF(q) \cup \{\infty\})^g$, $g \in PGL(2,q^i)$. Note that miquelian inversive planes are special cases of (ii).

It is not difficult to prove that the designs in (ii) have $P\Gamma L(2,q^i)$ as their full automorphism groups. For this reason, it seems as if the present situation should be much easier than that of 6A: if $k > 4$, G should be small.

Unfortunately, nothing is known here other than variations on the 2-transitive results of 6C and 6D. Thus, G_x acts on $S-\{x\}$ as a group satisfying the condition 6A. There is a natural definition for subspaces: sets Δ of points such that the block of $\mathcal{D}$ through any three points of Δ is again contained in Δ. There is always a subspace which is $AG_2(3,2)$ or is as in (ii) (where $k = q+1$); see KANTOR [105]. BUEKENHOUT [12,13] has proved other design versions of results related to 6C and 6D.

(2) According to a remarkable result of NAGAO [120], the case $t \geq 4$ does not occur. I will outline a proof, using an approach somewhat simpler than NAGAO's.

Suppose G exists; without loss of generality, $t = 4$. This time, G_B^B is sharply 4-transitive. There are thus just three cases (JORDAN [89, pp.245-361]; HALL [57, pp.72-73]):

(I) $k = 5$, $G_B^B \cong S_5$;

(II) $k = 6$, $G_B^B \cong A_6$; and

(III) $k = 11$, $G_B^B \cong M_{11}$.

(I) Here it is straightforward to use arguments of HALL [58] to find a subspace which is an extension of $AG_2(3,2)$ or the (miquelian) inversive plane of order 3 having an involution fixing a block pointwise. However, no such extensions exist. (This elementary, highly combinatorial approach was not used by NAGAO. In fact, case (I) was the hardest for him, requiring a complicated argument and involving the FEIT-THOMPSON theorem.)

(II) Let $t \in G$ be an involution and let f be its number of fixed points. Fix a 2-cycle (x,x^t) of t. If (y,y^t) is any other 2-cycle, then $\{x,x^t,y,y^t\}$ belongs to a unique block B, and t fixes B. Since t^B is in A_6, it fixes exactly two points z_1,z_2 of B. Conversely, any two fixed points z_1,z_2 of t uniquely determine a 2-cycle (y,y^t). Hence, t has exactly $\frac{1}{2}(v-f) - 1 = \frac{1}{2}f(f-1)$ 2-cycles other than (x,x^t). Thus, $v = f^2+2$. In particular, $f > 2$.

On the other hand, there are exactly $(v-3)/(k-3)$ blocks containing three fixed points of t, of which $(f-3)/(k-3)$ consist entirely of fixed points. Thus, $f^2+2 = v \equiv 0 \equiv f \pmod 3$, which is impossible.

(III) The same type of argument as in (II) shows that each involution t has exactly $f = \sqrt{v-2}$ fixed points. If (x,x^t) is a 2-cycle, then t commutes with some involution $u \in G_{xy}$. Here, t and u fix exactly $g < f$ common points.

Let Δ be the set of fixed points of t. Then Δ is a subspace of the design, and again as in (II), u fixes exactly $g = \sqrt{f-2}$ points of Δ. Here $g \geq 2$. There are $(v-2)(v-3)/9\cdot 8$ blocks containing two points fixed by t and u, of which $(f-2)(f-3)/9\cdot 8$ are fixed pointwise by t and $(g-2)(g-3)/9\cdot 8$ are fixed pointwise by both involutions. However, the conditions $v = f^2+2$, $f = g^2+2$, and $(v-2)(v-3) \equiv (f-2)(f-3) \equiv (g-2)(g-3) \equiv 0 \pmod 9$ cannot be met.

This contradiction proves NAGAO's theorem. Note that, in (II) and (III), the arguments were purely combinatorial, almost not requiring G.

7. JORDAN GROUPS

A. Situation

$\mathcal{D}$ is a design, $G \leq \mathrm{Aut}\,\mathcal{D}$ is 2-transitive on points and transitive on blocks, and $G(B)$ is transitive on $S-B$. Intuitively, this means that $\mathcal{D}$ has many "axial automorphisms".

JORDAN [88] (= [89, pp.313-338]) initiated the study of essentially this situation from the point of view of permutation groups. Almost 100 years later, HALL [58] noticed the geometric content of JORDAN's assumptions.

B. Examples

(1) $PG_e(d,q)$, $1 \leq e \leq d-1$.

(2) $AG_e(d,q)$, $1 \leq e \leq d-1$ if $q \neq 2$, and $2 \leq e \leq d-1$ if $q=2$. (This restriction is needed to eliminate the degenerate case $q = 2$, $e = 1$, where lines have only two points.)

(3) The Witt designs $\mathcal{W}_{22}$, $\mathcal{W}_{23}$ and $\mathcal{W}_{24}$ (see section 4).

For the latter designs, G must be M_{22}, $\mathrm{Aut}\,M_{22}$, M_{23}, or M_{24}. By PERIN's results (see 3B), if $\mathcal{D}$ is $PG_e(d,q)$ then $G \geq PSL(d+1,q)$, except perhaps if $e = 1$, $q = 2$, and d is odd. (The collineation group $G \cong A_7$ of $PG(3,2)$ is, in fact, an example of this exceptional situation; see 3A(3).) Similarly, 3C applies when $\mathcal{D}$ is $AG_e(d,q)$.

C. Basic properties

(1) First of all, $v \geq 2k$.

W. KNAPP has been kind enough to look into the history of this result. That $v \leq 2k$ implies the 3-transitivity of G was first proved by JORDAN [88, Théorème 1] (and not by MARGGRAFF [114], as stated on p.34 of WIELANDT [166]). KNAPP found that, in his two inaccessible papers, MARGGRAFF [114,115] proved the impossibility of $v < 2k$ (see WIELANDT [166, pp.34-38] for a proof), and also showed that $v \geq \frac{5}{2}k$ if $v-k$ is not a power of 2 (but obtained no characterizations of this exceptional case). Finally, KNAPP noted inaccuracies in the reference to MARGGRAFF in WIELANDT's bibliography.

For the case $v \leq 6k$, see 7D(2).

(2) *Now let* L *consist of the set of intersections of families of blocks. Certainly,* L *is a lattice* (this has nothing to do with $\mathcal{D}$). *In fact,* L *is a geometric lattice* (see 1A for the terminology). *Moreover,* G *is transitive on bases of* L*, and, if* $X \in L$*, then* $G(X)$ *is transitive on* $S-X$ (KANTOR [105], using different terminology).

PROOF. Let $\emptyset \neq X \in L$ and $X \subset B,C$ with B and C different blocks. Then $|S-(B \cup C)| = v-2k+ |B \cap C| > 0$ by (1). Since $G(B)$ is transitive on $S-B$ and $G(C)$ is transitive on $S-C$, $G(B \cap C)$ is transitive on $S-(B \cap C)$. It follows that $G(X)$ is transitive on $S-X$. Consequently, $G(X)$ is transitive on those $Y \in L$ in which X is maximal, so that X is maximal in $X \vee y$ for all $y \in S-X$. This proves that L is a geometric lattice, and the remaining assertions follow easily. □

(3) There is a great deal of information contained in (2). For example, G is 3-transitive if and only if lines have just two points, and is 4-transitive if and only if planes have just three points.

(4) If $X \in \mathcal{L}$, $G(X)$ induces an automorphism group $\overline{G(X)}$ on the interval $[X,S] = \{Y \in \mathcal{L} \mid X \le Y\}$. $\overline{G(X)}$ is 2-transitive on those elements of $[X,S]$ of dimension $1 + \dim(X)$. If $\dim(X) \le \dim(G) - 3$, then $\overline{G(X)}$ and the blocks in $[X,S]$ provide a group and a design satisfying the same conditions as G and $\mathcal{D}$.

