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Conditional Log-Laplace Functionals of Immigration Superprocesses with Dependent Spatial Motion

Zenghu Li [1]
School of Mathematical Sciences, Beijing Normal University,
Beijing 100875, P.R. China
E-mail: lizh@email.bnu.edu.cn

Hao Wang [2]
Department of Mathematics, University of Oregon,
Eugene OR 97403-1222, U.S.A.
E-mail: haowang@darkwing.uoregon.edu

Jie Xiong [3]
Department of Mathematics, University of Tennessee,
Knoxville, TN 37996-1300, U.S.A.
and
Department of Mathematics, Hebei Normal University,
Shijiazhuang 050016, P.R. China
E-mail: jxiong@math.utk.edu

Abstract

A non-critical branching immigration superprocess with dependent spatial motion is constructed and characterized as the solution of a stochastic equation driven by a time-space white noise and an orthogonal martingale measure. A representation of its conditional log-Laplace functionals is established, which gives the uniqueness of the solution and hence its Markov property. Some properties of the superprocess including an ergodic theorem are also obtained.

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1 Introduction

A class of superprocesses with dependent spatial motion (SDSM) over the real line $\mathbb{R}$ were introduced and constructed in Wang [27, 28]. A generalization of the model was then given in Dawson *et al* [8]. Let $c \in C_b^2(\mathbb{R})$ and $h \in C_b^2(\mathbb{R})$ and assume both h and h' are square-integrable. Let

$$\rho(x) = \int_{\mathbb{R}} h(y-x)h(y)dy, \quad x \in \mathbb{R},$$

and $a(x) = c(x)^2 + \rho(0)$. Let $\sigma \in C_b^2(\mathbb{R})^+$ be a strictly positive function. We denote by $M(\mathbb{R})$ the space of finite Borel measures on $\mathbb{R}$ endowed with a metric compatible with its topology of weak convergence. For $f \in C_b(\mathbb{R})$ and $\mu \in M(\mathbb{R})$ set $\langle f, \mu \rangle = \int f d\mu$. Then an SDSM $\{X_t : t \geq 0\}$ is characterized by the following martingale problem: For each $\phi \in C_b^2(\mathbb{R})$,

$$M_t(\phi) = \langle \phi, X_t \rangle - \langle \phi, X_0 \rangle - \frac{1}{2}\int_0^t \langle a\phi'', X_s \rangle ds, \quad t \geq 0, \tag{1.1}$$

is a continuous martingale with quadratic variation process

$$\langle M(\phi) \rangle_t = \int_0^t \langle \sigma\phi^2, X_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z-\cdot)\phi', X_s \rangle^2 dz. \tag{1.2}$$

Clearly, the SDSM reduces to a usual critical branching Dawson-Watanabe superprocess if $h(\cdot) \equiv 0$; see e.g. Dawson [6]. A general SDSM arises as the weak limit of critical branching particle systems with dependent spatial motion. Consider a family of independent Brownian motions $\{B_i(t) : t \geq 0, i = 1, 2, \cdots\}$, the individual noises, and a time-space white noise $\{W_t(B) : t \geq 0, B \in \mathcal{B}(\mathbb{R})\}$, the common noise. The migration of a particle in the approximating system with label i is defined by the stochastic equation

$$dx_i(t) = c(x_i(t))dB_i(t) + \int_{\mathbb{R}} h(y - x_i(t))W(dt, dy), \tag{1.3}$$

where $W(ds, dy)$ denotes the time-space stochastic integral relative to $\{W_t(B)\}$. The SDSM possesses properties very different from those of the usual Dawson-Watanabe superprocess. For example, a Dawson-Watanabe superprocess in $M(\mathbb{R})$ is usually absolutely continuous whereas the SDSM with $c(\cdot) \equiv 0$ is purely atomic; see [14] and [27, 29], respectively.

In this paper, we consider a further extension of the model of Wang [27, 28]. Let $b \in C_b^2(\mathbb{R})$ and let $m \in M(\mathbb{R})$. A modification of the above martingale problem is to replace (1.1) by

$$M_t(\phi) = \langle \phi, X_t \rangle - \langle \phi, X_0 \rangle - t\langle \phi, m \rangle - \frac{1}{2}\int_0^t \langle a\phi'', X_s \rangle ds + \int_0^t \langle b\phi, X_s \rangle ds. \tag{1.4}$$

We shall prove that there is indeed a solution $\{X_t : t \geq 0\}$ to the martingale problem given by (1.2) and (1.4). The process $\{X_t : t \geq 0\}$ may be regarded as a non-critical branching *SDSM with immigration* (SDSMI), where $b(\cdot)$ is the linear growth rate and $m(dx)$ gives the immigration rate. This modification is related to the recent work of Dawson and Li [7], where an interactive immigration given by

$$\int_0^t \langle q(\cdot, X_s)\phi, m \rangle ds \tag{1.5}$$

was considered, where $q(\cdot,\cdot)$ is a function on $\mathbb{R}\times M(\mathbb{R})$ representing a state dependent immigration density. However, it was assumed in [7] that $b(\cdot)\equiv c(\cdot)\equiv 0$ and the approach there relies essentially on the purely atomic property of the process, which is not available for the present model.

The main purpose of the paper is to give a representation of the conditional log-Laplace functionals of solution of (1.2) and (1.4) and to illustrate some applications of the representation. This approach was stimulated by Xiong [30], who established a similar characterization for the model of Skoulakis and Adler [25]. The key idea of the representation is to decompose the martingale (1.4) into two orthogonal components, which arise respectively from the migration and the branching. Since the decomposition uses additional information which is not provided by (1.2) and (1.4), we shall start with the corresponding particle system and consider the high density limit following [9]. In this way, we can easily separate the two kinds of noises. It turns out that the common migration noise $\{W(ds,dy)\}$ remains after the limit procedure and the limit process satisfies the following martingale problem: For each $\phi\in C_b^2(\mathbb{R})$,

$$\begin{aligned} Z_t(\phi) &= \langle\phi,X_t\rangle-\langle\phi,X_0\rangle-t\langle\phi,m\rangle-\frac{1}{2}\int_0^t\langle a\phi'',X_s\rangle ds\\ &\quad+\int_0^t\langle b\phi,X_s\rangle ds-\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\phi',X_s\rangle W(ds,dy) \end{aligned}\tag{1.6}$$

is a continuous martingale orthogonal to $\{W_t(\phi)\}$ with quadratic variation process

$$\langle Z(\phi)\rangle_t=\int_0^t\langle\sigma\phi^2,X_s\rangle ds.\tag{1.7}$$

This formulation suggests that we may regard $\{X_t:t\geq 0\}$ as a generalized inhomogeneous Dawson-Watanabe superprocess with immigration, where

$$\int_{\mathbb{R}}h(y-\cdot)W(dt,dy)$$

gives a generalized drift in the underlying migration. Based on the techniques developed in Kurtz and Xiong [15, 30], we prove that for each $\phi\in H_1(\mathbb{R})\cap C_b(\mathbb{R})$ there is a pathwise unique solution of the non-linear SPDE

$$\begin{aligned} \psi_{r,t}(x) &= \phi(x)+\int_r^t\left[\frac{1}{2}a(x)\psi''_{s,t}(x)-\frac{1}{2}\sigma(x)\psi_{s,t}(x)^2\right]ds\\ &\quad-\int_r^t b(x)\psi_{s,t}(x)ds+\int_r^t\int_{\mathbb{R}}h(y-x)\psi'_{s,t}(x)\cdot W(ds,dy), \end{aligned}\tag{1.8}$$

where the last term on the right hand side denotes the backward stochastic integral with respect to the white noise. Then we show that the conditional log-Laplace functionals of $\{X_t:t\geq 0\}$ given $\{W(ds,dy)\}$ can be represented by the solution of (1.8). The representation of the conditional log-Laplace functionals is proved by direct analysis based on (1.6), (1.7) and (1.8). This approach is different from that of Xiong [30], where a Wong-Zakai type approximation was used. The idea of conditional log-Laplace approach has also been used by Crisan [5] for a different model. In fact, the approach in Section 5 is adapted from [5] which simplifies our original arguments. It is well-known that non-conditional log-Laplace functionals play very important roles in the study of classical Dawson-Watanabe superprocesses.

We shall see that conditional Laplace functionals are almost as efficient as the non-conditional Laplace functionals in studying some properties of the SDSMI. In particular, the characterization of the conditional Laplace functionals gives immediately the uniqueness of solution of (1.6) and (1.7), which in turn implies the Markov property of $\{X_t : t \geq 0\}$. It follows that $\{X_t : t \geq 0\}$ is a diffusion process with generator $\mathcal{L}$ given by

$$\begin{aligned}\mathcal{L}F(\mu) &= \frac{1}{2}\int_{\mathbb{R}^2} \rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy) \\ &\quad +\frac{1}{2}\int_{\mathbb{R}} a(x)\frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) + \frac{1}{2}\int_{\mathbb{R}} \sigma(x)\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx) \\ &\quad -\int_{\mathbb{R}} b(x)\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) + \int_{\mathbb{R}} \frac{\delta F(\mu)}{\delta\mu(x)}m(dx), \end{aligned} \tag{1.9}$$

where

$$\frac{\delta F(\mu)}{\delta\mu(x)} = \lim_{r\to 0^+}\frac{1}{r}[F(\mu + r\delta_x) - F(\mu)] \tag{1.10}$$

and $\delta^2 F(\mu)/\delta\mu(x)\delta\mu(y)$ is defined in the same way with F replaced by $(\delta F/\delta\mu(y))$ on the right hand side; see Section 3. We also prove some properties of the SDSMI including an ergodic theorem. There are also some other applications of the conditional log-Laplace functional. For instance, based on this characterization the conditional excursion theory of the SDSM have been developed in [19]. However, consideration of the interactive immigration (1.5) for this present process seems sophisticated.

The remainder of the paper is organized as follows. In Section 2 we give a formulation of the system of branching particles with dependent spatial motions and immigration. Some useful estimates of the moments of the system are also given. In Section 3 we obtain a solution of the martingale problem (1.6) and (1.7) as the high density limit of a sequence of particle systems. The existence and uniqueness of the solution of (1.8) is established in Section 4. In Section 5 we give the representation of the conditional log-Laplace functionals of the solution of (1.6) and (1.7). Some properties of the SDSMI are discussed in Section 6.

2 Branching particle systems

The main purpose of this section is to give an explicit construction for the immigration branching particle system with dependent spatial motion by modifying the constructions of [9, 26]. This construction provides a useful set up of the process.

We start with a simple interacting particle system. Let $\theta > 0$ be a constant and (c, h) be given as in the introduction. Let $N(\mathbb{R}) \subset M(\mathbb{R})$ be the set of integer-valued measures on $\mathbb{R}$ and let $M_\theta(\mathbb{R}) := \{\theta^{-1}\sigma : \sigma \in N(\mathbb{R})\}$. Given $\{a_i : i = 1, \cdots, n\}$, let $\{x_i(t) : t \geq 0, i = 1, \cdots, n\}$ be given by

$$x_i(t) = a_i + \int_0^t c(x_i(s))dB_i(s) + \int_0^t\int_{\mathbb{R}} h(y - x_i(s))W(dy, ds). \tag{2.1}$$

We may define a measure-valued process $\{X_t : t \geq 0\}$ by

$$\langle\phi, X_t\rangle = \sum_{i=1}^n \theta^{-1}\phi(x_i(t)), \qquad t \geq 0. \tag{2.2}$$

By the discussions in [8, 27, 28], the process $\{X_t : t \geq 0\}$ is a diffusion in $M_\theta(\mathbb{R})$. Let $\mathcal{A}_\theta$ denote the generator of this diffusion process. If $F_{f,\{\phi_i\}}(\mu) := f(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle)$ for $f \in C_0^2(\mathbb{R}^n)$

and $\{\phi_i\} \subset C_b^2(\mathbb{R})$, by Itô's formula it is easy to see that

$$\begin{aligned}\mathcal{A}_\theta F_{f,\{\phi_i\}}(\mu) &= \frac{1}{2}\sum_{i,j=1}^n f''_{ij}(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\int_{\mathbb{R}^2}\rho(x-y)\phi'_i(x)\phi'_j(y)\mu(dx)\mu(dy)\\ &\quad+\frac{1}{2}\sum_{i=1}^n f'_i(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle a\phi''_i,\mu\rangle\\ &\quad+\frac{1}{2\theta}\sum_{i,j=1}^n f''_{ij}(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle c^2\phi'_i\phi'_j,\mu\rangle. \end{aligned}\tag{2.3}$$

More generally, if F is a function on $M_\theta(\mathbb{R})$ that can be extended to a sufficiently smooth function on $M(\mathbb{R})$, then

$$\begin{aligned}\mathcal{A}_\theta F(\mu) &= \frac{1}{2}\int_{\mathbb{R}^2}\rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy)\\ &\quad+\frac{1}{2}\int_{\mathbb{R}}a(x)\frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx)\\ &\quad+\frac{1}{2\theta}\int_{\mathbb{R}^2}c(x)c(y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\delta_x(dy)\mu(dx), \end{aligned}\tag{2.4}$$

where $\delta F(\mu)/\delta\mu(x)$ and $\delta^2 F(\mu)/\delta\mu(x)\delta\mu(y)$ are defined as in the introduction. This can be seen by approximating the function F by functions of the form $F_{f,\{\phi_i\}}$.

A more interesting particle system involves branching and immigration. Let $\gamma > 0$ be a constant and let $m \in M(\mathbb{R})$. Let $p(x,\cdot) = \{p_0(x), p_1(x), p_2(x), \cdots\}$ be a family of discrete probability distributions which measurably depends on the index $x \in \mathbb{R}$ and satisfies $p_1(\cdot) \equiv 0$. In addition, we assume that

$$q(x) := \sum_{i=1}^\infty i p_i(x), \quad x \in \mathbb{R}, \tag{2.5}$$

is a bounded function. We shall construct an immigration branching particle system with parameters $(a, \rho, \gamma, p, \theta m, 1/\theta)$.

