Additive Rule of Probability Given two events A and B,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

If A and B are mutually exclusive, then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$$

Multiplicative Rule of Probability Given two events A and B, then

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A) = \mathbb{P}(B)\mathbb{P}(A|B).$$

If A and B are independent,then

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Similarly, if A, B, and C are independent (?), then

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C).$$

Remark: Events A, B, and C are independent if and only if

$$\begin{aligned}
\mathbb{P}(A \cap B) &= \mathbb{P}(A)\mathbb{P}(B), \\
\mathbb{P}(A \cap C) &= \mathbb{P}(A)\mathbb{P}(C), \\
\mathbb{P}(B \cap C) &= \mathbb{P}(B)\mathbb{P}(C), \\
\mathbb{P}(A \cap B \cap C) &= \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)
\end{aligned}$$

Example 5.9 Suppose that in a box there are 2 distinct, white balls and 3 distinct, black balls. We successively, randomly draw two balls with replacement. Define the typical classical probability on the sample space and define $A = \{$ first drawing is a white ball $\}$ and $B = \{$ Second drawing is a black ball $\}$. Find $\mathbb{P}(A \cup B)$ and $\mathbb{P}(A \cap B)$. Does $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ hold? How about changing the experiment into without replacement?

Solution: The sample space is

$$\Omega = \{ \ (W_1, W_1), (W_1, W_2), (W_1, B_1), \\ \cdots, (B_3, B_3)\}$$

There are $5 \times 5 = 25$ simple events. Since

$$\mathbb{P}(A) = \frac{C_1^2 C_1^5}{25},$$

$$\mathbb{P}(B) = \frac{C_1^5 C_1^3}{25},$$

and

$$\mathbb{P}(A \cap B) = \frac{C_1^2 C_1^3}{25},$$

we have

$$\begin{aligned} \mathbb{P}(A \cup B) = & \ \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ & = \frac{C_1^2 C_1^5}{25} + \frac{C_1^5 C_1^3}{25} - \frac{C_1^2 C_1^3}{25} = \frac{19}{25}. \end{aligned}$$

With replacement,

$$\begin{aligned}\mathbb{P}(A\cap B) = &\ \frac{6}{25} = \frac{10}{25}\times\frac{15}{25}\\ &= \mathbb{P}(A)\mathbb{P}(B).\end{aligned}$$

If without replacement,

$$\Omega = \ \{(W_1, W_2), \cdots, \}$$

There are $5\times 4 = 20$ **simple events. Since**

$$\mathbb{P}(A) = \frac{C_1^2 C_1^4}{20},$$

$$\mathbb{P}(B) = \frac{C_1^2 C_1^3 + C_1^3 C_1^2}{20},$$

and

$$\mathbb{P}(A\cap B) = \frac{C_1^2 C_1^3}{20},$$

therefore,

$$\begin{aligned}\mathbb{P}(A\cap B) = &\ \frac{6}{20} \neq \frac{6}{25} = \frac{8}{20}\times\frac{12}{20}\\ &= \mathbb{P}(A)\mathbb{P}(B).\end{aligned}$$

<u>Law of Total Probability</u> Given a sequence of mutually exclusive events $S_1, S_2, \cdots, S_n$**. If event** $A \subset \cup_{i=1}^n S_i$ **and** $\mathbb{P}(S_i) > 0$**, then**

$$\mathbb{P}(A) = \mathbb{P}(S_1)\mathbb{P}(A|S_1) + \cdots + \mathbb{P}(S_n)\mathbb{P}(A|S_n)$$

Proof: Since $(A \cap S_i) \cap (A \cap S_j) = \phi$ for any $i \neq j$, this means that the sequence $\{(A \cap S_i), i = 1, \cdots, n\}$ are mutually exclusive, according to the second condition of the definition of probability, we have

$$\mathbb{P}(A) = \mathbb{P}(A \cap S_1) + \cdots + \mathbb{P}(A \cap S_n).$$

Since

$$\mathbb{P}(A \cap S_i) = \mathbb{P}(S_i)\mathbb{P}(A|S_i) \text{ for } i = 1, \cdots, n,$$

the law of total probability is proved.

Example 7.1 A fair coin is flipped. If a head turns up, a fair die is tossed; if a tail turns up, two fair dice are tossed. What is the probability of the event B that the sum of the appearing number(s) is equal to 6?

Solution: Since $\mathbb{P}(H) = \frac{1}{2}$ and $\mathbb{P}(T) = \frac{1}{2}$, and

$$\mathbb{P}(B|H) = \frac{1}{6}$$

$$\mathbb{P}(B|T) = \frac{5}{36}$$

$\{(1,5), (2,4), (3,3), (4,2), (5,1)\}$. Therefore,

$$\begin{aligned} \mathbb{P}(B) = &\ \mathbb{P}(H)\mathbb{P}(B|H) + \mathbb{P}(T)\mathbb{P}(B|T) \\ &= \frac{1}{2} \times \frac{1}{6} + \frac{1}{2} \times \frac{5}{36} = 0.15. \end{aligned}$$

Bayes' Rule Given a sequence of mutually exclusive

events $S_1, S_2, \cdots, S_n$ **and** $\mathbb{P}(S_i) > 0$**. If event** $A \subset \cup_{i=1}^{n} S_i$**, then**

$$\mathbb{P}(S_i|A) = \frac{\mathbb{P}(S_i)\mathbb{P}(A|S_i)}{\sum_{k=1}^{n} \mathbb{P}(S_k)\mathbb{P}(A|S_k)}$$

Proof: According to the definition of conditional probability, we have

$$\mathbb{P}(S_i|A) = \frac{\mathbb{P}(S_i \cap A)}{\mathbb{P}(A)}.$$

Since

$$\mathbb{P}(S_i \cap A) = \mathbb{P}(S_i)\mathbb{P}(A|S_i),$$

and according to the law of total probability,

$$\mathbb{P}(A) = \mathbb{P}(S_1)\mathbb{P}(A|S_1) + \cdots + \mathbb{P}(S_n)\mathbb{P}(A|S_n),$$

this proves the Bayes' rule.