Similarly, suppose for simplicity that G is not 3-transitive. Let $X \in \mathcal{L}$ be neither $\emptyset$, a point, nor a line. Then G_X^X also acts on the interval $[\emptyset,X]$ as in 7A.

(5) By (4) and some classical geometry (or KANTOR [105], or DOYEN & HUBAUT [43]), if suitable intervals $[X,S]$ or $[\emptyset,X]$ are of known type, $\mathcal{D}$ is essentially known. (See KANTOR [105, 6.5] for a precise statement.) This fact provides very nice inductive possibilities.

D. Characterizations

(1) *If* $k = 3$, *then* $\mathcal{D}$ *is* $PG_1(d,2)$ *or* $AG_1(d,3)$. (This is the HALL-BRUCK theorem; see 6C(4).)

(2) *If* $v \le 6k$, *then* $\mathcal{D}$ *is a projective or affine space*, $\mathcal{W}_{22}$, $\mathcal{W}_{23}$, *or* $\mathcal{W}_{24}$ (KANTOR [105]). *Moreover, in this case,* G *is even known.*

(3) *If* G_B *is 2-transitive on* $S-B$, *then* $\mathcal{D}$ *is* $PG_{d-1}(d,q)$, $AG_{d-1}(d,2)$, $\mathcal{W}_{22}$, $\mathcal{W}_{23}$, *or* $\mathcal{W}_{24}$, *and* G *is known* (KANTOR [105]).

(4) *If* $G(B)$ *has an abelian subgroup transitive on* $S-B$, *the conclusions of* (3) *hold.*

PROOF. BY 7C(5), without loss of generality G is not 3-transitive, so lines have $h > 2$ points. Fix $x \in B$. Then the given abelian group $A \le G(B)$ is transitive on the $(v-k)/(h-1) < |A|$ lines on x not in B. It follows that some $a \ne 1$ in A fixes all lines through x. Now a result of O'NAN [130] (see 5B(2)) completes the proof. □

Special cases of (4) are found in KANTOR [105,106], and MCDONOUGH [117,118].

(5) *If* $v-k$ *is a prime power, the conclusions of* (3) *hold.* (KANTOR [104]; special cases are in KANTOR [105,106], and MCDONOUGH [117,118]. Stronger results are proved in KANTOR [104].)

PROOF. By 7C(5), without loss of generality $\lambda = 1$. Let p be the prime dividing $v-k$. Let $B \cap C = x$. A Sylow p-subgroup P of $G(B)$ is transitive on $S-B$. Since $|P:P_C| = r-1 < v-k$, P_C fixes no point of $S-B$. Since P_C normalizes a Sylow p-subgroup Q of $G(C)$, it centralizes some $q \ne 1$ in the center of Q. Then q fixes the set B of fixed points of P_C. Now the transitivity of Q on $S-C$ implies that q fixes all lines through x. Once again, O'NAN's theorem 5B(2) completes the proof. □

(6) *If* $G(B)$ *has a subgroup normal in* G_B *and regular on* $S-B$, *then the conclusions of* (3) *hold or* $\mathcal{D}$ *is* $PG_1(3,2)$ *or* $AG_2(4,2)$. (KANTOR [97]; special cases have already been mentioned in 6C(4). This result, and its proof, were motivated by the HERING-KANTOR-SEITZ-SHULT theorem, already mentioned in 5A.)

E. Applications

(1) KANTOR & MCDONOUGH [106] showed that, *if* G *is a permutation group of degree* $v = (q^n-1)/(q-1)$ *containing the 2-transitive group* $PSL(n,q)$, $n \ge 3$, *then either* G *contains the alternating group or* $PSL(n,q) \le G \le P\Gamma L(n,q)$.

PROOF. If G is as much as $k = (q^{n-1}-1)/(q-1)$ transitive, results of WIELANDT [164] imply that G is alternating or symmetric. If G is not k-transitive, let $\mathcal{D}$ have as blocks $\{H^g \mid g \in G\}$, where H is a hyperplane. Now use any one of D(3, 4, or 5). □

Unfortunately, the preceding proof does not apply when $n = 2$. That case is far more interesting than the case $n \ge 3$, since $PSL(2,11) < M_{12} < A_{12}$ and $PSL(2,23) < M_{24} < A_{24}$. In fact, the study of groups G satisfying $PSL(2,p) < G < A_{p+1}$, with p prime, is precisely what led MATHIEU to the discovery of M_{12} and M_{24}. NEUMANN [124] has recently proved that G is necessarily 4-transitive here. For an application of this problem to coding theory, see SHAUGHNESSY [144].

(2) Several of the classification theorems concerning Jordan groups can be interpreted as stating that certain natural attempts at generalizing M_{22}, M_{23} and M_{24} lead to nothing new.

(3) PRAEGER [141] has recently used D(2) in the course of proving some general results concerning 2-transitive groups. Another recent appli-

cation of JORDAN's original situation is made in the beautiful paper of SCOTT [142].

F. Problem

Besides the obvious problem of determining all designs admitting Jordan automorphism groups, there is a natural, interesting type of problem these designs lead to.

First, can G be acting on the set S of points of $PG(d,q)$ or $AG(d,q)$ without $\mathcal{D}$ being $PG_e(d,q)$ or $AG_e(d,q)$ for some e? The answer is no for $PG(d,q)$, $q > 2$, by results of PERIN [139] (see 3B).

Now let's forget the group, and just consider the remaining geometric situation. Can a design with $\lambda = 1$ be constructed using all the points, and some but not all e-spaces, of a projective or affine space? Such designs are probably rare. There is an obvious generalization of this question in which a generalization of t-designs is involved.

Next, can a design with $\lambda > 1$ be constructed using some but not all e-spaces, in which the lines of the design consist of all the lines of the underlying geometry? I conjecture that this is impossible.

8. 2-TRANSITIVE SYMMETRIC DESIGNS

A. Situation

$\mathcal{D}$ is a symmetric design, and $G \leq \mathrm{Aut}\,\mathcal{D}$ is 2-transitive on points.

2A(2) indicates the group-theoretic interpretation of this situation. Note that the complementary design $\mathcal{D}'$ satisfies the same conditions as $\mathcal{D}$.

B. Examples

There are several very interesting examples of 2-transitive symmetric designs. It is only necessary to describe one of $\mathcal{D},\mathcal{D}'$. In each case, $\mathcal{D}$ has polarities.

(1) Projective spaces: $PG_{d-1}(d,q)$. Of course, $\mathrm{Aut}\,\mathcal{D} = P\Gamma L(d+1,q)$. In view of section 3, from this example it should already be clear that there will be serious obstacles to the study of G.

(2) The unique 11-point Hadamard design $\mathcal{W}_{11}$. Here $v = 11$, $k = 5$, $\lambda = 2$ (compare 4B(3)). The only possible G is $G = \mathrm{Aut}\,\mathcal{W}_{11} \cong PSL(2,11)$. Here, $G_B \cong A_5$ acts as A_5 on B and as $PSL(2,5)$ on $S-B$. $\mathcal{W}_{11}$ has polarities θ, and $G\langle\theta\rangle \cong PGL(2,11)$.

(3) G. HIGMAN's design $\mathcal{W}_{176}$ (see G.HIGMAN [73]; SIMS [150]; SMITH [152, 153]; CONWAY [35]). Here, $v = 176$, $k = 50$, and $\lambda = 14$. The only possible G is $G = HS$, the sporadic simple group of D.G. HIGMAN & C.C. SIMS [72]. $G_B \cong PSU(3,5)$ has rank 3 on B (and $G_{xB} \cong A_7$ if $x \in B$), while G_B acts on $S-B$ in its usual 2-transitive representation of degree 5^3+1. Also, $\mathcal{W}_{176}$ has polarities ϕ, and $G\langle\phi\rangle \cong \mathrm{Aut}\,HS$.