Let $\mathcal{A}$ be the set of all strings of the form $\alpha = n_0 n_1 \cdots n_{l(\alpha)}$, where $l(\alpha)$ is the length of α and the n_j are non-negative integers with $0 \le n_0 \le 1$ and $n_j \ge 1$ for $j \ge 1$. We shall label the particles by the strings in $\mathcal{A}$. We here use the first digit n_0 in the string to distinguish the aboriginal and the immigratory particles. More precisely, strings started with 0 refer to descendants of aboriginal ancestors and strings started with 1 refer to descendants of immigratory ancestors. (Note that the first digit is not counted in the length $l(\alpha)$.) We provide $\mathcal{A}$ with the arboreal ordering, that is, $m_0 \cdots m_p \prec n_0 \cdots n_q$ if and only if $p \le q$ and $m_0 = n_0, \cdots, m_p = n_p$. Then α has exactly $l(\alpha)$ predecessors, which we denote respectively by $\alpha - 1$, $\alpha - 2$, $\cdots$, $\alpha - l(\alpha)$. For example, if $\alpha = 12431$, then $\alpha - 2 = 124$ and $\alpha - 4 = 1$.

We need a collection of random variables to construct the immigration branching particle system. Let $\{a_{01}, \cdots, a_{0n}\}$ be a finite sequence of real-valued random variables. Let $\{W(ds, dx) : s \ge 0, x \in \mathbb{R}\}$ be a time-space white noise and $\{N(ds, dx) : s \ge 0, x \in \mathbb{R}\}$ a Poisson random measure with intensity $\theta ds m(dx)$. We shall assume $\langle 1, m\rangle > 0$, otherwise the construction of the immigration part is trivial. In this case, we can enumerate the atoms of $N(ds, dx)$ as

$$\{(s_i, a_{1i}) : 0 < s_1 < s_2 < \cdots, a_{1i} \in \mathbb{R}\}. \tag{2.6}$$

We also define the families

$$\{B_\alpha(t) : t \geq 0, \alpha \in \mathcal{A}\}, \quad \{S_\alpha : \alpha \in \mathcal{A}\}, \quad \{\eta_{a,\alpha} : a \in \mathbb{R}, \alpha \in \mathcal{A}\}, \tag{2.7}$$

where $\{B_\alpha\}$ are independent standard Brownian motions, $\{S_\alpha\}$ are i.i.d. exponential random variables with parameter γ, and $\{\eta_{a,\alpha}\}$ are independent random variables with distribution $p(a, \cdot)$. We assume that the families $\{W(ds, dx)\}$, $\{N(ds, dx)\}$, $\{a_{0i}\}$, $\{B_\alpha\}$, $\{S_\alpha\}$ and $\{\eta_{a,\alpha}\}$ are independent.

We define $\beta_{0n_1} = 0$ if $1 \leq n_1 \leq n$ and $\beta_{0n_1} = \infty$ if $n_1 > n$, and define $\beta_{1n_1} = s_{n_1}$ for all $n_1 \geq 1$. For $\alpha \in \mathcal{A}$ with $l(\alpha) = 1$ we let $\zeta_\alpha = \beta_\alpha + S_\alpha$. Heuristically, S_α is the life-span of the particle with label α, β_α is its birth time and ζ_α is its death time. The random variables a_α defined above can be interpreted as the birth place of the particle with label α. The trajectory $\{x_\alpha(t) : t \geq \beta_\alpha\}$ of the particle is the solution of the equation

$$x(\beta_\alpha + t) = a_\alpha + \int_{\beta_\alpha}^{\beta_\alpha + t} c(x(s)) dB_\alpha(s) + \int_{\beta_\alpha}^{\beta_\alpha + t} \int_{\mathbb{R}} h(y - x(s)) W(ds, dy). \tag{2.8}$$

For $\alpha \in \mathcal{A}$ with $l(\alpha) > 1$ the trajectory $\{x_\alpha(t) : t \geq \beta_\alpha\}$ is defined by the above equation with $a_\alpha = x_{\alpha-1}(\zeta_{\alpha-1}^-)$, $\zeta_\alpha = \beta_\alpha + S_\alpha$ and

$$\beta_\alpha = \begin{cases} \zeta_{\alpha-1} & \text{if } n_{l(\alpha)} \leq \eta_{x_{\alpha-1}(\zeta_{\alpha-1}-), \alpha-1} \\ \infty & \text{if } n_{l(\alpha)} > \eta_{x_{\alpha-1}(\zeta_{\alpha-1}-), \alpha-1}, \end{cases} \tag{2.9}$$

where $x_{\alpha-1}(\zeta_{\alpha-1}-)$ denotes the left limit of $x_{\alpha-1}(t)$ at $t = \zeta_{\alpha-1}$. Clearly,

$$\langle \phi, Y_t \rangle = \sum_{\alpha \in \mathcal{A}} \theta^{-1} \phi(x_\alpha(t)) 1_{[\beta_\alpha, \zeta_\alpha)}(t), \qquad t \geq 0. \tag{2.10}$$

defines an $M_\theta(\mathbb{R})$-valued process $\{Y_t : t \geq 0\}$. It is easy to see that $\{Y_t : t \geq 0\}$ has countably many jumps, and between those jumps it behaves just as the diffusion process $\{X_t : t \geq 0\}$ constructed by (2.2). We call $\{Y_t : t \geq 0\}$ an *immigration branching particle system* with parameters $(c, h, \gamma, p, \theta m, 1/\theta)$. Intuitively, $p(x, \cdot)$ gives the location dependent offspring distribution and $\{N(ds, dx)\}$ gives the landing times and sites of the immigrants.

Indeed, we may regard $\{Y_t : t \geq 0\}$ as a concatenation of a sequence of independent copies of $\{X_t : t \geq 0\}$. We refer the reader to [24] for discussions of concatenation of general Markov processes. As in [17] it can be seen that $\{Y_t : t \geq 0\}$ is a Markov process with generator $\mathcal{L}_\theta := \mathcal{A}_\theta + \mathcal{B}_\theta$, where

$$\begin{aligned} \mathcal{B}_\theta F(\mu) &= \sum_{j=0}^{\infty} \int_{\mathbb{R}} \theta \gamma p_j(x) [F(\mu + (j-1)\theta^{-1}\delta_x) - F(\mu)] \mu(dx) \\ &\quad + \int_{\mathbb{R}} \theta [F(\mu + \theta^{-1}\delta_x) - F(\mu)] m(dx). \end{aligned} \tag{2.11}$$

The first term on the right hand side of (2.11) represents the jumps given by the branching and the second terms represents the jumps given by the immigration. In particular, it is easy to show that

$$\begin{aligned} \mathcal{B}_\theta F_{f,\{\phi_i\}}(\mu) &= \sum_{j=0}^{\infty} \int_{\mathbb{R}} \theta \gamma p_j(x) [f(\langle \phi_1, \mu \rangle + \theta^{-1}\phi_1(x), \cdots, \langle \phi_n, \mu \rangle + \theta^{-1}\phi_n(x)) \\ &\qquad - f(\langle \phi_1, \mu \rangle, \cdots, \langle \phi_n, \mu \rangle)] \mu(dx) \\ &\quad + \int_{\mathbb{R}} \theta [f(\langle \phi_1, \mu \rangle + \theta^{-1}\phi_1(x), \cdots, \langle \phi_n, \mu \rangle + \theta^{-1}\phi_n(x)) \\ &\qquad - f(\langle \phi_1, \mu \rangle, \cdots, \langle \phi_n, \mu \rangle)] m(dx). \end{aligned} \tag{2.12}$$

Let $\mathcal{D}_1(\mathcal{L}_\theta)$ denote the collection of all functions $F_{f,\{\phi_i\}}$ with $f \in C_0^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C_b^2(\mathbb{R})$. By the general theory of Markov processes, we have the following

Theorem 2.1 *The process $\{Y_t : t \geq 0\}$ defined by (2.10) solves the $(\mathcal{L}_\theta, \mathcal{D}_1(\mathcal{L}_\theta))$-martingale problem, that is, for each $F \in \mathcal{D}_1(\mathcal{L}_\theta)$,*

$$F(X_t) - F(X_0) - \int_0^t \mathcal{L}_\theta F(X_s)ds, \qquad t \geq 0,$$

is a martingale.

Let us give another useful formulation of the immigration particle system. From (2.8), (2.10) and Itô's formula we get

$$\begin{aligned}
\langle \phi, Y_t \rangle &= \langle \phi, Y_0 \rangle + \sum_{i=1}^{\infty} \theta^{-1} \phi(a_{1i}) 1_{(0,t]}(s_i) \\
&\quad + \sum_{\alpha \in \mathcal{A}} [\eta_{x_\alpha(\zeta_\alpha -),\alpha} - 1] \theta^{-1} \phi(x_\alpha(\zeta_\alpha -)) 1_{(0,t]}(\zeta_\alpha) \\
&\quad + \sum_{\alpha \in \mathcal{A}} \int_0^t \theta^{-1} \phi'(x_\alpha(s)) 1_{[\beta_\alpha, \zeta_\alpha)}(s) c(x_\alpha(s)) dB_\alpha(s) \\
&\quad + \sum_{\alpha \in \mathcal{A}} \int_0^t \int_{\mathbb{R}} \theta^{-1} \phi'(x_\alpha(s)) 1_{[\beta_\alpha, \zeta_\alpha)}(s) h(y - x_\alpha(s)) W(ds, dy) \\
&\quad + \frac{1}{2} \sum_{\alpha \in \mathcal{A}} \int_0^t \theta^{-1} \phi''(x_\alpha(s)) 1_{[\beta_\alpha, \zeta_\alpha)}(s) a(x_\alpha(s)) ds,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\langle \phi, Y_t \rangle &= \langle \phi, Y_0 \rangle + \int_{(0,t]} \int_{\mathbb{R}} \theta^{-1} \phi(x) N(ds, dx) \\
&\quad + \sum_{\alpha \in \mathcal{A}} [\eta_{x_\alpha(\zeta_\alpha -),\alpha} - 1] \theta^{-1} \phi(x_\alpha(\zeta_\alpha -)) 1_{(0,t]}(\zeta_\alpha) \\
&\quad + \sum_{\alpha \in \mathcal{A}} \int_0^t \theta^{-1} \phi'(x_\alpha(s)) 1_{[\beta_\alpha, \zeta_\alpha)}(s) c(x_\alpha(s)) dB_\alpha(s) \\
&\quad + \int_0^t \int_{\mathbb{R}} \langle h(y - \cdot) \phi', Y_s \rangle W(ds, dy) + \frac{1}{2} \int_0^t \langle a\phi'', Y_s \rangle ds. \qquad (2.13)
\end{aligned}$$

On the right hand side, the second term comes from the immigration, the third term represents branching of the particles, and the last three terms are determined by the spatial motion. It is not hard to see that, for any $\psi \in C_b(\mathbb{R})$,

$$U_t(\psi) := \sum_{\alpha \in \mathcal{A}} \int_0^t \theta^{-1} \psi(x_\alpha(s)) 1_{[\beta_\alpha, \zeta_\alpha)}(s) c(x_\alpha(s)) dB_\alpha(s) \qquad (2.14)$$

is a continuous local martingale with quadratic variation process

$$\langle U(\psi) \rangle_t := \int_0^t \langle \theta^{-1} c^2 \psi^2, Y_s \rangle ds. \qquad (2.15)$$

In the sequel, we assume

$$\sigma(x) = \sum_{i=0}^{\infty} p_i(x)(i-1)^2, \qquad x \in \mathbb{R}, \tag{2.16}$$

is a bounded function on $\mathbb{R}$.

Proposition 2.1 *For any $\phi \in C_b(\mathbb{R})$,*

$$Z_t(\phi) := \sum_{\alpha\in\mathcal{A}} [\eta_{x_\alpha(\zeta_\alpha-),\alpha} - 1]\theta^{-1}\phi(x_\alpha(\zeta_\alpha-))1_{(0,t]}(\zeta_\alpha) - \int_0^t \langle \gamma(q-1)\phi, Y_s\rangle ds \tag{2.17}$$

is a local martingale with predictable quadratic variation process

$$\langle Z(\phi)\rangle_t = \int_0^t \langle \theta^{-1}\gamma\sigma\phi^2, Y_s\rangle ds. \tag{2.18}$$

Proof. Recall that $\{S_\alpha\}$ are i.i.d. exponential random variables with parameter γ. Let

$$J_t(\phi) = \sum_{\alpha\in\mathcal{A}} \theta^{-1}[\eta_{x_\alpha(\zeta_\alpha-),\alpha} - 1]\phi(x_\alpha(\zeta_\alpha-))1_{(0,t]}(\zeta_\alpha). \tag{2.19}$$

Observe that the process $\{J_t(\phi) : t \geq 0\}$ jumps only when a particle in the population splits. It is not hard to show that $\{(Y_t, J_t(\phi)) : t \geq 0\}$ is a Markov process with generator $\mathscr{J}_\theta$ such that

$$\begin{aligned}
\mathscr{J}_\theta F(\mu, z) &= \mathscr{A}_\theta F(\cdot, z)(\mu) + \int_{\mathbb{R}} \theta[F(\mu + \theta^{-1}\delta_x, z) - F(\mu, z)]m(dx) \\
&\quad + \sum_{j=0}^{\infty} \int_{\mathbb{R}} \theta\gamma p_j(x)[F(\mu + (j-1)\theta^{-1}\delta_x, z + (j-1)\theta^{-1}\phi(x)) - F(\mu, z)]\mu(dx).
\end{aligned}$$

