EXISTENCE AND UNIQUENESS OF CLASSICAL, NONNEGATIVE, SMOOTH SOLUTIONS OF A CLASS OF SEMI-LINEAR SPDES

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Abstract. In this paper, the regularity and the ψ-semigroup property of the solutions to a class of stochastic partial differential equations (SPDEs) derived from a class of interacting superprocesses are investigated.

Key words. Non-linear SPDE, classical smooth solution, ψ-semigroup, nonnegative solution.

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1. Introduction.

In order to investigate new properties of a class of superprocesses with dependent spatial motion (SDSMs) studied in Wang [7] and Dawson et al. [2], stochastic log-Laplace functionals for SDSMs have been constructed in [4] and we have derived the following semi-linear SPDE:

$$
\begin{aligned}
\psi_{r,t}(x) = \phi(x) &+ \int_r^t \left[\frac{1}{2}a(x)\partial_{xx}^2\psi_{r,s}(x) - \frac{1}{2}\sigma(x)\psi_{r,s}(x)^2\right] ds \\
&+ \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x\psi_{r,s}(x)W(ds,dy), \qquad t \geq r \geq 0,
\end{aligned}
\tag{1.1}
$$

where $\partial_x = \frac{\partial}{\partial x}$, $\partial_{xx}^2 = \frac{\partial^2}{\partial x^2}$, $W(ds,dx)$ is a space-time white noise or Brownian sheet (See Walsh [6] for the definition of Brownian sheet) and the last term in (1.1) is the Itô stochastic integral. To define the solution of SPDE (1.1) and describe conditions for the coefficients to guarantee the existence and the uniqueness of the solution of SPDE (1.1), first we have to introduce following notations. Let $L_2(\mathbb{R})$ be the Hilbert space of all square-integrable functions on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ with inner product $\langle \cdot, \cdot \rangle_0$ and norm $\|\cdot\|_0$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-field and λ is the Lebesgue measure on $\mathbb{R}$. Let $\{\mathcal{F}_t : t \geq 0\}$ be σ-fields generated by the Brownian sheet W. For a given Banach space $\mathbb{X}$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and any given $0 \leq T_0 < T$, let $\mathbb{L}_2(\Omega, \mathbb{X})$ be the set of all square-integrable $\mathbb{X}$-random variables, $\mathbb{L}_2([T_0,T] \times \Omega, \mathbb{X})$ be the set of all square-integrable (with respect to measure $\lambda \times \mathbb{P}$, where λ is the Lebesgue measure on $[T_0,T]$) $\mathbb{X}$-processes from $[T_0,T] \times \Omega$ into $\mathbb{X}$, $C([T_0,T], \mathcal{P}, \mathbb{X})$ be the set of all $\mathbb{X}$-processes, which are strongly continuous from $[T_0,T]$ into $\mathbb{X}$, and $\mathbb{L}_2([T_0,T], \mathcal{P}, \mathbb{X})$ be the set of all predictable representatives of $\mathbb{L}_2([T_0,T] \times \Omega, \mathbb{X})$. Let $\overline{\mathcal{B}([T_0,T]) \otimes \mathcal{F}}$ denote the completion of $\mathcal{B}([T_0,T]) \otimes \mathcal{F}$ with respect to measure $\lambda \times \mathbb{P}$. $\mathbb{L}_2^w([T_0,T]; \mathbb{X})$ is the set of all $\overline{\mathcal{B}([T_0,T]) \otimes \mathcal{F}}$ measurable mappings f from $[T_0,T] \times \Omega$ into $\mathbb{X}$ such that $f(\cdot, w) \in \mathbb{L}_2([T_0,T]; \mathbb{X})$, $(\mathbb{P} - a.s.)$. $\mathbb{L}_2^w([T_0,T]; \mathcal{P}; \mathbb{X})$ stands for the set of all predictable representatives of $\mathbb{L}_2^w([T_0,T]; \mathbb{X})$. $\mathbb{H}^m(\mathbb{R}^d)$ denotes the Sobolev space of classes of functions that, together with their partial derivatives in the sense of distribution up to order m, are square integrable on $\mathbb{R}^d$ with norm defined by

$$
\|\phi\|_m := \sqrt{\sum_{|\alpha| \leq m} \|\partial^\alpha \phi\|_0^2}, \qquad \phi \in \mathbb{H}^m(\mathbb{R}^d),
$$