$\mathcal{W}_{176}$ has a fascinating property: there is a 1-1-correspondence θ from 2-sets of points to 2-sets of blocks which is preserved by G. Here, θ is not induced by a polarity of $\mathcal{W}_{176}$. Moreover, $G_{\{x,y\}} = G_{\{x,y\}^\theta} \cong Z_2 \times \mathrm{Aut}\,A_6$.

(4) The symplectic symmetric designs $S^\varepsilon(2m)$, one for each $m \geq 2$ and $\varepsilon = \pm 1$. Here $v = 2^{2m}$, $k = 2^{m-1}(2^m+\varepsilon)$, $\lambda = 2^{m-1}(2^{m-1}+\varepsilon)$. $S^1(2m)$ and $S^{-1}(2m)$ are complementary designs.

Set $G = \mathrm{Aut}\,S^\varepsilon(2m)$. Then G has a regular normal elementary abelian 2-subgroup V of order $v = 2^{2m}$, and $G = VG_x$, $V \cap G_x = 1$, where $G_x \cong Sp(2m,2)$ is a symplectic group acting on V in the usual way. $G_B \cong Sp(2m,2)$ is 2-transitive on B and $S-B$. If $x \in B$, then G_{xB} is the orthogonal group $GO^\varepsilon(2m,2)$.

Moreover, by 2A, the preceding properties of G completely determine $S^\varepsilon(2m)$. It is remarkable that these properties were implicitly contained in work of JORDAN 100 years ago (see JORDAN [89, pp.XXI-XXIII] and [90, pp.229-249]).

Any subgroup of G of the form VT, with $T \leq G_x$ transitive on $V - \{1\}$, is 2-transitive on $S^\varepsilon(2m)$; for example, T can be $Sp(2e,2^f)$ whenever $ef = m$. The question of whether every 2-transitive automorphism group necessarily contains V leads to the same difficulties as in 3C.

In view of the action of G_x on V, there is an involution $t \in G_x$ fixing exactly $\frac{1}{2}v$ points (t is a transvection). If x_1 and x_2 are distinct points, there is a unique conjugate of t interchanging x_1 and x_2.

$S^\varepsilon(2m)$ has interesting combinatorial properties. Let $+$ denote the symmetric difference of sets of points. If B, C and D are any blocks, then B+C+D is either a block or the complement of a block. (This property alone does not quite characterize these designs.) If $B \neq C$, then V_{B+C} is transitive on B+C. (This property does characterize $S^{\varepsilon}(2m)$, assuming only that V is an automorphism group of a symmetric design regular on points; see KANTOR [101].)

Here's another description of $S^1(2m)$. Consider the dual of a completed conic in $PG(2,2^m)$. Use the dual of the knot as the line at infinity of $AG(2,2^m)$. Let B be the union (in $AG(2,2^m)$) of the remaining 2^m+1 lines. Then the translates of B are the blocks of $S^1(2m)$.

A similar description of $S^1(4(2e+1))$ can be given in terms of the LÜNEBURG-TITS affine planes of order $2^{2(2e+1)}$ (defined in LÜNEBURG [110,111]): once again, the dual of a suitable oval can be used, in which the dual of the line at infinity is the knot. I know of no other planes which yield any designs $S^1(2m)$ in this manner, but such planes undoubtedly exist (and merit study).

A (-1,1)-incidence matrix of $S^{\varepsilon}(2m)$ is a Hadamard matrix known since the last century: the tensor product of m Hadamard matrices of size 4. BLOCK [9] first noticed (using this incidence matrix) that Aut $S^{-1}(2m)$ is 2-transitive on points for each m. He pointed this out to me in 1968. All the properties of $S^{\varepsilon}(2m)$ just described were proved at that time, and eventually appeared in KANTOR [101]. The designs were later rediscovered by RUDVALIS (1969, unpublished), HILL [74], and CAMERON & SEIDEL [30]. The latter paper provides an interesting relationship between these designs and coding theory.

C. Basic properties

The most famous result concerning 2-transitive symmetric designs is the beautiful theorem of OSTROM & WAGNER [137]: *if* $\lambda = 1$, *then* $\mathcal{D}$ *is a desarguesian projective plane*. Consequently, I will assume $\lambda > 1$ throughout this section.

(1) G *is 2-transitive on blocks. If* B *is a block, then* G_B *is transitive on* B *and* S-B, *and dually. Moreover, if* $(v,k) = 1$, *then* G_{xB} *is transitive on* S-B (by 1C(6)), *and dually.*

(2) *If* G_B *is 2-transitive on both* B *and* S-B, *then the dual statements hold and* G_x *has rank* 3 *on* $S - \{x\}$. (More generally, in KANTOR [93] it is proved that, *if* G *is an automorphism group of a design 2-transitive on points and transitive on blocks, and if* G_B *is 2-transitive on both* B *and* S-B, *then the rank* ρ *of* $G_x^{S-\{x\}}$ *satisfies* $\rho \le 5$, *and even* $\rho \le 3$ *if* $v \neq 2k$.)

(3) *If* $\mathcal{D}$ *is a Hadamard design,* G_B *is necessarily 2-transitive on* S-B. This will be proved in 8C(5) below. Further special transitivity properties are found in KANTOR [93], especially Lemma 4.2.

(4) In KANTOR [93], a great deal of attention is paid to the case $k \mid v-1$ (which is equivalent to $(k,\lambda) = 1$, and which holds in $PG_{d-1}(d,q)$ and H_{11}). Assume this condition. *Then* G_B *must be primitive on* S-B. (In view of KANTOR [91, 4.7 and 4.8], the same conclusion holds under much weaker numerical restrictions.) Also, G has a simple normal subgroup 2-transitive on points.

Of course, the example $S^{\varepsilon}(2m)$ shows that the last assertion does not hold in general. KANTOR [93] showed that $\mathcal{D}$ has the parameters of $S^{\varepsilon}(2m)$ for some m,ε if G has a regular normal subgroup.

(5) As an example of the proofs of transitivity properties, I will prove: *if* $k-1 \mid v-1$ (*or equivalently, if* $\lambda \mid k$), *then* G_B *is 2-transitive on* B. (Note that this implies 8C(2) when $\mathcal{D}$ is the complementary design of a Hadamard design.)

PROOF. G_x is transitive on the v-1 points $\neq x$, and on the k blocks B on x. By 1C(6), each orbit of G_{xB} on $S - \{x\}$ has size divisible by $(v-1)/(v-1,k)$. But $k = \lambda\cdot(v-1)/(k-1)$ implies that $(v-1,k) = (v-1)/(k-1)$. Thus, G_{xB} has an orbit on $B - \{x\}$ of size divisible by k-1. □

In the next section it will be seen how desirable it is to have sufficiently strong transitivity results.

D. The DEMBOWSKI-WAGNER theorem

This theorem provides the basic characterization of projective spaces needed for the study of symmetric designs. Namely:

$\mathcal{D}$ *is a projective space if any one of the following holds:*

(i) *every line meets every block;*

(ii) *every line has at least* $1 + (v-1)/k$ *points; or*

(iii) G *is transitive on ordered triples of non-collinear points.*

Slightly stronger combinatorial characterizations are found in DEMBOWSKI [40, pp.65-67], and KANTOR [92,93]; in particular, the latter reference describes the relationship with geometric lattices.

PROOF. If L is a line of $\mathcal{D}$ (the intersection of the λ blocks containing two points), there are $v-\lambda-|L|(k-\lambda)$ blocks missing L. Since $(v-\lambda)/(k-\lambda) = 1+(v-1)/k$, this implies that (i) and (ii) are equivalent; assume both of them. If $x \notin L$, and if ρ blocks contain x and L, then there are $k-\rho = |L|(\lambda-\rho)$ blocks on x not containing L. Thus, ρ is a constant, so planes can be defined, and each is determined by any triangle in it. Suppose L and M are distinct lines of a plane E. Then some block $B \supset L$ does not contain E. Since B meets M, $L \cap M = E \cap (B \cap M) \neq \emptyset$. Thus, E is a projective plane, so $\mathcal{D}$ is a projective space (VEBLEN & YOUNG [161]).