In particular, if $F(\mu, z) = z$, then

$$\mathscr{J}_\theta F(\mu, z) = \sum_{j=0}^{\infty} \int_{\mathbb{R}} \gamma p_j(x)(j-1)\phi(x)\mu(dx) = \langle \gamma(q-1)\phi, \mu\rangle.$$

This shows that (2.17) is a local martingale. Let $\Delta_n := \{0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = t\}$ be a sequence of partitions of $[0, t]$ such that $D_n := \max_{1\leq i\leq n} |t_{n,i} - t_{n,i-1}| \to 0$ as $n \to \infty$. Since the second term on the right hand side of (2.17) is of locally finite variations, we have

$$\begin{aligned}
[Z(\phi)]_{t\wedge\tau_l} &:= \lim_{n\to\infty} \sum_{i=0}^{n} |Z_{t_{n,i}\wedge\tau_l}(\phi) - Z_{t_{i-1}\wedge\tau_l}(\phi)|^2 \\
&= \sum_{\alpha\in\mathcal{A}} \theta^{-2}[\eta_{x_\alpha(\zeta_\alpha-),\alpha} - 1]^2\phi(x_\alpha(\zeta_\alpha-))^2 1_{(0,t\wedge\tau_l]}(\zeta_\alpha).
\end{aligned}$$

By martingale theory, $Z_{t\wedge\tau_l}(\phi)^2 - [Z(\phi)]_{t\wedge\tau_l}$ is a martingale. Note that $[Z(\phi)]_{t\wedge\tau_l}$ has same jump times as $J_{t\wedge\tau_l}(\phi)$ but with squared jump sizes. By an argument similar to the beginning of this proof, we conclude that $[Z(\phi)]_{t\wedge\tau_l} - \langle Z(\phi)\rangle_{t\wedge\tau_l}$ is a martingale. Then $\langle Z(\phi)\rangle_{t\wedge\tau_l}$ is a predictable process such that $Z_{t\wedge\tau_l}(\phi)^2 - \langle Z(\phi)\rangle_{t\wedge\tau_l}$ is a martingale, implying the desired result. ∎

Let $\tilde{N}(ds,dx) = N(ds,dx) - \theta ds m(dx)$. Note that the assumptions on independence imply that the four martingale measures $\{W(ds,dx)\}$, $\{\tilde{N}(ds,dx)\}$, $\{Z(ds,dx)\}$ are $\{U(ds,dx)\}$ are orthogonal to each other. Now we may rewrite (2.13) into

$$\begin{aligned}\langle\phi, Y_t\rangle &= \langle\phi, Y_0\rangle + t\langle\phi, m\rangle + \int_{(0,t]}\int_{\mathbb{R}} \theta^{-1}\phi(x)\tilde{N}(ds,dx) \\ &\quad + \int_0^t \langle\gamma(q-1)\phi, Y_s\rangle ds + Z_t(\phi) + U_t(\phi') \\ &\quad + \int_0^t\int_{\mathbb{R}} \langle h(y-\cdot)\phi', Y_s\rangle W(ds,dy) + \frac{1}{2}\int_0^t \langle a\phi'', Y_s\rangle ds. \end{aligned} \tag{2.20}$$

Clearly, the third term on the right hand side of (2.20) has a càdlàg modification. By [10, p.69, Theorem VI.4], the martingale $\{Z_t(\phi) : t \geq 0\}$ has a càdlàg modification. All other terms on the right hand side have continuous modifications. Therefore, the measure-valued process $\{Y_t : t \geq 0\}$ has a càdlàg modification and (2.20) gives an SPDE formulation of this immigration branching particle system. The following result shows that (2.14) and (2.17) are in fact square-integrable martingales.

Proposition 2.2 *Let $B_1 := \|\gamma(q-1)\|$ and $B_2 := \|\theta\gamma\sigma\|$, where $\|\cdot\|$ denotes the supremum norm. Then there is a locally bounded function C_2 on $\mathbb{R}_+^3$ such that*

$$\mathbf{E}\{\sup_{0\leq s\leq t}\langle 1, Y_s\rangle^2\} \leq C_2(B_1, B_2, t)(1 + \langle 1,\mu\rangle^2 + \langle 1, m\rangle^2), \quad t \geq 0. \tag{2.21}$$

Proof. Applying (2.20) to $\phi \equiv 1$ we get

$$\langle 1, Y_t\rangle = \langle 1,\mu\rangle + \theta^{-1}N((0,t]\times\mathbb{R}) + \int_0^t \langle\gamma(q-1), Y_s\rangle ds + Z_t(1), \tag{2.22}$$

where $N((0,t]\times\mathbb{R})$ is a Poisson random variable with parameter $\theta t\langle 1, m\rangle$ and $\{Z_t(1) : t \geq 0\}$ is a local martingale with quadratic variation process

$$\langle Z(1)\rangle_t = \int_0^t \langle\theta^{-1}\gamma\sigma, Y_s\rangle ds. \tag{2.23}$$

Based on (2.22) and (2.23), the desired estimate follows by an application of Gronwall's inequality. ∎

3 Stochastic equation of the SDSMI

Let (c, h, σ, b, m) be given as in the introduction. Suppose that $W(ds,dx)$ is a time-space white noise. For $\mu \in M(\mathbb{R})$ we consider the stochastic equation:

$$\begin{aligned}\langle\phi, X_t\rangle &= \langle\phi,\mu\rangle + t\langle\phi, m\rangle + \frac{1}{2}\int_0^t \langle a\phi'', X_s\rangle ds - \int_0^t \langle b\phi, X_s\rangle ds \\ &\quad + \int_0^t\int_{\mathbb{R}} \phi(y) Z(ds,dy) + \int_0^t\int_{\mathbb{R}} \langle h(y-\cdot)\phi', X_s\rangle W(ds,dy), \end{aligned} \tag{3.1}$$

where $Z(ds,dy)$ is an orthogonal martingale measure which is orthogonal to the white noise $W(ds,dy)$ and has covariation measure $\sigma(y)X_s(dy)ds$. Clearly, this is equivalent to the martingale problem given by (1.6) and (1.7). We shall prove that (3.1) has a weak solution $\{X_t : t \geq 0\}$, which will serve as a candidate of the SDSMI with parameters (c, h, σ, b, m). For a function F on $M(\mathbb{R})$, let

$$\begin{aligned}\mathcal{A}F(\mu) &= \frac{1}{2}\int_{\mathbb{R}^2}\rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy) \\ &\quad+\frac{1}{2}\int_{\mathbb{R}}a(x)\frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) \qquad (3.2)\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}F(\mu) &= \frac{1}{2}\int_{\mathbb{R}}\sigma(x)\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx)-\int_{\mathbb{R}}b(x)\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) \\ &\quad+\int_{\mathbb{R}}\frac{\delta F(\mu)}{\delta\mu(x)}m(dx) \qquad (3.3)\end{aligned}$$

if the right hand sides are meaningful. We shall also prove that $\{X_t : t \geq 0\}$ solves a martingale problem associated with $\mathcal{L} := \mathcal{A} + \mathcal{B}$. It is easily seen that formally $\mathcal{A} = \lim_{\theta\to 0}\mathcal{A}_\theta$ and $\mathcal{B} = \lim_{\theta\to 0}\mathcal{B}_\theta$. Heuristically, $\{X_t : t \geq 0\}$ arises as the high density limit of the immigration branching particle system discussed in the last section. In particular, if $F_{f,\{\phi_i\}}(\mu) = f(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle)$ for $f \in C_0^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C_b^2(\mathbb{R})$, then

$$\begin{aligned}\mathcal{A}F_{f,\{\phi_i\}}(\mu) &= \frac{1}{2}\sum_{i,j=1}^n f''_{ij}(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\int_{\mathbb{R}^2}\rho(x-y)\phi'_i(x)\phi'_j(y)\mu(dx)\mu(dy) \\ &\quad+\frac{1}{2}\sum_{i=1}^n f'_i(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle a\phi''_i,\mu\rangle \qquad (3.4)\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}F_{f,\{\phi_i\}}(\mu) &= \frac{1}{2}\int_{\mathbb{R}}\sigma(x)\Big[\sum_{i,j=1}^n f''_{ij}(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\phi_i(x)\phi_j(x)\Big]\mu(dx) \\ &\quad-\int_{\mathbb{R}}b(x)\Big[\sum_{i=1}^n f'_i(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\phi_i(x)\Big]\mu(dx) \\ &\quad+\int_{\mathbb{R}}\Big[\sum_{i=1}^n f'_i(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\phi_i(x)\Big]m(dx). \qquad (3.5)\end{aligned}$$

Let $\mathcal{D}_1(\mathcal{L})$ denote the collection of all functions $F_{f,\{\phi_i\}}$ with $f \in C_0^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C_b^2(\mathbb{R})$.

We shall obtain (3.1) as the limit of a sequence of equations of immigration branching particle systems. Let $(c, h, \gamma_k, p^{(k)}, \theta_k m, \theta_k^{-1})$ be a sequence of parameters such that $\theta_k \to \infty$ as $k \to \infty$. Let q_k and σ_k be defined by (2.5) and (2.16) in terms of $(\gamma_k, p^{(k)}, \theta_k)$. We assume that $\{X_t^{(k)} : t \geq 0\}$ is a immigration particle system which satisfies

$$\begin{aligned}\langle\phi, X_t^{(k)}\rangle &= \langle\phi, X_0^{(k)}\rangle + t\langle\phi, m\rangle + \int_{(0,t]}\int_{\mathbb{R}}\theta_k^{-1}\phi(x)\tilde{N}^{(k)}(ds,dx) \\ &\quad+\int_0^t\langle\gamma_k(q_k-1)\phi, X_s^{(k)}\rangle ds + Z_t^{(k)}(\phi) + U_t^{(k)}(\phi') \\ &\quad+\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\phi', X_s^{(k)}\rangle W^{(k)}(ds,dy) + \frac{1}{2}\int_0^t\langle a\phi'', X_s^{(k)}\rangle ds, \qquad (3.6)\end{aligned}$$

where $(N^{(k)}, Z^{(k)}, M^{(k)}, W^{(k)})$ are as in (2.20) with parameters $(c, h, \gamma_k, p^{(k)}, \theta_k m, \theta_k^{-1})$. We assume that the $X_0^{(k)}$ are deterministic and $X_0^{(k)} \to \mu$ as $k \to \infty$.

Lemma 3.1 *Suppose that $B_1 := \sup_{k\geq 1} \|\gamma_k(q_k - 1)\| < \infty$ and $B_2 := \sup_{k\geq 1} \|\theta_k^{-1}\gamma_k\sigma_k\| < \infty$. Then for any $\phi \in C_b^2(\mathbb{R})$, the sequence $\{(\langle\phi, X_t^{(k)}\rangle)_{t\geq 0}, k = 1, 2, \cdots\}$ is tight in the Skorokhod space $D([0,\infty), \mathbb{R})$.*

Proof. Suppose that $\{\tau_k\}$ is a bounded sequence of stopping times. Let

$$V_t^{(k)}(\phi') = \int_0^t \int_{\mathbb{R}} \langle h(y - \cdot)\phi', X_s^{(k)}\rangle W^{(k)}(ds, dy)$$

and

$$Y_t^{(k)}(\phi) = \int_0^t \langle \gamma_k(q_k - 1)\phi, X_s^{(k)}\rangle ds.$$

It is easily seen that

$$\begin{aligned}
\mathbf{E}\{|V_{\tau_k+t}^{(k)}(\phi') - V_{\tau_k}^{(k)}(\phi')|^2\} &= \mathbf{E}\bigg\{\int_0^t ds \int_{\mathbb{R}} \langle h(y - \cdot)\phi', X_{\tau_k+s}^{(k)}\rangle^2 dy\bigg\} \\
&= \mathbf{E}\bigg\{\int_0^t ds \int_{\mathbb{R}^2} \rho(x - z)\phi'(x)\phi'(z) X_{\tau_k+s}^{(k)}(dx) X_{\tau_k+s}^{(k)}(dz)\bigg\} \\
&\leq \|\rho\| \int_0^t \mathbf{E}\{\langle\phi', X_{\tau_k+s}^{(k)}\rangle^2\} ds
\end{aligned}$$

and

$$\mathbf{E}\{|Y_{\tau_k+t}^{(k)}(\phi) - Y_{\tau_k}^{(k)}(\phi)|^2\} \leq B_1^2 t \int_0^t \mathbf{E}\{\langle\phi, X_{\tau_k+s}^{(k)}\rangle^2\} ds.$$

The remaining terms on the right hand side of (3.6) can be estimated by similar calculations. Combining those estimates and Proposition 2.2 we get

$$\sup_{0\leq t\leq T} \sup_{k\geq 1} \mathbf{E}\{\langle\phi, X_t^{(k)}\rangle^2\} < \infty \quad \text{and} \quad \sup_{k\geq 1} \mathbf{E}\{|\langle\phi, X_{\tau_k+t}^{(k)}\rangle - \langle\phi, X_{\tau_k}^{(k)}\rangle|^2\} \to 0$$

as $t \to 0$. Then the sequence $\{(\langle\phi, X_t^{(k)}\rangle)_{t\geq 0}, k = 1, 2, \cdots\}$ is tight in $D([0,\infty), \mathbb{R})$; see [1]. ∎

Lemma 3.2 *Suppose that $\gamma_k(1 - q_k(\cdot)) \to b(\cdot)$ and $\theta_k^{-1}\gamma_k\sigma_k(\cdot) \to \sigma(\cdot)$ uniformly for $b \in C_b(\mathbb{R})$ and $\sigma \in C_b(\mathbb{R})^+$. Then the sequence $\{X_t^{(k)} : t \geq 0, k = 1, 2, \cdots\}$ is tight in $D([0,\infty), M(\mathbb{R}))$. Moreover, the limit process $\{X_t : t \geq 0\}$ of any subsequence of $\{X_t^{(k)} : t \geq 0, k = 1, 2, \cdots\}$ is a.s. continuous and solves the $(\mathcal{L}, \mathcal{D}_1(\mathcal{L}))$-martingale problem, that is, for each $F \in \mathcal{D}_1(\mathcal{L}_\theta)$,*

$$F(X_t) - F(X_0) - \int_0^t \mathcal{L}F(X_s) ds, \qquad t \geq 0, \tag{3.7}$$

is a martingale.

