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# Immigration Superprocesses with Dependent Spatial Motion and Non-critical Branching


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**Abstract**

A class of immigration superprocess with dependent spatial motion is constructed by a passage to the limit from a sequence of superprocesses with positive jumps. A non-critical branching is then obtained by using a Girsanov transform of Dawson's type, which also gives a state-dependent spatial drift.



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## 1 Introduction

Let $B(\mathbb{R})$ be the totality of all bounded Borel functions on $\mathbb{R}$ and let $C(\mathbb{R})$ denote its subset comprising of continuous functions. Let $M(\mathbb{R})$ denote the space of finite Borel measures on $\mathbb{R}$ endowed with the topology of weak convergence. We write $\langle f, \mu \rangle$ for $\int f d\mu$ and for a function $F$ on $M(\mathbb{R})$ let

$$\frac{\delta F(\mu)}{\delta \mu(x)} = \lim_{r \to 0^+} \frac{1}{r}[F(\mu + r\delta_x) - F(\mu)], \qquad x \in \mathbb{R},$$

if the limit exists. Let $\delta^2 F(\mu)/\delta\mu(x)\delta\mu(y)$ be defined in the same way with $F$ replaced by $(\delta F/\delta\mu(y))$ on the right hand side. Suppose that $h$ is a continuously differentiable function on $\mathbb{R}$ such that both $h$ and $h'$ are square-integrable. Then the function

$$\rho(x) = \int_{\mathbb{R}} h(y-x)h(y)dy, \quad x \in \mathbb{R}, \tag{1.1}$$


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is twice continuously differentiable with bounded derivatives $\rho'$ and $\rho''$. Suppose that $c \in C(I\!\!R)$ is Lipschitz and $\sigma \in B(I\!\!R)^+$. We may define an operator $\mathcal{L}$ by

$$\begin{aligned}\mathcal{L}F(\mu) &= \frac{1}{2}\int_{I\!\!R} a(x)\frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) \\ &\quad+\frac{1}{2}\int_{I\!\!R^2}\rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy) \\ &\quad+\frac{1}{2}\int_{I\!\!R}\sigma(x)\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx), \qquad (1.2)\end{aligned}$$

which acts on a class of functions on $M(I\!\!R)$ to be specified. A Markov process with generator $\mathcal{L}$ was constructed in Dawson *et al* [2], generalizing the construction of Wang [9, 10]. The process generated by $\mathcal{L}$ is naturally called an *superprocess with dependent spatial motion* (SDSM) with parameters $(a, \rho, \sigma)$, where $a(\cdot)$ represents the rate of the underlying motion, $\rho(\cdot)$ represents the interaction between the "particles" and $\sigma(\cdot)$ represents the branching density. We shall also call the process simply a $(a, \rho, \sigma)$*-superprocess*. We refer the reader to [2, 9, 10] for detailed descriptions of the model. Given $\lambda \in M(I\!\!R)$, we may define another operator $\mathcal{J}$ by

$$\mathcal{J}F(\mu) = \mathcal{L}F(\mu) + \int_{I\!\!R}\frac{\delta F(\mu)}{\delta\mu(x)}\lambda(dx). \qquad (1.3)$$

A Markov process generated by $\mathcal{J}$ can be called an *SDSM with immigration* with parameters $(a, \rho, \sigma, \lambda)$ or simply a $(a, \rho, \sigma, \lambda)$*-superprocess*, where $\lambda$ represents the immigration rate.

In this work, we give a construction of the $(a, \rho, \sigma, \lambda)$-superprocess by a passage to the limit from a sequence of SDSM's with positive jumps. From the $(a, \rho, \sigma, \lambda)$-superprocess we shall use Girsanov transform of Dawson's type to derive an $M(I\!\!R)$-valued diffusion process with generator

$$\begin{aligned}\mathcal{J}^b F(\mu) &= \mathcal{J}F(\mu) - \int_{I\!\!R} b(x)\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) \\ &\quad-\int_{I\!\!R^2}\rho(x-y)b'(y)\frac{d}{dx}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx)\mu(dy), \qquad (1.4)\end{aligned}$$

where $b \in C^1(I\!\!R)$. Note that the generator $\mathcal{J}^b$ not only involves a non-critical branching given by the second term on the right hand side, it also involves a state-dependent drift in the spatial motion represented by the last term. This is different from the classical case where the Girsanov transform does not effect the spatial motion; see Dawson [1].

## 2 Function-valued dual processes

As in Dawson *et al* [2], we shall define a function-valued dual process and investigate its connection to the solution of the martingale problem for the immigration SDSM. For $\mu \in M(I\!\!R)$ and a subset $\mathcal{D}(\mathcal{J})$ of the domain of $\mathcal{J}$, we say an $M(I\!\!R)$-valued cádlág process $\{X_t : t \geq 0\}$ is a solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$*-martingale problem* if

$$F(X_t) - F(X_0) - \int_0^t \mathcal{J}F(X_s)ds, \qquad t \geq 0, \qquad (2.1)$$

is a martingale for each $F \in \mathcal{D}(\mathcal{J})$. Let $G^m$ denote the generator of the interacting particle system introduced in [2], and let $(P_t^m)_{t\geq 0}$ denote the transition semigroup generated by the operator $G^m$. Observe that, if $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ for $f \in C_\partial^2(I\!R^m)$, then