Now assume (iii). Then G_L is transitive on L and S-L. By the Orbit Theorem 1C(1), G_L has just two block-orbits. Since these must be the blocks containing L and the blocks meeting L once, (i) holds. □

E. Classification theorems

Many theorems have been proved classifying 2-transitive symmetric designs under suitable additional conditions. A catalogue of these follows.

(1) *If* $G(B) \neq 1$, *then* $\mathcal{D}$ *is a projective space* (ITO [81]). Thus, in the remainder of this section it may be assumed that $G(B) = 1$.

PROOF. G is 2-transitive on blocks. G(B) is a non-trivial normal subgroup of G_B. Each non-trivial element of G(B) fixes more than one point, and hence more than one block (1C(2)). A theorem of O'NAN [132] (see 5B(2)) now applies. (Of course, this wasn't ITO's original proof.) □

(2) *If* $\mathcal{D}$ *has the same parameters as* $PG_{d-1}(d,q)$, *then* $\mathcal{D}$ *is* $PG_{d-1}(d,q)$ (KANTOR [98]).

(3) *If* k *is prime, then* $\mathcal{D}$ *is* $\mathcal{W}_{11}$ *or a projective space* (KANTOR [93]; the case where v and k are prime is due to ITO [3]). From this it follows easily that $\mathcal{D}$ is $\mathcal{W}_{11}$ or a projective space if $(v-1)/2$ is prime.

(4) *If* $n = k-\lambda$ *is prime,* $\mathcal{D}$ *is* $\mathcal{W}_{11}$, $(\mathcal{W}_{11})'$, *or* $PG(2,n)'$ (KANTOR [93]).

(5) *If* k/2 *is prime, then* $\mathcal{D}$ *is a projective space,* $PG(2,2)'$, $(\mathcal{W}_{11})'$, $S^1(4)$, *or* $S^{-1}(4)$ (ITO & KANTOR [87]).

(6) *If* n/2 *is prime, then* $\mathcal{D}$ *is* $S^1(4)$ *or* $S^{-1}(4)$.

(7) *If* k-1 *is prime and* $\lambda > 2$, *then* $\mathcal{D}$ *is* $(\mathcal{W}_{11})'$ *or* $PG_{d-1}(d,2)'$.

PROOF. Write $k-1 = p$. Then $\lambda(v-1) = p(p+1)$ and $k > \lambda+1$ imply $p \mid v-1$, so $p \mid |G|$. A Sylow p-subgroup of G fixes a block B and a point x, and is transitive on $B - \{x\}$. Thus, G_B is 2-transitive on B. By 8E(10) (see below), it may be assumed that G_B is not 3-transitive on B. Also, by 8E(1), $G(B) = 1$. BURNSIDE [18, p.341] and classification theorems now yield the precise structure of G_B, from which $\mathcal{D} = (\mathcal{W}_{11})'$ is readily deduced. □

(8) *If* k-1 *and* v *are prime, then* $\mathcal{D}$ *is* $(\mathcal{W}_{11})'$ (ITO [83]).

Note that theorems 8E(2)-(8) all assume nothing more than *numerical* restrictions. In theorems 8E(9)-(14), further *transitivity* conditions will be imposed.

(9) *If* $\mathcal{D}$ *is a Hadamard design, and* G_B *is 2-transitive on* B, *then* $\mathcal{D}$ *is* $\mathcal{W}_{11}$ *or a projective space* (KANTOR [93]).

(10) *If* G_B *is 3-transitive on* B *and* $\lambda > 2$, *then* $\mathcal{D}$ *is* $PG_{d-1}(d,2)'$.

This is an unpublished result of CAMERON and KANTOR. The idea of the proof is as follows. As usual, $\mathcal{D}_B$ consists of the points of B and blocks $\neq$ B. Here, $\mathcal{D}_B$ is a 3-design. If $x \in B$, then $\mathcal{D}_B$ has the same number k-1 of points $\neq x$ as blocks on x. Thus, $\mathcal{D}_B$ is a symmetric 3-design, so a theorem of CAMERON [22] (see 10A,B) yields $k = 4\mu+4$, $\lambda = 2\mu+2$ or $k = (\mu+2)(\mu^2+4\mu+2) + 1$, $\lambda = \mu^2+3\mu+2$ (compare CAMERON [25]).

In the first case, $\lambda(v-1) = k(k-1)$ implies $v = 2k-1$, and 8E(9) applies to $\mathcal{D}'$. In the second case, if $x \notin B$ then G_{xB} has rank 3 on the blocks through x, and the parameter restrictions of HIGMAN [69] yield a contradiction.

(11) *If* G_B *is 2-transitive on both* B *and* S-B (compare 8C(2)), $\lambda > 2$, *and* 3 *points exist lying on no block, then* $\mathcal{D}$ *is* $PG(d,2)'$.

The proof is very similar to that of 8E(10). Note that the desired 3 points are easily shown to exist if $k \geq \lambda^2-\lambda+1$, except when $\mathcal{D}$ is $PG(3,\lambda-1)$.

(12) *If* $\lambda = 2$ *and* G_B *is 2-transitive on* S-B, *then* $\mathcal{D}$ *is* PG(2,2)', $\mathcal{W}_{11}$ *or* $S^{-1}(4)$ (CAMERON [29] and KANTOR [93]).

(13) *If* G_B *is 4-transitive on* B, *then* $\mathcal{D}$ *is* PG(2,2)', $\mathcal{W}_{11}$ *or* $S^{-1}(4)$. (This easy consequence of 8E(10) and 8E(12) is due to CAMERON [29].)

Further results of these types are found in KANTOR [3]. The following is quite a different sort of result, which (in spite of its technical nature) will be used in 8G.

(14) *Suppose* $k \mid v-1$, $x \notin B$, *and* G_{xB} *has a cyclic subgroup* A *regular on the points on* B *and the blocks on* x. *Then* $\mathcal{D}$ *is* $\mathcal{W}_{11}$ *or a projective space if either* (i) k *has no proper divisor* $\equiv 1 \pmod{\lambda}$, *or* (ii) $k < (\lambda+1)^2$ (KANTOR [93]). (In the projective space case, the given cyclic group is a Singer cycle of B.)

Some characterizations are also known for the designs $S^{\varepsilon}(2m)$ and $\mathcal{W}_{176}$.

(15) *If some* $g \neq 1$ *in* G *fixes at least* $\frac{1}{2}v$ *points, then* $\mathcal{D}$ *is* $S^{\varepsilon}(2m)$ (KANTOR [101]).

(16) *If some* $g \neq 1$ *fixes* S-(B+C) *pointwise for some* $B \neq C$, *then* $\mathcal{D}$ *is* $S^{\varepsilon}(2m)$ *or* $PG_{d-1}(d,2)$ (KANTOR [101]).

Both 8E(15) and 8E(16) rely heavily on FEIT's result 1C(3) and the DEMBOWSKI-WAGNER theorem 8C. The only possible automorphisms g which actually occur in 8E(15) and 8E(16) are elations of the underlying classical geometry.

(17) *If* G *has a regular normal subgroup, and if* G_B *is 2-transitive on both* B *and* S-B, *then* $\mathcal{D}$ *is* $S^{\varepsilon}(2m)$ (KANTOR [101]).

(18) *Suppose* G *preserves a 1-1-correspondence from 2-sets of points to 2-sets of blocks. If* $n = (\lambda-2)^2/4$, *then* $\mathcal{D}$ *is* $\mathcal{W}_{176}$ *or* $S^{1}(4)$. (KANTOR, unpublished; this was proved under additional transitivity assumptions by SMITH [152]).