Proof. By Lemma 3.1 and a result of [22], the sequence of processes $\{X_t^{(k)} : t \geq 0, k = 1, 2, \cdots\}$ is tight in $D([0,\infty), M(\bar{\mathbb{R}}))$. We write $\phi \in C_b^2(\bar{\mathbb{R}})$ if $\phi \in C_b^2(\mathbb{R})$ and its derivatives up to the second degree can be extended continuously to $\bar{\mathbb{R}}$. If $\{\phi_i\} \subset C^2(\bar{\mathbb{R}})$, we can extend $F_{f,\{\phi_i\}}$, $\mathcal{A}F_{f,\{\phi_i\}}$ and $\mathcal{B}F_{f,\{\phi_i\}}$ continuously to $M(\bar{\mathbb{R}})$. Let $\bar{F}_{f,\{\phi_i\}}$, $\bar{\mathcal{A}}\bar{F}_{f,\{\phi_i\}}$ and $\bar{\mathcal{B}}\bar{F}_{f,\{\phi_i\}}$ denote respectively those extensions. Let $(\mathcal{A}_k, \mathcal{B}_k)$ and $(\bar{\mathcal{A}}_k, \bar{\mathcal{B}}_k)$ denote the corresponding operators associated with $\{X_t^{(k)} : t \geq 0\}$. Clearly, if $\mu_k \in M_k(\bar{\mathbb{R}})$ and $\mu_k \to \mu$, then $\bar{\mathcal{A}}_k\bar{F}_{f,\{\phi_i\}}(\mu_k) \to \bar{\mathcal{A}}\bar{F}_{f,\{\phi_i\}}(\mu)$. By Taylor's expansion,

$$
\begin{aligned}
&\bar{\mathcal{B}}_k\bar{F}_{f,\{\phi_i\}}(\mu_k) \\
= \ &\sum_{j=0}^{\infty}\int_{\mathbb{R}} \theta_k\gamma_k p_j(x)[f(\langle\phi_1,\mu_k\rangle + (j-1)\theta_k^{-1}\phi_1(x), \cdots, \langle\phi_n,\mu_k\rangle + (j-1)\theta_k^{-1}\phi_n(x)) \\
&\qquad\qquad -f(\langle\phi_1,\mu_k\rangle, \cdots, \langle\phi_n,\mu_k\rangle)]\mu_k(dx) \\
&+\int_{\mathbb{R}} \theta_k[f(\langle\phi_1,\mu_k\rangle + \theta_k^{-1}\phi_1(x), \cdots, \langle\phi_n,\mu_k\rangle + \theta_k^{-1}\phi_n(x)) \\
&\qquad\qquad -f(\langle\phi_1,\mu_k\rangle, \cdots, \langle\phi_n,\mu_k\rangle)]m(dx) \\
= \ &\int_{\mathbb{R}} \gamma_k(q_k(x)-1)\Big[\sum_{i=1}^n f_i'(\langle\phi_1,\mu_k\rangle, \cdots, \langle\phi_n,\mu_k\rangle)\phi_i(x)\Big]\mu_k(dx) \\
&+\int_{\mathbb{R}} \frac{\gamma_k\sigma_k(x)}{2\theta_k}\Big[\sum_{i,j=1}^n f_{ij}''(\langle\phi_1,\mu_k\rangle + \eta_k\phi_1(x), \cdots, \langle\phi_n,\mu_k\rangle + \eta_k\phi_n(x))\phi_i(x)\phi_j(x)\Big]\mu_k(dx) \\
&+\int_{\mathbb{R}}\sum_{i=1}^n \Big[f_i'(\langle\phi_1,\mu_k\rangle + \zeta_k\phi_1(x), \cdots, \langle\phi_n,\mu_k\rangle + \zeta_k\phi_n(x))\phi_i(x)\Big]m(dx),
\end{aligned}
$$

where $0 < \eta_k, \zeta_k < \theta_k^{-1}$. Then $\bar{\mathcal{B}}_k\bar{F}_{f,\{\phi_i\}}(\mu_k) \to \bar{\mathcal{B}}\bar{F}_{f,\{\phi_i\}}(\mu)$ under the assumption. Let $\{X_t : t \geq 0\}$ be the limit of any subsequence of $\{X_t^{(k)} : t \geq 0, k = 1, 2, \cdots\}$. As in the proof of Lemma 4.2 of Dawson *et al* [8] one can show that

$$
\bar{F}_{f,\{\phi_i\}}(X_t) - \bar{F}_{f,\{\phi_i\}}(X_0) - \int_0^t \bar{\mathcal{L}}\bar{F}_{f,\{\phi_i\}}(X_s)ds
$$

is a martingale, where $\bar{\mathcal{L}} = \bar{\mathcal{A}} + \bar{\mathcal{B}}$. As in [28], it is not hard to check that the "gradient squared" operator associated with $\bar{\mathcal{L}}$ satisfies the derivation property of [2]. Then $\{X_t : t \geq 0\}$ is actually almost surely continuous as an $M(\bar{\mathbb{R}})$-valued process. By a modification of the proof of Theorem 4.1 of [8] one can show that $\{X_t : t \geq 0\}$ is almost surely supported by $\mathbb{R}$. Thus $\{X_t^{(k)} : t \geq 0, k = 1, 2, \cdots\}$ is tight in $D([0,\infty), M(\mathbb{R}))$ and $\{X_t : t \geq 0\}$ is a.s. continuous as an $M(\mathbb{R})$-valued process. ∎

Lemma 3.3 *If $\{X_t : t \geq 0\}$ is the continuous solution of the $(\mathcal{L}, \mathcal{D}_1(\mathcal{L}))$-martingale problem, then for each integer $n \geq 1$ there is a locally bounded function C_n on $\mathbb{R}_+^3$ such that*

$$
\mathbf{E}\{\sup_{0\leq s\leq t}\langle 1, X_s\rangle^n\} \leq C_n(\|b\|, \|\sigma\|, t)(1 + \langle 1,\mu\rangle^n + \langle 1, m\rangle^n), \quad t \geq 0. \tag{3.8}
$$

Proof. If $\{X_t : t \geq 0\}$ is the continuous solution of the $(\mathcal{L}, \mathcal{D}_1(\mathcal{L}))$-martingale problem, then

$$
Z_t(1) := \langle 1, X_t\rangle - \langle 1,\mu\rangle - t\langle 1, m\rangle + \int_0^t \langle b, X_s\rangle ds \tag{3.9}
$$

is a continuous local martingale with quadratic variation process

$$\langle Z(1)\rangle_t = \int_0^t \langle \sigma, X_s\rangle ds. \tag{3.10}$$

For $l > 0$ let $\tau_l = \inf\{s \geq 0 : \langle 1, X_s\rangle \geq l\}$. The inequalities for $n = 1$ and $n = 2$ can be proved as in the proof of Proposition 2.2. Now the Burkholder-Davis-Gundy inequality implies that

$$\begin{aligned}
\mathbf{E}\{\sup_{0\leq s\leq t}\langle 1, X_{s\wedge\tau_l}\rangle^{2n}\} &\leq C_n\bigg[\langle 1,\mu\rangle^{2n} + t^{2n}\langle 1, m\rangle^{2n} + \mathbf{E}\bigg\{\bigg(\int_0^{t\wedge\tau_l}\langle |b|, X_s\rangle ds\bigg)^{2n}\bigg\} \\
&\quad + \mathbf{E}\bigg\{\bigg(\int_0^{t\wedge\tau_l}\langle \sigma, X_s\rangle ds\bigg)^{n}\bigg\}\bigg] \\
&\leq C_n\bigg[\langle 1,\mu\rangle^{2n} + t^{2n}\langle 1, m\rangle^{2n} + \theta^{-n}t^n\langle 1, m\rangle^n \\
&\quad + \|b\|^{2n}t^{2n-1}\int_0^t \mathbf{E}\{\sup_{0\leq r\leq s}\langle 1, X_{r\wedge\tau_l}\rangle^{2n}\}ds\bigg] \\
&\quad + \|\sigma\|^n t^{n-1}\int_0^t \mathbf{E}\{\langle 1, X_s\rangle^n\}ds,
\end{aligned}$$

where $C_n \geq 0$ is a universal constant. By using the above estimate and Gronwall's inequality inductively, we get some estimates for $\mathbf{E}\{\sup_{0\leq s\leq t}\langle 1, X_{t\wedge\tau_l}\rangle^n\}$. Then we obtain the inequalities for $\mathbf{E}\{\sup_{0\leq s\leq t}\langle 1, X_t\rangle^n\}$ by Fatou's lemma. ∎

Lemma 3.4 *Suppose there are constants $d_0 > 0$ and $\delta > 1/2$ such that $h(x) \leq d_0(1+|x|)^{-\delta}$ for all $x \in \mathbb{R}$. If $\gamma_k(1 - q_k(\cdot)) \to b(\cdot)$ and $\theta_k^{-1}\gamma_k\sigma_k(\cdot) \to \sigma(\cdot)$ uniformly for $b \in C_b(\mathbb{R})$ and $\sigma \in C_b(\mathbb{R})^+$, then the limit process $\{X_t : t \geq 0\}$ of any subsequence of $\{X_t^{(k)} : t \geq 0, k = 1, 2, \cdots\}$ is a weak solution of (3.1).*

Proof. By the proof of Lemma 3.1 and the results of [21, 22], $\{(X_t^{(k)}, U_t^{(k)}, W_t^{(k)}, Z_t^{(k)}) : t \geq 0, k = 1, 2, \cdots\}$ is a tight sequence in $D([0,\infty), M(\bar{\mathbb{R}}) \times \mathcal{S}'(\mathbb{R})^3)$. By passing to a subsequence, we simply assume that $\{(X_t^{(k)}, U_t^{(k)}, W_t^{(k)}, Z_t^{(k)}) : t \geq 0\}$ converges in distribution to some process $\{(X_t, U_t, W_t, Z_t) : t \geq 0\}$. By Lemma 3.2, $\{X_t : t \geq 0\}$ is a.s. continuous and solves the $(\mathcal{L}, \mathcal{D}_1(\mathcal{L}))$-martingale problem. Considering the Skorokhod representation, we assume $\{(X_t^{(k)}, U_t^{(k)}, W_t^{(k)}, Z_t^{(k)}) : t \geq 0\}$ converges almost surely to the process $\{(X_t, U_t, W_t, Z_t) : t \geq 0\}$ in the topology of $D([0,\infty), M(\bar{\mathbb{R}}) \times \mathcal{S}'(\mathbb{R})^3)$. Since each $\{W_t^{(k)} : t \geq 0\}$ is a time-space white noise, so is $\{W_t : t \geq 0\}$. In view of (2.15), we have a.s. $U_t(\phi) = 0$ for all $t \geq 0$ and $\phi \in \mathcal{S}(\mathbb{R})$. Then the theorem follows once it is proved that $\{(X_t, W_t, Z_t) : t \geq 0\}$ satisfies (3.1). Clearly, it is sufficient to prove this for $\phi \in \mathcal{S}(\mathbb{R})$ with compact support supp(ϕ). Let $Y_t(y) = \langle h(y - \cdot)\phi', X_t\rangle$ and $Y_t^{(k)}(y) = \langle h(y - \cdot)\phi', X_t^{(k)}\rangle$. For $l > 0$ let $\tau_l = \inf\{s \geq 0 : \langle 1, X_s^{(k)}\rangle \geq l$ for some $k \geq 1\}$. Since the weak convergence of measures can be induced by the (Vasershtein) metric defined in [11, p.150], it is easy to show that $\{Y_t^{(k)}1_{\{t<\tau_l\}} : t \geq 0\}$ converges to $\{Y_t 1_{\{t<\tau_l\}} : t \geq 0\}$ in $D([0,\infty), C_0(\mathbb{R}))$, where $C_0(\mathbb{R})$ is furnished with the uniform norm. By [4, Theorem 2.1], for $\psi \in \mathcal{S}(\mathbb{R})$ we have almost surely

$$\lim_{k\to\infty}\int_0^t\int_{\mathbb{R}}\psi(y)Y_s^{(k)}(y)1_{\{s<\tau_l\}}W^{(k)}(ds,dy) = \int_0^t\int_{\mathbb{R}}\psi(y)Y_s(y)1_{\{s<\tau_l\}}W(ds,dy). \tag{3.11}$$

Let $\alpha = \sup\{|x|, x \in \mathrm{supp}(\phi)\}$. We have

$$\sup_{|z|\leq\alpha} |h(y-z)| \leq d(y) := d_0[1_{\{|y|\leq\alpha\}} + 1_{\{|y|>\alpha\}}(1+|y|-\alpha)^{-\delta}],$$

and hence

$$|Y_t(y)| \leq \langle|\phi'|, X_t\rangle d(y) \quad \text{and} \quad |Y_t^{(k)}(y)| \leq \langle|\phi'|, X_t^{(k)}\rangle d(y).$$