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where $\partial^\alpha = \partial^{\alpha_1\alpha_2\cdots\alpha_d}$ if $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$, $\|\cdot\|_0$ is the norm of $L_2(\mathbb{R}^d)$. $\{\mathbb{H}^m(\mathbb{R}^d) : m \geq 0\}$ are Hilbert spaces. In particular, we have $\mathbb{H}^0(\mathbb{R}^d) = L_2(\mathbb{R}^d)$. Let $C_b^k(\mathbb{R}) \bigcap \mathbb{H}^k(\mathbb{R})$ denote the set of functions that, together with their bounded, continuous derivatives [1] up to order k, are square integrable on $\mathbb{R}$. Since (1.1) is derived from a class of interacting superprocesses, the coefficients of SPDE (1.1) have to satisfy the following condition:

$$a(x) := c^2(x) + \rho(0), \qquad x \in \mathbb{R}, \tag{1.2}$$

where

$$\rho(x) := \int_{\mathbb{R}} h(y-x)h(y)dy. \tag{1.3}$$

Throughout this paper, we assume that $c(\cdot)$, $h(\cdot)$, $\sigma(\cdot) \geq 0$ are bounded, continuous on $\mathbb{R}$ and $h(\cdot)$ is square integrable with respect to Lebesgue measure λ on $\mathbb{R}$. A generalized solution of (1.1) is defined as follows.

Definition 1.1. For any given initial data $\phi \in \mathbb{L}_2(\Omega, \mathbb{R})$, a stochastic process $\psi_{r,\cdot} \in \mathbb{L}_2([r,T], \mathcal{P}, L_2(\mathbb{R}))$ is called a *generalized solution* of equation (1.1) if, for every $f \in C_c^\infty(\mathbb{R}^d)$, the space of infinitely differentiable functions with compact support, it satisfies the following equation:

$$\begin{aligned}\langle \psi_{r,t}, f\rangle_0 = \langle \phi, f\rangle_0 &+ \int_r^t \frac{1}{2}\bigg(- \langle a\psi'_{r,s}, f'\rangle_0 - \langle a'\psi'_{r,s} + \sigma\psi_s^2, f\rangle_0\bigg) ds \\ &+ \int_r^t \int_{\mathbb{R}} \langle h(y-\cdot)\psi'_{r,s}, f\rangle_0 dW(ds,dy), \qquad \text{for any } t \geq r \geq 0,\ \mathbb{P}\text{-a.s.},\end{aligned} \tag{1.4}$$

where g' denotes the derivative of g in the sense of distribution or in the classical sense according as g is a generalized function or regular differentiable function.
Now, let us give the basic assumption for the coefficients of (1.1) as follows.
Basic Condition: For a given integer $m \geq 1$, we assume that
(1) $c(x) \in C_b^{m+1}(\mathbb{R})$ and there exists an $\epsilon > 0$ such that $c^2(x) \geq \epsilon$, where $C_b^k(\mathbb{R})$ is the set of functions on $\mathbb{R}$ having bounded, continuous derivatives up to order k inclusive.
(2) $h(x) \in C_b^{m+1}(\mathbb{R}) \bigcap \mathbb{H}^{m+1}(\mathbb{R})$.
(3) $\sigma(x) \in C_b^{m+1}(\mathbb{R})$ and there exist two positive numbers $0 < \sigma_a < \sigma_b$ such that $\sigma_a \leq \sigma(x) \leq \sigma_b$ holds for all $x \in \mathbb{R}$.
For a given initial function $\phi \in \{C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})\}$, Li et al. ([4]) has given a proof of the existence and the uniqueness of a generalized, nonnegative solution of the equation (1.1). More precisely, Li et al. ([4]) has following Theorem.