F. Prime v and linked systems

(1) One of the main sources of interest in 2-transitive symmetric designs is permutation groups G of prime degree v. These are necessarily

solvable or 2-transitive (BURNSIDE [18, p.341]. Very few 2-transitive examples are known: $P\Gamma L(d+1,q) \geq G \geq PSL(d+1,q)$ acting on PG(d,q), for rare pairs d,q; PSL(2,11) with v = 11; and A_v, S_v, M_{11} and M_{23}. Here, the first two types yield symmetric designs (see 10A for the sense in which M_{23} produces a generalization of a symmetric design). This naturally leads to the study of symmetric designs with prime v. The reader is referred to NEUMANN [123] for an excellent survey of the general question of 2-transitive groups of prime degree.

(2) If $\mathcal{D}$ is a symmetric design, v is prime, and Aut $\mathcal{D}$ is transitive, then $\mathcal{D}$ is obviously a difference set design. See HALL [56,61] (and his talk at this conference)*), MANN [113], and DEMBOWSKI [40] for the definitions and basic properties of difference set designs.

Of importance in the present context is the well-known fact that, if A is an abelian automorphism group regular on the points of a symmetric design $\mathcal{D}$, and if v is odd, then the map $a \to a^{-1}$, $a \in A$, does not induce an automorphism of $\mathcal{D}$. More generally: *an involutory automorphism of a design cannot fix just one block* (NEUMANN [121]).

Also, if $\mathcal{D}$ and A are as above, then $\mathcal{D}$ admits polarities.

In the case of 2-transitive symmetric designs with v prime, the only other known way of using the primality of v is through modular character theory (as in ITO [82,83]).

(3) In 1955, WIELANDT posed the following problem: can a 2-transitive group of prime degree v have more than two conjugacy classes of subgroups of index v? Certainly, two are possible, as has been noted in 8F(1).

Thus, suppose G is 2-transitive on each of the sets $S_1,\dots,S_\mu$, $\mu > 2$, $|S_i| = v$ for each i, and the stabilizer of a point x_i in S_i fixes no point in any S_j, $j \neq i$. By 2A(2), each pair (S_i,S_j), $i \neq j$, determines a 2-transitive symmetric design. By 8C(1), G_{x_i} has two orbits on S_j. Thus:

(*) if $x_i \in S_i$ and $x_j \in S_j$, $i \neq j$, then the number of $x_h \in S_h$, $h \neq i,j$, incident with both x_i and x_j, depends only on i,j,h and whether x_i and x_j are incident or not.

CAMERON [24] considered this situation from a purely combinatorial point of view. *A system of linked symmetric designs* consists of sets $S_1,\dots,S_\mu$, $\mu > 2$, and an incidence relation between each pair of sets turning each pair into a symmetric design, such that (*) holds.

*) This volume, pp. 321-346.

Needless to say, there is a lot of arithmetic information in this situation. CAMERON rediscovered some such unpublished information due to WIELANDT and to ITO, but in the more general combinatorial setting. The conditions proved there are, however, too technical to reproduce. Additional numerical information has been obtained by ITO. For example, very recently, ITO [68] has shown that if v is prime, then for some design (S_i, S_j) neither k nor v-k can divide v-1.

Furthermore, NEUMANN [123] used a computer to show that WIELANDT's original situation cannot occur if $p < 2{,}000{,}000$. The proof of this provided a test for the available numerical data.

(4) WIELANDT has proved that, in the original situation in 8F(3), G can be the full automorphism group of at most one of the designs. (A proof is found in CAMERON [24].)

(5) The combinatorial setting is as interesting as WIELANDT's group-theoretic one: examples exist.

(a) Let V be a 2m-dimensional vector space over GF(q), $q = 2^e$. Let Sp(2m,q) act on V as usual. Then $G = V\cdot Sp(2m,q)$ has exactly q classes of complements to V (POLLATSEK [140]). Clearly, the scalar transformations act on this family of q sets, and it is not hard to see that Aut G is 2-transitive on these q sets. Since each pair of sets determines an $S^{\varepsilon}(2me)$, this is a linked system of designs having $v = 2^{2me}$ and $\mu = q$.

(b) A much larger system is possible for a given $v = 2^{2m}$. Namely, a system of linked symmetric designs with $\mu = 2^{2m-1}$ has been constructed by GOETHALS from the KERDOCK [108] codes (see CAMERON [24] and CAMERON & SEIDEL [30]).

(c) CAMERON [24] notes the following construction for examples (a) and (b) when v = 16. In the notation of 4A(5), S^*, S_{xy}, S_{xz}, S_{yz} (with x,y,z three points of B^*) form example (a) with m = 1, e = 2. S^*, together with the seven sets S_{xy}, $y \in B^* - \{x\}$, for a fixed $x \in B^*$, form example (b) with m = 2.

In each of examples (a)-(c), each symmetric design is isomorphic to $S^{\varepsilon}(2\ell)$ for some ℓ. No other examples are known of symmetric designs arising in linked systems.

(6) If $S_1,\ldots,S_\mu$ is a linked system, its *automorphism group* H consists of those permutations of $S_1 \cup \ldots \cup S_\mu$ which preserve both the partition and incidence. In example (a), H is 2-transitive on the q systems; the subgroup of H fixing each set S_i is 2-transitive on each S_i.

In example (b), H is known only for m = 2. Namely, from (c) it is clear that H contains $(M_{24})_{xB^*} \cong A_7\cdot V$, where $V = M_{24}(B^*)$ is elementary abelian of order 16. In fact (CAMERON & SEIDEL [30]), $H \cong A_8\cdot V \cong SL(4,2)\cdot V$, where V fixes S^* and each S_{xy}, while A_8 acts as usual on these 8 sets. The subgroup of H fixing 2 of the 8 sets is $A_6\cdot V$, and induces an automorphism group of the resulting design $S^{-1}(4)$.

Some properties of H for certain types of linked systems (e.g., when v is prime) are found in WIELANDT [167] and CAMERON [24].

G. Some difference set designs

In this section, a special class of difference set designs will be considered. These are of interest for both combinatorial and number-theoretic reasons (see HALL [61] and MANN [113]).

(1) Let v be an odd prime power, and set F = GF(v). Let $1 < k < v-1$ and $k \mid v-1$, and let B = B(v,k) be the subgroup of F^* of order k. Let $\mathcal{D}(v,k)$ have the elements of F as points and the translates B+a, $a \in F$, as blocks. B is a difference set in F^+ if and only if $\mathcal{D}(v,k)$ is a symmetric design.

The designs $\mathcal{D}(v,\frac{1}{2}(v-1))$ are the Hadamard designs of PALEY [138], where $v \equiv 3 \pmod 4$ can be any prime power.

By DEMBOWSKI [40, p.35] (or an easy Singer cycle argument), $\mathcal{D}(v,k)$ cannot be a projective space if $\lambda > 1$. If $\lambda = 1$, the only desarguesian exceptions are PG(2,2) and PG(2,8).

(2) PROBLEM: what is Aut $\mathcal{D}(v,k)$?

Clearly, Aut $\mathcal{D}(v,k)$ contains the group S(v,k) of all mappings $x \to bx^{\sigma}+a$, $b \in B$, $a \in F$, $\sigma \in$ Aut F. In only three cases is Aut $\mathcal{D}(v,k) > S(v,k)$ known, namely, $\mathcal{D}(11,5) = \mathcal{W}_{11}$, $\mathcal{D}(7,3) = PG(2,2)$ and $\mathcal{D}(73,9) = PG(2,8)$. These are almost certainly the only possibilities.

This problem can be reformulated in terms of permutation polynomials. Let $f(x), g(x) \in F[x]$, and assume that both polynomials act as permutations of F. If

$$f(x+b) - g(x) \in B \qquad \forall x \in F,\ \forall b \in B,$$

then the pair (f,g) determines an automorphism of $\mathcal{D}(v,k)$. Conversely, each automorphism determines such a pair (f,g), where f is the permutation induced on blocks and g the one on points.