By the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}\mathbf{E}\Big\{\Big(\int_0^t\int_{\mathbb{R}}\psi(y)Y_s^{(k)}(y)1_{\{s<\tau_l\}}W^{(k)}(ds,dy)\Big)^4\Big\} \\ \leq \mathrm{const}\cdot\mathbf{E}\Big\{\Big(\int_0^t\int_{\mathbb{R}}\psi(y)^2Y_s^{(k)}(y)^2 1_{\{s<\tau_l\}}dsdy\Big)^2\Big\} \\ \leq \mathrm{const}\cdot l^4\|\phi'\|^4\langle\psi^2d^2,\lambda\rangle^2t^2, \end{aligned} \tag{3.12}$$

where λ denotes the Lebesgue measure on $\mathbb{R}$. Since the right hand side of (3.12) is independent of $k \geq 1$, the convergence of (3.11) also holds in the L^2-sense. For each $\epsilon > 0$, it is not hard to choose $\psi \in \mathcal{S}(\mathbb{R})$ so that

$$\begin{aligned}\mathbf{E}\Big\{\Big(\int_0^t\int_{\mathbb{R}}(1-\psi(y))Y_s^{(k)}(y)1_{\{s<\tau_l\}}W^{(k)}(ds,dy)\Big)^2\Big\} \\ \leq \mathrm{const}\cdot l^2\|\phi'\|^2\langle|1-\psi|^2d^2,\lambda\rangle t \leq \epsilon. \end{aligned} \tag{3.13}$$

The same estimate is available with $Y^{(k)}$ and $W^{(k)}$ replaced respectively by Y and W. Clearly, (3.11) and (3.13) imply that

$$\lim_{k\to\infty}\int_0^t\int_{\mathbb{R}}Y_s^{(k)}(y)1_{\{s<\tau_l\}}W^{(k)}(ds,dy) = \int_0^t\int_{\mathbb{R}}Y_s(y)1_{\{s<\tau_l\}}W(ds,dy) \tag{3.14}$$

in the L^2-sense. Passing to a suitable subsequence we get the almost sure convergence for (3.14). Now letting $k \to \infty$ in (3.6) we get

$$\begin{aligned}\langle\phi, X_{t\wedge\tau_l}\rangle &= \langle\phi,\mu\rangle + (t\wedge\tau_l)\langle\phi,m\rangle + \frac{1}{2}\int_0^{t\wedge\tau_l}\langle a\phi'', X_s\rangle ds - \int_0^{t\wedge\tau_l}\langle b\phi, X_s\rangle ds \\ &\quad + \int_0^{t\wedge\tau_l}\int_{\mathbb{R}}\phi(y)Z(ds,dy) + \int_0^{t\wedge\tau_l}\int_{\mathbb{R}}\langle h(y-\cdot)\phi', X_s\rangle W(ds,dy),\end{aligned}$$

from which (3.1) follows. The extensions from $\phi \in \mathcal{S}(\mathbb{R})$ to $\phi \in C_b^2(\mathbb{R})$ is immediate. ∎

Theorem 3.1 *Suppose there are constants $d_0 > 0$ and $\delta > 1/2$ such that $h(x) \leq d_0(1+|x|)^{-\delta}$ for all $x \in \mathbb{R}$. Then the stochastic equation (3.1) has a continuous weak solution $\{X_t : t \geq 0\}$. Moreover, $\{X_t : t \geq 0\}$ also solves the $(\mathcal{L}, \mathcal{D}_1(\mathcal{L}))$-martingale problem.*

Proof. Given $b \in C_b(\mathbb{R})$ and $\sigma \in C_b(\mathbb{R})^+$, we set $\theta_k = k$, $\gamma_k = \sqrt{k}$ and

$$p_0^{(k)} = 1 - p_2^{(k)} - p_k^{(k)}, \quad p_2^{(k)} = \frac{(k-1)^2(1-b/\sqrt{k}) - k\sigma_k}{2(k-1)^2-k}, \quad p_k^{(k)} = \frac{2\sigma_k - 1 + b/\sqrt{k}}{2(k-1)^2-k},$$

where $\sigma_k(\cdot) = \sqrt{k}\sigma(\cdot)+1$. Then the sequence $(\gamma_k, p^{(k)}, \theta_k)$ satisfies the conditions of Lemma 3.4. By Lemmas 3.2 and 3.4, equation (3.1) has a continuous weak solution $\{X_t : t \geq 0\}$ which solves the $(\mathcal{L}, \mathcal{D}_1(\mathcal{L}))$-martingale problem. ∎

4 Stochastic log-Laplace equations

In this section, we establish the existence and uniqueness of solution of the stochastic log-Laplace equation (1.8). The techniques here are based on the results of [15] and have been stimulated by [5, 30]. Let (c, h, σ, b, m) be given as in the introduction. Suppose that $W(ds, dx)$ is a time-space white noise. The main objective is to discuss the non-linear SPDE:

$$\begin{aligned}\psi_t(x) &= \phi(x) + \int_0^t \left[\frac{1}{2}a(x)\partial_x^2\psi_s(x) - b(x)\psi_s(x) - \frac{1}{2}\sigma(x)\psi_s(x)^2\right]ds \\ &\quad + \int_0^t \int_{\mathbb{R}} h(y-x)\partial_x\psi_s(x)W(ds, dy), \qquad t \geq 0. \end{aligned} \tag{4.1}$$

Let $\{H_k(\mathbb{R}) : k = 0, \pm 1, \pm 2, \cdots\}$ denote the Sobolev spaces on $\mathbb{R}$. Let "$\|\cdot\|_0$" and "$\langle\cdot,\cdot\rangle_0$" denote respectively the norm and the inner product in $H_0(\mathbb{R}) = L^2(\mathbb{R})$. For $\phi \in H_k(\mathbb{R})$ let

$$\|\phi\|_k^2 = \sum_{i=0}^k \|\partial_x^i\phi\|_0^2. \tag{4.2}$$

Following Xiong [30], we first consider a smoothed version of equation (4.1). Let $(T_t)_{t\geq 0}$ denote the transition semigroup of a standard Brownian motion. Let $\{h_j : j = 1, 2, \cdots\}$ be a complete orthonormal system of $H_0(\mathbb{R})$. Then

$$W_j(t) = \int_0^t \int_{\mathbb{R}} h_j(y)W(ds, dy), \qquad t \geq 0 \tag{4.3}$$

defines a sequence of independent standard Brownian motions $\{W_j : j = 1, 2, \cdots\}$. For $\epsilon > 0$ let

$$W^\epsilon(dt, dx) = \sum_{j=1}^{[1/\epsilon]} h_j(x)W_j(dt)dx, \qquad s \geq 0, y \in \mathbb{R}. \tag{4.4}$$

For $\phi \in H_0(\mathbb{R})$ we set $d_\epsilon(\phi) = (\|T_\epsilon\phi\| \wedge \epsilon^{-1})\|T_\epsilon\phi\|^{-1}$. By the general results of [15, Theorem 3.5] and [23, p.133], for any $\phi \in H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$ there is a pathwise unique $H_2(\mathbb{R})$-valued solution $\{\psi_t^\epsilon : t \geq 0\}$ of the equation

$$\begin{aligned}\psi_t^\epsilon(x) &= T_\epsilon\phi(x) + \int_0^t \left[\frac{1}{2}a(x)\partial_x^2\psi_s^\epsilon(x) - b(x)\psi_s^\epsilon(x) - \frac{1}{2}\sigma(x)\psi_s^\epsilon(x)d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon(x)\right]ds \\ &\quad + \int_0^t \int_{\mathbb{R}} h(y-x)\partial_x\psi_s^\epsilon(x)W^\epsilon(ds, dy), \qquad t \geq 0. \end{aligned} \tag{4.5}$$

Lemma 4.1 *The solution $\{\psi_t^\epsilon : t \geq 0\}$ of (4.5) is non-negative and satisfies a.s. $\|\psi_t^\epsilon\|_{ess} \leq e^{-b_0 t}\|\phi\|_{ess}$ for all $t \geq 0$, where $b_0 = \inf_x b(x)$.*

Proof. Indeed, for any non-negative and non-trivial function $\phi \in H_0(\mathbb{R})$, the solution of (4.5) can be obtained in the following way. Let $\{B_i(t)\}$ be a sequence of independent Brownian motions which are also independent of the white noise $\{W(ds, dy)\}$. As in [15, Theorems 2.1 and 2.2], one can show that there is a pathwise unique solution $\psi_t^\epsilon(x)$ of the stochastic system

$$\begin{aligned}\xi_i(t) - \xi_i(0) &= \int_0^t c(\xi_i(s))dB_i(s) + 2\int_0^t c(\xi_i(s))c'(\xi_i(s))ds \\ &\quad - \int_0^t \int_{\mathbb{R}} h(y - \xi_i(s))W^\epsilon(ds, dy), \end{aligned} \tag{4.6}$$

$$
\begin{aligned}
m_i(t)-m_i(0) &= \int_0^t \Big[\frac{1}{2}a''(\xi_i(s))-b(\xi_i(s))\Big]m_i(s)ds \\
&\quad -\frac{1}{2}\int_0^t \sigma(\xi_i(s))d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon(\xi_i(s))m_i(s)ds \\
&\quad -\int_0^t\int_{\mathbb{R}} h'(y-\xi_i(s))m_i(s)W^\epsilon(ds,dy), \qquad (4.7)
\end{aligned}
$$

and

$$
\psi_t^\epsilon(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n m_i(t)\delta_{\xi_i(t)}(dx), \qquad t\geq 0, x\in\mathbb{R}, \qquad (4.8)
$$

where $\{(m_i(0),\xi_i(0)) : i=1,2,\cdots\}$ is a sequence of exchangeable random variables on $[0,\infty)\times\mathbb{R}$ which are independent of $\{B_i(t)\}$ and $\{W(ds,dy)\}$ and satisfy

$$
\lim_{n\to\infty} n^{-1}\sum_{i=1}^n m_i(0)\delta_{\xi_i(0)}(dx) = T_\epsilon\phi(x)dx.
$$

By the arguments of [15, Theorems 3.1-3.5], it can be proved that $\psi_t^\epsilon(x)$ is also the pathwise unique solution of (4.5). By a duality argument similar to the proof of [30, Lemma 2.2] we get $\|\psi_t^\epsilon\|_{\text{ess}}\leq e^{-b_0t}\|\phi\|_{\text{ess}}$. ∎

Lemma 4.2 *There is a locally bounded function $K(\cdot)$ on $[0,\infty)$ such that*

$$
\mathbf{E}\Big\{\sup_{0\leq r\leq t}\|\psi_r^\epsilon\|_0^4\Big\}\leq K(t), \qquad t\geq 0. \qquad (4.9)
$$

Proof. Although the arguments are similar to those of [30], we shall give the detailed proof for the convenience of the reader. For any $f\in C^\infty(\mathbb{R})$ with compact support,

$$
\begin{aligned}
\langle\psi_t^\epsilon,f\rangle_0 &= \langle T_\epsilon\phi,f\rangle_0+\int_0^t\Big[\frac{1}{2}\langle a\partial_x^2\psi_s^\epsilon,f\rangle_0-\langle b\psi_s^\epsilon,f\rangle_0-\frac{1}{2}\langle\sigma\psi_s^\epsilon d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon,f\rangle_0\Big]ds \\
&\quad +\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\partial_x\psi_s^\epsilon,f\rangle_0 W^\epsilon(ds,dy).
\end{aligned}
$$

By Itô's formula,

$$
\begin{aligned}
\langle\psi_t^\epsilon,f\rangle_0^2 &= \langle T_\epsilon\phi,f\rangle_0^2+\int_0^t\langle\psi_s^\epsilon,f\rangle_0\langle a\partial_x^2\psi_s^\epsilon-2b\psi_s^\epsilon-\sigma\psi_s^\epsilon d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon,f\rangle_0 ds \\
&\quad +2\int_0^t\int_{\mathbb{R}}\langle\psi_s^\epsilon,f\rangle_0\langle h(y-\cdot)\partial_x\psi_s^\epsilon,f\rangle_0 W^\epsilon(ds,dy) \\
&\quad +\sum_{j=1}^{[1/\epsilon]}\int_0^t\Big[\int_{\mathbb{R}}h_j(y)\langle h(y-\cdot)\partial_x\psi_s^\epsilon,f\rangle_0 dy\Big]^2 ds.
\end{aligned}
$$

Then we may add f over in a complete orthonormal system of $H_0(\mathbb{R})$ to get

$$
\|\psi_t^\epsilon\|_0^2 = \|T_\epsilon\phi\|_0^2+\int_0^t\langle a\partial_x^2\psi_s^\epsilon-2b\psi_s^\epsilon-\sigma\psi_s^\epsilon d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon,\psi_s^\epsilon\rangle_0 ds
$$

$$
\begin{aligned}
&+2\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\partial_x\psi_s^\epsilon,\psi_s^\epsilon\rangle_0 W^\epsilon(ds,dy)\\
&+\sum_{j=1}^{[1/\epsilon]}\int_0^t ds\int_{\mathbb{R}}\left[\int_{\mathbb{R}}h_j(y)h(y-z)\partial_x\psi_s^\epsilon(z)dy\right]^2dz\\
\le\ & \|T_\epsilon\phi\|_0^2+\int_0^t\langle c^2\partial_x^2\psi_s^\epsilon,\psi_s^\epsilon\rangle_0 ds+\int_0^t\langle -2b\psi_s^\epsilon-\sigma\psi_s^\epsilon d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon,\psi_s^\epsilon\rangle_0 ds\\
&+2\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\partial_x\psi_s^\epsilon,\psi_s^\epsilon\rangle_0 W^\epsilon(ds,dy)\\
&+\int_0^t\langle\rho(0)\partial_x^2\psi_s^\epsilon,\psi_s^\epsilon\rangle_0 ds+\int_0^t ds\int_{\mathbb{R}}\left[\int_{\mathbb{R}}h(y-z)^2(\partial_x\psi_s^\epsilon(x))^2dy\right]dx. \qquad (4.10)
\end{aligned}
$$

Since $\psi_s^\epsilon\in H_2(\mathbb{R})$, there exists a sequence $f_n\in C_0^\infty(\mathbb{R})$ such that $f_n\to\psi_s^\epsilon$ in $H_2(\mathbb{R})$. Note that

$$\langle c^2 f_n'',f_n\rangle=\langle (c^2)'',f_n^2\rangle/2-\langle c^2,(f_n')^2\rangle\le K\|f_n\|_0^2.$$

