Tanaka Representation and Joint Continuity of Local Time for a Class of Interacting Branching Measure-valued Diffusions *

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Abrstract

In [28], [27], a class of superprocesses with dependent spatial motion (SDSMs) with starting at a finite measure on $\mathbb{R}$ was introduced and constructed and extended in [6] and [7] to the d-dimensional ($d = 1, 2$ or 3) Euclidean spaces $\mathbb{R}^d$ and starting at some unbounded positive measure on $\mathbb{R}^d$. In our SDSMs model, due to the particle motion dependence and interacting, the powerful log-Laplace functional and evolution equation techniques approach are intractable because the dependence and the interacting particle motion destroy the multiplicative property which is the basic independent feature of Dawson-Watanabe superprocesses. Even more challenges present in our model because the high order singularities involved in the stochastic convolution integral, the higher moment estimation of these stochastic integral terms raise technique difficulties and the missing tools directly leads to the proof of the joint Hölder continuity of the SDSM local time becomes tough and challenging and this problem has been standing open more than 20 years. In the present paper, thanks to the idea of Tanaka representation of the SDSM local time, based on this Representation of the SDSM local times and the dual identity, by the comparison and the equivalence of the fundamental solutions of uniformly parabolic PDEs and the sharp estimation techniques, the joint Hölder continuity in time and space of this class of

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local times follows from the Kolmogorov's continuity criterion when $d \leq 3$ and finally this long standing problem is solved.

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1 Introduction

In our previous paper [7], we have constructed local time for a class of interacting superprocesses on Eucliidean space $\mathbb{R}^d$ when $d \leq 3$. Known as SDSM which was indtroduced in Wang [28], [27] and extended in Dawson et al. [6]. We have also provided the Tanaka representation of the SDSM local time based on the estimation of first two moments of the SDSM. In other words, in [7] we have identify conditions which are sufficient to prove the existence of the SDSM local time and establish the Tanaka representation of the SDSM local time. However, due to the singularities and interaction of SDSM particle model, we left an open problem how to identify conditions which guarantee the joint Hölder continuity of the SDSM local time. In the present paper, we provide solution to the open problem and give sufficient conditions which guarantee the joint Hölder continuity of the SDSM local time based on the Tanaka representation of the SDSM local time.

Since if the initial state of the SDSM is a finite measure, then according the result of Branching process, the total mass process of the SDSM is equivalent to a one-dimensional continuous state critical branching process, thus extinction occurs almost surely. So unbounded measure as the initial state may raise more interesting phenomena and based on this reason, in [7] we have reconstructed SDSM with unbounded measure as initial state on $\mathbb{R}^d, d = 1, 2, 3$ by the duality argument in which the transition probability density and the semigroup operator of the SDSM are directly defined by dual processes and dual moments. See [7] for more details.

Clearly the class of SDSM includes the critical branching Dawson-Watanabe superprocesses when $h \equiv 0$. The literature on these is extensive and the reader may consult the lecture notes by Dawson [3], Dawson [4] and Perkins [22] for some historical insights into the evolution of the field, as well as the more recent books by Li [18] and Xiong [29] for thorough updates on the subject.

Intuitively speaking, the local time of SDSM $\{\mu_t\}$ is the density process of the occupation time process $\int_0^t \mu_s ds$ of SDSM, a time-averaging which gives rise to a new measure-valued process with more regular paths and, in some cases, a density with respect to

Lebesgue measure, even when SDSM itself does not have one. Since from a regular way to discuss the construction of the local time of a measure-valued process, we are concerned about the existence of the density processes of the occupation times of SDSM and from Wang [27], we know that in the degenerate case, the SDSM is a purely-atomic-measure-valued processes, the density of the occupation times of this degenerate SDSM process may not exist, especially for $d \geq 2$. However, Li and Xiong [19] introduced an interesting way to define the local time for this degenerate case, a class of purely-atomic-measure-valued processes along each particle's path. Then, the local time of the degenerate SDSM with immigration in this sense is constructed, its Hölder joint continuity, excursion representation, and scaling limit theorem are discussed.

A Borel measurable process which maps any $(t, x) \in [0, \infty) \times \mathbb{R}^d$ to $\Lambda_t^x \in [0, \infty)$ is called the local time of SDSM $\{\mu_t\}$ if for any continuous function with compact support $\phi \in C_c(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \phi(x) \Lambda_t^x dx = \int_0^t \langle \phi, \mu_s \rangle ds. \tag{1.1}$$

In the case of Super-Brownian motion (where $h \equiv 0$ and c is the identity matrix) the existence and the joint space-time continuity of paths for its local time when $d \leq 3$ go back to Iscoe [14] and Sugitani [25]. These results, as well as further path properties, were generalized to superdiffusions (still $h \equiv 0$) in Krone [16]. In these and many other papers where the finer aspects of the superprocesses are analyzed, the argumentation largely depends on a multiplicative property of branching processes and the availability of a manageable closed form for the log-Laplace functional, a powerful tool to estimate the higher moments of $\{\mu_t\}$. Unfortunately, in our model the dependency of motion (h is no longer the null function) destroys the multiplicative property in question and makes this approach largely intractable, as it relies intimately on the independence structure built into Dawson-Watanabe superprocesses. This was the method applied by Adler and Lewin [1] in their proof of the Tanaka formula for the local time of Super-Brownian motion and super stable processes. This also occurs when trying the approach proposed in López-Mimbela and Villa [21] for Super-Brownian motion, where an alternative representation of the local time simplifies the proof of its joint continuity. Fortunately, we found that dependency does not affect the construction of Tanaka representation of a local time. However, the higher order singularity of the Green function and its derivative, in our case, raises some new technical difficulties in the higher moment estimation of the interacting term, as well as in the handling of a stochastic convolution integral term appearing in the corresponding Tanaka formula.

Nevertheless, only based on the estimation of the second moments of the SDSM $\{\mu_u\}$, Theorem 2.4 of Section 2 of the main result in [7] establishes the existence and the Tanaka representation of the local time Λ_t^x for SDSM directly through the characterization provided by (2.17), an explicit Tanaka formula expressed through a Green function with a singularity at the origin, in the spirit of the approach proposed in López-Mimbela and Villa [21] in their Theorem 3.1, of which our Theorem 2.4 is an extension. However, in order to make sense of it, we have to approximate this singular Green function- and its derivatives by smooth functions to ensure that the various stochastic integrals in (2.17) are well-defined and square-integrable martingales. Although these are tedious work, they are fundamental basis of present paper. A Tanaka formula for SDSM emerges and it is used to prove the existence of the local time. In the present paper, its joint continuity in (t, x) ensues, using Kolmogorov's continuity criterion on the estimate of higher moments of the interacting stochastic integral term with a singular integrant by the equivalence of heat kernel and the fundamental solutions of m SDSM particles associated uniformly parabolic partial differential equations (See Dressel [8] [9]) and by taking advantage of sharp estimates (2.7) for the Green function of SDSM m-particles and its associated Tanaka formula. The evaluation of the moments of $\{\mu_t\}$ proceeds from a duality argument, inspired by Sugitani [25] and Krone [16], as well as some sharp inequalities for the fundamental solutions of associated parabolic partial differential equations.

The remainder of this paper is organized as follows. Section 1 describes the connection of our previous paper [7] and the current paper. Section 2 states the main results of paper [7] and the current paper and assembles all the notation required for their formulation. Section 3 introduces basic notation in the dual construction of SDSM. Finally, Section 4 is devoted to the proof of the Hölder joint continuity in (t, x) of the local time Λ_t^x for SDSM $\{\mu_t\}$ and some important technical results in the estimation of dual moments have their proof also included in this section.

2 Sufficient Conditions, Sharp Estimate and Main Results

2.1 Basic Notation

For any Polish space S, that is, a topologically complete and separable metric space, $\mathcal{B}(S)$ denotes its Borel σ-field, $B(S)$ the Banach space of real valued bounded Borel measurable functions on S with the supremum norm $\|\cdot\|_\infty$ and $C(S)$ the space of real valued continuous functions on S. Subscripts b or c on any space of functions will always refer to its subspace of bounded or compactly supported functions, respectively, as in $C_b(S)$ and $C_c(S)$ here. S^m denotes the m-fold product of S.

The spaces of continuous $C([0,\infty),S)$ and càdlàg $D([0,\infty),S)$ trajectories into Polish space S are respectively equipped with the topology of uniform convergence on compact time sets and the usual Skorohod topology; they are themselves also Polish spaces (see Ethier and Kurtz [10]).

Given any positive Radon measure $\mu \in M(\mathbb{R}^d)$ and any $p \in [1,\infty)$, we write $L^p(\mu)$ for the Banach space of real valued Borel measurable functions on $\mathbb{R}^d$, with finite norm $\|\phi\|_{\mu,p} := \{\int_{\mathbb{R}^d} |\phi(x)|^p d\mu(x)\}^{1/p} < \infty$ and $|x|^2 = \sum_{i=1}^d x_i^2$. When $\mu = \lambda_0$ is the Lebesgue measure we use the standard notation $L^p(\mathbb{R}^d) = L^p(\lambda_0)$ and $\|\phi\|_p := \|\phi\|_{\lambda_0,p}$.

We need various subspaces of continuous functions inside $C(\mathbb{R}^d)$, notably $C^k(\mathbb{R}^d)$ the space of continuous functions on $\mathbb{R}^d$ with continuous derivatives up to and including order $k \geq 0$, with $C^\infty(\mathbb{R}^d)$ their common intersection (the smooth functions) and noticing that $C^0(\mathbb{R}^d) = C(\mathbb{R}^d)$; $C_b^k(\mathbb{R}^d)$ their respective subspace of bounded continuous functions with bounded derivatives up to and including order k, again with $C_b^\infty(\mathbb{R}^d)$ their common intersection and $C_b^0(\mathbb{R}^d) = C_b(\mathbb{R}^d)$; $C_0^k(\mathbb{R}^d)$ those bounded continuous functions vanishing at ∞ together with their derivatives up to and including order k, with $C_0^\infty(\mathbb{R}^d)$ their common intersection and $C_0^0(\mathbb{R}^d) = C_0(\mathbb{R}^d)$, this last a Banach space when equipped with finite supremum norm; $C_c^k(\mathbb{R}^d)$ the further subspace of those with compact support, again with $C_c^\infty(\mathbb{R}^d)$ their common intersection and $C_c^0(\mathbb{R}^d) = C_c(\mathbb{R}^d)$. We use $\mathrm{Lip}(\mathbb{R}^d)$ to denote the space of Lipschitz functions on $\mathbb{R}^d$, that is, $\phi \in \mathrm{Lip}(\mathbb{R}^d)$ if there is a constant $M > 0$ such that $|\phi(x) - \phi(y)| \leq M|x-y|$ for every $x, y \in \mathbb{R}^d$. The class of bounded Lipschitz functions on $\mathbb{R}^d$ will be denoted by $\mathrm{Lip}_b(\mathbb{R}^d)$.

We will also need $C_b^{1,2}([0,t] \times (\mathbb{R}^d)^m)$, the space of bounded continuous functions with all derivatives bounded, up to and including order 1 in the time variable up to time t and order 2 in the md space variables, including mixed derivatives of that order. When no ambiguity is present we also write the partial derivatives (of functions and distributions) in abridged form

$$\partial_p = \frac{\partial}{\partial x_p} \quad \text{and} \quad \partial_p \partial_q = \frac{\partial}{\partial x_p}\frac{\partial}{\partial x_q} \quad \text{and so on.}$$

The main set of functions of interest here is

$$K_a(\mathbb{R}^d) = \{\phi : \phi = h + \beta I_a, \beta \in \mathbb{R}, h \in C_c^\infty(\mathbb{R}^d)\},$$

defined for any real number $a \geq 0$ with $I_a(x) = (1+|x|^2)^{(-a/2)}$. Since $C_c^\infty(\mathbb{R}^d)$ is uniformly dense in $C_c(\mathbb{R}^d)$ (with $C_0(\mathbb{R}^d)$ as common closure), the uniform closure of $K_a(\mathbb{R}^d)$ remains unchanged if we replace $C_c^\infty(\mathbb{R}^d)$ by $C_c(\mathbb{R}^d)$. Both are also $\|\cdot\|_p$-dense in $L^p(\mathbb{R}^d)$ for every $p \in [1,\infty)$ (see Lemma 2.19 in Lieb and Loss [20]), a fact that will come in handy later.