(3) Write $G = \mathrm{Aut}\ \mathcal{D}(v,k)$, and assume $G > S(v,k)$. If v is prime, then G is 2-transitive on points by BURNSIDE's theorem on groups of prime degree (see BURNSIDE [18, p.341]). If $k = \frac{1}{2}(v-1)$, G must also be 2-transitive (KANTOR [93]; compare CARLITZ [31]; MCCONNEL [116]; BRUEN & LEVINGER [10]).

However, it is not known in any other cases that G must be 2-transitive if $G > S(v,k)$.

(4) *If* $G > S(v,k)$ *and* $k = \frac{1}{2}(v-1)$, *then* $\mathcal{D} = PG(2,2)$ *or* $\mathcal{W}_{11}$ (KANTOR [93]; for some small values of v, this was proved by TODD [159] and F. HERING [67]).

More generally, *if* G *is 2-transitive then* $\mathcal{D} = PG(2,2)$ *or* $\mathcal{W}_{11}$ *provided that either* $1 + \sqrt{k} > (v-1)/k$ *or* k *has no proper divisor* $\equiv 1 \pmod{\lambda}$.

PROOF. 8E(14) applies with $A = \{x \to bx \mid b \in B\}$. □

Further information when G is 2-transitive (but when the above numerical conditions do not hold) is found in KANTOR [93]. The fact that, even for these specific designs, it is not known whether Aut $\mathcal{D}$ can be 2-transitive, indicates the sad state of affairs concerning 2-transitive symmetric designs!

H. An application to the irreducibility of polynomials

A very unexpected sort of occurrence of 2-transitive symmetric designs has recently been found by M. FRIED. Let K be a subfield of the complex field C. If $f(x) \in K[x]$ and $g(x) \in C[x]$, it is natural to study the irreducibility of $f(x) - g(y)$ in $C[x,y]$. This question leads to difference set designs having 2-transitive automorphism groups!

The following discussion is based primarily on FRIED [49,50] (see also CASSELS [33]). $f(x)$ is called *indecomposable* over K if it is not possible to write $f(x) = f_1(f_2(x))$ with $f_i \in K[x]$ and $\deg f_i > 1$, $i=1,2$; assume that this is the case. Assume further that $g(x)$ cannot be written $g(x) = f(ax+b)$ for some $a,b \in C$, $a \neq 0$. Finally, assume that $f(x) - g(y) = \prod_{i=1}^{t} h_i(x,y)$ with $h_i(x,y) \in C[x,y]$ irreducible and $t > 1$.

FRIED shows that it may be assumed that $\deg f = \deg g = v$, say. Then $g(x)$ is indecomposable over C. Moreover, $t = 2$. Write $k = \deg h_1(x,y)$. Then there is a difference set mod v with k elements. The corresponding symmetric design $\mathcal{D}$ admits a 2-transitive automorphism group G. (Here, G can be interpreted as the Galois group of a suitable extension field of $C(x)$.)

Furthermore, G is generated by permutations $s_1,\ldots,s_\mu$, with $\mu \le 3$, such that (i) $s_1\ldots s_\mu$ is a v-cycle on points, and (ii) $\sum_i \ell(s_i) = v-1 = \ell(s_1\ldots s_\mu)$. (Here, $\ell(s_i)$ is the smallest integer ℓ such that s_i is the product of ℓ transpositions.)

Of course, $PG_{d-1}(d,q)$ and $\mathcal{W}_{11}$ are the only known cyclic difference set designs $\mathcal{D}$ for which Aut $\mathcal{D}$ is 2-transitive. (Examples 8B(3) and 8B(4) do not admit transitive cyclic automorphism groups.) FEIT [50] enumerated all cases in which these designs can arise in FRIED's situation; each case produces a pair of polynomials $f(x)$, $g(x)$.

Needless to say, conditions (i) and (ii) are weird from a geometric or group-theoretic point of view. Nevertheless, it should be clear that they merit further study.

Note that the study of the polynomial $f(x) - g(y)$ is remarkably reminiscent of the situation in 8G(2).

In more recent work of FRIED [51], 2-transitive designs have arisen in which $b = 2v$ and some element of order v has one v-cycle on points and two on blocks.

I. 2-transitive suborbits

One recent occurrence of 2-transitive symmetric designs has been in work of CAMERON [19,20,21,26], on multiply-transitive suborbits (i.e., orbits of G_x) of primitive permutation groups. Since these will be discussed in CAMERON's talk at this conference, the reader is referred to that talk *) and the above papers.

*) This volume, pp. 419-450.

J. Problems

(1) The case $\lambda = 2$ should be feasible. The combinatorial structure here is extremely rich (see HUSSAIN [78,79], HALL [62], and CAMERON [23,29]). So, for that matter, is the permutation structure: G_B must be 2-transitive on B; if $x,y \in B$, $x \neq y$, then either G_B is 3-transitive on B, or G_{xyB} has two orbits of length $(k-2)/2$ on $B - \{x,y\}$ (KANTOR [3], CAMERON [23]). CAMERON [23,29] has indicated a possible approach to this problem.

Note that only three examples are known: PG(2,2)', $\mathcal{W}_{11}$ and $S^{-1}(4)$.

(2) In the situation of 8C(2), there is a natural strongly regular graph structure on $S - \{x\}$. Unfortunately, the parameter restrictions on this graph and the tactical decomposition relations of DEMBOWSKI [38; 40, pp.60-61] involve too many unknowns. The latter relations were studied by KANTOR [93,101]; the former, in a purely combinatorial setting, by CAMERON [25] (using a method of GOETHALS & SEIDEL [54]). All the results thus far are very inconclusive.

(3) Prove that $\mathcal{D}$ is $S^{\varepsilon}(2m)$ if G has a regular normal subgroup. As already mentioned in 8C(4), in this case $\mathcal{D}$ has the same parameters as some $S^{\varepsilon}(2m)$.

(4) No satisfactory characterization of $\mathcal{W}_{176}$ is known. $\mathcal{W}_{176}$ and $(\mathcal{W}_{176})'$ are probably the only 2-transitive symmetric designs with $\lambda > 2$ and $v-2k+\lambda > 2$ in which G preserves a correspondence θ as in 8E(18); no numerical restrictions should be needed. (The main reason for the restriction in 8E(18) is to prevent k from being too large relative to λ.) If such a θ exists, $\mathcal{D}$ can be replaced (if necessary) by $\mathcal{D}'$ in order to obtain $\{x,y\} \subset X \cap Y$ if $\{x,y\}^{\theta} = \{X,Y\}$. Then $2(v-1)/k$ is an integer τ (so this situation is similar to the one considered in KANTOR [93], where $k|v-1$). If τ is odd, $G_{\{x,y\}}$ is transitive on $\{x,y\}^{\theta}$, and if $x \in B$, G_{xB} is transitive on the τ points $y \in B - \{x\}$ for which $B \in \{x,y\}^{\theta}$.

SMITH [152] has proposed a reasonable axiom one can assume in addition to the existence of θ in order to try to characterize $\mathcal{W}_{176}$, but this is too technical to state.

(5) Each of the known 2-transitive symmetric designs has polarities. Study these, and find some way to use them in the characterization of self-dual designs.

When v is prime, $\mathcal{D}$ automatically has "natural" polarities. However, no effective use has been found for them.

(6) The proof of 8E(2) in KANTOR [93] indicates that, when n is a power of a prime not dividing λ, $\mathcal{D}$ should be $\mathcal{W}_{11}$ or a projective space.

(7) Remove the numerical restrictions (i) and (ii) of 8E(14) and 8G(4).