Taking $n\to\infty$ we have

$$\langle c^2\partial_x^2\psi_s^\epsilon,\psi_s^\epsilon\rangle_0\le K\|\psi_s^\epsilon\|_0^2. \qquad (4.11)$$

It is easy to see that

$$\langle -2b\psi_s^\epsilon-\sigma\psi_s^\epsilon d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon,\psi_s^\epsilon\rangle_0\le K\|\psi_s^\epsilon\|_0^2.$$

Therefore, we can continue (4.10) with

$$
\begin{aligned}
\|\psi_t^\epsilon\|_0^2\ \le\ & \|T_\epsilon\phi\|_0^2+K\int_0^t\|\psi_s^\epsilon\|_0^2ds+2\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\partial_x\psi_s^\epsilon,\psi_s^\epsilon\rangle_0W^\epsilon(ds,dy)\\
&+\rho(0)\int_0^t\langle\partial_x^2\psi_s^\epsilon,\psi_s^\epsilon\rangle_0ds+\rho(0)\int_0^t ds\int_{\mathbb{R}}(\partial_x\psi_s^\epsilon(z))^2dz\\
\le\ & \|\phi\|_0^2+K\int_0^t\|\psi_s^\epsilon\|_0^2ds+2\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\partial_x\psi_s^\epsilon,\psi_s^\epsilon\rangle_0W^\epsilon(ds,dy).
\end{aligned}
$$

By Burkholder's inequality we get

$$
\begin{aligned}
\mathbf{E}\Big\{\sup_{0\le r\le t}\|\psi_r^\epsilon\|_0^4\Big\}\ \le\ & 4\|\phi\|_0^2+K\mathbf{E}\Big\{\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\partial_x\psi_s^\epsilon,\psi_s^\epsilon\rangle_0^2dyds\Big\}\\
&+K\mathbf{E}\Big\{\int_0^t\|\psi_s^\epsilon\|_0^4ds\Big\}\\
\le\ & 4\|\phi\|_0^2+K\mathbf{E}\Big\{\int_0^t\|\psi_s^\epsilon\|_0^4ds\Big\}. \qquad (4.12)
\end{aligned}
$$

where the last inequality follows from the same arguments as those leading to (4.11). Using stopping times if necessary, we may assume that $\mathbf{E}\{\|\psi_t^\epsilon\|_0^4\}<\infty$ for each $t\ge 0$. Then we obtain (4.9) by Gronwall's inequality. ∎

Lemma 4.3 *There is a locally bounded function $K(\cdot)$ on $[0,\infty)$ such that*

$$\mathbf{E}\Big\{\sup_{0\le r\le t}\|\psi_r^\epsilon\|_1^4\Big\}\le K(t),\qquad t\ge 0. \qquad (4.13)$$

Proof. We shall omit some details since they are similar to those in the proof of Lemma 4.2. From (4.5) it follows that

$$\begin{aligned}\partial_x\psi_t^\epsilon(x) &= \partial_x T_\epsilon\phi(x)+\int_0^t\Big[\frac{1}{2}a'(x)\partial_x^2\psi_s^\epsilon(x)+\frac{1}{2}a(x)\partial_x^3\psi_s^\epsilon(x)-b'(x)\psi_s^\epsilon(x)-b(x)\partial_x\psi_s^\epsilon(x)\\ &\quad-\frac{1}{2}\sigma'(x)\psi_s^\epsilon(x)d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon(x)-\frac{1}{2}\sigma(x)\partial_x\psi_s^\epsilon(x)d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon(x)\\ &\quad-\frac{1}{2}\sigma(x)\psi_s^\epsilon(x)d_\epsilon(\psi_s^\epsilon)T_\epsilon\partial.\psi_s^\epsilon(x)\Big]ds\\ &\quad+\int_0^t\int_{\mathbb{R}}[h(y-x)\partial_x^2\psi_s^\epsilon(x)-h'(y-x)\partial_x\psi_s^\epsilon(x)]W^\epsilon(ds,dy).\end{aligned}$$

Then we have

$$\begin{aligned}\|\partial_x\psi_t^\epsilon\|_0^2 &= \|T_\epsilon\partial_x\phi\|_0^2+\int_0^t\Big[\langle\partial_x\psi_s^\epsilon,a'\partial_x^2\psi_s^\epsilon+a\partial_x^3\psi_s^\epsilon\rangle_0-2\langle\partial_x\psi_s^\epsilon,b'\psi_s^\epsilon+b\partial_x\psi_s^\epsilon\rangle_0\\ &\quad-d_\epsilon(\psi_s^\epsilon)\langle\partial_x\psi_s^\epsilon,\sigma'\psi_sT_\epsilon\psi_s^\epsilon+\sigma\partial_x\psi_s^\epsilon T_\epsilon\psi_s^\epsilon+\sigma\psi_s^\epsilon T_\epsilon\partial_x\psi_s^\epsilon\rangle_0\Big]ds\\ &\quad+2\int_0^t\int_{\mathbb{R}}\langle\partial_x\psi_s^\epsilon,h(y-\cdot)\partial_x^2\psi_s^\epsilon-h'(y-\cdot)\partial_x\psi_s^\epsilon\rangle_0W^\epsilon(ds,dy)\\ &\quad+\int_0^t ds\int_{\mathbb{R}}\|h(y-\cdot)\partial_x^2\psi_s^\epsilon-h'(y-\cdot)\partial_x\psi_s^\epsilon\|_0^2dy.\end{aligned}$$

As in the proof of the previous lemma, we have that

$$\mathbf{E}\Big\{\sup_{0\le r\le t}\|\partial_x\psi_t^\epsilon\|_0^4\Big\}\le 4\|\partial_x\phi\|_0^4+K\mathbf{E}\int_0^t\left(\|\psi_s^\epsilon\|_0^4+\|\partial_x\psi_s^\epsilon\|_0^4\right)ds. \tag{4.14}$$

Again, we may assume $\mathbf{E}\Big\{\sup_{0\le r\le t}\|\partial_x\psi_r^\epsilon\|_0^4\Big\}<\infty$ for all $t\ge 0$. Then we obtain (4.13) by Gronwall's inequality. ∎

Theorem 4.1 *For any $\phi\in H_1(\mathbb{R})\cap C_b(\mathbb{R})^+$, equation (4.1) has a pathwise unique $H_0(\mathbb{R})^+$-valued solution $\{\psi_t:t\ge 0\}$. We have a.s. $\|\psi_t\|\le e^{-b_0t}\|\phi\|$ for all $t\ge 0$, where $b_0=\inf_x b(x)$. Moreover, there is a locally bounded function $K(\cdot)$ on $[0,\infty)$ such that*

$$\mathbf{E}\Big\{\sup_{0\le r\le t}\|\psi_r\|_1^4\Big\}\le K(t), \tag{4.15}$$

and so $\{\psi_t(\cdot):t\ge 0\}$ has an $H_1(\mathbb{R})\cap C_b(\mathbb{R})^+$-valued version.

Proof. Let $z_t(x)=\psi_t^\epsilon(x)-\psi_t^\eta(x)$. As for (4.12), by the same arguments leading to (2.12) of [30] we have

$$\begin{aligned}\mathbf{E}\Big\{\sup_{0\le s\le t}\|z_s\|_0^4\Big\} &\le K\int_0^t\mathbf{E}\{\|z_r\|_0^4\}dr+3\|\phi\|^4\mathbf{E}\Big\{\int_0^t\Big(\int|T_\epsilon\psi_r^\eta(x)-T_\eta\psi_r^\eta(x)|^2dx\Big)^2dr\Big\}\\ &\quad+K\mathbf{E}\Big\{\int_0^t|d_\epsilon(\psi_r^\epsilon)-d_\eta(\psi_r^\eta)|^4dr\Big\}\\ &\quad+K\mathbf{E}\Big\{\sum_{j=[1/\eta]+1}^{[1/\epsilon]}\int_0^t\Big(\int_{\mathbb{R}}\langle h(y-\cdot)\partial_x\psi_s^\eta,z_s\rangle\,h_j(y)dy\Big)^2ds\Big\}.\end{aligned} \tag{4.16}$$

As in Section 2.4 of [30], the second and third terms on the right hand side of (4.16) converge to 0 as $\epsilon,\ \eta\to 0$. On the other hand, the last term is bounded by

$$\int_0^t\int_{\mathbb{R}}\sum_{j=[1/\eta]+1}^{[1/\epsilon]}\Big(\int_{\mathbb{R}}h_j(y)h(y-x)dy\Big)^2\mathbf{E}\{z_s(x)^2\}dx\int_{\mathbb{R}}\mathbf{E}\{(\partial_x\psi_s^\eta)^2\}dxds,$$

which tends to zero as $\epsilon,\eta\to 0$. As in Section 2.4 of [30] we can show that ψ^ϵ is a Cauchy sequence in $H_0(\mathbb{R})$ and its limit ψ is the pathwise unique solution of (4.1). The second assertion follows from Lemma 4.1 and Fatou's lemma. Finally, we obtain (4.15) by Lemma 4.3 and Sobolev's result. ∎

Based on Theorem 4.1, let us consider the following more useful backward SPDE:

$$\begin{aligned}\psi_{r,t}(x) &= \phi(x)+\int_r^t\Big[\frac{1}{2}a(x)\partial_x^2\psi_{s,t}(x)-b(x)\psi_{s,t}(x)-\frac{1}{2}\sigma(x)\psi_{s,t}(x)^2\Big]ds\\ &\quad+\int_r^t\int_{\mathbb{R}}h(y-x)\partial_x\psi_{s,t}(x)\cdot W(ds,dy),\qquad t\geq r\geq 0,\end{aligned}\tag{4.17}$$

where "$\cdot$" denotes the backward stochastic integral.

Theorem 4.2 *For any $\phi\in H_1(\mathbb{R})\cap C_b(\mathbb{R})^+$, the backward equation (4.17) has a pathwise unique $H_1(\mathbb{R})\cap C_b(\mathbb{R})^+$-valued solution $\{\psi_{r,t}:t\geq r\geq 0\}$. Further, we have a.s. $\|\psi_{r,t}\|\leq e^{-b_0(t-r)}\|\phi\|$ for all $t\geq r\geq 0$.*

Proof. For fixed $t>0$, define the white noise

$$W_t([0,s]\times B)=-W([t-s,t]\times B),\qquad 0\leq s\leq t, B\in\mathcal{B}(\mathbb{R}).\tag{4.18}$$

By Theorem 4.1, there is a pathwise unique solution $\{\phi_{r,t}:0\leq r\leq t\}$ of the equation

$$\begin{aligned}\phi_{r,t}(x) &= \phi(x)+\int_0^r\Big[\frac{1}{2}a(x)\partial_x^2\phi_{s,t}(x)-b(x)\phi_{s,t}(x)-\frac{1}{2}\sigma(x)\phi_{s,t}(x)^2\Big]ds\\ &\quad+\int_0^r\int_{\mathbb{R}}h(y-x)\partial_x\phi_{s,t}(x)W_t(ds,dy).\end{aligned}\tag{4.19}$$

Setting $\psi_{r,t}(x):=\phi_{t-r,t}(x)$, we have

$$\begin{aligned}\psi_{r,t}(x) &= \phi(x)+\int_0^{t-r}\Big[\frac{1}{2}a(x)\partial_x^2\psi_{t-s,t}(x)-b(x)\psi_{t-s,t}(x)-\frac{1}{2}\sigma(x)\psi_{t-s,t}(x)^2\Big]ds\\ &\quad+\int_0^{t-r}\int_{\mathbb{R}}h(y-x)\partial_x\psi_{t-s,t}(x)W_t(ds,dy)\\ &= \phi(x)+\int_r^t\Big[\frac{1}{2}a(x)\partial_x^2\psi_{s,t}(x)-b(x)\psi_{s,t}(x)-\frac{1}{2}\sigma(x)\psi_{s,t}(x)^2\Big]ds\\ &\quad+\int_r^t\int_{\mathbb{R}}h(y-x)\partial_x\psi_{s,t}(x)\cdot W(ds,dy).\end{aligned}$$

That is, $\{\psi_{r,t}:t\geq r\geq 0\}$ solves (4.17). The remaining assertions are immediate by Theorem 4.1. ∎

We may regard the white noise $\{W(ds,dy)\}$ as a random variable taking values in the Schwartz apace $\mathcal{S}'([0,\infty)\times\mathbb{R})$. As in the classical situation of [13, p.163], the result of Theorem 4.2 implies the existence of a measurable mapping $F:(\phi,\nu)\mapsto\psi_{r,t}^\nu(\phi,\cdot)$ from $(H_1(\mathbb{R})\cap C_b(\mathbb{R})^+)\times\mathcal{S}'([0,\infty)\times\mathbb{R})$ to $H_1(\mathbb{R})\cap C_b(\mathbb{R})^+$ such that $\psi_{r,t}^W(\phi,\cdot)$ is the pathwise unique solution of (4.17).