$$\mathcal{J}F_{m,f}(\mu) = F_{m,G^m f}(\mu) + \frac{1}{2}\sum_{i,j=1,i\neq j}^{m} F_{m-1,\Phi_{ij}f}(\mu) + \sum_{i=1}^{m} F_{m-1,\Psi_i f}(\mu), \tag{2.2}$$

with $\Phi_{ij}f \in C_\partial^2(I\!R^{m-1})$ defined by

$$\Phi_{ij}f(x_1,\cdots,x_{m-1}) = \sigma(x_{m-1})f(x_1,\cdots,x_{m-1},\cdots,x_{m-1},\cdots,x_{m-2}), \tag{2.3}$$

where $x_{m-1}$ is in the places of the $i$th and the $j$th variables of $f$ on the right hand side, and $\Psi_i f \in C_\partial^2(I\!R^{m-1})$ defined by

$$\Psi_i f(x_1,\cdots,x_{m-1}) = \int_{I\!R} f(x_1,\cdots,x_{i-1},x,x_i,\cdots,x_{m-1})\lambda(dx), \quad x_j \in I\!R, \tag{2.4}$$

where $x \in I\!R$ is the $i$th variable of $f$ on the right hand side.

Let $\{M_t : t \geq 0\}$ be a nonnegative integer-valued cádlág Markov process with transition intensities $\{q_{i,j}\}$ such that $q_{i,i-1} = -q_{i,i} = i(i+1)/2$ and $q_{i,j} = 0$ for all other pairs $(i,j)$. Let $\tau_0 = 0$ and $\tau_{M_0+1} = \infty$, and let $\{\tau_k : 1 \leq k \leq M_0\}$ be the sequence of jump times of $\{M_t : t \geq 0\}$. Let $\{\Gamma_k : 1 \leq k \leq M_0\}$ be a sequence of random operators which are conditionally independent given $\{M_t : t \geq 0\}$ and satisfy

$$\boldsymbol{P}\{\Gamma_k = \Phi_{i,j} | M(\tau_k^-) = l\} = \frac{1}{l(l+1)}, \qquad 1 \leq i \neq j \leq l, \tag{2.5}$$

and

$$\boldsymbol{P}\{\Gamma_k = \Psi_i | M(\tau_k^-) = l\} = \frac{2}{l(l+1)}, \qquad 1 \leq i \leq l. \tag{2.6}$$

Let $\boldsymbol{B}$ denote the topological union of $\{B(I\!R^m) : m = 1,2,\cdots\}$ endowed with pointwise convergence on each $B(I\!R^m)$. Then

$$Y_t = P_{t-\tau_k}^{M_{\tau_k}} \Gamma_k P_{\tau_k - \tau_{k-1}}^{M_{\tau_{k-1}}} \Gamma_{k-1} \cdots P_{\tau_2-\tau_1}^{M_{\tau_1}} \Gamma_1 P_{\tau_1}^{M_0} Y_0, \quad \tau_k \leq t < \tau_{k+1}, 0 \leq k \leq M_0, \tag{2.7}$$

defines a Markov process $\{Y_t : t \geq 0\}$ taking values from $\boldsymbol{B}$. Clearly, $\{(M_t, Y_t) : t \geq 0\}$ is also a Markov process. To simplify the presentation, we shall suppress the dependence of $\{Y_t : t \geq 0\}$ on $\sigma$ and let $\boldsymbol{E}_{m,f}^{\sigma}$ denote the expectation given $M_0 = m$ and $Y_0 = f \in C(I\!R^m)$, just as we are working with a canonical realization of $\{(M_t, Y_t) : t \geq 0\}$.

**Theorem 2.1** *Let $\mathcal{D}(\mathcal{J})$ be the set of all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ with $f \in C_\partial^2(I\!R^m)$. Suppose that $\{X_t : t \geq 0\}$ is a continuous $M(I\!R)$-valued process and that $\boldsymbol{E}\{\langle 1, X_t \rangle^m\}$ is locally bounded in $t \geq 0$ for each $m \geq 1$. If $\{X_t : t \geq 0\}$ is a solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem with $X_0 = \mu$, then*

$$\boldsymbol{E}\langle f, X_t^m \rangle = \boldsymbol{E}_{m,f}^{\sigma}\left[\langle Y_t, \mu^{M_t} \rangle \exp\left\{\frac{1}{2}\int_0^t M_s(M_s+1)ds\right\}\right] \tag{2.8}$$

*for any $t \geq 0$, $f \in B(I\!R^m)$ and integer $m \geq 1$. Consequently, the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem has at most one solution possessing locally bounded moments of all degrees.*

*Proof.* The general equality follows by bounded pointwise approximation once it is proved for $f \in C^2_\partial(I\!R^m)$. Set $F_\mu(m,f) = F_{m,f}(\mu) = \langle f, \mu^m\rangle$. From the construction (2.7), it is not hard to see that $\{(M_t, Y_t) : t \geq 0\}$ has generator $\mathcal{L}^*$ given by

$$\begin{aligned}\mathcal{L}^* F_\mu(m,f) &= F_\mu(m, G^m f) \\ &\quad + \frac{1}{2}\sum_{i,j=1,i\neq j}^{m} [F_\mu(m-1, \Phi_{ij} f) - F_\mu(m,f)] \\ &\quad + \sum_{i=1}^{m} [F_\mu(m-1, \Psi_i f) - F_\mu(m,f)].\end{aligned}$$