(8) Answer WIELANDT's question (see 8F(3)). More generally, decide exactly what parameters can occur for linked systems (compare 8F(5)).

9. SYMMETRIC 3-DESIGNS

A. CAMERON's theorem

A *symmetric 3-design* is a 3-design $\mathcal{D}$ such that $\mathcal{D}_x$ is a symmetric design for each x. CAMERON [22] proved that the parameters of $\mathcal{D}$ must satisfy one of the following conditions (where μ is the number of blocks on any three points):

(i) $v = 4\mu + 4$, $k = 2\mu + 2$ (Hadamard 3-design);

(ii) $v = (\mu+2)(\mu^2+4\mu+2) + 1 = (\mu+1)(\mu^2+5\mu+5)$, $k = \mu^2+3\mu+2$;

(iii) $v = 112$, $k = 12$, $\mu = 1$ (extension of a projective plane $\mathcal{D}_x$ of order 10); or

(iv) $v = 496$, $k = 40$, $\mu = 3$.

Note that the λ for $\mathcal{D}$ is given by $\lambda = k-1$. Case (i) occurs if and only if there is a $v \times v$ Hadamard matrix. The only other case known to occur is $\mu = 1$ in **(ii)**, when $\mathcal{D}$ is $\mathcal{W}_{22}$.

For a generalization of CAMERON's theorem, see CAMERON [27].

B. 3-transitive automorphism groups

(1) Now suppose $G \le \mathrm{Aut}\,\mathcal{D}$ is 3-transitive on points. Then G_x is a 2-transitive automorphism group of the symmetric design $\mathcal{D}_x$ (cf. section 8).

It is not hard to show that cases (iii) and (iv) cannot occur. Cases (i) and (ii) remain open. Some special values of μ have, however, been ruled out by CAMERON [19], such as when $2 \le \mu < 103$ or $\mu+1$ is a prime power.

For a remarkable occurrence of case (ii) -which originally led CAMERON to his theorem- see CAMERON [20,26].

(2) *If* G_B *is 3-transitive on* B, *then* $\mathcal{D}$ *is* $AG_{d-1}(d,2)$, *the unique Hadamard 3-design with 12 points, or* $\mathcal{W}_{22}$. (This follows readily from 8E(9) and 8E(11).)

(3) Suppose next that $\mathcal{D}$ is a Hadamard 3-design. NORMAN [127] proved that $v = 12$ if μ is even. A slight modification of his argument shows that the same conclusion holds if G is 3-transitive on parallel classes of blocks. Note that, by 5B(4), the unique Hadamard 3-design having 12 points satisfies these conditions. The case n even -where $\mathcal{D}$ should be $AG_{d-1}(d,2)$- remains open.

C. Hadamard matrices

An automorphism of a Hadamard matrix H of size n is a pair (P,Q) of monomial $n\times n$ matrices such that $PHQ = H$. The automorphisms form a group $G = \text{Aut } H$ containing $1 = (I,I)$ and $-1 = (-I,-I)$ in its center. $\bar{G} = G/\langle -1\rangle$ acts faithfully as a permutation group on the union of the sets of rows and columns of H.

It may be assumed that the first row r and column c of H consist of 1's. Deleting columns 1 and n+1 of (H,-H) produces the (-1,1) incidence matrix of a Hadamard 3-design $\mathcal{D}$. Then $\bar{G}_c$ is the automorphism group of $\mathcal{D}$. In view of this, the results in B(2) and B(3) apply to $\mathcal{D}$. These in turn yield results about H. For example, if G is 4-transitive on rows, then n = 4 or 12. Another characterization of the case n = 12 follows from 6G(4) (KANTOR [94]).

Suppose n = 12. Then B(2) and the discussion of $\bar{G}_c$ imply that $\bar{G}_c \cong M_{11}$, from which $\bar{G} \cong M_{12}$ follows easily. However, $G \not\cong M_{12} \times \langle -1\rangle$. At the end of 4B(2) it was noted that $|\text{Aut } M_{12}| = 2|M_{12}|$. The resulting outer automorphism can be visualized in the present context as follows. $(P,Q) \in G$ implies that $PHQ = H$, and hence (since H is symmetric) that $Q^tHP^t = H$, so $(Q^t,P^t) \in G$. Thus, $(P,Q) \to (Q^t,P^t)$ is an automorphism of G, and induces one of $\bar{G}$; these are both outer automorphisms (see HALL [59]).

10. FURTHER TOPICS AND PROBLEMS

A. Block intersections

Let $\mathcal{D}$ be a t-design, $t \ge 2$. According to a generalization of FISHER's inequality $b \ge v$, if $v \ge k+\frac{1}{2}t$ then $b \ge \binom{v}{[\frac{1}{2}t]}$ (WILSON & RAY-CHAUDHURI [168]). Equality holds only if t = 2s for an integer s, and then $\mathcal{D}$ is called a *tight t-design*. (This is evidently a generalization of symmetric designs.) WILSON & RAY-CHAUDHURI also proved that, if $\mathcal{D}$ is a 2s-design, then $\mathcal{D}$ is tight if and only if there are at most s different intersection sizes $|B \cap C|$, where B and C run through all pairs of distinct blocks (cf. CAMERON [25]).

It is natural to consider 2s-transitive automorphism groups of tight 2s-designs. Partly motivated by the group-theoretic context, ITO [85] has just completed a proof that the only tight 4-designs are degenerate (v = k-2), $\mathcal{W}_{23}$, or its complementary design $(\mathcal{W}_{23})'$. The case s > 2 remains completely open in both the combinatorial and group-theoretic contexts.

One way to guarantee that a t-design $\mathcal{D}$ has few intersection sizes $|B \cap C|$ is to assume that $G = \text{Aut } \mathcal{D}$ is block-transitive and has small block-rank ρ; thus, G_B has exactly ρ block orbits (so there are at most $\rho-1$ different sizes $|B \cap C|$ with $B \ne C$). This was considered by NODA [126] when $\mathcal{D}$ is a Steiner system S(t,k,v). He assumed t = 3 or 4 and ρ = 3 or 4, and showed $\mathcal{D}$ must be $\mathcal{W}_{22}$, $\mathcal{W}_{23}$, $\mathcal{W}_{24}$ or $AG_2(3,2)$. The proofs are very similar to tight design arguments. (In fact, the case t = 4, ρ = 3 follows from the aforementioned results of WILSON & RAY-CHAUDHURI.)

It should also be possible to handle the case t = 2, λ = 1 and ρ = 3. Here, G_B is transitive on the lines disjoint from B, and G_x is 2-transitive on the lines through x. Presumably, $\mathcal{D}$ must be AG(2,k) or $PG_1(d,k-1)$. NODA has observed that $\mathcal{D}$ is AG(2,k) if G is not line-primitive; moreover, in unpublished work, he has used an argument of HIGMAN [70] to show that $\mathcal{D}$ is $PG_1(d,k-1)$ if $v > k^2(k-1)^2(k-2)^2 + k^2 - k + 1$.

B. Parallel relations

Let $\mathcal{D}$ be a design. A parallel relation on $\mathcal{D}$ is an equivalence relation $\parallel$ partitioning the blocks into classes, each of which partitions the points of $\mathcal{D}$. Each parallel class has v/k blocks, and there are exactly r parallel classes.

Relatively little is known about subgroups G of Aut $\mathcal{D}$ which preserve $\parallel$. If the classical affine space (or plane) case is excluded, little is known beyond NORMAN's theorem (see 9B(3)) and the following result of CAMERON [28].