5 Conditional log-Laplace functionals

Let (c, h, σ, b, m) be given as in the introduction. Let $\{X_t : t \geq 0\}$ be a continuous solution of the SPDE:

$$\begin{aligned}\langle \phi, X_t \rangle &= \langle \phi, \mu \rangle + t\langle \phi, m \rangle + \frac{1}{2}\int_0^t \langle a\phi'', X_s \rangle ds - \int_0^t \langle b\phi, X_s \rangle ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \phi(y) Z(ds, dy) + \int_0^t \int_{\mathbb{R}} \langle h(y - \cdot)\phi', X_s \rangle W(ds, dy), \qquad (5.1)\end{aligned}$$

where $W(ds, dx)$ is a time-space white noise and $Z(ds, dy)$ is an orthogonal martingale measure which is orthogonal to $W(ds, dy)$ and has covariation measure $\sigma(y)X_s(dy)ds$. Let $(\mathcal{F}_t)_{t\geq 0}$ denote the filtration generated by $\{W(ds, dy)\}$ and $\{Z(ds, dy)\}$. Since σ is strictly positive, the process $\{X_t : t \geq 0\}$ can be represented in terms of the covariation measure of $Z(ds, dy)$, so it is adapted to the $(\mathcal{F}_t)_{t\geq 0}$. By Theorem 4.2, for $\phi \in H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$ the equation

$$\begin{aligned}\psi_{r,t}(x) &= \phi(x) + \int_r^t \Big[\frac{1}{2}a(x)\psi''_{s,t}(x) - b(x)\psi_{s,t}(x) - \frac{1}{2}\sigma(x)\psi_{s,t}(x)^2\Big] ds \\ &\quad + \int_r^t \int_{\mathbb{R}} h(y - x)\psi'_{s,t}(x) \cdot W(ds, dy), \qquad t \geq r \geq 0, \qquad (5.2)\end{aligned}$$

has a pathwise unique solution $\psi_{r,t} = \psi^W_{r,t}$ in $H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$. Let $\mathbf{P}^W$ and $\mathbf{E}^W$ denote respectively the conditional probability and expectation given the white noise $\{W(ds, dy)\}$. The main result of this section is the following

Theorem 5.1 *For $t \geq r \geq 0$ and $\phi \in H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$ we have a.s.*

$$\mathbf{E}^W\{e^{-\langle \phi, X_t \rangle} | \mathcal{F}_r\} = \exp\Big\{ - \langle \psi^W_{r,t}, X_r \rangle - \int_r^t \langle \psi^W_{s,t}, m \rangle ds \Big\}, \qquad (5.3)$$

where $\psi^W_{r,t}$ is defined by (5.2). Consequently, $\{X_t : t \geq 0\}$ is a diffusion process with Feller transition semigroup $(Q_t)_{t\geq 0}$ given by

$$\int_{M(\mathbb{R})} e^{-\langle \phi, \nu \rangle} Q_t(\mu, d\nu) = \mathbf{E} \exp\Big\{ - \langle \psi^W_{0,t}, \mu \rangle - \int_0^t \langle \psi^W_{s,t}, m \rangle ds \Big\}. \qquad (5.4)$$

Our proof of the theorem are based on direct calculations derived from (5.1) and (5.2). The argument is different from that of [30], where the Wong-Zakai approximation was used to get the result. We shall give four lemmas which together with the proof of the theorem show clearly the key steps of the calculations.

Suppose that α and β are bounded measurable functions on $[0, \infty) \times \mathbb{R}$ and that

$$\int_0^t \int_{\mathbb{R}} \alpha(s, y)^2 ds dy < \infty.$$

For $t \geq r \geq 0$, define

$$\theta_\alpha(r, t) = \exp\Big\{ \int_r^t \int_{\mathbb{R}} \alpha(s, y) W(ds, dy) - \frac{1}{2}\int_r^t \int_{\mathbb{R}} \alpha(s, y)^2 ds dy \Big\}, \qquad (5.5)$$

and

$$\zeta_\beta(r, t) = \exp\Big\{ \int_r^t \int_{\mathbb{R}} \beta(s, y) Z(ds, dy) - \frac{1}{2}\int_r^t \langle \sigma\beta(s, \cdot)^2, X_s \rangle ds \Big\}. \qquad (5.6)$$

Indeed, we have the following

Lemma 5.1 *Under the conditional probability measure $\mathbf{P}^W$, the process $\{\zeta_\beta(0,t) : t \geq 0\}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.*

Proof. Clearly, both $\{\theta_\alpha(0,t) : t \geq 0\}$ and $\{\zeta_\beta(0,t) : t \geq 0\}$ are martingales on $(\Omega, \mathcal{F}, \mathbf{P})$. Recall that the martingale measures $\{W(ds,dy)\}$ and $\{Z(ds,dy)\}$ are orthogonal. By integration by parts it is easy to see that $\{\theta_\alpha(0,t)\zeta_\beta(0,t) : t \geq 0\}$ is a martingale. Since α is arbitrary, for any $u \geq t \geq r \geq 0$ and any bounded $\mathcal{F}_r$-measurable random variable Z we obtain

$$\mathbf{E}\{\theta_\alpha(0,u)\zeta_\beta(0,t)Z\} = \mathbf{E}\{\theta_\alpha(0,r)\zeta_\beta(0,r)Z\} = \mathbf{E}\{\theta_\alpha(0,u)\zeta_\beta(0,r)Z\}.$$

Note that the linear span of the functionals $\{\theta_\alpha(0,u)\}$ is dense in the space of squared-integrable and $\sigma(W)$-measurable random variables; see e.g. [3, p.81] and [5]. Then we have the desired equality $\mathbf{E}^W\{\zeta_\beta(0,t)|\mathcal{F}_r\} = \zeta_\beta(0,r)$. ∎

By the property of independent increments of the white noise $\{W(ds,dy)\}$ we have

$$\xi_{r,t}(x) := \mathbf{E}\{\psi_{r,t}(x)\theta_\alpha(r,t)\} = \mathbf{E}\{\psi_{r,t}(x)\theta_\alpha(r,t)|\mathcal{F}_r\} \tag{5.7}$$

and

$$\eta_{r,t}(x) := \mathbf{E}\{\psi_{r,t}(x)^2\theta_\alpha(r,t)\} = \mathbf{E}\{\psi_{r,t}(x)^2\theta_\alpha(r,t)|\mathcal{F}_r\}. \tag{5.8}$$

Lemma 5.2 *For $t \geq r \geq 0$, we have a.s.*

$$\mathbf{E}\{\langle\psi_{r,t}, X_r\rangle\theta_\alpha(0,t)\zeta_\beta(0,t)|\mathcal{F}_r\} = \langle\xi_{r,t}, X_r\rangle\theta_\alpha(0,r)\zeta_\beta(0,r) \tag{5.9}$$

and

$$\mathbf{E}\{\langle\sigma\psi_{r,t}^2, X_r\rangle\theta_\alpha(0,t)\zeta_\beta(0,t)|\mathcal{F}_r\} = \langle\sigma\eta_{r,t}, X_r\rangle\theta_\alpha(0,r)\zeta_\beta(0,r). \tag{5.10}$$

Proof. By Lemma 5.1 it is easy to see that $\mathbf{E}^W[\zeta_\beta(r,t)|\mathcal{F}_r] = 1$. Since $\theta_\alpha(0,r)\zeta_\beta(0,r)$ is $\mathcal{F}_r$-measurable and $\langle\psi_{r,t}, X_r\rangle\theta_\alpha(r,t)$ is $\sigma(W, \mathcal{F}_r)$-measurable, we have

$$\begin{aligned}\mathbf{E}\{\langle\psi_{r,t}, X_r\rangle\theta_\alpha(0,t)\zeta_\beta(0,t)|\mathcal{F}_r\} &= \mathbf{E}\{\langle\psi_{r,t}, X_r\rangle\theta_\alpha(r,t)\zeta_\beta(r,t)|\mathcal{F}_r\}\theta_\alpha(0,r)\zeta_\beta(0,r)\\ &= \mathbf{E}\{\langle\psi_{r,t}, X_r\rangle\theta_\alpha(r,t)\mathbf{E}^W[\zeta_\beta(r,t)|\mathcal{F}_r]|\mathcal{F}_r\}\theta_\alpha(0,r)\zeta_\beta(0,r)\\ &= \mathbf{E}\{\langle\psi_{r,t}, X_r\rangle\theta_\alpha(r,t)|\mathcal{F}_r\}\theta_\alpha(0,r)\zeta_\beta(0,r)\\ &= \langle\xi_{r,t}, X_r\rangle\theta_\alpha(0,r)\zeta_\beta(0,r).\end{aligned}$$

A similar calculation gives (5.10). ∎

Lemma 5.3 *For $t \geq r \geq 0$ and $x \in \mathbb{R}$, we have a.s.*

$$\begin{aligned}\xi_{r,t}(x) - \phi(x) &= \int_r^t \left[\frac{1}{2}a(x)\xi_{s,t}''(x) - b(x)\xi_{s,t}(x) - \frac{1}{2}\sigma(x)\eta_{s,t}(x)\right]ds\\ &\quad + \int_r^t \langle h(\cdot - x), \alpha(s,\cdot)\rangle\xi_{s,t}'(x)ds,\end{aligned} \tag{5.11}$$

where the derivatives are taken in the classical sense.

Proof. Note that the backward and forward integrals coincide for deterministic integrands. Then we may fix $t > 0$ and apply Itô's formula to the process $\{\theta_\alpha(r,t) : r \in [0,t]\}$ to get

$$\theta_\alpha(r,t) = 1 + \int_r^t \int_{\mathbb{R}} \theta_\alpha(s,t)\alpha(s,y) \cdot W(ds,dy). \tag{5.12}$$

By (5.2), (5.12) and backward Itô formula, for any $f \in C_b^\infty(\mathbb{R})$ we have

$$\begin{aligned}
\langle \psi_{r,t}, f\rangle \theta_\alpha(r,t) &= \langle \phi, f\rangle + \int_r^t \left[\frac{1}{2}\langle a\psi''_{s,t}, f\rangle - \langle b\psi_{s,t}, f\rangle - \frac{1}{2}\langle \sigma\psi^2_{s,t}, f\rangle\right]\theta_\alpha(s,t)ds \\
&\quad + \int_r^t \int_{\mathbb{R}} [\langle h(y-\cdot)\psi'_{s,t}, f\rangle - \langle \psi_{s,t}, f\rangle \alpha(s,y)]\theta_\alpha(s,t) \cdot W(ds,dy) \\
&\quad + \int_r^t \int_{\mathbb{R}} \langle h(y-\cdot)\psi'_{s,t}, f\rangle \theta_\alpha(s,t)\alpha(s,y)dsdy. \qquad (5.13)
\end{aligned}$$

(See e.g. [3, p.124] for the backward Itô formula.) Observe that for fixed $t > 0$, the process

$$\int_r^t \int_{\mathbb{R}} [\langle h(y-\cdot)\psi'_{s,t}, f\rangle - \langle \psi_{s,t}, f\rangle \alpha(s,y)]\theta_\alpha(s,t) \cdot W(ds,dy)$$

is a backward martingale in $r \leq t$. Taking the expectation in (5.13) we obtain

$$\begin{aligned}
\langle \xi_{r,t}, f\rangle - \langle \phi, f\rangle &= \int_r^t \left[\frac{1}{2}\langle a\xi''_{s,t}, f\rangle - \langle b\xi_{s,t}, f\rangle - \frac{1}{2}\langle \sigma\eta_{s,t}, f\rangle\right] ds \\
&\quad + \int_r^t \int_{\mathbb{R}} \langle h(y-\cdot)\xi'_{s,t}, f\rangle a(s,y)dsdy.
\end{aligned}$$

Then $\{\xi_{r,t}\}$ must coincides with the classical solution of the parabolic equation (5.11). ∎

Lemma 5.4 *For any $t \geq r \geq 0$, we have a.s.*

$$\langle \phi, X_t\rangle = \langle \psi_{r,t}, X_r\rangle + \int_r^t \int_{\mathbb{R}} \psi_{s,t}(x)Z(ds,dx) + \frac{1}{2}\int_r^t \langle \sigma\psi^2_{s,t}, X_s\rangle ds + \int_r^t \langle \psi_{s,t}, m\rangle ds. \tag{5.14}$$

Proof. In view of (5.1) and (5.11), we may integrate $\xi_{s,t}$ backward relative to X_s to see that

$$\begin{aligned}
d\langle \xi_{s,t}, X_s\rangle &= \frac{1}{2}\langle \sigma\eta_{s,t}, X_s\rangle ds - \int_{\mathbb{R}} \langle h(y-\cdot)\xi'_{s,t}, X_s\rangle \alpha(s,y)dsdy + \langle \xi_{s,t}, m\rangle ds \\
&\quad + \int_{\mathbb{R}} \xi_{s,t}(y)Z(ds,dy) + \int_{\mathbb{R}} \langle h(y-\cdot)\xi'_{s,t}, X_s\rangle W(ds,dy),
\end{aligned}$$

where the first two terms from (5.11) cancelled out with the second and third terms from (5.1). Since the two martingale measures $\{W(ds,dy)\}$ and $\{Z(ds,dy)\}$ are orthogonal, by Itô's formula we have

$$\begin{aligned}
d\langle \xi_{s,t}, X_s\rangle \theta_\alpha(0,s)\zeta_\beta(0,s) &= \frac{1}{2}\langle \sigma\eta_{s,t}, X_s\rangle \theta_\alpha(0,s)\zeta_\beta(0,s)ds + \langle \xi_{s,t}, m\rangle \theta_\alpha(0,s)\zeta_\beta(0,s)ds \\
&\quad + \int_{\mathbb{R}} \langle h(y-\cdot)\xi'_{s,t}, X_s\rangle \theta_\alpha(0,s)\zeta_\beta(0,s)W(ds,dy) \\
&\quad + \int_{\mathbb{R}} \xi_{s,t}(y)\theta_\alpha(0,s)\zeta_\beta(0,s)Z(ds,dy)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}}\langle\xi_{s,t},X_s\rangle\theta_\alpha(0,s)\zeta_\beta(0,s)\alpha(s,x)W(ds,dy)\\
&+\int_{\mathbb{R}}\langle\xi_{s,t},X_s\rangle\theta_\alpha(0,s)\zeta_\beta(0,s)\beta(s,y)Z(ds,dy)\\
&+\langle\sigma\xi_{s,t}\beta(s,\cdot),X_s\rangle\theta_\alpha(0,s)\zeta_\beta(0,s)ds.
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
&\mathbf{E}\{\langle\phi,X_t\rangle\theta_\alpha(0,t)\zeta_\beta(0,t)\}-\mathbf{E}\{\langle\xi_{r,t},X_r\rangle\theta_\alpha(0,r)\zeta_\beta(0,r)\}\\
=\;&\frac{1}{2}\mathbf{E}\Big\{\int_r^t\langle\sigma\eta_{s,t},X_s\rangle\theta_\alpha(0,s)\zeta_\beta(0,s)ds\Big\}+\mathbf{E}\Big\{\int_r^t\langle\xi_{s,t},m\rangle\theta_\alpha(0,s)\zeta_\beta(0,s)ds\Big\}\\
&+\mathbf{E}\Big\{\int_r^t\langle\sigma\xi_{s,t}\beta(s,\cdot),X_s\rangle\theta_\alpha(0,s)\zeta_\beta(0,s)ds\Big\}.
\end{aligned}
\tag{5.15}
$$