In view of (2.2) we have

$$\mathcal{L}F_{m,f}(\mu) = \mathcal{L}^* F_\mu(m,f) + \frac{1}{2}m(m+1)F_\mu(m,f). \tag{2.9}$$

Guided by (2.9) one can prove (2.8) using similar calculations as in [2]. To show the last assertion of the theorem, we may first consider the special case $\sigma(x) \equiv \sigma_0$ for a constant $\sigma_0$. In this case, (2.1) implies that $\{\langle 1, X_t\rangle : t \geq 0\}$ is a one-dimensional diffusion with generator $2^{-1}\sigma_0 x d^2/dx^2 + \langle 1, \lambda\rangle d/dx$. As in [5, pp.236-237] one sees that

$$\boldsymbol{E}\exp\{z\langle 1, X_t\rangle\} = [1 - \sigma_0 z t/2]^{-2\langle 1,\lambda\rangle/\sigma_0} \exp\left\{\frac{\langle 1,\mu\rangle z}{1 - \sigma_0 z t/2}\right\}, \qquad t \geq 0, |z| < 2/\sigma_0 t.$$

The remaining arguments are similar to those in the proof of Theorem 2.2 in [2]. □

# 3 SDSM with discrete immigration

Suppose that $(P_t)_{t\geq 0}$ is a Feller transition semigroup on some metric space $E$ which has a Hunt process realization $\xi$. Suppose that $K(x,dy)$ is a bounded kernel on $E$. We assume that $K(x,\cdot)$ depends on $x \in E$ continuously. Let $\beta(x) = K(x,E)$. Let $K_0(x,dy) = \beta(x)^{-1}K(x,dy)$ if $\beta(x) > 0$ and $K_0(x,dy) = \delta_x(dy)$ if $\beta(x) = 0$. By the concatenation argument described in Sharpe [7, p.82] it is not hard to construct a Markov process $\eta$ with the following properties:

(3A) The process evolves in $E$ according to the law given by the transition probabilities of $\xi$ until the random time $\tau_1$ with $\boldsymbol{P}\{\tau_1 > t\} = \exp\{-\int_0^t \beta(\eta_s)ds\}$.

(3B) At time $\tau_1$ the particle jumps from $\eta_{\tau_1^-}$ to a new place in $E$ according to the probability distribution $K(\eta_{\tau_1^-}, dy)$, and then moves randomly according to the transition probabilities of $\xi$ again until the random time $\tau_1 + \tau_2$ with $\boldsymbol{P}\{\tau_2 > t\} = \exp\{-\int_{\tau_1}^{\tau_1+t} \beta(\eta_s)ds\}$; and so on.

**Lemma 3.1** *Suppose that $\xi$ has generator $(A, \mathcal{D}(A))$, where $\mathcal{D}(A) \subset C(E)$. Then $\eta$ has generator $(B, \mathcal{D}(B))$, where $\mathcal{D}(B) = \mathcal{D}(A)$ and*

$$Bf(x) = Af(x) + \int_E [f(y) - f(x)]K(x,dy), \quad x \in E, f \in \mathcal{D}(B). \tag{3.1}$$

*Moreover, the transition semigroup of $\eta$ is Feller.*

*Proof.* Let $(Q_t)_{t\geq 0}$ denote the transition semigroup of $\eta$. The properties (3A) and (3B) imply that

$$\begin{aligned} Q_t f(x) &= \boldsymbol{P}_x\bigg[f(\xi_t)\exp\bigg\{-\int_0^t \beta(\xi_s)ds\bigg\}\bigg] \\ &\quad + \boldsymbol{P}_x\bigg[\int_0^t \exp\bigg\{-\int_0^s \beta(\xi_u)du\bigg\}\beta(\xi_s)ds\int_E Q_{t-s}f(y)K_0(\xi_s,dy)\bigg] \\ &= \boldsymbol{P}_x\bigg[f(\xi_t)\exp\bigg\{-\int_0^t \beta(\xi_s)ds\bigg\}\bigg] \\ &\quad + \boldsymbol{P}_x\bigg[\int_0^t \exp\bigg\{-\int_0^s \beta(\xi_u)du\bigg\}K(\xi_s,Q_{t-s}f)ds\bigg] \\ &= \boldsymbol{P}_x\bigg[f(\xi_t)\exp\bigg\{-\int_0^t \beta(\xi_s)ds\bigg\}\bigg] \\ &\quad + \int_0^t \boldsymbol{P}_x\bigg[\exp\bigg\{-\int_0^s \beta(\xi_u)du\bigg\}K(\xi_s,Q_{t-s}f)\bigg]ds. \end{aligned} \tag{3.2}$$