Let $M(\mathbb{R}^d)$ be the space of all positive Radon measures on $\mathbb{R}^d$ and $M_0(\mathbb{R}^d)$, its subspace of finite positive Radon measures. For any real number $a \geq 0$, define the main set of measures of interest here as

$$M_a(\mathbb{R}^d) = \{\mu \in M(\mathbb{R}^d) : \langle I_a, \mu\rangle = \int_{\mathbb{R}^d} I_a(x)\mu(dx) < \infty\}.$$

The topology τ_a of $M_a(\mathbb{R}^d)$ is defined in the following way: $\mu_n \in M_a(\mathbb{R}^d) \Rightarrow \mu \in M_a(\mathbb{R}^d)$ as $n \to \infty$, iff $\lim_{n\to\infty}\langle \phi, \mu_n\rangle = \langle \phi, \mu\rangle$ holds for every $\phi \in K_a(\mathbb{R}^d)$. Then, $(M_a(\mathbb{R}^d), \tau_a)$ is a Polish space (see Iscoe [13] and Konno and Shiga [15]). For instance, the Lebesgue measure λ_0 on $\mathbb{R}^d$ belongs to $M_a(\mathbb{R}^d)$ for any $a > d$. Furthermore, both $dx = \lambda_0(dx)$ are used indifferently when calculating Lebesgue integrals.

2.2 Sufficient Conditions

To avoid repetitions, we make the following basic assumptions, valid throughout this paper. The first ones purport to the coefficients of our equations and the second ones, to the properties of the processes themselves as well as the filtered probability spaces they are constructed on.

Hypothesis 1. *The vector $h = (h_1, \cdots, h_d)$ satisfies $h_i \in L^1(\mathbb{R}^d) \cap \mathrm{Lip}_b(\mathbb{R}^d)$ and the $d \times d$ matrix $c = (c_{ij})$ satisfies $c_{ij} \in \mathrm{Lip}_b(\mathbb{R}^d)$, for $i, j = 1, \cdots, d$.*

Hypothesis 2. *The vector $h = (h_1, \cdots, h_d)$ satisfies $h_i \in C_c^2(\mathbb{R}^d)$ for $i, j = 1, \cdots, d$ and the $d \times d$ matrix $c = (c_{ij})$ satisfies $c_{ij} \in \mathrm{Lip}_b(\mathbb{R}^d)$, for $i, j = 1, \cdots, d$. The m particles diffusion matrix $(\Gamma^{kl}_{pq})_{1\le k,l\le m; 1\le p,q\le d}$ is defined by*

$$\Gamma^{kl}_{pq}(x_1, \cdots, x_m) := \begin{cases} (a_{pq}(x_k) + \rho_{pq}(0)) & \text{if } k = l, \\ \rho_{pq}(x_k - x_l) & \text{if } k \neq l, \end{cases} \tag{2.2}$$

where

$$a_{pq}(x) := \sum_{r=1}^{d} c_{pr}(x)c_{qr}(x) \tag{2.3}$$

and

$$\rho_{pq}(x - y) := \int_{\mathbb{R}^d} h_p(u - x)h_q(u - y)du.$$

$(\Gamma^{kl}_{pq})_{1\le k,l\le m; 1\le p,q\le d}$ *defined by* (2.2) *is uniformly elliptic or strictly positive definite everywhere on $(\mathbb{R}^d)^m$ or more precisely there exist two positive constants λ^* and Λ^* such that for any $\xi = (\xi^{(1)}, \cdots, \xi^{(m)}) \in (\mathbb{R}^d)^m$ we have a positive definite form which satisfies*

$$0 < \lambda^*|\xi|^2 \le \sum_{k,l=1}^{m} \sum_{p,q=1}^{d} \Gamma^{kl}_{pq}\xi^{(k)}_p \xi^{(l)}_q \le \Lambda^*|\xi|^2 < \infty. \tag{2.4}$$

Remark: Hypothesis (2) implies that the coefficients of the uniformly elliptic operator G_m are bounded continuous in $(\bar{\mathbb{R}}^d)^m$ and are Hölder continuous or more precisely

$$\Gamma_{pq}^{kl} \in C^{\alpha}(\bar{\mathbb{R}}^d)^m, \qquad\qquad k,l = 1,\cdots,m; p,q = 1,\cdots,d.$$

Hypothesis 3. *Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *be a filtered probability space with a right continuous filtration* $\{\mathcal{F}_t\}_{t\geq 0}$*, satisfying the usual hypotheses and upon which all our processes are built, notably an* $\mathbb{R}^1$*-valued Brownian sheet* W *on* $\mathbb{R}^d$ *(see below) and a countable family* $\{B_k, k \geq 1\}$ *of independent,* $\mathbb{R}^d$*-valued, standard Brownian motions written* $B_k = (B_{k1}, \cdots, B_{kd})$*. The family* $\{B_k, k \geq 1\}$ *is assumed independent of* W*.*

Example 2.1. *Let* p_t *is the transition density of a Brownian motion particle on* $\mathbb{R}^d$*. Since for initial measure* $\mu_0(dx) = \delta(dx)$*, the Dirac delta measure at* 0*, and for any* $t > 0$*,*

$$\int_0^t \int_{\mathbb{R}^d} p_s(y,x)\delta(dy)ds \leq \int_0^t \sup_{0\leq s\leq t} \langle p_s(y,x), \mu_0\rangle ds \begin{cases} = \infty & \text{if } x = 0, \quad d = 1,2,3 \\ < \infty & \text{if } x \neq 0 \quad d = 1,2,3 \end{cases} \tag{2.5}$$

Let q_t be the transition density of one SDSM particle on $\mathbb{R}^d$. Based on the Hypothesis 2 and (2.4), Dressel (see p62-63, (6), (7), (8) of [9]) proved that there exist four positive constants a^*, b, c and A^* such that

$$a^* \cdot p_{bt}(x,y) \leq q_t(x,y) \leq A^* \cdot p_{ct}(x,y)$$

holds for any $x, y \in \mathbb{R}^d$. From above example, we have seen that if the initial measure has an atom and $d = 1,2,3$, then, the existence of the continuous SDSM local time is questionable. So we put an additional hypothesis in the following.

Hypothesis 4. (4.1) *For any* $\mu_0 \in M_a(\mathbb{R}^d)$*, there exists a positive constant* $c(\mu_0)$*, which depends on* μ_0*, such that for any* $w \in \mathbb{R}^d$ *the following inequality*

$$\langle \mathbb{I}(x+w), \mu_0(dx)\rangle \leq c(\mu_0)\langle \mathbb{I}(x), \mu_0(dx)\rangle$$

holds, where

$$\mathbb{I} \in \{I_a(\cdot) \text{ or } p_r(\cdot) : I_a(x) = (1+|x|^2)^{-a/2}, a \geq 0, p_r(x) = \frac{1}{(2\pi r)^{d/2}}\exp[-\frac{|x|^2}{2r}], r > 0, x \in \mathbb{R}^d\}.$$

Define

$$M_a^T(\mathbb{R}^d) \triangleq \{\mu \in M_a(\mathbb{R}^d) \text{ and satifies Hypothesis (4.1) }\}.$$

(4.2) *There exists a positive* $\epsilon > 0$ *such that holds*

$$\sup_{0\leq r\leq \epsilon} \langle p_r(x,y), \mu_0\rangle < \infty, \tag{2.6}$$

where $p_r(x,y) = \frac{1}{(2\pi r)^{d/2}} \exp\{-\frac{|x-y|^2}{2r}\}$ *and* $x, y \in \mathbb{R}^d$*. for any* $r > 0$ *and any* $x \in \mathbb{R}^d, d = 1, 2,$ *or* 3*, where* μ_0 *is the initial measure and* p_t *is a d-dimensional Brownian motion particle's transition density on* $\mathbb{R}^d$*.*

Notice that Hypothesis (4.1) is equivalent to the translation invariance. The class of measures satisfies Hypothesis (4.1) is denoted by $M_a^T(\mathbb{R}^d)$. Then, $M_a^T(\mathbb{R}^d) \subset M_a(\mathbb{R}^d)$ and $M_a^T(\mathbb{R}^d)$ contains any finite measures and measure which is absolutely continuous with respect to the Lebesgue measure with bounded measurable density function. Any measure μ_0 which is absolutely continuous with respect to Lebesgue measure λ_0 satisfies Hypothesis (4), by the Radon-Nikodym theorem. Also $I_a(x)dx$ and $I_a^{-1}(x)dx$ satisfy Hypothesis (4).

Following Walsh [26, Chapter 2], a random set function W on $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}_+)$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is called an $\mathbb{R}^1$-valued Brownian sheet on $\mathbb{R}^d$ (or space-time white noise) if both of the following statements hold: for every $A \in \mathcal{B}(\mathbb{R}^d)$ having finite Lebesgue measure $\lambda_0(A)$, the process $M(A)_t := W(A \times [0,t])$ is a square-integrable $\{\mathcal{F}_t\}$-martingale; and for every pair $A_i \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}_+)$, $i = 1, 2$, having finite Lebesgue measure with $A_1 \cap A_2 = \emptyset$, the random variables $W(A_1)$ and $W(A_2)$ are independent, Gaussian random variables with mean zero, respective variance $\lambda_0(A_i)$ and $W(A_1 \cup A_2) = W(A_1) + W(A_2)$ holds $\mathbb{P}$-a.s. (see Walsh [26], Dawson [4, Section 7.1] and Perkins [22] for further details).

2.3 Sharp Estimation

Since we need important properties and estimations of fundamental solution of a general parabolic partial differential equations. In the following, we introduce some related notation.

Lemma 2.1. *Let* $l = 2r + |s|$, $|s| = s_1 + s_2 + \cdots + s_d$, $r, s_i = 0, 1$ *or* 2 *for* $1 \leq i \leq d$ *and* $\partial^l = \partial_t^r \partial_x^s$*. Let*

$$\mathbb{G}u = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j}u + \sum_{i=1}^{d} b_i(x,t)\frac{\partial}{\partial x_i}u + c(x,t)u,$$

and

$$\mathbb{L} = \mathbb{G} - \frac{\partial}{\partial t}.$$

We say operator $\mathbb{L}$ *is uniformly parabolic if* $\mathbb{G}$ *is uniformly elliptic. Suppose that the coefficients of the uniformly parabolic operator* $\mathbb{L}$ *are bounded continuous in* $\bar{\mathbb{R}}^d \times [0,T]$ *and are Hölder continuous or more precisely*

(i)

$$a_{ij} \in C^{\alpha,\alpha/2}(\bar{\mathbb{R}}^d \times [0,T]), \qquad i, j = 1, \cdots, d$$

(ii)

$$b_i \in C^{\alpha,\alpha/2}(\bar{\mathbb{R}}^d \times [0,T]), \qquad\qquad i = 1,\cdots,d$$

where $1 < \alpha < 1$*. Then, there exists a unique fundamental solution,* $\Gamma(x,t;\xi,\tau)$*, of the forward parabolic partial differential equation* $\mathbb{L}u = 0$*, where* $\Gamma(x,t;\xi,\tau)$ *corresponds to both time and spatial inhomogeneous (the coefficients of the elliptic operator depend on* t *and* x*)* $t > \tau$ *and* ξ *is the starting position,* τ *is the starting time,* x *is the new position after spending time* $t-\tau$*. Then, there exist two positive constants* $c > 0$ *and* $c_0 > 0$*, such that*

$$\begin{aligned}
&|\partial^l\Gamma(x,\xi,t,\tau) - \partial^l\Gamma(x,\xi',t,\tau')| \\
&\le c(|\xi-\xi'|^\alpha + |\tau-\tau'|^{\alpha/2})\Big[(t-\tau)^{-(d+l)/2}\exp\{-c_0\frac{|x-\xi|^2}{t-\tau}\} \\
&+(t-\tau')^{-(d+l)/2}\exp\{-c_0\frac{|x-\xi'|^2}{t-\tau'}\}\Big], \\
&\qquad\qquad 0<\alpha<1, \qquad\qquad l = 0,1,2. \qquad (2.7)
\end{aligned}$$

Proof: This lemma is same as the Theorem 3.5 of section 3 of Chapter V of Garroni and Menaldi [11]. Check the ideas and the proof of the Theorem 3.5. The formula can be directly proved based on a sequence of estimations of section 11,12, and 13 of Chapter IV of Ladyženskaja et al. [17]. The reference Dressel [8], [9], and Aroson [2] contain very useful estimations. □

For any integer $m \ge 1$, write $Z_m(t) := (z_1(t),\cdots,z_m(t))$ for the motion of the cloud of m-SDSM particles, $\mathbb{P}_x$ for the law of Z_m with initial point $x \in (\mathbb{R}^d)^m$ and $\mathbb{E}_x$ for the expectation with respect to $\mathbb{P}_x$. Since Z_m is a time-homogeneous $\{\mathcal{F}_t\}_{t\ge0}$-Markov process, let $\{P_t^m : t \ge 0\}$ be the corresponding Markov semigroup on $B((\mathbb{R}^d)^m)$ for Z_m, that is

$$P_t^m f(x) := \mathbb{E}_x\left[f(Z_m(t))\right] \qquad \text{for } t \ge 0 \text{ and } f \in B((\mathbb{R}^d)^m). \qquad (2.8)$$

Note that P_t^m is a Feller semigroup and maps each of $B((\mathbb{R}^d)^m)$, $C_b((\mathbb{R}^d)^m)$ and $C_0((\mathbb{R}^d)^m)$ into itself.