Theorem 1.1. ([4]) *Suppose that the basic condition holds. Then, for any $\phi \in \{C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})\}$, equation* (1.1) *has a unique $C_b(\mathbb{R}) \bigcap \mathbb{H}^1(\mathbb{R})$-valued, non-negative, strong solution $\{\psi_{r,t} : t \geq r \geq 0\}$. Furthermore, for any $\phi \in \{C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})\}$, $\|\psi_{r,t}\|_a \leq \|\phi\|_a$ holds $\mathbb{P}-a.s.$ for all $t \geq r$, where $\|\phi\|_a$ is the supremum norm of ϕ.*

Remark. Above strong solution is in the probability sense, which means that for the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as well as Brownian sheet W defined on it, (1.4) is satisfied.
This naturally raises a question: If we assume that m can be any positive integer in the basic condition and the initial function ϕ is an infinitely differentiable, square integrable function, can we prove the existence and the uniqueness of the classical smooth solution $\psi_{r,t}(x)$ of equation (1.1)? (where the classical smooth solution roughly means that $\psi_{r,t}(x)$ is infinitely differentiable in x. The precise definition will be given later.)

To answer this question, we have checked currently existing results. Even though the initial function and the coefficients are smooth functions, Kurtz and Xiong [3] can only give a unique $L_2(\mathbb{R})$-valued solution due to the nonlinearity of the SPDEs. Da Prato

[1] Here we make a convention. If we do not clearly indicate that the derivative is in the sense of distribution, it means that the derivative is in the classical sense.

and Zabczyk [1] and Rozovskii [5] cannot directly handle this type of nonlinear SPDE with random term which is an Itô integral with respect to a Brownian sheet. However, for linear SPDEs with random terms which are the Itô integrals with respect to finite dimensional Brownian motions, Rozovskii [5] does obtain smooth solution for this kind of linear SPDEs. In order to generalize Rozovskii's results to our semi-linear SPDE case, first let us give notations and more details of Rozovskii's results for the linear SPDE case.

2. Linear SPDE.

Consider the following Cauchy problem

$$u_{r,t}(x,\omega)=\phi(x,\omega)+I+II+III,\qquad (t,x,\omega)\in(r,T]\times\mathbb{R}^d\times\Omega, \tag{2.1}$$

where

$$I=\int_r^t\sum_{i,j=1}^d\partial_{x_j}[a^{ij}(s,x,\omega)\partial_{x_i}u_{r,s}(x,\omega)]ds,$$

$$II=\int_r^t\left\{\sum_{i=1}^d b^i(s,x,\omega)\partial_{x_i}u_{r,s}(x,\omega)+c(s,x,\omega)u_{r,s}(x,\omega)+f(s,x,\omega)\right\}ds,$$

$$III=\int_r^t\sum_{l=1}^{d_1}\left\{\sum_{i=1}^d\sigma^{il}(s,x,\omega)\partial_{x_i}u_{r,s}(x,\omega)+h^l(s,x,\omega)u_{r,s}(x,\omega)+g^l(s,x,\omega)\right\}dB^l(s),$$

and $\{B^l(t),l=1,\cdots,d_1\}$ are one-dimensional Brownian motions. In the following we often denote $\mathbb{H}^m(\mathbb{R}^d)$ by $\mathbb{H}^m$ if $\mathbb{R}^d$ is clear from context.