This equation follows as we think about the behavior of the particle. It either moves according to $\xi$ without jumping until time $t$, or it first jumps at some time $s \in (0,t]$. The first event happens with probability $\exp\{-\int_0^t \beta(\xi_s)ds\}$ and the second happens with probability $\exp\{-\int_0^t \beta(\xi_u)du\}\beta(\xi_s)ds$, giving the two terms of on the right hand side. For $f \in \mathcal{D}(A)$, we get from (3.2) that

$$\begin{aligned} Bf(x) &= \lim_{t\downarrow 0} t^{-1}\boldsymbol{P}_x\bigg[f(\xi_t)\exp\bigg\{-\int_0^t \beta(\xi_s)ds\bigg\} - f(x)\bigg] \\ &\quad + \lim_{t\downarrow 0} t^{-1}\int_0^t \boldsymbol{P}_x\bigg[\exp\bigg\{-\int_0^s \beta(\xi_u)du\bigg\}K(\xi_s,Q_{t-s}f)\bigg]ds \\ &= \lim_{t\downarrow 0} t^{-1}\boldsymbol{P}_x[f(\xi_t)-f(x)] + \lim_{t\downarrow 0} t^{-1}\boldsymbol{P}_x\bigg[f(\xi_t)\exp\bigg\{-\int_0^t \beta(\xi_s)ds\bigg\} - f(\xi_t)\bigg] \\ &\quad + \lim_{t\downarrow 0} t^{-1}\int_0^t \boldsymbol{P}_x\bigg[\exp\bigg\{-\int_0^s \beta(\xi_u)du\bigg\}K(\xi_s,Q_{t-s}f)\bigg]ds \\ &= Af(x) - \beta(x)f(x) + K(x,f) \\ &= Af(x) + \int_E [f(y)-f(x)]K(x,dy). \end{aligned}$$

Since $(A,\mathcal{D}(A))$ generates a Feller transition semigroup, so does $(B,\mathcal{D}(B))$; see e.g. [4, p.37]. □

For a fixed non-trivial measure $\lambda \in M(I\!\!R)$ we consider a random variable $\zeta$ in $I\!\!R$ with distribution $\lambda(1)^{-1}\lambda$. For $\mu \in M(I\!\!R)$, let $K(\mu,d\nu)$ denote the distribution of the random measure

$$X := \mu + \theta^{-1}\delta_\zeta.$$

Observe that

$$\int_{M(I\!\!R)} [F(\nu)-F(\mu)]K(\mu,d\nu) = \lambda(1)^{-1}\int_{I\!\!R} [F(\mu+\theta^{-1}\delta_y)-F(\mu)]\lambda(dy). \tag{3.3}$$

For $\theta > 0$ we can define the generator $\mathcal{J}_\theta$ by

$$\mathcal{J}_\theta F(\mu) = \mathcal{L}F(\mu) + \theta \int_{\mathbb{R}} [F(\mu + \theta^{-1}\delta_x) - F(\mu)]\lambda(dx). \tag{3.4}$$

By the result in [2], $\mathcal{L}$ generates a Feller semigroup on $M(\mathbb{R})$, then so does $\mathcal{J}_\theta$ by Lemma 3.1. We shall call the process generated by $\mathcal{J}_\theta$ a *SDSM with discrete immigration* with parameters $(a, \rho, \sigma, \lambda)$ and unit mass $1/\theta$. Intuitively, the immigrants come to $\mathbb{R}$ by cliques with mass $1/\theta$ with time-space configuration given by a Poisson random measure with intensity $\theta ds\lambda(dx)$. A more general immigration model for superprocesses with independent spatial motions has been considered in Li [6].

# 4 SDSM with continuous immigration

In this section, we construct a solution of the $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$-martingale problem by using an approximation by the SDSM with discrete immigration. Observe that, if

$$F_{f,\{\phi_i\}}(\mu) = f(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle), \qquad \mu \in M(\mathbb{R}), \tag{4.1}$$

for $f \in C_0^2(\mathbb{R}^n)$ and $\phi_i \in C_\partial^2(\mathbb{R})$, then

$$\begin{aligned} \mathcal{J}F_{f,\{\phi_i\}}(\mu) \;=\; & \frac{1}{2}\sum_{i=1}^n f_i'(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle)\langle a\phi_i'', \mu\rangle \\ & + \frac{1}{2}\sum_{i,j=1}^n f_{ij}''(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle) \int_{\mathbb{R}^2} \rho(x-y)\phi_i'(x)\phi_j'(y)\mu(dx)\mu(dy) \\ & + \frac{1}{2}\sum_{i,j=1}^n f_{ij}''(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle)\langle\sigma\phi_i\phi_j, \mu\rangle \\ & + \sum_{i=1}^n f_i'(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle)\langle\phi_i, \lambda\rangle. \end{aligned} \tag{4.2}$$

Let $\{\theta_k\}$ be any sequence such that $\theta_k \to \infty$ as $k \to \infty$. For $k \geq 1$, let $\{X_t^{(k)} : t \geq 0\}$ be a cádlág SDSM with discrete immigration with parameters $(a, \rho, \sigma, m)$, unit $1/\theta_k$ and initial state $X_0^{(k)} = \mu_k \in M_{\theta_k}(\mathbb{R})$.

**Lemma 4.1** *If the sequence $\{\langle 1, \mu_k\rangle\}$ is bounded, then $\{X_t^{(k)} : t \geq 0\}$ form a tight sequence in $D([0,\infty), M(\hat{\mathbb{R}}))$.*

*Proof.* Let $H(\nu) = \langle 1, \nu\rangle$. By (3.4), it is not hard to see that $\mathcal{J}_{\theta_k}H(\nu) = \langle 1, \lambda\rangle$. It follows that

$$\boldsymbol{E}_{\mu_k}\{\langle 1, X_t^{(k)}\rangle\} = \langle 1, \mu_k\rangle + \langle 1, \lambda\rangle t, \qquad t \geq 0.$$

