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# Valuation of a Barrier European Option on Jump-Diffusion Underlying Stock Price

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*Dedicated to Professor Donald A. Dawson*

ABSTRACT. In this paper, a formula for the fair price of a barrier European option on jump-diffusion underlying stock price is given. The model for the discontinuous underlying stock price is assumed to be the same as that introduced by Merton [8] (1976). A dynamic programming method is used in the derivation of the fair price formula.

## 1. Introduction

In this paper, based on the same model and assumptions as in Merton [8] on jump-diffusion stock price, an analytic option pricing formula for an European barrier option is derived. It is difficult to derive an analytic formula for this option because the distribution of the lifetime maximum of the underlying jump-diffusion stock price is not available. To overcome this, a dynamic programming method for the conditional expectations of Markov processes is introduced.

In his classic paper on option pricing on discontinuous underlying stock price, Merton [8] introduced a model for jump-diffusion stock price and presented an analytic pricing formula for an European option on the jump-diffusion stock price. This formula has no closed-form solution and is very complicated. However, in recent years, there have been considerable number of papers on discontinuous price models (e.g. Zhou [12], [11], Cox-Ross [3], Ahn-Thompson [1], Amin [2], Das-Tufano [4], Duffie-Singleton [5], Jarrow-Turnbull [7], Jarrow-Lando-Turnbull [6], Page-Sander [10], $\cdots$). Many of them showed by market data that in some situations the diffusion approach produces very disappointing results, while the jump-diffusion approach gives results which are consistent with many empirical facts.

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## 2. Stock price dynamic and preliminary

The dynamic of the underlying stock price is assumed to be driven by the following stochastic differential equation

$$dS_t = (\alpha - \lambda k)S_t dt + \sigma S_t dW_t + (Y_t - 1)S_t dq_t. \tag{2.1}$$

For above equation, we have following assumptions.

(1) $k, \alpha, \lambda, \sigma$ are positive fixed numbers.
(2) For any different $t, s > 0$, $Y_t$ and $Y_s$ are independent. $Y_t > 0$ is a random variable with distribution $F$ and expectation $k+1$.
(3) $\{q_t\}$ is a Poisson process with intensity $\lambda$.
(4) $\{W_t\}$ is a standard Brownian motion.
(5) Here $\{W_t\}$,$\{q_t\}$, and $\{Y_t\}$ are assumed to be independent.

Then, an explicit solution to the above equation is

$$S_t = S_0 Y(n_t) \exp\{(\alpha - \frac{\sigma^2}{2} - \lambda k)t + \sigma W_t\}, \tag{2.2}$$

where $n_t$ is the number of jumps of the Poisson process until time $t$ and $Y(n_t)$ is defined by

$$Y(n_t) \equiv \prod_{i=1}^{n_t} Y_i,$$

where $\{Y_i\}$ are i.i.d random variables with distribution $F$.

To derive the fair pricing formula for the barrier European option on the jump-diffusion stock price, we have to find distributions of certain functionals of a Brownian motion. First, let us cite following basic result.

LEMMA 2.1. *Let $\{B_t\}$ be a standard Brownian motion. Define $M(T) \equiv \max_{0\le t\le T} B_t$. Then*

$$\mathbb{P}\{M(T) \in dm, B(T) \in db\} = \frac{2(2m-b)}{T\sqrt{2\pi T}} \exp\left\{-\frac{(2m-b)^2}{2T}\right\} dmdb \quad \text{for } m > 0,\ b < m. \tag{2.3}$$

PROOF. The result follows from the reflection principle of Brownian motion. □

For a more general process – namely, a Brownian motion with non-zero drift. Consider $X_t = \sigma B_t + \nu t$, where $\{B_t\}$ is a standard Brownian motion, $\sigma > 0$, $\nu$ are real numbers. Denote $M^X(T) \equiv \max_{0\le t\le T} X_t$ and $m^X(T) \equiv \min_{0\le t\le T} X_t$. By virtue of Girsanov's theorem, we have the following lemma.