Definition 2.1. A function $u_{r,\cdot}\in\mathbb{L}_2^w([r,T];\mathcal{P};\mathbb{H}^1)$ is called a *generalized solution* to problem (2.1) if for each $y\in C_c^\infty(\mathbb{R}^d)$, the following equality holds $\mathbb{P}-a.s.$:

$$\begin{aligned}\langle u_{r,t}(\cdot),y\rangle_0=\langle\phi,y\rangle_0+\int_r^t\Bigg\{&\sum_{i,j=1}^d\langle-[a^{ij}(s,x,\omega)\partial_{x_i}u_{r,s}(x,\omega)],\partial_{x_j}y\rangle_0\\&+\sum_{i=1}^d\langle b^i(s,x,\omega)\partial_{x_i}u_{r,s}(x,\omega),y\rangle_0\\&+\langle c(s,x,\omega)u_{r,s}(x,\omega),y\rangle_0+[f(s,x,\omega),y]_0\Bigg\}ds\\&+\sum_{l=1}^{d_1}\int_r^t\langle\sum_{i=1}^d\sigma^{il}(s,x,\omega)\partial_{x_i}u_{r,s}(x,\omega)+h^l(s,x,\omega)u_{r,s}(x,\omega)\\&+g^l(s,x,\omega),\quad y\rangle_0dB^l(s),\end{aligned}\tag{2.2}$$

where $[\cdot,\cdot]_0$ is the canonical bilinear functional of the normal triple $(\mathbb{H}^{-1},\mathbb{L}_2,\mathbb{H}^1)$.

Remark. For more details of canonical bilinear functional and the normal triple, the reader is refereed to Chapter 3 of Rozovskii [5].

Then, Rozovskii [5] has following theorem.

Theorem 2.1. *Suppose that there exists a $\delta>0$, which is independent of $t,x,\omega,$, and ξ, such that*

$$2\sum_{i,j=1}^d a^{ij}(t,x,\omega)\xi^i\xi^j-\sum_{l=1}^{d_1}|\sum_{i=1}^d\sigma^{il}(t,x,\omega)\xi^i|^2\geq\delta\sum_{i=1}^d|\xi^i|^2,\qquad\forall\,\xi\in\mathbb{R}^d,t\geq0, \tag{2.3}$$

holds (Above (2.3) is called superparabolic condition) and for a positive integer m, the following conditions are satisfied:

(a) *The functions* a^{ij}, b^i, c, σ^{il},h^l $(i,j = 1,2,\cdots,d, l = 1,2,\cdots,d_1)$ *are* $\mathcal{B}([0,T]\times\mathbb{R}^d)\otimes\mathcal{F}$ *measurable, bounded, predictable (for each* $x\in\mathbb{R}^d$*), real functions, and* ϕ *is an* $\mathcal{F}_r$*-measurable function taking value in* $L_2(\mathbb{R}^d)$.
(b) *The functions* a^{ij}, b^i, c, σ^{il},h^l $(i,j = 1,2,\cdots,d, l = 1,2,\cdots,d_1)$ *are differentiable in* x *up to order* m *for all* $t\geq 0$ *and* ω. *They, together with their derivatives, are uniformly bounded with respect to* t, x, *and* ω *by a constant* $K(m)$.
(c) $\phi\in\mathbb{L}_2(\Omega,\mathbb{H}^m)$, $f\in\mathbb{L}_2([0,T]\times\Omega,\mathbb{H}^{m-1})$, $g^l\in\mathbb{L}_2([0,T]\times\Omega,\mathbb{H}^m)$, *where* $l=1,2,\cdots,d_1$. *Then, there exists a unique generalized solution* u *of* (2.1)*, which belongs to the class* $\mathbb{L}_2([r,T];\mathcal{P};\ \mathbb{H}^{m+1})\cap C([r,T];\mathcal{P};\mathbb{H}^m)$ *and satisfies equality* (2.2) *for all* $t\in[r,T]$ *and almost surely with respect to probability* $\mathbb{P}$. *There exists an* $N>0$ *depending only on* $K(m), d, d_1, m, r$, *and* T *such that*

$$\begin{aligned}\mathbb{E}\sup_{t\in[r,T]}\|u_{r,t}(\cdot)\|_m^2 &+ \mathbb{E}\int_r^T\|u_{r,t}(\cdot)\|_{m+1}^2dt \\ &\leq N\mathbb{E}\left(\|\phi\|_m^2+\int_r^T\{\|f(t,\cdot)\|_{m-1}^2+\sum_{l=1}^{d_1}\|g^l(t,\cdot)\|_m^2\}dt\right).\end{aligned}\tag{2.4}$$

Proof. See the proof of (Rozovskii [5], pp133, Theorem 2). □

For the classical solution of the Cauchy problem (2.1), first let us give a precise definition.