Then $\{\langle 1, X_t^{(k)}\rangle - \langle 1, \lambda\rangle t : t \geq 0\}$ is a martingale. By a martingale inequality, for $u > 0$ and $\eta > \langle 1, \lambda\rangle u$ we have

$$\boldsymbol{P}\Big\{\sup_{0\leq t\leq u}\langle 1, X_t^{(k)}\rangle > 2\eta\Big\} \;\leq\; \boldsymbol{P}\Big\{\sup_{0\leq t\leq u}|\langle 1, X_t^{(k)}\rangle - \langle 1, \lambda\rangle t| > \eta\Big\}$$

$$\leq 3\eta^{-1} \sup_{0\leq t\leq u} \boldsymbol{E}_{\mu_k}\{|\langle 1, X_t^{(k)}\rangle - \langle 1,\lambda\rangle t|\}$$
$$\leq 3\eta^{-1}(\langle 1,\mu_k\rangle + 2\langle 1,\lambda\rangle u);$$

see e.g. [3, p.66]. That is, $\{X_t^{(k)} : t \geq 0\}$ satisfies the compact containment condition of [4, p.142]. Let $\mathcal{J}_k$ denote the generator of $\{X_t^{(k)} : t \geq 0\}$ and let $F_{f,\{\phi_i\}}$ be given by (4.1) with $f \in C_0^2(I\!\!R^n)$ and with each $\phi_i \in C_\partial^2(I\!\!R)$ bounded away from zero. Then

$$F_{f,\{\phi_i\}}(X_t^{(k)}) - F_{f,\{\phi_i\}}(X_0^{(k)}) - \int_0^t \mathcal{J}_k F_{f,\{\phi_i\}}(X_s^{(k)})ds, \qquad t \geq 0,$$

is a martingale and the desired tightness follows from the result of [4, p.145]. □

Now suppose that all functions in $C_\partial(I\!\!R)$ are extended to $\hat{I\!\!R}$ by continuity. If $\sigma \in C_\partial(I\!\!R)^+$, we may regard $F$ given by (4.1) and the right hand side of (4.2) as functions on $M(\hat{I\!\!R})$. Let $\hat{\mathcal{J}}F(\mu)$ be defined by the right hand side of (4.2) as a function on $M(\hat{I\!\!R})$. Let $\mathcal{D}(\hat{\mathcal{J}})$ be the totality of all functions of the form (4.1) with $f \in C_0^2(I\!\!R^n)$ and with each $\phi_i \in C_\partial^2(I\!\!R)$ bounded away from zero. Suppose that $\mu_k \to \mu \in M(\hat{I\!\!R})$ as $k \to \infty$ and let $\boldsymbol{Q}_\mu$ be a limit point of the distributions of $\{X_t^{(k)} : t \geq 0\}$. As in the proof of Lemma 4.2 in [2], we may see that $\boldsymbol{Q}_\mu$ is supported by $C([0,\infty), M(\hat{I\!\!R}))$ and

$$F_{f,\{\phi_i\}}(w_t) - F_{f,\{\phi_i\}}(w_0) - \int_0^t \hat{\mathcal{J}} F_{f,\{\phi_i\}}(w_s)ds, \qquad t \geq 0, \tag{4.3}$$

is a martingale for each $F_{f,\{\phi_i\}} \in \mathcal{D}(\hat{\mathcal{J}})$, where $\{w_t : t \geq 0\}$ denotes the coordinate process of $C([0,\infty), M(\hat{I\!\!R}))$.

**Lemma 4.2** *Let $\boldsymbol{Q}_\mu$ be given as the above. Then for $n \geq 1$, $t \geq 0$ and $\mu \in M(I\!\!R)$ we have*

$$\boldsymbol{Q}_\mu\{\langle 1, w_t\rangle^n\} \leq \langle 1,\mu\rangle^n + n[(n-1)\|\sigma\|/2 + \langle 1,\lambda\rangle] \int_0^t \boldsymbol{Q}_\mu\{\langle 1, w_s\rangle^{n-1}\}ds.$$

*Consequently, $\boldsymbol{Q}_\mu\{\langle 1, w_t\rangle^n\}$ is a locally bounded function of $t \geq 0$. Let $\mathcal{D}(\hat{\mathcal{J}})$ be the union of all functions of the form (4.1) with $f \in C_0^2(I\!\!R^n)$ and $\phi_i \in C_\partial^2(I\!\!R)$ and all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ with $f \in C_\partial^2(I\!\!R^m)$. Then (4.3) under $\boldsymbol{Q}_\mu$ is a martingale for each $F \in \mathcal{D}(\hat{\mathcal{J}})$.*

*Proof.* For any $k \geq 1$, take $f_k \in C_0^2(I\!\!R))$ such that $f_k(z) = z^n$ for $0 \leq z \leq k$ and $f_k''(z) \leq n(n-1)z^{n-2}$ for all $z \geq 0$. Let $F_k(\mu) = f_k(\langle 1,\mu\rangle)$. It s easy to see that

$$\hat{\mathcal{J}} F_k(\mu) \leq n[(n-1)\|\sigma\|/2 + \langle 1,\lambda\rangle]\langle 1,\mu\rangle^{n-1}.$$

Since

$$F_k(X_t) - F_k(X_0) - \int_0^t \hat{\mathcal{J}} F_k(\langle 1, X_s\rangle)ds, \qquad t \geq 0,$$

is a martingale, we get

$$\boldsymbol{Q}_\mu f_k(\langle 1, X_t\rangle^n) \leq f_k(\langle 1,\mu\rangle) + n[(n-1)\|\sigma\|/2 + \langle 1,\lambda\rangle] \int_0^t \boldsymbol{Q}_\mu(\langle 1, X_s\rangle^{n-1})ds$$
$$\leq \langle 1,\mu\rangle^n + n[(n-1)\|\sigma\|/2 + \langle 1,\lambda\rangle] \int_0^t \boldsymbol{Q}_\mu(\langle 1, X_s\rangle^{n-1})ds.$$