LEMMA 2.2. *For every $t > 0$, the joint distribution of $X_t$ and $M^X(t)$ is given by the formula*

$$\mathbb{P}\{M^X(t) \in dy, X_t \in dx\} = f_t(x, y, \sigma, \nu)dxdy, \tag{2.4}$$

*where*

$$f_t(x,y,\sigma,\nu) \equiv \left[-\frac{2(x-2y-\nu t)}{\sigma^3 t\sqrt{2\pi t}} - \frac{2\nu}{\sigma^3\sqrt{2\pi t}}\right] \exp\left\{2\nu y\sigma^{-2} - \frac{(2y-x+\nu t)^2}{2t\sigma^2}\right\} \quad \text{for } y > 0,\ x < y.$$

*The joint distribution of $X_t$ and $m^X(t)$ is given by the formula*

$$\mathbb{P}\{m^X(t) \in dy, X_t \in dx\} = \tilde{f}_t(x, y, \sigma, \nu)dxdy, \tag{2.5}$$

*where*

$$\tilde{f}_t(x,y,\sigma,\nu) \equiv \left[\frac{2(x-2y-\nu t)}{\sigma^3 t\sqrt{2\pi t}} + \frac{2\nu}{\sigma^3\sqrt{2\pi t}}\right]\exp\left\{2\nu y\sigma^{-2} - \frac{(2y-x+\nu t)^2}{2t\sigma^2}\right\}$$
$$\textit{for } y \le 0,\ y \le x.$$

PROOF. See the proof of Lemma B.3.2 and Corollary B.3.3 of Musiela and Rutkowski [9]. □

LEMMA 2.3. *Let $\{B_t\}$ be a Brownian motion starting at $B_0 = x \neq 0$. Define $M(T) \equiv \max_{0\le t\le T} B_t$. Then*

$$\mathbb{P}\{M(T) \in dm, B(T) \in db\} = g_T^x(m,b)dmdb, \tag{2.6}$$

*where*

$$g_T^x(m,b) \equiv \begin{cases} \dfrac{2(2m-b-x)}{T\sqrt{2\pi T}}\exp\left\{-\dfrac{(2m-b-x)^2}{2T}\right\} & \textit{for } m > x,\ b < m \\ 0 & \textit{otherwise.} \end{cases} \tag{2.7}$$

PROOF. This can be directly derived from Lemma 2.1. □

Barrier options have payoffs that depend on whether the underlying stock price reaches a barrier, $L$ before expiry $T$. Barrier options can be classified as either knock-out options or knock-in options. Knock-out options are options that become worthless if the stock price reaches a barrier but otherwise have payoffs identical to a standard option. Knock-in options are options that become standard options if the stock price reaches a barrier but otherwise are worthless. A down-and-out call option is a knock-out option that becomes worthless if the stock price falls below the barrier $L$, but otherwise is the same as a call option with payoff $[S_T - K]^+$ at expiry, where $K$ is the strike price. The "down" means that the barrier level $L$ is below the initial stock price. A down-and-out put option has payoff $[K - S_T]^+$ provided the underlying stock price does not fall below the barrier $L$, otherwise it becomes worthless. An up-and-out call is a regular call option that ceases to exist if the stock price reaches a barrier $L$ which is higher than the current stock price. According to the probability complementary relationship, the price of a regular call option equals the value of a down-and-in call plus the value of a down-and-out call.That is

$$c = c_{di} + c_{do}, \tag{2.8}$$

and

$$c = c_{ui} + c_{uo}. \tag{2.9}$$

for up-and-in call and up-and-out call, respectively. Similarly, For put options, we have

$$p = p_{ui} + p_{uo}, \tag{2.10}$$

$$p = p_{di} + p_{do}. \tag{2.11}$$

Since the formulae for regular call or put options are already given by Merton's paper [8], from the above identities (2.8) - (2.11), we only need to give formulae for $c_{uo}$, $c_{do}$, $p_{uo}$, $p_{do}$. The derivation of these formulae follows from Lemma 2.2. Hence it is sufficient to only calculate $c_{uo}$ for this paper.