Itô's formula yields the following generator for $\{P_t^m : t \ge 0\}$: for all $f \in C_b^2((\mathbb{R}^d)^m)$,

$$G_m f(x) := \frac{1}{2}\sum_{i,j=1}^m \sum_{p,q=1}^d \Gamma_{pq}^{ij}(x_1,\cdots,x_m)\frac{\partial^2}{\partial x_{ip}\partial x_{jq}} f(x_1,\cdots,x_m) \qquad (2.9)$$

where $x = (x_1,\cdots,x_m) \in (\mathbb{R}^d)^m$ has components $x_i = (x_{i1},\cdots,x_{id}) \in \mathbb{R}^d$ for $1 \le i \le m$ and Γ_{pq}^{ij} is defined by

$$\Gamma_{pq}^{ij}(x_1,\cdots,x_n) := \begin{cases} (a_{pq}(x_i) + \rho_{pq}(0)) & \text{if } i = j, \\ \rho_{pq}(x_i - x_j) & \text{if } i \ne j, \end{cases} \qquad (2.10)$$

Following Stroock and Varadhan [24], it is useful to view process $\{Z_m(t) : t \geq 0\}$ as a solution to the $(G_m, \delta_{Z_m(0)})$-martingale problem on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ for any fixed starting point $Z_m(0) \in (\mathbb{R}^d)^m$, meaning that, for every choice of $f \in C_c^\infty((\mathbb{R}^d)^m)$, the process $f(Z_m(t)) - \int_0^t G_m f(Z_m(s))ds$ is an $\mathcal{F}_t$-martingale. We say this martingale problem is well-posed (or has a unique solution) if any two solutions have the same finite dimensional distributions.

We also need the following summary of several known results from the literature.

Lemma 2.2. *Under Hypotheses* (2) *and* (3)*, the following statements hold with* $m \geq 1$.

- *For any initial value* $Z_m(0) \in (\mathbb{R}^d)^m$*, the* $(G_m, \delta_{Z_m(0)})$*-martingale problem is well-posed. The trajectories of* $\{Z_m(t) : t \geq 0\}$ *are in* $C([0,\infty), (\mathbb{R}^d)^m)$.

- *For any* $T > 0$, $P_t^m f(x)$*, as a function of* (t, x)*, belongs to* $C_b^{1,2}([0,T] \times (\mathbb{R}^d)^m)$*, for every choice of* $f \in C_0((\mathbb{R}^d)^m)$ *and* $\{P_t^m\}$ *is a Feller semigroup mapping* $C_0^2((\mathbb{R}^d)^m)$ *into itself.*

- $\{P_t^m : t \geq 0\}$ *has a transition probability density when* $t > 0$*, i.e., there is a function* $q_t^m(x, y) > 0$ *which is jointly continuous in* $(t, x, y) \in (0,\infty) \times (\mathbb{R}^d)^m \times (\mathbb{R}^d)^m$ *everywhere and such that there holds* $P_t^m f(\cdot) = \int_{(\mathbb{R}^d)^m} f(y) q_t^m(\cdot, y)dy$ *when* $t > 0$*, for every* $f \in C_0((\mathbb{R}^d)^m)$.

- *For each choice of* $T > 0$, $d \geq 1$ *and* $m \geq 1$*, there are positive constants* a_1 *and* a_2 *such that, for any choice of* $1 \leq p \leq dm$ *and nonnegative integers* r *and* s *such that* $0 \leq 2r + s \leq 2$,

$$\left| \frac{\partial^r}{\partial t} \frac{\partial^s}{\partial y_p} q_t^m(x, y) \right| \leq \frac{a_1}{t^{(dm+2r+s)/2}} \exp\left\{ -a_2 \left(\frac{|y-x|^2}{t} \right) \right\} \tag{2.11}$$

  holds everywhere in $(t, x, y) \in (0,T) \times (\mathbb{R}^d)^m \times (\mathbb{R}^d)^m$ *with* $y = (y_1, \ldots, y_{dm})$.

One important consequence of Lemma 2.2 is that $C_0^2((\mathbb{R}^d)^m)$ is a core for generator G_m (see Propositions 1.3.3 and 8.1.6 in Ethier and Kurtz [10]).

2.4 SPDE and Tanaka Representation

Let us now turn our attention to the characterization of SDSM through the formulation of a well-posed martingale problem, which is defined as follows (see Ethier and Kurtz [10] for this infinite dimensional formulation). The pregenerator of SDSM $\{\mu_t\}$ is defined for smooth real valued functions F on $M(\mathbb{R}^d)$, the set of all positive Radon measures on $\mathbb{R}^d$, as the second order differential operator :

$$\mathcal{L}F(\mu) := \mathcal{A}F(\mu) + \mathcal{B}F(\mu), \tag{2.12}$$

where

$$\mathcal{B}F(\mu) := \frac{\gamma\sigma^2}{2}\int_{\mathbb{R}^d} \frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx)$$

and

$$\mathcal{A}F(\mu) := \frac{1}{2}\sum_{p,q=1}^{d}\int_{\mathbb{R}^d}(a_{pq}(x)+\rho_{pq}(0))\left(\frac{\partial^2}{\partial x_p \partial x_q}\right)\frac{\delta F(\mu)}{\delta\mu(x)}\,\mu(dx)$$

$$+\frac{1}{2}\sum_{p,q=1}^{d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_{pq}(x-y)\left(\frac{\partial}{\partial x_p}\right)\left(\frac{\partial}{\partial y_q}\right)\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy).$$

Here, with the above mappings $h = (h_p)$ and $c = (c_{pq})$ expressed coordinatewise, we write, for $p, q = 1, \ldots, d$, the local (or individual) diffusion coefficient as

$$a_{pq}(x) := \sum_{r=1}^{d} c_{pr}(x)c_{qr}(x) \tag{2.13}$$

and the global (or common) interactive diffusion coefficient as

$$\rho_{pq}(x-y) := \int_{\mathbb{R}^d} h_p(u-x)h_q(u-y)du.$$

Parameter $\gamma > 0$ is related to the branching rate of the particle system and $\sigma^2 > 0$ is the variance of the limiting offspring distribution. The variational derivative is defined by

$$\frac{\delta F(\mu)}{\delta\mu(x)} := \lim_{\epsilon\downarrow 0}\frac{F(\mu+\epsilon\delta_x)-F(\mu)}{\epsilon}$$

where δ_x stands for the Dirac measure at x and the domain $\mathcal{D}(\mathcal{L}) \subset C_b(M(\mathbb{R}^d))$ of the pregenerator $\mathcal{L}$ includes all functions of the form $F(\mu) = g(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_k,\mu\rangle)$ with $g \in C_b^2(\mathbb{R}^k)$ for some $k \geq 1$ and $\phi_i \in C_c^\infty(\mathbb{R}^d)$ for every $1 \leq i \leq k$. For any $\mu \in M(\mathbb{R}^d)$ and any μ-integrable ϕ we write $\langle\phi,\mu\rangle = \int_{\mathbb{R}^d}\phi(x)\mu(dx)$ here and henceforth.

Theorem 2.3 of Section 2 shows that the operator $\mathcal{L}$ and its (full) domain $\mathcal{D}(\mathcal{L})$ jointly determine the law of a diffusion process $\{\mu_t\}$, hereafter called SDSM, by way of a well-posed martingale problem. The construction of SDSM directly on an enlarged space of trajectories, with a wider class of initial measures on $\mathbb{R}^d$ including Lebesgue measure, using a duality argument explained in [7].

Clearly the class of SDSM includes the critical branching Dawson-Watanabe super-processes when $h \equiv 0$. The literature on these is extensive and the reader may consult the lecture notes by Dawson [3], Dawson [4] and Perkins [22] for some historical insights into the evolution of the field, as well as the more recent books by Li [18] and Xiong [29] for thorough updates on the subject.

A solution to the $(\mathcal{L}, \delta_{\mu_0})$-martingale problem for $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is a stochastic process μ with values in $M_a(\mathbb{R}^d)$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with initial value $\mu_0 \in M_a(\mathbb{R}^d)$ such that, for every $F \in \mathcal{D}(\mathcal{L})$, the process $F(\mu_t) - \int_0^t \mathcal{L}F(\mu_s)ds$ is an $\mathcal{F}_t$-martingale. We say this martingale problem is well-posed (or has a unique solution) if any two solutions have the same finite dimensional distributions. This unique solution is our SDSM.

Since all our measure-valued processes will have continuous trajectories almost surely, we can select $\Omega = C([0, \infty), M_a(\mathbb{R}^d))$ in Hypothesis (3) as the space upon which our constructions are carried out, in a canonical way.

Theorem 2.3. *Assume both Hypotheses* (1) *and* (3). *For any $a \geq 0$ and any initial value $\mu_0 \in M_a^T(\mathbb{R}^d)$, the $(\mathcal{L}, \delta_{\mu_0})$-martingale problem for the operator given by* (2.12) *is well-posed and its unique solution μ_t is a diffusion process which satisfies*

$$\langle \phi, \mu_t \rangle - \langle \phi, \mu_0 \rangle = X_t(\phi) + M_t(\phi) + \int_0^t \langle G_1\phi,\ \mu_s \rangle\, ds \tag{2.14}$$

for every $t > 0$ and $\phi \in K_a(\mathbb{R}^d)$, with $G_1 = \sum_{p,q=1}^d \frac{1}{2}(a_{pq}(x) + \rho_{pq}(0))\partial_p\partial_q$ from (2.9) *and where both*

$$X_t(\phi) := \sum_{p=1}^d \int_0^t \int_{\mathbb{R}^d} \langle h_p(y - \cdot)\partial_p\phi(\cdot), \mu_s \rangle\, W(dy, ds)$$

and

$$M_t(\phi) := \int_0^t \int_{\mathbb{R}^d} \phi(y) M(ds, dy)$$

are continuous square-integrable $\{\mathcal{F}_t\}$-martingales, mutually orthogonal for every choice of $\phi \in K_a(\mathbb{R}^d)$ and driven respectively by a Brownian sheet W and a square-integrable martingale measure M with

$$\langle M(\phi) \rangle_t = \gamma\sigma^2 \int_0^t \langle \phi^2, \mu_s \rangle ds \qquad \text{for every } t > 0 \text{ and } \phi \in K_a(\mathbb{R}^d).$$

Here the filtration of choice is $\mathcal{F}_t := \sigma\{\langle \phi, \mu_s \rangle, M_s(\phi), X_s(\phi) : \phi \in K_a(\mathbb{R}^d), s \leq t\}$. Part of the statement of course is that all the integrals involved do make sense. The proof is provided in Section 3.

For the single particle transition density $q_t^1(x, 0)$ (from 0) exhibited in Lemma 2.2 for the semigroup P_t^1 associated with generator G_1 from (2.9), its Laplace transform (in the time variable) is given by

$$Q^\lambda(x) := \int_0^\infty e^{-\lambda t} q_t^1(x, 0) dt, \tag{2.15}$$

for any $\lambda > 0$. Formally Q^0 is known as Green's function for density q_t^1 and exhibits a potential singularity at $x = 0$. By Lemma 2.2, for all $x \in \mathbb{R}^d \setminus \{0\}$ we can also write

$$\partial_{x_i} Q^\lambda(x) = \partial_{x_i} \int_0^\infty e^{-\lambda t} q_t^1(x,0)dt = \int_0^\infty e^{-\lambda t} \partial_{x_i} q_t^1(x,0)dt < \infty \tag{2.16}$$

for any $i \in \{1, 2, \cdots, d\}$, with the derivative taken in the classical sense.

We now state our main result, under some restriction on the family of initial measures.