Then the desired estimate follows by Fatou's Lemma. The last assertion is immediate. $\square$

By the martingale problem (4.3) and the last lemma, it is easy to find that for each $\phi \in C^2_\partial(I\!\!R)$,

$$M_t(\phi) := \langle \phi, w_t\rangle - \langle \phi, \mu\rangle - \langle \phi, m\rangle t - \frac{1}{2}\int_0^t \langle a\phi'', w_s\rangle ds, \qquad t \geq 0, \tag{4.4}$$

is a $\boldsymbol{Q}_\mu$-martingale with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t \langle \sigma\phi^2, w_s\rangle ds + \int_0^t ds \int_{\hat{I\!\!R}} \langle h(z-\cdot)\phi', w_s\rangle^2 dz. \tag{4.5}$$

For a continuous branching density function $\sigma \in C_\partial(I\!\!R)^+$, the existence of a SDSM with immigration is given by the following

**Theorem 4.1** *Let $\mathcal{D}(\mathcal{J})$ be the union of all functions of the form (4.1) with $f \in C^2_0(I\!\!R^n)$ and $\phi_i \in C^2(I\!\!R)$ and all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ with $f \in C^2(I\!\!R^m)$. Let $\{w_t : t \geq 0\}$ denote the coordinate process of $C([0,\infty), M(I\!\!R))$. Then for each $\mu \in M(I\!\!R)$ there is a unique probability measure $\boldsymbol{Q}_\mu$ on $C([0,\infty), M(I\!\!R))$ such that $\boldsymbol{Q}_\mu\{w_0 = \mu\} = 1$, the moments $\boldsymbol{Q}_\mu\{\langle 1, w_t\rangle^m\}$ are locally bounded and $\{w_t : t \geq 0\}$ under $\boldsymbol{Q}_\mu$ is a solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem.*

*Proof.* Let $\boldsymbol{Q}_\mu$ be as in Lemma 4.2. By Theorem 2.1, the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem has at most one solution possessing locally bounded moments of all degrees. Then the desired result follows once it is proved that

$$\boldsymbol{Q}_\mu\{w_t(\{\partial\}) = 0 \text{ for all } t \in [0,u]\} = 1, \qquad u > 0. \tag{4.6}$$

Let $M(ds,dx)$ denote the stochastic integral relative to the martingale measure defined by (4.4) and (4.5). As in [2], we have

$$\langle \phi, w_t\rangle = \langle \hat{P}_t\phi, \mu\rangle + \int_0^t \langle \hat{P}_{t-s}\phi, \lambda\rangle ds + \int_0^t \int_{\hat{I\!\!R}} \hat{P}_{t-s}\phi(x) M(ds,dx)$$

for $t \geq 0$ and $\phi \in C^2_\partial(I\!\!R)$. For any fixed $u > 0$, we have that

$$\begin{aligned} M^u_t(\phi) &:= \langle \hat{P}_{u-t}\phi, w_t\rangle - \langle \hat{P}_u\phi, \mu\rangle - \int_0^t \lambda(\hat{P}_{u-s}\phi) ds \\ &= \int_0^t \int_{\hat{I\!\!R}} \hat{P}_{u-s}\phi M(ds,dx), \qquad t \in [0,u], \end{aligned}$$

is a continuous martingale with quadratic variation process

$$\begin{aligned} \langle M^u(\phi)\rangle_t &= \int_0^t \langle \sigma(\hat{P}_{u-s}\phi)^2, w_s\rangle ds + \int_0^t ds \int_{\hat{I\!\!R}} \langle h(z-\cdot)\hat{P}_{u-s}(\phi'), w_s\rangle^2 dz \\ &= \int_0^t \langle \sigma(\hat{P}_{u-s}\phi)^2, w_s\rangle ds + \int_0^t ds \int_{\hat{I\!\!R}} \langle h(z-\cdot)(\hat{P}_{u-s}\phi)', w_s\rangle^2 dz. \end{aligned}$$

By a martingale inequality we have

$$\begin{aligned}
&\boldsymbol{Q}_\mu\bigg\{\sup_{0\le t\le u}\bigg|\langle\hat{P}_{u-t}\phi,w_t\rangle-\langle\hat{P}_u\phi,\mu\rangle-\int_0^t\lambda(\hat{P}_{u-s}\phi)ds\bigg|^2\bigg\}\\
\le\;& 4\int_0^u\boldsymbol{Q}_\mu\{\langle\sigma(\hat{P}_{u-s}\phi)^2,w_s\rangle\}ds+4\int_0^u ds\int_{\hat{I\!R}}\boldsymbol{Q}_\mu\{\langle h(z-\cdot)\hat{P}_{u-s}(\phi'),w_s\rangle^2\}dz\\
\le\;& 4\int_0^u\langle\sigma(\hat{P}_{u-s}\phi)^2,\mu\hat{P}_s\rangle ds+4\int_{\hat{I\!R}}h(z)^2dz\int_0^u\boldsymbol{Q}_\mu\{\langle 1,w_s\rangle\langle\hat{P}_{u-s}(\phi')^2,w_s\rangle\}ds\\
\le\;& 4\int_0^u\langle\sigma(\hat{P}_{u-s}\phi)^2,\mu\hat{P}_s\rangle ds+4\|\phi'\|^2\int_{\hat{I\!R}}h(z)^2dz\int_0^u\boldsymbol{Q}_\mu\{\langle 1,w_s\rangle^2\}ds.
\end{aligned}$$