In the remaining section, we propose our method to derive the desired fair up-and-down call option price formula. This method is similar to Dynamic Programming in Stochastic Optimization and Control Theory, and hence we call it, the dynamic programming method. Roughly speaking, the method is as follows. For a given option maturity time $T$ and the stock price $S_t$ given by (2.2), it is difficult to directly find the joint distribution function of $\max_{0\le t\le T} S_t$ and $W_T$ due to the jump perturbation. However we can divide the time interval $[0,T]$ into $n$ subintervals $[0,\tau_1)$,$[\tau_1,\tau_2)$, ... , $[\tau_n,T]$ if there are $n$ jumps exactly before $T$, where $\tau_i$ denotes the $i^{th}$ jump time. If $W_{\tau_i}$ already known, then from Lemma 2.2 it is not very difficult to find the joint distribution function of $\max_{\tau_i\le t<\tau_{i+1}} S_t$ and $W_{\tau_{i+1}}$. In short, the idea is to transform an evaluation of an expectation into an evaluation of a sequence of one-period conditional expectations, working forward in time in a recursive manner.

Another concept we need to introduce is the risk-neutral measure or martingale measure. That is, a risk-neutral measure (martingale measure) is any probability measure, equivalent to the market measure, which makes all discounted stock prices martingales.

To simplify our notation, we assume that $\mathbb{P}$ is the risk-neutral measure. Then

$$S_t = S_0 Y(n_t)\exp\left\{(r-\frac{\sigma^2}{2}-\lambda k)t+\sigma W_t\right\}, \tag{2.12}$$

where $r$ is the risk-free rate and $\{W_t\}$ is a standard Brownian motion. Since $\{W_t\}$,$\{q_t\}$, and $\{Y_t\}$ are assumed to be independent, we can use Girsanov's theorem again to define

$$\tilde{\mathbb{P}}(A) \equiv \int_A Z(T)d\mathbb{P}, \tag{2.13}$$

where $A\in\mathcal{F}_T$, $\{\mathcal{F}_t\}$ is the $\sigma$-field generated by $\{W_s, s\le t\}$,

$$Z(t) = \exp\left\{-\theta W_t - \frac{\theta^2 t}{2}\right\}, \tag{2.14}$$

and $\theta = (r-\lambda k-\sigma^2/2)/\sigma$. Therefore, $\tilde{W}_t = \theta t + W_t$ is a standard Brownian motion under $\tilde{\mathbb{P}}$ and

$$S_t = S_0 Y(n_t)\exp\{\sigma\tilde{W}_t\}. \tag{2.15}$$

## 3. Up-and-out European call

Before deriving the barrier option formula, we should mention again that this paper is based on the same assumptions as that given in Merton's paper [8]. Given $0<K<L$ and assume $S_0<L$, the central aim of this section is to evaluate the following present value of an up-and-out European call option:

$$v(0,S_0) = e^{-rT}\mathbb{E}[(S_T-K)^+ 1_{\{S_T^*<L\}}], \tag{3.1}$$

where $S_t$ is defined by (2.12) and $S_T^* \equiv \max_{0\le t\le T} S_t$. Recall that $\tau_i$ is the $i^{th}$ jump time of the Poisson process with $\tau_0\equiv 0$. By the independence of the Poisson process and the Brownian motion, we have

$$v(0,S_0) = e^{-(r+\lambda)T}\sum_{n\ge 0}\frac{(\lambda T)^n}{n!}\mathbb{E}[(S_T-K)^+ 1_{\{S_T^*<L\}}|\tau_n\le T<\tau_{n+1}]. \tag{3.2}$$