(1) *Let $\mathcal{D}$ be the degenerate design with* $k = 2$ *and* $\lambda = 1$, *whose blocks are just the 2-sets of points. Assume that* $v > 3$, G *is 3-transitive, and* G *preserves* $\parallel$. *Then either* $v = 6$ *and* $G \cong PGL(2,5)$, *or* $v = 2^d$ *for some* d *and* $\mathcal{D}$ *can be regarded as the design* $AG_1(d,2)$ *with the obvious parallel relation.*

PROOF. Let x,y,z be any three points. Then G_{xyz} fixes the block through z parallel to $\{x,y\}$. Hence, G_{xyz} fixes $k \geq 4$ points. If $k = v$ then ZASSENHAUS [172] can be used to show that $v = 6$ and G is PGL(2,5). If $k < v$, 6D(1) can be applied to yield $k = 4$. If B and C are two blocks of this $S(3,4,v)$, and if $|B \cap C| = 2$, then $B - B \cap C$, $B \cap C$, and $C - B \cap C$ are parallel. Hence, $B+C$ is a block of the $S(3,4,v)$. It follows easily that the $S(3,4,v)$ is $AG_2(d,2)$ (compare 4C(3)). □

Actually, CAMERON's proof does not use 6D(1). In fact, it was while I was eliminating one case of CAMERON's situation that 6D(1) and 6E(1) were born.

More recently, CAMERON has obtained a generalization of 10B(1) to groups preserving a parallelism of the trivial design of all k-sets of a v-set ($1 < k < v$).

The natural extension of 10B(1) to the case of triangle-transitive automorphism groups of more general designs $\mathcal{D}$ (with $\parallel$) remains open.

(2) *If $\mathcal{D}$ and $\parallel$ are as before, then* $b \geq v+r-1$; *moreover,* $b = v+r-1$ *if and only if any two blocks meet in* 0 *or* k^2/v *points* (see DEMBOWSKI [40, pp.72-73]). When $b = v+r-1$, $\mathcal{D}$ is called an *affine design*. Clearly, affine designs provide a common generalization of Hadamard 3-designs and affine spaces. A theorem of DEMBOWSKI [40, p.74] characterizes affine spaces $AG_{d-1}(d,q)$, $q > 2$, among affine designs; this result is similar to the DEMBOWSKI-WAGNER theorem (see 8C). But relatively little attention has been paid to automorphism groups, so perhaps a few additional remarks are worthwhile.

Consider $\mathcal{D}$, $\parallel$, and $G \leq \text{Aut}\,\mathcal{D}$ preserving $\parallel$. Let G have t_p point-orbits, t_b block-orbits, and $t_\parallel$ parallel-class orbits. If $\mathcal{D}$ is an affine design, then $t_b + 1 = t_p + t_\parallel$ (NORMAN [127]). In general, it turns out that one can at least say $t_b + 1 \geq t_p + t_\parallel$. Also, if $\mathcal{D}$ is affine and $g \in G$, then $f_p + 1 = f_b + f_\parallel$, where f_p, f_b and $f_\parallel$ are the numbers of points, blocks and parallel-classes fixed by g. From these facts, further results can be deduced as in KANTOR [91].

Incidentally, it should be noted that the arguments on pp.113-114 of DEMBOWSKI [40] show that the number of non-isomorphic affine designs having the same parameters as $AG_{d-1}(d,q)$, $d \geq 3$, is enormous (and in fact $\to \infty$, as $d \to \infty$ or $q \to \infty$). However, I conjecture that affine spaces are the only affine designs which are not Hadamard 3-designs and whose automorphism groups are transitive on ordered pairs of non-parallel blocks.

C. Transitive extensions

Let H be a given group, possibly given together with a specific transitive permutation representation on a set S'. A *transitive extension* of H is a 2-transitive group G on a set S such that, for some $x \in S$, $G_x \cong H$; if, moreover, H is given as acting on S', then it is also required that $|S| = |S'|+1$ and that G_x acts on $S-\{x\}$ as H does on S'.

A basic open problem concerning 2-transitive groups is: if H is known as an abstract group, find all transitive extensions of H. Needless to say, very few groups H have transitive extensions.

Transitive extensions have been studied geometrically by DEMBOWSKI [39], HUGHES [75,76], and TITS [158]. Their approach was to extend designs associated with groups such as the collineation group of AG(d,q) or PG(d,q), given as acting 2-transitively on the points of the corresponding affine or projective space.

Much more generally, TITS (unpublished) has shown that a Chevalley group over GF(q), acting on a class of parabolic subgroups, has no transitive extensions if q is not very small. Still more generally, SEITZ (unpublished) has obtained the same conclusion if H is isomorphic to a Chevalley group over GF(q) and $(q,|S|-1) = 1$.

D. Some maximal subgroups of alternating or symmetric groups

Let H be a transitive permutation group on S, about which a lot is known. PROBLEM: determine all permutation groups G on S containing H.

Here, I have in mind some "geometric" group H and set S. The case $H = PSL(n,q)$, $n \geq 2$, with S the set of points of $PG(n-1,q)$, has been discussed in 7E(1). In general, if H is chosen "large" enough, and $G > H$, then G will presumably have to be 2-transitive. PROBLEM: handle the case $H = PSL(n,q)$, $n \geq 4$, and S the set of e-spaces of $PG(n-1,q)$, where $1 \leq e \leq n-2$.

I have settled the case $H = Sp(2m,2)$, in its 2-transitive representations of degree $2^{m-1}(2^m \pm 1)$: if $G > H$ then G is alternating and symmetric. The elementary proof uses transvections and the geometry of $GO^{\pm}(2m,2)$.

The reader should have no difficulty in listing many other, similar questions. Perhaps the most intriguing general question of this type concerns a Chevalley group H acting on a set S of parabolic subgroups.

E. Sp(2m,2) and .3

SHULT [148] has obtained some graph-theoretic characterizations of $Sp(2m,2)$ in its 2-transitive representations of degree $2^{m-1}(2^m \pm 1)$. However, no characterization is known in terms of designs. The difficulty is that no really interesting designs seem to have $Sp(2m,2)$ as a 2-transitive automorphism group.

Precisely the same difficulty occurs in the case of CONWAY's smallest group .3, in its 2-transitive representation of degree 276 (see CONWAY [35]). In both cases, the 2-graph approach seems more relevant than the design one (cf. SEIDEL [143]).

APPENDIX

The known 2-transitive groups

The following is a list of all the known 2-transitive groups G having no regular normal subgroup.

(1) $G = A_n$ or S_n, $|S| = n$.

(2) $PSL(d+1,q) \leq G \leq P\Gamma L(d+1,q)$; S is the set of points or hyperplanes of $PG(d,q)$.

(3) $PSU(3,q) \leq G \leq P\Gamma U(3,q)$; S is the set of q^3+1 points of the corresponding unital.

(4) G has a normal Ree subgroup; S is the set of q^3+1 points of the corresponding unital ($q = 3^{2e+1}$). When $e = 0$, $G \cong P\Gamma L(2,8)$, acting on the points of $\mathcal{D}(4)$ (see 6B(3)).

(5) $Sz(2^{2e+1}) \leq G \leq \mathrm{Aut}\, Sz(2^{2e+1})$; S is the set of $(2^{2e+1})^2 + 1$ points of the corresponding inversive plane or ovoid (see LÜNEBURG [111]).

(6) $G = Sp(2m,2)$, $|S| = 2^{m-1}(2^m \pm 1)$, $G_x = GO^{\pm}(2m,2)$.

(7) $G = PSL(2,11)$ acting on the 11 points or blocks of $\mathcal{W}_{11}$ (see 8B(2)).

(8) $G = A_7$ acting on the 15 points or planes of $PG(3,2)$ (see 4A).

(9) The Mathieu groups M_{11}, M_{12}, M_{22}, Aut M_{22}, M_{23} and M_{24} in their usual representations on the points of the corresponding Steiner systems.

(10) $G = M_{11}$ acting 3-transitively on the 12 points of a Hadamard 3-design (see 4B(3), 9B and 9C).

(11) $G = HS$ acting on the 176 points or blocks of $\mathcal{W}_{176}$ (see 8B(4)).

(12) $G = .3$, $|S| = 276$.

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