Choose a sequence $\{\phi_k\}\subset C^2_\partial(I\!R)$ such that $\phi_k(\cdot)\to 1_{\{\partial\}}(\cdot)$ boundedly and $\|\phi_k'\|\to 0$ as $k\to\infty$. Replacing $\phi$ by $\phi_k$ in the above and letting $k\to\infty$ we obtain (4.6). □

For a general $\sigma\in B(I\!R)^+$, we may choose a bounded sequence of functions $\{\sigma_k\}\subset C_\partial(I\!R)^+$ such that $\sigma_k\to\sigma$ pointwise out of a Lebesgue null set. Suppose that $\{\mu_k\}\subset M(I\!R)$ and $\mu_k\to\mu\in M(I\!R)$ as $k\to\infty$. For each $k\ge 1$, let $\{X_t^{(k)}:t\ge 0\}$ be an immigration SDSM with parameters $(a,\rho,\sigma_k,m)$ and initial state $\mu_k\in M(I\!R)$ and let $\boldsymbol{Q}_k$ denote the distribution of $\{X_t^{(k)}:t\ge 0\}$ on $C([0,\infty),M(I\!R))$. By the arguments in the proofs of Theorems 5.1 and 5.2 in [2] we get

**Theorem 4.2** *As $k\to\infty$, the sequence $\boldsymbol{Q}_k$ converges to a probability $\boldsymbol{Q}_\mu$ on $C([0,\infty),M(I\!R))$. Let $\mathcal{D}(\mathcal{J})$ be as in Theorem 4.1 for the more general $\sigma\in B(I\!R)^+$. Then $\boldsymbol{Q}_\mu$ is the unique probability measure on $C([0,\infty),M(I\!R))$ such that $\boldsymbol{Q}_\mu\{w_0=\mu\}=1$ and $\{w_t:t\ge 0\}$ under $\boldsymbol{Q}_\mu$ solves the $(\mathcal{J},\mathcal{D}(\mathcal{J}))$-martingale problem. Consequently, $\{w_t:t\ge 0\}$ under $\boldsymbol{Q}_\mu$ is a diffusion process with transition semigroup $(Q_t)_{t\ge 0}$ defined by*

$$\int_{M(I\!R)}\langle f,\nu^m\rangle Q_t(\mu,d\nu)=\boldsymbol{E}^\sigma_{m,f}\bigg[\langle Y_t,\mu^{M_t}\rangle\exp\bigg\{\frac{1}{2}\int_0^t M_s(M_s+1)ds\bigg\}\bigg].\tag{4.7}$$

This gives the existence of the SDSM with continuous immigration for a bounded measurable branching density $\sigma\in B(I\!R)^+$. Clearly, we have that for each $\phi\in C^2(I\!R)$,

$$M_t(\phi):=\langle\phi,w_t\rangle-\langle\phi,\mu\rangle-\langle\phi,m\rangle t-\frac{1}{2}\int_0^t\langle a\phi'',w_s\rangle ds,\qquad t\ge 0,\tag{4.8}$$

is a $\boldsymbol{Q}_\mu$-martingale with quadratic variation process

$$\langle M(\phi)\rangle_t=\int_0^t\langle\sigma\phi^2,w_s\rangle ds+\int_0^t ds\int_{I\!R}\langle h(z-\cdot)\phi',w_s\rangle^2dz.\tag{4.9}$$

Conversely, if $\boldsymbol{Q}_\mu$ is the unique probability measure on $C([0,\infty),M(I\!R))$ such that (4.8) is a martingale with quadratic variation process (4.9), by Itô's formula one can show that $\boldsymbol{Q}_\mu$ is a solution of the $(\mathcal{J},\mathcal{D}(\mathcal{J}))$-martingale problem. Then (4.8) and (4.9) give an alternate definition of the immigration SDSM.

## 5 Non-critical branching mechanism

Let $\boldsymbol{Q}_\mu$ denote the distribution on $C([0,\infty), M(I\!R))$ of an $(a,\rho,\sigma,\lambda)$-superprocess with initial state $\mu \in M(I\!R)$. Let $M(ds,dx)$ denote the martingale measure defined by (4.8) and (4.9). Then for any $b \in C^1(I\!R)$ the stochastic integral

$$M_t(b) := \int_0^t b(x)M(ds,dx), \qquad t \geq 0, \tag{5.1}$$

is well-defined and

$$\langle M(b)\rangle_t = \int_0^t \langle \sigma b^2, w_s\rangle ds + \int_0^t ds \int_{I\!R} \langle h(z-\cdot)b', w_s\rangle^2 dz. \tag{5.2}$$

We consider the exponential martingale

$$Z_t(b) := \exp\bigg\{ - M_t(b) - \frac{1}{2}\langle M(b)\rangle_t \bigg\}, \qquad t \geq 0. \tag{5.3}$$