Theorem 2.4. *Under Hypotheses* (2) *and* (3)*, with* $d = 1$*,* 2 *or* 3*, select any* $a \geq 0$ *and* $\mu_0 \in M_a^T(\mathbb{R}^d)$ *satisfying Hypothesis* (4)*, with Brownian sheet* W *from Hypothesis* (3) *and martingale measure* M *from Theorem* 2.3*. For every* $(t,x) \in [0,\infty) \times \mathbb{R}^d$*,*

$$\begin{aligned}\Lambda_t^x &:= \langle Q^\lambda(x-\cdot), \mu_0\rangle - \langle Q^\lambda(x-\cdot), \mu_t\rangle + \lambda \int_0^t \langle Q^\lambda(x-\cdot), \mu_s\rangle ds \\ &+ \sum_{p=1}^d \int_0^t \int_{\mathbb{R}^d} \langle h_p(y-\cdot)\partial_p Q^\lambda(x-\cdot), \mu_s\rangle W(dy,ds) \\ &+ \int_0^t \int_{\mathbb{R}^d} Q^\lambda(x-y) M(dy,ds)\end{aligned} \tag{2.17}$$

is the local time for SDSM $\{\mu_t\}$ *from Theorem* 2.3 *and satisfies* (1.1) $\mathbb{P}_{\mu_0}$*-almost surely for every choice of* $\phi \in C_c(\mathbb{R}^d)$*. There exists a version of* Λ_t^x *which is Hölder jointly continuous in* $(t,x) \in [0,\infty) \times \mathbb{R}^d$ *and is square-integrable for every* $(t,x) \in [0,\infty) \times \mathbb{R}^d$*, that is* $\mathbb{E}_{\mu_0}\left[|\Lambda_t^x|^2\right] < \infty$ *no matter what value* $a \geq 0$ *takes.*

Equation (2.17) is called the **Tanaka formula for SDSM** μ_t.
The proof of the Theorem 2.4 is directly following from the Theorem 4.4 and Theorem 4.5.
Remark:

(1) By Lemma 4.1 and Corollary 3.4, each term on the right hand side of (2.17) is well-defined.

(2) (2.17) can be understood as a distribution-valued processes. Then, see the proof of the Tanaka formula (2.17) and (4.12) and (4.13) of [7].

(3) Λ_t^x defined by (2.17) is the SDSM local time corresponds to the non-degenerate case ($d \leq 3$) or uniformly elliptic case. For the degenerate case and $d = 1$, the SDSM is a purely atomic measure-valued process whose local time constructed and its joint Hölder continuity in $(t,x) \in [0,T] \times \mathbb{R}^1$ proved along each particle's path in Li and Xiong [19].

(4) A simple way to construct the Λ_t^x is defining

$$Q_\epsilon^\lambda(x) = \int_0^\infty e^{-\lambda u} q_{u+\epsilon}(x,0)du = e^{\lambda\epsilon}\int_\epsilon^\infty e^{-\lambda t} q_t(x,0)dt,$$

which is a bounded smooth function and belongs to $L^p(\mathbb{R}^d)$ for any $p \in [1,\infty]$. Then, replacing each $Q^\lambda(x)$ by $Q_\epsilon^\lambda(x)$ in the right hand side of (2.17) and replacing the left hand side of (2.17) by $\Lambda_t^{\epsilon,x}$.Then, we get

$$\begin{aligned}\Lambda_t^{\epsilon,x} &:= \langle Q_\epsilon^\lambda(x-\cdot),\mu_0\rangle - \langle Q_\epsilon^\lambda(x-\cdot),\mu_t\rangle + \lambda\int_0^t \langle Q_\epsilon^\lambda(x-\cdot),\mu_s\rangle ds \\ &+ \sum_{p=1}^d \int_0^t \int_{\mathbb{R}^d} \langle h_p(y-\cdot)\partial_p Q_\epsilon^\lambda(x-\cdot),\mu_s\rangle W(dy,ds) \\ &+ \int_0^t \int_{\mathbb{R}^d} Q_\epsilon^\lambda(x-y)M(dy,ds) \qquad (2.18)\end{aligned}$$

It is clear that each term on the right hand side is well-defined. It is quite easy to prove that each term on the right hand side of (2.18) satisfies the conditions of Kolmogorov and Čentsov Theorem. Thus, the right hand side of (2.18) defines an approximate SDSM local times which are jointly Hölder continuous. Then, we can prove that

$$\lim_{\epsilon\downarrow 0+} \Lambda_t^{\epsilon,x} = \Lambda_t^x \qquad \text{in } L^2(\mathbb{P}_{\mu_0}),$$

which defines a joint continuous SDSM local time.

(5) Note that the value of the local time does not depend on parameter $\lambda > 0$ (although it does vary with the dimension d of the space).

3 Dual Construction of SDSM

In [7], we use the dual construction of a function-valued dual process, in the sense of Dawson and Kurtz [5], as a way to directly exhibit the transition probability of SDSM, thus immediately giving an elegant construction of SDSM as a unique probability measure on the space $C([0,\infty), M_a(\mathbb{R}^d))$, since duality also yields the full characterization of the law of SDSM by way of the martingale problem formulation. Part of the interest of this section lies with the uncommon use of the existence of a dual function-valued process, in order to construct a transition function for SDSM and show the existence of associated measure-valued processes of interest on richer spaces of trajectories resulting from the inclusion of infinite starting measures. The technique of duality was developed in order to identify the more complex measure-valued one uniquely and find some of its mathematical features, after first showing its existence through some other means, often by way of a tightness argument or some other limiting scheme. (Note that some technical aspects of the treatment in this section are required due to the topology on $M_a(\mathbb{R}^d)$. The reader can refer to the appendix in Konno and Shiga [15] for additional clarifications.)

Let us begin with the construction of the function-valued process that will serve our purpose, namely an extension of the ones built in Ren et al. [23] and Dawson et al. [6].

In order to facilitate some of the calculations required henceforth, notably because infinite starting measures lying in $M_a(\mathbb{R}^d)$ impose restrictions on the set of functions needed for a full description of the dual process, the domain $\mathcal{D}(\mathcal{L})$ of operator $\mathcal{L}$ in (2.12) — the set of functions in $B(M_a(\mathbb{R}^d))$ upon which $\mathcal{L}$ is well-defined — is enlarged to comprise all bounded continuous functions of the form

$$F(\mu) = g(\langle f_1, \mu^{m_1}\rangle, \cdots, \langle f_k, \mu^{m_k}\rangle) \tag{3.1}$$

with $g \in C^2(\mathbb{R}^k)$ for some $k \geq 1$, any choice of positive integers $m_1, \ldots, m_k$ and, for every $1 \leq i \leq k$, $f_i \in \mathcal{D}_a(G_{m_i})$. For instance the choice $g(x) = |x|^2$ will be used later.

We describe the space $\mathcal{D}_a(G_m)$ next. For the generator G_m from (2.9) of strongly continuous contraction semigroup $\{P_t^m\}$ on Banach space $C_0((\mathbb{R}^d)^m)$, the domain $\mathcal{D}(G_m)$ — the set of functions in $B((\mathbb{R}^d)^m)$ upon which G_m is well-defined — is simply the set of those functions f such that the limit

$$\lim_{t\to 0+} \frac{1}{t}(P_t^m f - f)$$

exists, so we write $f \in \mathcal{D}(G_m)$ if and only if this limit exists and equals $G_m f$.

In order to ensure integrability with respect to some infinite measures, our statements about functions in this domain $\mathcal{D}(G_m)$ are restricted to its subspace defined by

$$\mathcal{D}_a(G_m) := \{f \in \mathcal{D}(G_m) : \|\mathcal{I}_{a,m}^{-1} f\|_\infty < \infty \text{ and } \|\mathcal{I}_{a,m}^{-1} G_m f\|_\infty < \infty.\} \tag{3.2}$$

The short form $\mu^{\otimes m} = \mu \otimes \ldots \otimes \mu$ denotes the m-fold product measure of $\mu \in M_a(\mathbb{R}^d)$ by itself and we write $\mathcal{I}_{a,m}$ for the product $\mathcal{I}_{a,m}(x) = I_a(x_1) \cdot \ldots \cdot I_a(x_m)$, keeping in mind that $\mathcal{I}_{a,m}^{-1} f(x) = \mathcal{I}_{a,m}^{-1}(x) f(x)$ means the product, not the composition of functions.

Observe first that both

$$\mathcal{I}_{a,m} \in C_b^\infty((\mathbb{R}^d)^m) \tag{3.3}$$

and

$$\mathcal{I}_{a,m}^{-1} G_m \mathcal{I}_{a,m} \in C_b((\mathbb{R}^d)^m) \tag{3.4}$$

hold, under Hypothesis (1), hence so do $\|\mathcal{I}_{a,m}^{-1} G_m \mathcal{I}_{a,m}\|_\infty < \infty$ and $\mathcal{I}_{a,m} \in \mathcal{D}_a(G_m)$. A quick sketch of proof of these facts is supplied in [7]. The useful inclusions $C_c^2((\mathbb{R}^d)^m) \subset \mathcal{D}_a(G_m) \subset \mathcal{D}(G_m)$ and $\mathcal{I}_{a,m}^{-1} G_m\{C_c^2((\mathbb{R}^d)^m)\} \subset C_c((\mathbb{R}^d)^m)$ are also clearly valid for every choice of $a \geq 0$.

It is important to note at this point that, for every positive value of $a > 0$ and $m \geq 1$, while $\mathcal{I}_{a,m} \in C_0^\infty((\mathbb{R}^d)^m)$ holds (this is false when $a = 0$), we also have $\mathcal{I}_{a,m} \notin \mathcal{D}_b(G_m)$ for any $b > a$. Therefore $C_0^\infty((\mathbb{R}^d)^m) \not\subset \mathcal{D}_a(G_m)$ for any $a > 0$, so the core $C_0^2((\mathbb{R}^d)^m)$ of G_m does not lie inside $\mathcal{D}_a(G_m)$ even though $C_c^2((\mathbb{R}^d)^m)$ is uniformly dense in $C_0^2((\mathbb{R}^d)^m)$.

More generally, we also get the following results pertaining to the preservation of the semigroup property under the rescaling induced by function $\mathcal{I}_{a,m}$, the proof of which may also be found in [7].

Lemma 3.1. *Assume that Hypothesis* (1) *is satisfied. For every $a \geq 0$, $f \in \mathcal{D}_a(G_m)$ and $T > 0$, there holds $P_T^m f \in \mathcal{D}_a(G_m)$, $\sup_{0 \leq t \leq T} \|\mathcal{I}_{a,m}^{-1} P_t^m f\|_\infty < \infty$ and*

$$\sup_{0 \leq t \leq T} \|\mathcal{I}_{a,m}^{-1} \frac{\partial}{\partial t} P_t^m f\|_\infty = \sup_{0 \leq t \leq T} \|\mathcal{I}_{a,m}^{-1} G_m P_t^m f\|_\infty = \sup_{0 \leq t \leq T} \|\mathcal{I}_{a,m}^{-1} P_t^m G_m f\|_\infty < \infty.$$

The construction of the function-valued process can now proceed, as follows.

Let $\{J_t : t \geq 0\}$ be a decreasing càdlàg Markov jump process on the nonnegative integers $\{0, 1, 2, \ldots\}$, started at $J_0 = m$ and decreasing by 1 at a time, with Poisson waiting times of intensity $\gamma\sigma^2 l(l-1)/2$ when the process has reached value $l \geq 2$. The process is frozen in place when it reaches value 1 and never moves if it is started at either $m = 0$ or 1. Write $\{\tau_k : 0 \leq k \leq J_0 - 1\}$ for the sequence of jump times of $\{J_t : t \geq 0\}$ with $\tau_0 = 0$ and $\tau_{J_0} = \infty$. At each such jump time a randomly chosen projection is effected on the function-valued process of interest, as follows. Let $\{S_k : 1 \leq k \leq J_0\}$ be a sequence of random operators which are conditionally independent given $\{J_t : t \geq 0\}$ and satisfy

$$\mathbb{P}\{S_k = \Phi_{ij}^m | J_{\tau_k -} = m\} = \frac{1}{m(m-1)}, \qquad 1 \leq i \neq j \leq m,$$

as long as $m \geq 2$. Here $\Phi_{ij}^m f$ is a mapping from $\mathcal{D}_a(G_m)$ into $\mathcal{D}_a(G_{m-1})$ defined by

$$\Phi_{ij}^m f(y) := f(y_1, \cdots, y_{j-1}, y_i, y_{j+1}, \cdots, y_m), \tag{3.5}$$

for any $m \geq 2$ and $y = (y_1, \cdots, y_{j-1}, y_{j+1}, \cdots, y_m) \in (\mathbb{R}^d)^{m-1}$. When $m = 0$ we simply write $\mathcal{D}_a(G_0) = \mathbb{R}^d$ and P_t^0 acts as the identity mapping on constant functions.

That Φ^m_{ij} is well-defined follows from the observation that the sets $\mathcal{D}_a(G_m)$ form an increasing sequence in m, in this last case when interpreting any function of $m \le n$ variables also as the restriction of a function of n variables. Details are in Subsection **??**.

Given $J_0 = m$ for some $m \ge 0$, define process $Y := \{Y_t : t \ge 0\}$, started at some point $Y_0 \in \mathcal{D}_a(G_m)$ within the (disjoint) topological union $\boldsymbol{B} := \cup_{m=0}^{\infty} \mathcal{D}_a(G_m)$, by

$$Y_t = P^{J_{\tau_k}}_{t-\tau_k} S_k P^{J_{\tau_{k-1}}}_{\tau_k-\tau_{k-1}} S_{k-1} \cdots P^{J_{\tau_1}}_{\tau_2-\tau_1} S_1 P^{J_0}_{\tau_1} Y_0, \quad \tau_k \le t < \tau_{k+1}, 0 \le k \le J_0 - 1. \tag{3.6}$$

By Lemma 3.1, the process Y is a well-defined $\boldsymbol{B}$-valued strong Markov process for any starting point $Y_0 \in \boldsymbol{B}$. Clearly, $\{(J_t, Y_t) : t \ge 0\}$ is also a strong Markov process.