DEFINITION 2.2. A function $v_{r,\cdot}(\cdot,\cdot)$ mapping from $(t,x,\omega)\in[r,T]\times\mathbb{R}^d\times\Omega$ to $v_{r,t}(x,\omega)\in\mathbb{R}$, which belongs to $C^{0,2}([r,T]\times\mathbb{R}^d)$, $(\mathbb{P}-a.s.)$, is predictable stochastic process for each $x\in\mathbb{R}^d$, and satisfies equation (2.1), is called a *classical solution* of problem (2.1).

THEOREM 2.2. *If the conditions of Theorem 2.1 are fulfilled for any* $m\in\mathbb{N}$*, then the classical solution of problem* (2.1) *is a classical smooth solution or, in other words, it is infinitely differentiable in* x $(\mathbb{P}-a.s.)$.
Proof. See Chapter 4 of Rozovskii [5]. □

3. Semi-linear SPDE. Based on previous section's results and notations, now we prove that the problem (1.1) has a unique, smooth classical solution. According to Theorem 1.1, for any given $r\geq 0$ and $\phi\in\{C_b(\mathbb{R})^+\bigcap\mathbb{H}^1(\mathbb{R})\}$, problem (1.1) has a unique $C_b(\mathbb{R})\bigcap\mathbb{H}^1(\mathbb{R})$-valued, non-negative, strong solution $\{\psi_{r,t}: t\geq r\geq 0\}$ if the basic condition holds. Furthermore, for any $\phi\in\{C_b(\mathbb{R})^+\bigcap\mathbb{H}^1(\mathbb{R})\}$, $\|\psi_{r,t}\|_{\mathrm{a}}\leq\|\phi\|_{\mathrm{a}}$ holds $\mathbb{P}-a.s.$ for all $t\geq r\geq 0$, where $\|\phi\|_{\mathrm{a}}$ is the supremum of ϕ. Now we want to generalized this result such that if we assume that the coefficients and the initial function of (1.1) have better regularity, then the solution of (1.1) also has better regularity. After that, the nonnegativity of the solution with a better regularity can also be derived.
First, let us give the ψ-semigroup property of the solution of (1.1). Since the solution of (1.1) depends on the initial function $\phi(\cdot)$, we can rewrite the solution of (1.1) as $\psi_{r,t}(x)=\psi_{r,t}(x,\phi)$. Based on this new notation, we say that $\psi_{r,t}(x,\phi)$, the solution of (1.1), defines a ψ-semigroup if there exists a set $N\subset\Omega$ such that $\mathbb{P}(N)=0$ and for any $\phi\in C_b(\mathbb{R})^+\bigcap\mathbb{H}^1(\mathbb{R})$ and $0\leq r\leq s\leq t$,

$$\psi_{r,t}(x,\phi)=\psi_{s,t}(x,\psi_{r,s}(\cdot,\phi))\tag{3.1}$$

holds for all $\omega\notin N$.

Remark. (3.1) defines a forward ψ-semigroup. This corresponds that (1.1) is a forward SPDE.
Based on this definition, we have following theorem.