Since $Y_t > 0$, the joint distribution of $(Y(1), Y(2), \cdots, Y(n))$ can be derived by a random vector transformation. Denote its joint distribution by $G(y_1, y_2, \cdots, y_n)$. Define $Y(0) \equiv y_0 \equiv 1$ and

$$R(x,y,z) \equiv (zS_0 \exp\{\sigma x\} - K) \exp\left\{\theta x - \frac{\theta^2 T}{2}\right\} \times 1_{\{y < \frac{\ln(L/(zS_0))}{\sigma}, x > \frac{\ln(K/(zS_0))}{\sigma}\}}, \tag{3.3}$$

$$h(y,z) \equiv 1_{\{y < \frac{\ln(L/(zS_0))}{\sigma}\}}. \tag{3.4}$$

Then, for $n = 0$

$$\mathbb{E}[(S_T - K)^+ 1_{\{S_T^* < L\}} | \tau_n \leq T < \tau_{n+1}] = \tilde{\mathbb{E}}[R(\tilde{W}_T, \max_{t\in[0,T]} \tilde{W}_t, y_n)],$$

and for $n \geq 1$

$$\mathbb{E}[(S_T - K)^+ 1_{\{S_T^* < L\}} | \tau_n \leq T < \tau_{n+1}] = \int_0^T \int_{t_1}^T \cdots \int_{t_{n-1}}^T \int_{\mathbb{R}^n} D dG(y_1, y_2, \cdots, y_n) \lambda^n e^{-\lambda t_n} dt_1 \cdots dt_n, \tag{3.5}$$

where

$$D \equiv \tilde{\mathbb{E}}[R(\tilde{W}_T, \max_{t\in[t_n,T]} \tilde{W}_t, y_n) h(\max_{t\in[t_{n-1},t_n)} \tilde{W}_t, y_{n-1}) \times h(\max_{t\in[t_1,t_2)} \tilde{W}_t, y_1) \cdots h(\max_{t\in[0,t_1)} \tilde{W}_t, 1)]. \tag{3.6}$$

Above $\tilde{\mathbb{E}}$ is the expectation with respect to the probability measure $\tilde{\mathbb{P}}$. Recall that $\{\tilde{W}_t\}$ is a standard Brownian motion under the probability $\tilde{\mathbb{P}}$. Using the Markov property, we have

$$\begin{aligned}
D &= \tilde{\mathbb{E}} \underbrace{\{}_{(0)} h(\max_{0\leq t<t_1} \tilde{W}_t, 1) \tilde{\mathbb{E}}^{\tilde{W}_{t_1}} \underbrace{\{}_{(1)} h(\max_{0\leq t<t_2-t_1} \tilde{W}_t, y_1) \tilde{\mathbb{E}}^{\tilde{W}_{t_2-t_1}} \underbrace{\{}_{(2)} \cdots \\
&\quad \tilde{\mathbb{E}}^{\tilde{W}_{t_{n-1}-t_{n-2}}} \underbrace{\{}_{(n-1)} h(\max_{0\leq t<t_n-t_{n-1}} \tilde{W}_t, y_{n-1}) \\
&\quad \times \tilde{\mathbb{E}}^{\tilde{W}_{t_n-t_{n-1}}} \underbrace{\{}_{(n)} R(\tilde{W}_{T-t_n}, \max_{0\leq t\leq T-t_n} \tilde{W}_t, y_n) \underbrace{\}}_{(n)} \cdots \underbrace{\}}_{(0)} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} h(z_1, y_0) g_{t_1}^0(z_1, x_1) dz_1 dx_1 \int_{\mathbb{R}} \int_{\mathbb{R}} h(z_2, y_1) g_{t_2-t_1}^{x_1}(z_2, x_2) dz_2 dx_2 \cdots \\
&\quad \int_{\mathbb{R}} \int_{\mathbb{R}} h(z_n, y_{n-1}) g_{t_n-t_{n-1}}^{x_{n-1}}(z_n, x_n) dz_n dx_n \times \\
&\quad \int_{\mathbb{R}} \int_{\mathbb{R}} R(x_{n+1}, z_{n+1}, y_n) g_{T-t_n}^{x_n}(z_{n+1}, x_{n+1}) dz_{n+1} dx_{n+1}.
\end{aligned}$$

**Comments:** In spite of concise definitions of the density functions in above pricing formula, it is not easy to evaluate these multiple integrals. Therefore, in the future we need to find a simpler, approximate formula for the barrier option on jump-diffusion underlying stock prices.

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