Example 7.2 A man takes either a bus or the subway to work with probabilities 0.3 and 0.7 , respectively. When he takes the bus, he is late 30% of the days. When he takes the subway, he is late 20% of the days. If the man is late for work on a particular day, what is the probability that he took the bus?

Solution: Define $B = \{$ **The man takes a bus** $\}$, $S = \{$ **The man takes the subway** $\}$**, and** $L = \{$ **The man is late on the day** $\}$**. Then,**

$$\mathbb{P}(B) = 0.3, \mathbb{P}(S) = 0.7, \mathbb{P}(L|B) = 0.3, \mathbb{P}(L|S) = 0.2.$$

By Bayes' rule, we have

$$\begin{aligned}\mathbb{P}(B|L) &= \frac{\mathbb{P}(B)\mathbb{P}(L|B)}{\mathbb{P}(S)\mathbb{P}(L|S)+\mathbb{P}(B)\mathbb{P}(L|B)} \\ &= \frac{0.3\times 0.3}{0.7\times 0.2+0.3\times 0.3} = \frac{0.09}{0.23} = 0.3913.\end{aligned}$$

Example 7.3 To evaluate the effectiveness of a screening procedure, we will evaluate the probability of a false negative or a false positive using the following notation:
T^+ : **The test is positive and indicate that the person has the disease.**
T^- : **The test is negative and indicate that the person does not have the disease.**
D^c : **The person really does not have the disease.**
D : **The person really has the disease.**
According to the test results, we found that the sensitivity of the test has following conditional probabilities:

$$\mathbb{P}(T^+|D) = 0.98,$$

and

$$\mathbb{P}(T^-|D^c) = 0.99.$$

If the proportion of the general population infected with this disease is 2 per million, what is
(a) the probability of a false positive,

$$\mathbb{P}(D^c|T^+) \quad ?$$

(b) the probability of a false negative,

$$\mathbb{P}(D|T^-) \quad ?$$

<u>Solution:</u> **From the given information, we know the following:**

$$\mathbb{P}(D) = 0.000002, \qquad \mathbb{P}(D^c) = 0.999998$$

$$\mathbb{P}(T^+|D) = 0.98 \qquad \mathbb{P}(T^-|D) = 0.02$$

$$\mathbb{P}(T^+|D^c) = 0.01 \qquad \mathbb{P}(T^-|D^c) = 0.99$$

(a) From Bayes' Rule,

$$\begin{aligned}\mathbb{P}(D^c|T^+) &= \frac{\mathbb{P}(D^c \cap T^+)}{\mathbb{P}(T^+)} \\ &= \frac{\mathbb{P}(D^c)\mathbb{P}(T^+|D^c)}{\mathbb{P}(D^c)\mathbb{P}(T^+|D^c) + \mathbb{P}(D)\mathbb{P}(T^+|D)}\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}(D^c|T^+) &= \frac{0.00999998}{0.01000194} \\ &= 0.999804038\end{aligned}$$

(b) Using a similar calculation,

$$\begin{aligned}\mathbb{P}(D|T^-) &= \frac{\mathbb{P}(D \cap T^-)}{\mathbb{P}(T^-)} \\ &= \frac{\mathbb{P}(D)\mathbb{P}(T^-|D)}{\mathbb{P}(D^c)\mathbb{P}(T^-|D^c) + \mathbb{P}(D)\mathbb{P}(T^-|D)}\end{aligned}$$

Therefore,

$$\mathbb{P}(D|T^-) = \frac{0.00000004}{0.98999806}$$
$$= 0.00000004$$

Hence, the probability of a false positive is near 1 and very likely, while the probability of a false negative is quite small and very unlikely.

Example 7.4 **If men constitute** 47% **of the population and tell the truth** 78% **of the time, while women tell the truth** 63% **of the time, what is the probability that a person selected at random will answer a question truthfully?**

Solution: Define

$$B = \{\text{The person interviewd answers truthfully }\}$$

$$A = \{\text{ The person interviewed is a man }\}$$

According to the law of total probability, we have

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(A)\mathbb{P}(B|A) + \mathbb{P}(A^c)\mathbb{P}(B|A^c) \\ &= (0.47)(0.78) + (0.53)(0.63) = 0.70\end{aligned}$$

Example 7.4 **A worker-operated machine produces a defective item with probability 0.01 if the worker follows the machine's operating instructions exactly, and with probability 0.03 if he does not . If the worker follows the instructions 90% of time, what proportion of all items produced by the machine will be defective?**

Given that a defective item is produced, what is the conditional probability of the event that the worker exactly follows the machine operating instructions?