Lemma 3.2. *Assume that Hypothesis* (1) *is satisfied. Given any* $a \ge 0$, $J_0 = m \ge 1$ *and* $T > 0$, *there exists a constant* $c = c(a, d, m, T) > 0$ *such that, for every* $Y_0 \in \mathcal{D}_a(G_m)$ *we have* $\mathbb{P}$*-almost surely*

$$\sup_{0 \le t \le T} \|\mathcal{I}^{-1}_{a,J_t} Y_t\|_\infty \le c \|\mathcal{I}^{-1}_{a,m} Y_0\|_\infty.$$

The proof is found in [7]. This was the last integrability requirement needed, prior to proceeding with the proof of Theorem 2.3. We can now build the transfer function that lifts the finite measure-valued processes to infinite ones.

For each $a \ge 0$, the map $T_{I_a} : M_a(\mathbb{R}^d) \to M_0(\mathbb{R}^d)$ defined by

$$T_{I_a}(\mu)(A) = \int_A I_a(x)\mu(dx) = \int_A (1 + |x|^2)^{-a/2} \mu(dx), \tag{3.7}$$

for any $A \in \mathcal{B}(\mathbb{R}^d)$, is clearly homeomorphic (continuous and bijective, with a continuous inverse), hence Borel measurable.

Theorem 3.3. *Assume Hypotheses* (1) *and* (3) *are satisfied. For any* $a \ge 0$, $m \ge 1$, $f \in C_0((\mathbb{R}^d)^m)$, $\mu_0 \in M^T_a(\mathbb{R}^d)$ *and* $t \in [0, \infty)$, *there exists a time homogeneous transition function* $\{Q_t(\mu, \Gamma) : t \in [0, \infty), \mu \in M_a(\mathbb{R}^d), \Gamma \in \mathcal{B}(M_a(\mathbb{R}^d))\}$, *given by*

$$\begin{aligned} &\int_{M_a(\mathbb{R}^d)} \langle f, \nu^m \rangle Q_t(\mu, d\nu) \\ = \quad &\mathbb{E}_{\mu_0}\left[\langle \mathcal{I}^{-1}_{a,J_t} Y_t, (T_{I_a}(\mu))^{J_t} \rangle \exp\left(\frac{\gamma\sigma^2}{2} \int_0^t J_s(J_s - 1) ds\right) \Big| (J_0, Y_0) = (m, f), \right] \end{aligned} \tag{3.8}$$

for which the associated probability measure Q^{μ_0} *on* $C([0, \infty), M_a(\mathbb{R}^d))$ *of the form:*

$$\begin{aligned} &Q^{\mu_0}(\{w \in C([0, \infty), M_a(\mathbb{R}^d)) : w_{t_i} \in \Gamma_i, i = 0, \cdots, n\}) \\ = \quad &\int_{\Gamma_0} \cdots \int_{\Gamma_{n-1}} Q_{t_n - t_{n-1}}(\mu_{n-1}, \Gamma_n) Q_{t_{n-1}-t_{n-2}}(\mu_{n-2}, d\mu_{n-1}) \cdots Q_{t_1}(\mu_0, d\mu_1), \end{aligned} \tag{3.9}$$

for any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ *and* $\Gamma_i \in \mathcal{B}(M_a(\mathbb{R}^d)), i = 0, 1, \cdots, n$, *is the unique probability measure on* $C([0,\infty), M_a(\mathbb{R}^d))$ *which satisfies* (3.9). *Probability measure* Q^{μ_0} *is a solution to the* $(\mathcal{L}, \delta_{\mu_0})$*-martingale problem.*

Remark: The SDSM on $\mathbb{R}^d$ was already constructed in Ren et al. [23], using a tightness argument for the laws on $D([0,\infty), M_0(\mathbb{R}^d))$ of the trajectories of high-density particles, but only when these particles move in a bounded domain $D \subset \mathbb{R}^d$ with killing boundary and the initial data is a finite measure $\mu_0 \in M_0(\mathbb{R}^d)$. Here instead we adapt the approach used for the case $a = 0$ and $d = 1$ in Dawson et al. [6] by exhibiting a transition function, built by using the law of function-valued process Y and charging space $C([0,\infty), M_a(\mathbb{R}^d))$ with a probability measure fitting our needs. In the circumstances, we only give a quick sketch of the main ideas but provide details for overcoming the new difficulties arising from the larger space.

One of the consequences of Theorem 3.3 is the following useful technical result.

Corollary 3.4. *Assume Hypotheses* (1) *and* (3) *are satisfied. For any* $p \geq 1$, $a \geq 0$ *and initial data* $\mu_0 \in M_a(\mathbb{R}^d)$ *for SDSM* μ_t, *every* $\phi \in L^p(\mu_0)$ *also belongs* $\mathbb{P}$*-almost surely to* $L^p(\mu_t)$, *for all* $t > 0$. *Any nonnegative* $\phi \in L^1(\mu_0)$ *for which there holds* $\langle \phi, \mu_0 \rangle = 0$ *also verifies* $\mathbb{P}(\int_0^\infty \langle \phi, \mu_s \rangle ds = 0) = 1$. *In particular, every Borel measurable set* $N \in \mathcal{B}(\mathbb{R}^d)$ *initially null remains so, that is,* $\mu_0(N) = 0$ *implies* $\mathbb{P}(\int_0^\infty \mu_s(N) ds = 0) = 1$. *Moreover, the duality identity (3.8) holds true for all functions* $f \in C_0((\mathbb{R}^d)^m)$ *and extends to all nonnegative integrable* $f \in L^1(\mu_0^m)$ *as an inequality, with* $=$ *replaced by* $\leq$.

The proof can be found in in our paper [7].
The stage is now set for the proof of our main result (Theorem 2.4 of Section 2), the object of the next section.

4 Joint Continuity of the SDSM Local Time

We need the following result about the Laplace transform Q^λ defined in (2.15).

Lemma 4.1. *Assume Hypothesis* (2) *is satisfied. For any* $\lambda > 0$ *there holds:*
(i) *For all* $d \geq 1$, *we have* $Q^\lambda \in L^1(\mathbb{R}^d)$ *and* $\partial_{x_i} Q^\lambda \in L^1(\mathbb{R}^d)$ *for any* $i \in \{1, 2, \cdots, d\}$.
(ii) *For* $d = 1$, *we also have* $\partial_x Q^\lambda \in L^2(\mathbb{R})$.
(iii) *For* $d = 1$, 2 *or* 3, *we finally have* $Q^\lambda \in L^2(\mathbb{R}^d)$.

The proof is technical and found in [7].

Lemma 4.2. *Let*

$$p(u,x) := p_u(x) := \frac{1}{(2\pi u)^{d/2}} \exp\{-\frac{|x|^2}{2u}\}$$

be the transition density of the d-dimensional Brownian motion and the corresponding transition semigroup is denoted by T_t^1. Then, there exists a positive constant c such that

$$p(u,x)p(v,x) := p_u(x)p_v(x) \leq c(u+v)^{-d/2} p(\frac{uv}{u+v}, x) = c(u+v)^{-d/2} p_{\frac{uv}{u+v}}(x). \tag{4.1}$$

Proof: Since

$$\begin{aligned} p(u,x)p(v,x) &= \frac{1}{(2\pi u)^{d/2}} \exp\{-\frac{|x|^2}{2u}\} \frac{1}{(2\pi v)^{d/2}} \exp\{-\frac{|x|^2}{2v}\} \\ &= \frac{1}{(2\pi u)^{d/2}} \exp\{-\frac{v|x|^2}{2uv}\} \frac{1}{(2\pi v)^{d/2}} \exp\{-\frac{u|x|^2}{2uv}\} \\ &= \frac{1}{(2\pi)^{d/2}} \frac{1}{(2\pi uv)^{d/2}} \exp\{-\frac{|x|^2}{(\frac{2uv}{u+v})}\} \\ &= \frac{1}{(2\pi)^{d/2}} \cdot \frac{1}{(u+v)^{d/2}} \cdot \frac{1}{(2\pi(\frac{uv}{u+v}))^{d/2}} \exp\{-\frac{|x|^2}{(\frac{2uv}{u+v})}\}, \end{aligned} \tag{4.2}$$

if we choose $c > \frac{1}{(2\pi)^{d/2}}$, then we get (4.1). □

Further, for each $\lambda > 0$ and $x \in \mathbb{R}^d$, $Q^\lambda(x - \cdot)$ solves equation $(-G_1 + \lambda)\, u = \delta_x$ in the distributional sense, so the Green operator $Q^\lambda * \phi(x) = \int_{\mathbb{R}^d} \phi(y) Q^\lambda(x-y)dy$ for Markov semigroup P_t^1, is a well-defined convolution for any $\phi \in C_b(\mathbb{R}^d)$ and solves

$$(-G_1 + \lambda)u = \phi. \tag{4.3}$$

Correspondingly, let $\tilde{Q}^\lambda$ be the Green function for each $\lambda > 0$ and $x \in \mathbb{R}^d$, $\tilde{Q}^\lambda(x - \cdot)$ solves equation $\left(-\frac{1}{2}\Delta + \lambda\right) u = \delta_x$ in the distributional sense, so the Green operator $\tilde{Q}^\lambda * \phi(x) = \int_{\mathbb{R}^d} \phi(y)\tilde{Q}^\lambda(x-y)dy$ for Markov semigroup T_t^1, is a well-defined convolution for any $\phi \in C_b(\mathbb{R}^d)$ and solves

$$\big(-\frac{1}{2}\Delta + \lambda\big)u = \phi. \tag{4.4}$$

Precisely,

$$\tilde{Q}^\lambda(x) := \int_0^\infty e^{-\lambda r} p_r(x) dr, \qquad x \in \mathbb{R}^d \tag{4.5}$$

Let

$$\begin{aligned} G(x,w) &:= \int_0^\infty \frac{e^{-\lambda r}}{r^{(d+1)/2}} \Big(\exp\{-a_0 \frac{|w-x|^2}{r}\} \Big) dr, \\ f(\cdot) &:= G(\cdot, w). \end{aligned} \tag{4.6}$$

Then, for any $w \in \mathbb{R}^d$, by (2.11) there exists a positive constant K_c such that

$$|\partial_p Q^\lambda(w-\cdot)| \leq K_c G(\cdot, w). \tag{4.7}$$

$$G(\cdot, w) \in L^1(\mathbb{R}^d) \cap C_0^\infty(\mathbb{R}^d \setminus \{w\}), \qquad d = 1, 2, 3.$$

where a_0 is a positive constant and set $s = \frac{r}{2a_0}$ and $\hat{\lambda} = 2a_0\lambda$, then we have

$$\begin{aligned}
&G(x,w) + G(x,v) \\
&= \int_0^\infty \frac{e^{-\lambda r}}{r^{(d+1)/2}} \Big(\exp\{-a_0 \frac{|w-x|^2}{r}\} + \exp\{-a_0 \frac{|v-x|^2}{r}\} \Big) dr \\
&= \int_0^\infty \frac{2a_0 e^{-\hat{\lambda} s}}{(2a_0 s)^{(d+1)/2}} \Big(\exp\{-\frac{|w-x|^2}{2s}\} + \exp\{-\frac{|v-x|^2}{2s}\} \Big) ds \\
&= \int_0^\infty \frac{e^{-\hat{\lambda} s}}{\sqrt{s}(2a_0)^{(d-1)/2}} \Big(p_s(w,x) + p_s(v,x) \Big) ds
\end{aligned} \tag{4.8}$$

$$g(\cdot) := (G(\cdot, w) + G(\cdot, v)). \tag{4.9}$$

Then, for any $w, v \in \mathbb{R}^d$,

$$G(\cdot, w) \in L^1(\mathbb{R}^d) \cap C_0^\infty(\mathbb{R}^d \setminus \{w\}), G(\cdot, v) \in L^1(\mathbb{R}^d) \cap C_0^\infty(\mathbb{R}^d \setminus \{v\}) \qquad d = 1, 2, 3.$$

We have

$$g(\cdot) \in C_c^\infty(\mathbb{R}^d \setminus \{w, v\}) \cap L^1(\mathbb{R}^d) \subset C_0^\infty(\mathbb{R}^d \setminus \{w, v\}).$$