¿From (5.6) it is easy to see that

$$
\zeta_\beta(0,t)=1+\int_0^t\int_{\mathbb{R}}\zeta_\beta(0,s)\beta(s,y)Z(ds,dy),
$$

and hence

$$
\begin{aligned}
&\mathbf{E}\Big\{\int_r^t\int_{\mathbb{R}}\psi_{s,t}(y)Z(ds,dy)\theta_\alpha(0,t)\zeta_\beta(0,t)\Big\}\\
=\;&\mathbf{E}\Big\{\mathbf{E}^W\Big[\int_r^t\int_{\mathbb{R}}\psi_{s,t}(y)Z(ds,dy)\zeta_\beta(0,t)\Big]\theta_\alpha(0,t)\Big\}\\
=\;&\mathbf{E}\Big\{\mathbf{E}^W\Big[\int_r^t\langle\sigma\psi_{s,t}\beta(s,\cdot),X_s\rangle\zeta_\beta(0,s)ds\Big]\theta_\alpha(0,t)\Big\}\\
=\;&\int_r^t\mathbf{E}\Big[\langle\sigma\psi_{s,t}\beta(s,\cdot),X_s\rangle\theta_\alpha(0,t)\zeta_\beta(0,s)\Big]ds\\
=\;&\int_r^t\mathbf{E}\Big[\langle\sigma\xi_{s,t}\beta(s,\cdot),X_s\rangle\theta_\alpha(0,s)\zeta_\beta(0,s)\Big]ds.
\end{aligned}
\tag{5.16}
$$

By a calculation similar to the proof of Lemma 5.2 we get

$$
\mathbf{E}\{\langle\psi_{s,t},m\rangle\theta_\alpha(0,t)\zeta_\beta(0,t)|\mathcal{F}_s\}=\langle\xi_{s,t},m\rangle\theta_\alpha(0,s)\zeta_\beta(0,s).
\tag{5.17}
$$

Combining (5.10) and (5.15) – (5.17) gives

$$
\begin{aligned}
&\mathbf{E}\{\langle\phi,X_t\rangle\theta_\alpha(0,t)\zeta_\beta(0,t)\}-\mathbf{E}\{\langle\xi_{r,t},X_r\rangle\theta_\alpha(0,r)\zeta_\beta(0,r)\}\\
=\;&\frac{1}{2}\mathbf{E}\Big\{\int_r^t\langle\sigma\psi_{s,t}^2,X_s\rangle\theta_\alpha(0,t)\zeta_\beta(0,t)ds\Big\}+\mathbf{E}\Big\{\int_r^t\langle\psi_{s,t},m\rangle\theta_\alpha(0,t)\zeta_\beta(0,t)ds\Big\}\\
&+\mathbf{E}\Big\{\int_r^t\int_{\mathbb{R}}\psi_{s,t}(y)Z(ds,dy)\theta_\alpha(0,t)\zeta_\beta(0,t)\Big\}.
\end{aligned}
$$

But by (5.9) we have

$$
\begin{aligned}
&\mathbf{E}\{[\langle\phi,X_t\rangle-\langle\psi_{r,t},X_r\rangle]\theta_\alpha(0,t)\zeta_\beta(0,t)\}\\
=\;&\mathbf{E}\{\langle\phi,X_t\rangle\theta_\alpha(0,t)\zeta_\beta(0,t)\}-\mathbf{E}\{\langle\xi_{r,t},X_r\rangle\theta_\alpha(0,r)\zeta_\beta(0,r)\}
\end{aligned}
$$

It follows that

$$\mathbf{E}\Big\{\Big[\langle\phi, X_t\rangle - \langle\psi_{r,t}, X_r\rangle - \frac{1}{2}\int_r^t \langle\sigma\psi_{s,t}^2, X_s\rangle ds - \int_r^t \langle\psi_{s,t}, m\rangle ds - \int_r^t\int_{\mathbb{R}} \psi_{s,t}(x)Z(ds,dx)\Big]\theta_\alpha(0,t)\zeta_\beta(0,t)\Big\} = 0.$$

Then we have the desired equation; see e.g. [3, p.81] and [5]. ■

Proof of Theorem 5.1. Recall that $Z(ds,dy)$ is an orthogonal martingale measure with covariation measure $\sigma(y)X_s(dy)ds$. By Lemma 5.1, for any fixed $u \geq r$ the process

$$\exp\Big\{-\int_r^t\int_{\mathbb{R}} \psi_{s,u}(y)Z(ds,dy) - \frac{1}{2}\int_r^t \langle\sigma\psi_{s,u}^2, X_s\rangle ds\Big\}, \qquad r \leq t \leq u,$$

is a martingale under $\mathbf{P}^W$. By Lemma 5.4 we get a.s.

$$\begin{aligned} &\mathbf{E}^W\{e^{-\langle\phi,X_t\rangle}|\mathcal{F}_r\} \\ = \;& \mathbf{E}^W\Big[\exp\Big\{-\langle\psi_{r,t}, X_r\rangle - \int_r^t\int_{\mathbb{R}} \psi_{s,t}(y)Z(ds,dy) \\ &\qquad -\frac{1}{2}\int_r^t \langle\sigma\psi_{s,t}^2, X_s\rangle ds - \int_r^t \langle\psi_{s,t}, m\rangle ds\Big\}\Big|\mathcal{F}_r\Big] \\ = \;& \exp\Big\{-\langle\psi_{r,t}, X_r\rangle - \int_r^t \langle\psi_{s,t}, m\rangle ds\Big\}, \end{aligned}$$

giving (5.3). In particular, we have

$$\mathbf{E}\{e^{-\langle\phi,X_t\rangle}\} = \mathbf{E}\exp\Big\{-\langle\psi_{0,t}, \mu\rangle - \int_0^t \langle\psi_{s,t}, m\rangle ds\Big\}. \tag{5.18}$$

The distribution of X_t is uniquely determined by (5.18) and the uniqueness of solution of (5.1) follows. This in turn implies the strong Markov property of $\{X_t : t \geq 0\}$. Since $\psi_{r,t}(x)$ is continuous in $x \in \mathbb{R}$, the transition semigroup $(Q_t)_{t\geq 0}$ defined by (5.4) is Feller. ■

6 Some properties of the SDSMI

We here investigate some properties of the SDSMI. Let (c, h, σ, b, m) be given as in the introduction. As in the last section, let $\mathbf{P}^W$ and $\mathbf{E}^W$ denote respectively the conditional probability and expectation given the white noise $\{W(ds,dy)\}$. The equality (5.3) shows that $\{X_t : t \geq 0\}$ under $\mathbf{P}^W$ is a Markov process with transition semigroup $(Q_{r,t}^W)_{t\geq r}$ satisfying a.s.

$$\int_{M(E)} e^{-\langle\phi,\nu\rangle} Q_{r,t}^W(X_r, d\nu) = \exp\Big\{-\langle\psi_{r,t}^W, X_r\rangle - \int_r^t \langle\psi_{s,t}^W, m\rangle ds\Big\}. \tag{6.1}$$

In other words, the SDSMI conditioned upon $\{W(ds,dy)\}$ should be an inhomogeneous immigration superprocess. This observation suggests a number of applications of the conditional log-Laplace functional. For instance, based on the results in the last section, the conditional excursion theory of the SDSM have been developed in [19]. Moreover, some moment formulas can be also derived from (5.3) in a similar way as [30].

As another application of the conditional Laplace functionals, we prove the following ergodicity property of the SDSMI.

Theorem 6.1 *Suppose that there is a constant $\epsilon > 0$ such that $b(x) \geq \epsilon$ for all $x \in \mathbb{R}$. Then the SDSMI has a unique stationary distribution Q_∞ given by*

$$\int_{M(\mathbb{R})} e^{-\langle\phi,\nu\rangle} Q_\infty(d\nu) = \mathbf{E}\exp\bigg\{-\int_0^\infty \langle\psi_t^W, m\rangle dt\bigg\}, \tag{6.2}$$

where $\psi_t^W(x)$ is the solution of (4.1). Moreover, we have $\lim_{t\to\infty} Q_t(\mu,\cdot) = Q_\infty(\cdot)$ in the topology of weak convergence for each $\mu \in M(\mathbb{R})$.

Proof. Using the notation of the proof of Theorem 4.2, for any $t \geq r \geq 0$ we have

$$\begin{aligned}
\mathbf{E}\bigg\{\int_{M(\mathbb{R})} e^{-\langle\phi,\nu\rangle} N_{r,t}^W(d\nu)\bigg\} &= \mathbf{E}\exp\bigg\{-\int_r^t \langle\psi_{s,t}^W, m\rangle ds\bigg\} \\
&= \mathbf{E}\exp\bigg\{-\int_r^t \langle\phi_{t-s,t}^W, m\rangle ds\bigg\} \\
&= \mathbf{E}\exp\bigg\{-\int_0^{t-r} \langle\phi_{s,t}^W, m\rangle ds\bigg\} \\
&= \mathbf{E}\exp\bigg\{-\int_0^{t-r} \langle\psi_s^W, m\rangle ds\bigg\}.
\end{aligned}$$

By Theorem 4.2 we have $\|\psi_{s,t}^W\| \leq e^{-\epsilon(t-s)}\|\phi\|$ for $s \leq t$. It follows that

$$\begin{aligned}
\lim_{t\to\infty}\int_{M(\mathbb{R})} e^{-\langle\phi,\nu\rangle} Q_t(\mu, d\nu) &= \lim_{t\to\infty}\mathbf{E}\exp\bigg\{-\langle\psi_{0,t}^W, \mu\rangle - \int_0^t \langle\psi_{s,t}^W, m\rangle ds\bigg\} \\
&= \lim_{t\to\infty}\mathbf{E}\exp\bigg\{-\int_0^t \langle\psi_{s,t}^W, m\rangle ds\bigg\} \\
&= \mathbf{E}\exp\bigg\{-\int_0^\infty \langle\psi_s^W, m\rangle ds\bigg\}.
\end{aligned}$$

On the other hand, by Theorem 4.1 it is easy to get

$$\lim_{\|\phi\|\to 0}\mathbf{E}\exp\bigg\{-\int_0^\infty \langle\psi_s^W, m\rangle ds\bigg\} = 1.$$

Then (6.2) defines a probability measure Q_∞ on $M(\mathbb{R})$ and $\lim_{t\to\infty} Q_t(\mu,\cdot) = Q_\infty(\cdot)$ in the topology of weak convergence; see e.g. [16, Lemma 2.1]. ∎

The properties of the SDSMI varies sharply for different choices of the parameters. The special case where $b(\cdot) \equiv 0$ and $\langle 1, m\rangle = 0$ was discussed in [7, 8, 9, 27, 29]. In this case, we have

$$\begin{aligned}
\langle\phi, X_t\rangle &= \langle\phi,\mu\rangle + \frac{1}{2}\int_0^t \langle a\phi'', X_s\rangle ds + \int_0^t\int_{\mathbb{R}} \phi(y) Z(ds,dy) \\
&\quad + \int_0^t\int_{\mathbb{R}} \langle h(y-\cdot)\phi', X_s\rangle W(ds,dy).
\end{aligned} \tag{6.3}$$

The solution of (6.3) is a critical branching SDSM without immigration. In particular, if $c(\cdot)$ is bounded away from zero, then $\{X_t : t > 0\}$ is absolutely continuous for any initial state X_0; see [8, 9, 27]. On the other hand, if $c(\cdot) \equiv 0$, then $\{X_t : t > 0\}$ is purely atomic for any initial state X_0; see [7, 27, 29].

Another special case is where $\sigma(\cdot) \equiv 0$ and $\langle 1, m\rangle = 0$. In this case, we get from (6.3) the linear equation

$$\langle \phi, X_t\rangle = \langle \phi, \mu\rangle + \frac{1}{2}\int_0^t \langle a\phi'', X_s\rangle ds - \int_0^t \langle b\phi, X_s\rangle ds + \int_0^t \int_{\mathbb{R}} \langle h(y-\cdot)\phi', X_s\rangle W(ds, dy). \quad (6.4)$$

The process defined in this way is closely related to the superprocesses arising from isotropic stochastic flows investigated by [20]. The following theorem shows that $\{X_t : t \geq 0\}$ is absolutely continuous for a large class of absolutely continuous initial states.

Theorem 6.2 *If $\{X_t : t \geq 0\}$ is a solution of (6.4) with $X_0(dx) = v_0(x)dx$ for some $v_0 \in H_0(\mathbb{R})$, then there is an $H_0(\mathbb{R})$-valued process $\{v_t : t \geq 0\}$ such that $X_t(dx) = v_t(x)dx$ a.s. holds.*

Proof. By [15, Theorem 3.5], the equation

$$v_t(x) = v_0(x) + \int_0^t \left[\frac{1}{2}(av_s)''(x) - b(x)v_s(x)\right] ds - \int_0^t \int_{\mathbb{R}} (h(y-\cdot)v_s)'(x) W(ds, dy) \quad (6.5)$$

has a unique $H_0(\mathbb{R})$-valued solution $\{v_t : t \geq 0\}$. Let $X_t(dx) = v_t(x)dx$. Clearly, $\{X_t : t \geq 0\}$ solves (6.4). ∎

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