Fix a constant $T>0$ and let $\boldsymbol{Q}_\mu^b(dw) = Z_T(w,b)\boldsymbol{Q}_\mu(dw)$. By Girsanov's theorem,

$$\begin{aligned} N_t(\phi) \;\; := \;\; & \langle \phi, w_t\rangle - \langle \phi, \mu\rangle - \langle \phi, m\rangle t - \frac{1}{2}\int_0^t \langle a\phi'', w_s\rangle ds - \int_0^t \langle \sigma b\phi, w_s\rangle ds \\ & + \int_0^t ds \int_{I\!R} \langle h(z-\cdot)b', w_s\rangle \langle h(z-\cdot)\phi', w_s\rangle dz, \qquad 0 \leq t \leq T, \end{aligned} \tag{5.4}$$

is a $\boldsymbol{Q}_\mu^b$-martingale with quadratic variation process

$$\langle N(\phi)\rangle_t = \int_0^t \langle \sigma\phi^2, w_s\rangle ds + \int_0^t ds \int_{I\!R} \langle h(z-\cdot)\phi', w_s\rangle^2 dz, \quad 0 \leq t \leq T. \tag{5.5}$$

As usual, the coordinate process $\{w_t : 0 \leq t \leq T\}$ under $\boldsymbol{Q}_\mu^b$ is a diffusion process; see e.g. [5, pp.190-197]. We call the new process a $(a,\rho,\sigma,b,\lambda)$-*superprocess.* Intuitively, the term $\int_0^t \langle \sigma b\phi, w_s\rangle ds$ in (5.5) represents a linear growth with growth rate $\sigma(\cdot)b(\cdot)$. Girsanov transformations of this type were introduced by Dawson [1] to get non-critical superprocesses for critical ones. Note that we have on the right hand side of (5.5) an extra term

$$\int_0^t ds \int_{I\!R} \langle h(z-\cdot)b', w_s\rangle \langle h(z-\cdot)\phi', w_s\rangle dz, \tag{5.6}$$

which may be interpreted as a spatial drift with state-dependent coefficient

$$\int_{I\!R}\int_{I\!R} h(z-y)b'(y)h(z-\cdot)w_s(dy)dz = \int_{I\!R} b'(y)\rho(y-\cdot)w_s(dy). \tag{5.7}$$

This is different from the classical case where the Girsanov transform does not effect the spatial motion; see [1]. Let $\mathcal{D}(\mathcal{J}^b)$ be the union of all functions of the form (4.1) with $f \in C_0^2(I\!R^n)$ and $\phi_i \in C^2(I\!R)$ and all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ with $f \in C^2(I\!R^m)$.

**Theorem 5.1** *The $(a,\rho,\sigma,b,\lambda)$-superprocess solves the $(\mathcal{J}^b, \mathcal{D}(\mathcal{J}^b))$-martingale problem.*

*Proof.* If $F_{f,\{\phi_i\}}$ is given by (4.1), we have

$$\begin{aligned}\mathcal{J}^b F_{f,\{\phi_i\}}(\mu) &= \frac{1}{2}\sum_{i=1}^n f_i'(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle a\phi_i''-2b\phi,\mu\rangle \\ &+\frac{1}{2}\sum_{i,j=1}^n f_{ij}''(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\int_{\mathbb{R}^2}\rho(x-y)\phi_i'(x)\phi_j'(y)\mu(dx)\mu(dy) \\ &+\frac{1}{2}\sum_{i,j=1}^n f_{ij}''(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle\sigma\phi_i\phi_j,\mu\rangle \\ &-\sum_{i=1}^n f_i'(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\int_{\mathbb{R}^2}\rho(x-y)b'(y)\phi_i'(x)\mu(dx)\mu(dy) \\ &+\sum_{i=1}^n f_i'(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle\phi_i,\lambda\rangle. \end{aligned} \tag{5.8}$$

Based on (5.4) and (5.5), it is easy to check by Itô's formula that

$$F_{f,\{\phi_i\}}(w_t) - F_{f,\{\phi_i\}}(w_0) - \int_0^t \mathcal{J}^b F_{f,\{\phi_i\}}(w_s)ds, \qquad 0 \geq t \leq T, \tag{5.9}$$

is a martingale under $\boldsymbol{Q}_\mu^b$. Then the theorem follows by an approximation of an arbitrary $F \in \mathcal{D}(\mathcal{L})$. □

If $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ for $f \in C_\partial^2(\mathbb{R}^m)$, then

$$\begin{aligned}\mathcal{J}F_{m,f}(\mu) &= F_{m,G_b^m f}(\mu) + \frac{1}{2}\sum_{i,j=1,i\neq j}^m F_{m-1,\Phi_{ij}f}(\mu) \\ &\quad + \sum_{i=1}^m F_{m-1,\Psi_i f}(\mu) + \sum_{i=1}^m F_{m+1,\Gamma_i f}(\mu),\end{aligned} \tag{5.10}$$

where

$$G_b^m f(x_1,\cdots,x_m) = G^m f(x_1,\cdots,x_m) - \sum_{i=1}^m b(x_i)f(x_1,\cdots,x_m), \tag{5.11}$$

and

$$\Gamma_i f(x_1,\cdots,x_m,x_{m+1}) = -\rho(x_{m+1}-x_i)b'(x_{m+1})f_i'(x_1,\cdots,x_m). \tag{5.12}$$

In view of this expression of the generator, we may construct a dual process which gives expressions for the moments of the $(a,\rho,\sigma,b,\lambda)$-superprocess.

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