Lemma 4.3. *Assume that the non-negative function f, g and G are defined by (4.6) and (4.9). For any $\mu_0 \in M_a^T(\mathbb{R}^d)$, let $\{\mu_t, t \geq 0\}$ be the SDSM on the filtered probability space $(\Omega.\mathcal{F}_t, \mathbb{P}_{\mu_0})$ with the initial state μ_0. Then, for any $u > 0$ and any positive integer $n \in \mathbb{N}$, we have*

$$\sup_{w \in \mathbb{R}^d, v \in \mathbb{R}^d} \mathbb{E}_{\mu_0} \langle [2\tilde{Q}^\lambda(w-\cdot)^2 + 2\tilde{Q}^\lambda(v-\cdot)^2], \mu_u \rangle^{2n} < \infty,$$

and

$$\sup_{w \in \mathbb{R}^d, v \in \mathbb{R}^d} \mathbb{E}_{\mu_0} \langle g, \mu_u \rangle^{2n} < \infty.$$

Proof: To prove

$$\sup_{w \in \mathbb{R}^d, v \in \mathbb{R}^d} \mathbb{E}_{\mu_0} \langle [2\tilde{Q}^\lambda(w-\cdot)^2 + 2\tilde{Q}^\lambda(v-\cdot)^2], \mu_u \rangle^{2n} < \infty$$

we only need to prove that

$$\sup_{w \in \mathbb{R}^d} \mathbb{E}_{\mu_0} \langle [\tilde{Q}^\lambda(w-\cdot)^2], \mu_u \rangle^{2n} < \infty. \tag{4.10}$$

Similarly to prove

$$\sup_{w\in\mathbb{R}^d, v\in\mathbb{R}^d} \mathbb{E}_{\mu_0}\langle g, \mu_u\rangle^{2n} < \infty$$

we only need to prove that

$$\sup_{w\in\mathbb{R}^d} \mathbb{E}_{\mu_0}\langle f, \mu_u\rangle^{2n} < \infty. \tag{4.11}$$

To prove (4.11), first, we need to prove that $\langle f(\cdot), \mu_0\rangle < \infty$ or $f \in L^1(\mu_0)$. By the Hypothesis (4) for any $\mu_0 \in M_a^T(\mathbb{R}^d)$, there exists a positive $\epsilon > 0$ such that

$$\sup_{w\in\mathbb{R}^d}(\sup_{0\le r\le\epsilon} \langle p_r(w,\cdot), \mu_0\rangle) \le c(\mu_0) \sup_{0\le r\le\epsilon} \langle p_r(x), \mu_0(dx)\rangle) < \infty,$$

where $p_r(w,\cdot)$ is the transition density of a d-dimensional Brownian motion. Thus, for any $\mu_0 \in M_a^T(\mathbb{R}^d)$, there exists a positive constant $k_2 > 0$ such that

$$\begin{aligned}
\langle f(\cdot), \mu_0\rangle &\le \langle \int_0^\infty \frac{e^{-\hat{\lambda} r}}{r^{(d+1)/2}}\left(\exp\{-a_0\frac{|w-\cdot|^2}{r}\}\right)dr, \mu_0\rangle \\
&\le k_2\langle\left[\int_0^\infty \frac{e^{-\lambda r}}{\sqrt{r}} p_r(w,\cdot)dr\right], \mu_0\rangle \\
&\le k_2\langle\left[\int_0^\epsilon \frac{e^{-\lambda r}}{\sqrt{r}} p_r(w,\cdot)dr + \int_\epsilon^\infty \frac{e^{-\lambda r}}{\sqrt{r}} p_r(w,\cdot)dr\right], \mu_0\rangle \\
&\le k_2\left[\sup_{w\in\mathbb{R}^d}(\sup_{0\le r\le\epsilon} \langle p_r(w,\cdot), \mu_0\rangle) \int_0^\epsilon \frac{e^{-\lambda r}}{\sqrt{r}} dr + k_\epsilon\right] \\
&< \infty,
\end{aligned} \tag{4.12}$$

where

$$\sup_{w\in\mathbb{R}^d}(\sup_{0\le r\le\epsilon} \langle p_r(w,\cdot), \mu_0\rangle) < \infty,$$

since $\mu_0 \in M_a^T(\mathbb{R}^d)$.

$$\begin{aligned}
&k_\epsilon := \langle \int_\epsilon^\infty \frac{e^{-\lambda r}}{\sqrt{r}} p_r(w,\cdot)dr, \mu_0\rangle \\
&= \int_\epsilon^\infty \frac{e^{-\lambda r}}{\sqrt{r}} \langle p_r(w,\cdot), \mu_0\rangle dr \\
&< \infty,
\end{aligned}$$

since for $r \ge \epsilon$ and $M_a^T(\mathbb{R}^d)$

$$\sup_{w\in\mathbb{R}^d} \langle p_r(w,x), \mu_0(dx)\rangle \le \sup_{w\in\mathbb{R}^d} \langle p_\epsilon(w,x), \mu_0(dx)\rangle < \infty.$$

This proves that $f \in L^1(\mu_0)$. Then, by Corollary 3.4 we have $f(\cdot) \in L^1(\mu_u)$ for any $u > 0$. Now we are going to prove that $\mathbb{E}_{\mu_0}\langle f, \mu_u\rangle^m < \infty$. Define

$$\mu_0^{\bigotimes m}(dx_1, dx_2, \cdots, dx_m) := \mu_0(dx_1)\cdots\mu_0(dx_m) \tag{4.13}$$

$$f^{\bigotimes m}(x_1, \cdots, x_m) := f(x_1)\cdots f(x_m). \tag{4.14}$$

$$M_k := \mathbb{E}_{m,f^{\bigotimes m}}\left[\langle Y_t, \mu_0^{J_t}\rangle \exp\{\frac{1}{2}\int_0^t J_s(J_s-1)ds\} 1_{(\tau_k < t \le \tau_{k+1})}\right] \tag{4.15}$$

Then,

$$\mathbb{E}_{\mu_0}\langle f, \mu_u\rangle^m = \mathbb{E}_{m,f^{\bigotimes m}}\left[\langle Y_t, \mu_0^{J_t}\rangle \exp\{\frac{1}{2}\int_0^t J_s(J_s-1)ds\}\right] = \sum_{k=0}^{m-1} M_k. \tag{4.16}$$

$$M_0 = \langle P_t^m f, \mu_0^{\bigotimes m}\rangle. \tag{4.17}$$

$$\begin{aligned} M_k &= \frac{m!(m-1)!}{2^k(m-k)!(m-k-1)!}\int_{(0,t]} dr_1 \int_{(r_1,t]} dr_2 \cdots \int_{(r_{(k-1)},t]} \\ &\cdot\mathbb{E}_{m,f^{\bigotimes m}}\left[\langle P_{t-r_k}^{m-k}\Gamma_k \cdots P_{r_2-r_1}^{m-1}\Gamma_1 P_{r_1}^m f^{\bigotimes(m)}, \mu_0^{\bigotimes(m-k)},\rangle \middle| \tau_j = r_j : 1 \le j \le k\right] dr_k \\ &\le \frac{m!(m-1)!}{2^k(m-k)!(m-k-1)!}\int_{(0,t]} dr_1 \int_{(r_1,t]} dr_2 \cdots \int_{(r_{(k-1)},t]} \\ &\mathbb{E}_{m,f^{\bigotimes m}}\Bigg[c_0\langle (T_{t-r_k}^1)^{\bigotimes(m-k)}\Gamma_k \cdots (T_{r_2-r_1}^1)^{\bigotimes(m-1)}\Gamma_1 \\ &(T_{r_1}^1)^{\bigotimes(m)} f^{\bigotimes(m)}, \mu_0^{\bigotimes(m-k)}\rangle \Bigg| \tau_j = r_j : 1 \le j \le k\Bigg] dr_k \\ &\le \frac{m!(m-1)!}{2^k(m-k)!(m-k-1)!}\int_{(0,t]} dr_1 \int_{(r_1,t]} dr_2 \cdots \int_{(r_{(k-1)},t]} \\ &\mathbb{E}_{m,f^{\bigotimes m}}\left[c_0c_1c_2\cdots c_k\langle (T_t^1)^{\bigotimes(m-k)} f^{\bigotimes(m-k)}, \mu_0^{\bigotimes(m-k)}\rangle \middle| \tau_j = r_j : 1 \le j \le k\right] dr_k, \end{aligned} \tag{4.18}$$

where for $1 \le i \le k$, $c_i := \sup_{x\in\mathbb{R}^d} |T_{r_i}^1 f|(x)$ and we will prove that $c_i < \infty$. Indeed, since operator ∂_p can commute with the operator T_t^1, we get,

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}|T_t^1 f|(x) &\leq \sup_{x\in\mathbb{R}^d}\int_0^\infty k_0\frac{e^{-\lambda r}}{\sqrt{r}}\langle p_r(w-\xi), p_t(x-\xi)d\xi\rangle dr \\ &\leq k_0\sup_{x\in\mathbb{R}^d}|T_t^1\partial_p\tilde{Q}^\lambda(w-x)| \\ &= k_0\sup_{x\in\mathbb{R}^d}|\partial_p T_t^1\tilde{Q}^\lambda(w-x)| \\ &= k_0\sup_{x\in\mathbb{R}^d}|\partial_p\tilde{Q}_t^{\lambda}(w-x)| \\ &\leq k_0\sup_{z\in\mathbb{R}^d}|\partial_p\tilde{Q}_t^{\lambda}(z)| \\ &< \infty. \end{aligned} \tag{4.19}$$

Since for any fixed $t>0$, we have $p_t(\cdot)\in C_b^\infty(\mathbb{R}^d)\cap C_0^\infty(\mathbb{R}^d)$ and $\mu_0\in M_a^T(\mathbb{R}^d)$ thus

$$\sup_{\xi\in\mathbb{R}^d}\langle p_t(x-\xi), \mu_0(dx)\rangle < \infty. \tag{4.20}$$

Since there exists a positive constant c such that

$$\begin{aligned}G(x,w) &:= \int_0^\infty \frac{e^{-\lambda r}}{r^{(d+1)/2}}\exp\{-a_0\frac{|w-x|^2}{r}\}dr \\ &\leq c\partial_p\tilde{Q}^\lambda(w-x)\end{aligned} \tag{4.21}$$

holds and operator ∂_p can commute with the operator T_t^1, we get

$$\begin{aligned}\langle (T_t^1 f)(x), \mu_0\rangle &\leq \langle\int_0^\infty \frac{e^{-\lambda r}}{\sqrt{r}}\int_{\mathbb{R}^d} p_r(w-\xi)p_t(x-\xi)d\xi dr, \mu_0(dx)\rangle \\ &\leq c\langle T_t^1\partial_p\tilde{Q}^\lambda(w-x), \mu_0(dx)\rangle \\ &\leq c\langle \partial_p T_t^1\tilde{Q}^\lambda(w-x), \mu_0(dx)\rangle \\ &\leq c\sup_{w\in\mathbb{R}^d}\langle\partial_p\tilde{Q}_t^\lambda(w-x), \mu_0(dx)\rangle < \infty. \end{aligned} \tag{4.22}$$

Then, we have proved $c_i<\infty$, $i=1,2,\cdots,k$ and by (4.22) we have

$$\langle (T_t^1)^{\bigotimes(m-k)} f^{\bigotimes(m-k)}, \mu_0^{\bigotimes(m-k)}\rangle = \langle (T_t^1 f)(x), \mu_0(dx)\rangle^{(m-k)} < \infty.$$

by (4.18) we get

$$\begin{aligned}M_k &\leq \frac{m!(m-1)!}{2^k(m-k)!(m-k-1)!}\int_{(0,t]}dr_1\int_{(r_1,t]}dr_2\cdots\int_{(r_{(k-1)},t]} \\ &\quad \mathbb{E}_{m,f^{\otimes m}}\left[c_0c_1c_2\cdots c_k\langle (T_t^1)^{\bigotimes(m-k)} f^{\bigotimes(m-k)}, \mu_0^{\bigotimes(m-k)}\rangle\Big|\tau_j=r_j : 1\leq j\leq k\right]dr_k \\ &< \infty.\end{aligned} \tag{4.23}$$

Finally, we get

$$\begin{aligned}
\mathbb{E}_{\mu_0}\langle f, \mu_u\rangle^m &= \mathbb{E}_{m,f^{\otimes m}}\Big[\langle Y_t, \mu_0^{J_t}\rangle \exp\{\frac{1}{2}\int_0^t J_s(J_s-1)ds\}\Big] \\
&= \sum_{k=0}^{m-1} M_k < \infty. \qquad (4.24)
\end{aligned}$$

Now we are going to prove that (4.10). Since by Lemma 4.2 there exists a constant k_0 such that

$$\begin{aligned}
\tilde{Q}^\lambda(w-\cdot)^2 &= \int_0^\infty\int_0^\infty e^{-\lambda(u+v)} p_u(w-\cdot)p_v(w-\cdot)dudv \\
&\leq \int_0^\infty\int_0^\infty \frac{k_0 e^{-\lambda(u+v)}}{(u+v)^{d/2}} p_{(\frac{uv}{u+v})}(w-\cdot)dudv.
\end{aligned} \qquad (4.25)$$

First, let us consider the estimation of

$$\sup_{w\in\mathbb{R}^d}\langle [Q^\lambda(w-x)]^2, \mu_0(dx)\rangle$$

or $Q^\lambda(w-\cdot)\in L^2(\mu_0)$. Since for any $\epsilon\geq 0$ and by (2.11), there exists positive constants c and k such that

$$\begin{aligned}
&\langle [Q^\lambda(w-x)]^2, \mu_0(dx)\rangle \\
&\leq k\langle\int_0^\infty\int_0^\infty e^{-\lambda(u+v)}p_{cu}(w-x)p_{cv}(w-x)dudv, \mu_0(dx)\rangle \\
&\leq \frac{k}{c^2}\langle\int_0^\infty\int_0^\infty e^{-\hat{\lambda}(\xi+\eta)}p_\xi(w-x)p_\eta(w-x)d\xi d\eta, \mu_0(dx)\rangle \\
&\leq \frac{k}{c^2}\langle [\tilde{Q}^{\hat{\lambda}}(w-x)]^2, \mu_0(dx)\rangle \\
&\leq \frac{k}{c^2}\big(\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4\big) \qquad (4.26)
\end{aligned}$$

where $\tilde{Q}^\lambda(x) = \int_0^\infty e^{-\lambda u}p_u(x)du$ and we have set $\hat{\lambda} = \lambda/c$, $\xi = cu$, $\eta = cv$. Define

$$\begin{aligned}
\mathcal{I}_1 &:= \langle\int_0^\epsilon\int_0^\epsilon e^{-\lambda(u+v)}p_u(w-x)p_v(w-x)dudv, \mu_0(dx)\rangle \\
\mathcal{I}_2 &:= \langle\int_0^\epsilon\int_\epsilon^\infty e^{-\lambda(u+v)}p_u(w-x)p_v(w-x)dudv, \mu_0(dx)\rangle \\
\mathcal{I}_3 &:= \langle\int_\epsilon^\infty\int_0^\epsilon e^{-\lambda(u+v)}p_u(w-x)p_v(w-x)dudv, \mu_0(dx)\rangle \\
\mathcal{I}_4 &:= \langle\int_\epsilon^\infty\int_\epsilon^\infty e^{-\lambda(u+v)}p_u(w-x)p_v(w-x)dudv, \mu_0(dx)\rangle. \qquad (4.27)
\end{aligned}$$

Since by Lemma 4.2

$$\begin{aligned}
\mathcal{I}_1 &:= \langle \int_0^\epsilon \int_0^\epsilon e^{-\lambda(u+v)} p_u(w-x) p_v(w-x) dudv, \mu_0(dx)\rangle \\
&\le \langle \frac{1}{(2\pi)^{d/2}} \int_0^\epsilon \int_0^\epsilon e^{-\lambda(u+v)} \frac{1}{(u+v)^{d/2}} \cdot \frac{1}{(2\pi(\frac{uv}{u+v}))^{d/2}} \exp\{-\frac{|w-x|^2}{(\frac{2uv}{u+v})}\} dudv, \mu_0(dx)\rangle \\
&\le \sup_{0\le u\le\epsilon, 0\le v\le\epsilon} \left[\langle \frac{1}{(2\pi(\frac{uv}{u+v}))^{d/2}} \exp\{-\frac{|w-x|^2}{(\frac{2uv}{u+v})}\}, \mu_0(dx)\rangle \right] \cdot \\
&\qquad \left[\frac{1}{(2\pi)^{d/2}} \int_0^\epsilon \int_0^\epsilon e^{-\lambda(u+v)} \frac{1}{(u+v)^{d/2}} \cdot dudv \right] \\
&\le \sup_{0\le r\le\epsilon/2} \left[\langle \frac{1}{(2\pi r)^{d/2}} \exp\{-\frac{|w-x|^2}{2r}\}, \mu_0(dx)\rangle \right] \cdot \\
&\qquad \left[\langle \frac{1}{(2\pi)^{d/2}} \int_0^\epsilon \int_0^\epsilon e^{-\lambda(u+v)} \frac{1}{(u+v)^{d/2}} \cdot dudv \right] \\
&\le \frac{1}{(2\pi)^{d/2}} \sup_{w\in\mathbb{R}^d} \sup_{0\le r\le\epsilon} \langle p_r(w-x), \mu_0(dx)\rangle \int_0^\epsilon \int_0^\epsilon e^{-\lambda(u+v)} \frac{1}{(u+v)^{d/2}} dudv, \qquad (4.28)
\end{aligned}$$

where $r = \frac{uv}{u+v}$. It is easy to check that

$$\int_0^\epsilon \int_0^\epsilon e^{-\lambda(u+v)} \frac{1}{(u+v)^{d/2}} dudv < \infty$$

and by Hypothesis (4), we have

$$\sup_{w\in\mathbb{R}^d} \sup_{0\le r\le\epsilon} \langle p_r(w-x), \mu_0(dx)\rangle < \infty.$$

Thus we have $\mathcal{I}_1 < \infty$. Now let us consider the estimation of the $\mathcal{I}_2$. Since for any fixed $\epsilon > 0$

$$\mathcal{I}_2 := \langle \int_0^\epsilon e^{-\lambda u} p_u(w-x) du \int_\epsilon^\infty e^{-\lambda v} p_v(w-x) dv, \mu_0(dx)\rangle$$

and for $v \ge \epsilon$, $p_v(z)$ is bounded, smooth function. Thus, we have

$$\langle \int_0^\epsilon e^{-\lambda u} p_u(w-x) du, \mu_0(dx)\rangle \le \sup_{w\in\mathbb{R}^d} \sup_{0\le u\le\epsilon} \langle p_u(w-x), \mu_0(dx)\rangle \int_0^\epsilon e^{-\lambda u} du < \infty \quad (4.29)$$

and

$$
\begin{aligned}
\mathcal{I}_2 &:= \langle \int_0^{\epsilon} e^{-\lambda u} p_u(w-x) du \int_{\epsilon}^{\infty} e^{-\lambda v} p_v(w-x) dv, \mu_0(dx) \rangle \\
&\le \sup_{v\ge\epsilon} \sup_{x\in\mathbb{R}^d} p_v(w-x) \left[\int_{\epsilon}^{\infty} e^{-\lambda v} dv \right] \\
&\quad \cdot \sup_{0\le u\le\epsilon} \langle p_u(w-x), \mu_0(dx) \rangle \left[\int_0^{\epsilon} e^{-\lambda u} du \right] \\
&\le \sup_{v\ge\epsilon} \sup_{z\in\mathbb{R}^d} p_v(z) \left[\int_{\epsilon}^{\infty} e^{-\lambda v} dv \right] \\
&\quad \cdot \sup_{w\in\mathbb{R}^d} \sup_{0\le u\le\epsilon} \langle p_u(w-x), \mu_0(dx) \rangle \left[\int_0^{\epsilon} e^{-\lambda u} du \right] < \infty. \qquad (4.30)
\end{aligned}
$$

Similarly, we can prove that $\mathcal{I}_3 < \infty$ and $\mathcal{I}_4 < \infty$. Thus, by (4.26) we have proved that

$$
\sup_{w\in\mathbb{R}^d} \langle [Q^{\lambda}(w-x)]^2, \mu_0(dx) \rangle < \infty. \qquad (4.31)
$$

Let $\gamma = \frac{uv}{u+v}$. For any $t > 0$ fixed, similar to (4.19) and (4.22) we can prove that

$$
\sup_{w\in\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} |T_t^1 p_{\gamma}(w-\cdot)|(x) < \infty, \qquad (4.32)
$$

and

$$
\sup_{w\in\mathbb{R}^d} \langle |T_t^1 p_{\gamma}(w-\cdot)|(x), \mu_0(dx) \rangle < \infty. \qquad (4.33)
$$

In (4.16), we replace f by $p_{\gamma}(w-\cdot)$, bsaed on (4.32) and (4.33), we get (4.23) and (4.24). □

Theorem 4.4. *Assume Hypotheses* (2), (3) *and* (4) *are satisfied. For either* $d = 1$, 2 *or* 3, *the random field*

$$
\Xi_t(x) := \int_0^t \int_{\mathbb{R}^d} \langle h_p(y-\cdot) \partial_p Q^{\lambda}(x-\cdot), \mu_s \rangle W(dy, ds) \qquad (4.34)
$$

is a square-integrable $\mathcal{F}_t$*-martingale. There exists a version of* $\{\Xi_t(x), t \ge 0, x \in \mathbb{R}^d\}$ *which is continuous in* $t \in [0,\infty)$ *for every fixed* $x \in \mathbb{R}^d$ *and Hölder jointly continuous in* $(t,x) \in [0,\infty) \times \mathbb{R}^d$, *for every* $\lambda > 0$ *and* $p \in \{1, 2, \ldots, d\}$.

Proof: In the following, based on the Lemma 2.1, we will give a short and elegant proof of the jointly Hölder continuity of the stochastic interacting term (4.34). For any choice of $w, v, x, y, \xi \in \mathbb{R}^d$, define

$$
\Psi_{y,w}(\xi) := h_p(y-\xi) \partial_p Q^{\lambda}(w-\xi)
$$

Then,

$$\Xi_t(w) = \int_0^t \int_{\mathbb{R}^d} \langle \Psi_{y,w}(\cdot), \mu_s \rangle W(dy, ds) \tag{4.35}$$

We can get

$$\mathbb{E}_{\mu_0} \left[\Xi_t(w) - \Xi_s(v) \right]^{2n} \quad \leq \quad 2^{2n-1} I_{ts}(ww) + 2^{2n-1} I_{ss}(wv), \tag{4.36}$$

where

$$\begin{aligned} I_{ts}(ww) &:= \mathbb{E}_{\mu_0} \left[\Xi_t(w) - \Xi_s(w) \right]^{2n}, \\ I_{ss}(wv) &:= \mathbb{E}_{\mu_0} \left[\Xi_s(w) - \Xi_s(v) \right]^{2n}. \end{aligned}$$

By the moment inequalities for martingales (See p110, Theorem 3.1 of Ikeda and Watanabe [12]), there exists a positive number k_0 such that

$$I_{ts}(ww) = \mathbb{E}_{\mu_0} \left[\Xi_t(w) - \Xi_s(w) \right]^{2n} \leq k_0 \mathbb{X}(t, s, w) \tag{4.37}$$

where

$$\mathbb{X}(t, s, w) := \mathbb{E}_{\mu_0} \bigg(\int_s^t \int_{\mathbb{R}^d} \langle \Psi_{y,w}(\cdot), \mu_u \rangle^2 dy du \bigg)^n \tag{4.38}$$

By the moment inequalities for martingales again, there exists a positive number k_1 such that

$$I_s(wv) = \mathbb{E}_{\mu_0} \left[\Xi_s(w) - \Xi_s(v) \right]^{2n} \leq k_1 \mathbb{Z}(s, w, v) \tag{4.39}$$

where

$$\mathbb{Z}(s, w, v) := \mathbb{E}_{\mu_0} \bigg[\int_0^s \int_{\mathbb{R}^d} \langle \Psi_{y,w}(\cdot) - \Psi_{y,v}(\cdot), \mu_u \rangle^2 dy du \bigg]^n.$$

Then, there exists a constant $\alpha \in (0, 1)$ such that

$$\begin{aligned} I_{ss}(wv) \quad &:= \quad \mathbb{E}_{\mu_0} \left[\Xi_s(w) - \Xi_s(v) \right]^{2n} \\ &\leq \quad k_1 \mathbb{E}_{\mu_0} \bigg[\int_0^s \int_{\mathbb{R}^d} \langle \Psi_{y,w}(\cdot) - \Psi_{y,v}(\cdot), \mu_u \rangle^2 dy du \bigg]^n \\ &\leq \quad k_1 \mathbb{E}_{\mu_0} \bigg[\int_0^s \int_{\mathbb{R}^d} \langle |h_p(y - \cdot)| \big(|\partial_p Q^\lambda (w - \cdot) - \partial_p Q^\lambda (v - \cdot)| \big), \mu_u \rangle^2 dy du \bigg]^n \end{aligned} \tag{4.40}$$

Remark: Based on the Lemma 2.1, we have

$$\begin{aligned}
&\leq k_1\rho_{pp}(0,0)|w-v|^{2n\alpha}\mathbb{E}_{\mu_0}\left[\int_0^s\langle\int_0^\infty \frac{e^{-\lambda r}}{r^{(d+1)/2}}\Bigg(\exp\{-a_0\frac{|w-\cdot|^2}{r}\}\right.\\
&+\left.\exp\{-a_0\frac{|v-\cdot|^2}{r}\}\Bigg)dr,\mu_u\rangle^2 du\right]^n\\
&= k_1\rho_{pp}(0,0)|w-v|^{2n\alpha}\mathbb{E}_{\mu_0}\left[\int_0^s\langle(G(\cdot,w)+G(\cdot,v)),\mu_u\rangle^2 du\right]^n\\
&\leq k_1\rho_{pp}(0,0)|w-v|^{2n\alpha}\mathbb{E}_{\mu_0}\left[\int_0^s\langle g(\cdot),\mu_u\rangle^2 du\right]^n
\end{aligned}$$

Remark: By the Hölder's inequality with $p=n/(n-1)$ and $q=n$, we have

$$\begin{aligned}
&\leq k_1|w-v|^{2n\alpha}\mathbb{E}_{\mu_0}\left[\int_0^s\langle g(\cdot),\mu_u\rangle^{2n} du\right]\left[\int_0^s 1^p du\right]^{n-1}\\
&= k_1|w-v|^{2n\alpha}\left[\int_0^s \mathbb{E}_{\mu_0}\langle g(\cdot),\mu_u\rangle^{2n} du\right]\left[\int_0^s 1^p du\right]^{n-1}. \qquad (4.41)
\end{aligned}$$

where a_0 is a positive constant and $G(x,w)$ and $g(\cdot)$ are defined by (4.6) and (4.9). Then, for any $w,v\in\mathbb{R}^d$,

$$G(\cdot,w)\in L^1(\mathbb{R}^d)\cap C_0^\infty(\mathbb{R}^d\setminus\{w\}), G(\cdot,v)\in L^1(\mathbb{R}^d)\cap C_0^\infty(\mathbb{R}^d\setminus\{v\}) \qquad d=1,2,3.$$

We have

$$g(\cdot)\in C_c(\mathbb{R}^d\setminus\{w,v\})\cap L^1(\mathbb{R}^d)\subset C_0(\mathbb{R}^d\setminus\{w,v\}).$$

Then, by Lemma 4.3, we have

$$\sup_{w\in\mathbb{R}^d,v\in\mathbb{R}^d}\mathbb{E}_{\mu_0}\langle g(\cdot),\mu_u\rangle^{2n}<\infty, \qquad (4.42)$$

where $0\leq u\leq s\leq T$. Thus, by (4.40) there exists a positive constant k_3 which is independent of s such that

$$\mathbb{E}_{\mu_0}\left[\Xi_s(w)-\Xi_s(v)\right]^{2n}\leq k_1k_3\cdot s^n\cdot|w-v|^{2n\alpha}\leq k_1k_3\cdot T^n\cdot|w-v|^{2n\alpha}. \qquad (4.43)$$

Now let us consider the estimation of (4.37).

$$\begin{aligned}
I_{ts}(ww) &:= \mathbb{E}_{\mu_0}\left[\Xi_t(w)-\Xi_s(w)\right]^{2n} \\
&\leq c_1\rho_{pp}(0,0)\mathbb{E}_{\mu_0}\left[\int_s^t \langle\big(|\partial_p Q^\lambda(w-\cdot)|\big),\mu_u\rangle^2 du\right]^n \\
&\leq c_1\rho_{pp}(0,0)\mathbb{E}_{\mu_0}\left[\int_s^t \langle\int_0^\infty \frac{e^{-\lambda r}}{r^{(d+1)/2}}\bigg(\exp\{-a_0\frac{|w-x|^2}{r}\}\bigg)dr,\mu_u\rangle^2 du\right]^n \\
&= c_1\mathbb{E}_{\mu_0}\left[\int_s^t \langle G(\cdot,w),\mu_u\rangle^2 du\right]^n \\
&\leq c_1\rho_{pp}(0,0)\mathbb{E}_{\mu_0}\left[\int_s^t \langle f(\cdot),\mu_u\rangle^{2n} du\right]\left[\int_s^t 1^{(\frac{n}{n-1})}du\right]^{n-1} \\
&= c_1\rho_{pp}(0,0)\left[\int_s^t \mathbb{E}_{\mu_0}\langle f(\cdot),\mu_u\rangle^{2n} du\right]\left[\int_s^t 1\cdot du\right]^{n-1}. \qquad (4.44)
\end{aligned}$$

where a_0 is a positive constant and

$$\begin{aligned}
&G(x,w) := \int_0^\infty \frac{e^{-\lambda r}}{r^{(d+1)/2}}\bigg(\exp\{-a_0\frac{|w-x|^2}{r}\}\bigg)dr, \\
&f(\cdot) := G(\cdot,w). \qquad (4.45)
\end{aligned}$$

Then, for any $w\in\mathbb{R}^d$,

$$G(\cdot,w)\in L^1(\mathbb{R}^d)\cap C_0^\infty(\mathbb{R}^d\setminus\{w\}), \qquad d=1,2,3.$$

So here f is just defined by (4.6), By Lemma 4.3, we have $f\in L^1(\mu_0)$. By Corollary 3.4 we have $f(\cdot)\in L^1(\mu_u)$ for any $u>0$. By Lemma 4.3, we have

$$\sup_{w\in\mathbb{R}^d}\mathbb{E}_{\mu_0}\langle f(\cdot),\mu_u\rangle^{2n}<\infty,$$

where $0\leq s\leq u\leq t\leq T$. Thus, by (4.44) there exists a positive constant c_2 which is independent of w such that

$$\mathbb{E}_{\mu_0}\left[\Xi_t(w)-\Xi_s(w)\right]^{2n}\leq c_1c_2|t-s|^n. \qquad (4.46)$$

From notation,

$$\begin{aligned}
I_{ts}(ww) &:= \mathbb{E}_{\mu_0}\left[\Xi_t(w)-\Xi_s(w)\right]^{2n}, \\
I_{ss}(wv) &:= \mathbb{E}_{\mu_0}\left[\Xi_s(w)-\Xi_s(v)\right]^{2n},
\end{aligned}$$

finally, from (4.43) and (4.46) we obtain

$$\begin{aligned}
&\mathbb{E}_{\mu_0}\left[\Xi_t(w)-\Xi_s(v)\right]^{2n} \\
&\leq 2^{2n-1}I_{ts}(ww)+2^{2n-1}I_{ss}(wv) \\
&\leq const[|t-s|^n+|w-v|^{2n\alpha}], \qquad (4.47)
\end{aligned}$$

where *const* is a constant which is independent of s, t, w, v, and T. Then, the joint Hölder continuity of $\Xi_t(x)$ in (t,x) follows from Kolmogorov and Čentsov's continuity Theorem.

□

Theorem 4.5. *Assume Hypotheses (2), (3) and (4) are satisfied. For either $d = 1$, 2 or 3, the random field*

$$Y_t(x) := \int_0^t \int_{\mathbb{R}^d} Q^\lambda(x-y)M(dy,ds) \tag{4.48}$$

is a square-integrable $\mathcal{F}_t$-martingale with

$$\langle Y(x)\rangle_t = \gamma\sigma^2 \int_0^t \langle Q^\lambda(x-y)^2, \mu_s(dy)\rangle ds$$

. There exists a version of $\{Y_t(x), t \geq 0, x \in \mathbb{R}^d\}$ which is continuous in $t \in [0,\infty)$ for every fixed $x \in \mathbb{R}^d$ and Hölder jointly continuous in $(t,x) \in [0,\infty) \times \mathbb{R}^d$, for every $\lambda > 0$.

Proof:

First, from the moment inequality for the martingales we have

$$\begin{aligned}
&\mathbb{E}_{\mu_0}|Y_t(w) - Y_s(v)|^{2n} \\
&\leq 2^{2n-1}\mathbb{E}_{\mu_0}[Y_t(w) - Y_s(w)|^{2n} + 2^{2n-1}\mathbb{E}_{\mu_0}[Y_s(w) - Y_s(v)|^{2n} \\
&\leq 2^{2n-1}\mathbb{E}_{\mu_0}[\int_s^t \langle [Q^\lambda(w-\cdot)]^2, \mu_u(dy)\rangle du]^n \\
&+2^{2n-1}\mathbb{E}_{\mu_0}[\int_0^s \langle [Q^\lambda(w-\cdot) - Q^\lambda(v-\cdot)]^2, \mu_u(dy)\rangle du]^n \\
&= 2^{2n-1}\mathbb{K}(s,t,w) + 2^{2n-1}\mathbb{H}(s,w,v)
\end{aligned} \tag{4.49}$$

where

$$\begin{aligned}
\mathbb{K}(s,t,w) &:= \mathbb{E}_{\mu_0}[\int_s^t \langle [Q^\lambda(w-x)]^2, \mu_u(dx)\rangle du]^n \\
\mathbb{H}(s,w,v) &:= \mathbb{E}_{\mu_0}[\int_0^s \langle [Q^\lambda(w-x) - Q^\lambda(v-x)]^2, \mu_u(dx)\rangle du]^n
\end{aligned} \tag{4.50}$$

and by the Hölder's inequality we get

$$\begin{aligned}
&\mathbb{K}(s,t,w) := \mathbb{E}_{\mu_0}[\int_s^t \langle [Q^\lambda(w-x)]^2, \mu_u(dx)\rangle du]^n \\
&\leq \mathbb{E}_{\mu_0}[\int_s^t 1^{(\frac{n}{n-1})} du]^{n-1}[\int_s^t \langle [Q^\lambda(w-x)]^2, \mu_u(dx)\rangle^n du] \\
&= |t-s|^{n-1}\int_s^t \mathbb{E}_{\mu_0}\langle [Q^\lambda(w-x)]^2, \mu_u(dx)\rangle^n du
\end{aligned} \tag{4.51}$$

In the following, we will prove that

$$\sup_{w\in\mathbb{R}^d} \mathbb{E}_{\mu_0}\langle [Q^\lambda(w-x)]^2, \mu_u(dx)\rangle^n < \infty.$$

Now let us consider

$$\mathbb{H}(s,w,v) := \mathbb{E}_{\mu_0}[\int_0^s \langle [Q^\lambda(w-x) - Q^\lambda(v-x)]^2, \mu_u(dx)\rangle du]^n \tag{4.52}$$

and by the Hölder's inequality with ($p = \frac{n}{n-1}$, $q = n$) and by Lemma 2.2 we get

$$\begin{aligned}
&\mathbb{H}(s,w,v) := \mathbb{E}_{\mu_0}[\int_0^s \langle [Q^\lambda(w-x) - Q^\lambda(v-x)]^2, \mu_u(dx)\rangle du]^n \\
&\leq \mathbb{E}_{\mu_0}[\int_0^s 1^p du]^{n-1}[\int_0^s \langle [Q^\lambda(w-x) - Q^\lambda(v-x)]^2, \mu_u(dx)\rangle^n du] \\
&\leq c|s|^{n-1}|w-v|^{2n\alpha}\mathbb{E}_{\mu_0}\int_0^s \langle [\int_0^\infty e^{-\lambda r}(p_r(w-x) + p_r(v-x))dr]^2, \mu_u(dx)\rangle^n du] \\
&\text{Define } \tilde{Q}^\lambda(x) = \textstyle\int_0^\infty e^{-\lambda u} p_u(x)du \\
&= c|s|^{n-1}|w-v|^{2n\alpha}\int_0^s \mathbb{E}_{\mu_0}\langle [\tilde{Q}^\lambda(w-x) + \tilde{Q}^\lambda(v-x)]^2, \mu_u(dx)\rangle^n du] \\
&\leq c|s|^{n-1}|w-v|^{2n\alpha}\int_0^s \mathbb{E}_{\mu_0}\langle [2\tilde{Q}^\lambda(w-x)^2 + 2\tilde{Q}^\lambda(v-x)^2], \mu_u(dx)\rangle^n du]
\end{aligned} \tag{4.53}$$

By (4.26) we have

$$\sup_{w\in\mathbb{R}^d} \langle [\tilde{Q}^\lambda(w-x)]^2, \mu_0(dx)\rangle < \infty.$$

$$\sup_{v\in\mathbb{R}^d} \langle [\tilde{Q}^\lambda(v-x)]^2, \mu_0(dx)\rangle < \infty.$$

Then, by Corollary 3.4 we have

$$\sup_{w\in\mathbb{R}^d} \langle [\tilde{Q}^\lambda(w-x)]^2, \mu_u(dx)\rangle < \infty.$$

Then, by Lemma 4.3 we have that

$$\sup_{w\in\mathbb{R}^d, v\in\mathbb{R}^d} \mathbb{E}_{\mu_0}\langle [2\tilde{Q}^\lambda(w-x)^2 + 2\tilde{Q}^\lambda(v-x)^2], \mu_u(dx)\rangle^n < \infty.$$

Then, the joint Hölder continuity of $Y_t(x)$ in (t,x) follows from (4.49), (4.51), (4.53) and Kolmogorov and Čentsov's continuity Theorem.

□

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