THEOREM 3.1. *Suppose that the basic condition holds for* $m\geq 1$. *Then, for any* $\phi\in\{C_b^m(\mathbb{R})\bigcap\mathbb{H}^m(\mathbb{R})\}$, *equation* (1.1) *with* $\sigma\equiv 0$ *has a unique* $\{C_b^m(\mathbb{R})\bigcap\mathbb{H}^m(\mathbb{R})\}$*-valued, strong solution* $\{\psi_{r,t}: t\geq r\geq 0\}$. *Moreover, the solution defines a* ψ*-semigroup.*

Proof. First, with $\sigma \equiv 0$, SPDE (1.1) becomes the following linear SPDE:

$$\begin{aligned} T_{r,t}(x) = \phi(x) + \int_r^t \left[\frac{1}{2} a(x)\partial_{xx}^2 T_{r,s}(x)\right] ds \\ + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x T_{r,s}(x) W(ds,dy), \qquad t \geq r. \end{aligned} \tag{3.2}$$

In order to use Theorem 2.1 , here we decompose the Brownian sheet into a sequence of one-dimensional Brownian motions first introduced in [4]. Let $\{h_j : j = 1, 2, \cdots\}$ be a complete orthonormal system of $L_2(\mathbb{R})$. Then, for any $t \geq 0$ and $j \geq 1$,

$$W_j(t) = \int_0^t \int_{\mathbb{R}} h_j(y) W(ds, dy)$$

defines a sequence of independent standard Brownian motions $\{W_j : j = 1, 2, \cdots\}$. For $\epsilon > 0$ let

$$W^\epsilon(dt, dx) = \sum_{j=1}^{[1/\epsilon]} h_j(x) W_j(dt) dx, \qquad t \geq 0, x \in \mathbb{R},$$

where $[1/\epsilon]$ denotes the maximum integer less than $1/\epsilon$.

By assumption, we have that $c \in C_b^{m+1}(\mathbb{R})$, $c^2(x) \geq \epsilon > 0$, $h \in C_b^{m+1}(\mathbb{R}) \bigcap \mathbb{H}^{m+1}(\mathbb{R})$. Now we consider the equation

$$\begin{aligned} T_{r,t}^\epsilon(x) = \phi(x) + \int_r^t \left[\frac{1}{2} a(x)\partial_x^2 T_{r,s}^\epsilon(x)\right] ds \\ + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x T_{r,s}^\epsilon(x) W^\epsilon(ds,dy), \qquad t \geq r \geq 0. \end{aligned} \tag{3.3}$$

Now we check whether equation (3.3) satisfies the superparabolic condition (2.3). For equation (3.3), the left hand side of (2.3) becomes

$$2a(x)\xi^2 - \sum_{l=1}^{d_1} \left| \int_{\mathbb{R}} h(y-x) h_l(y) dy \cdot \xi \right|^2,$$

where $\xi \in \mathbb{R}$ and $d_1 = [1/\epsilon]$. Then, by Parseval equality and the basic condition, we have

$$\begin{aligned} a(x)\xi^2 - \sum_{l=1}^{d_1} \left| \int_{\mathbb{R}} h(y-x) h_l(y) dy \cdot \xi \right|^2 &\geq \xi^2 \left\{ a(x) - \sum_{l=1}^{\infty} \left| \int_{\mathbb{R}} h(y-x) h_l(y) dy \right|^2 \right\} \\ &= \xi^2 \{a(x) - \rho(0)\} = \xi^2 c^2(x) \geq \epsilon \cdot \xi^2. \end{aligned} \tag{3.4}$$

Therefore, for any $\phi \in \{C_b^m(\mathbb{R}) \bigcap \mathbb{H}^m(\mathbb{R})\}$, by Theorem 2.1, equation (3.3) has a unique solution $T_{r,t}^\epsilon(x) \in \mathbb{L}_2([r,T], \mathcal{P}, \mathbb{H}^{m+1}) \bigcap C([r,T], \mathcal{P}, \mathbb{H}^m)$ and the following inequality

$$\mathbb{E} \sup_{s \in [r,T]} \|T_{r,s}^\epsilon\|_m^2 \leq K \mathbb{E} \|\phi\|_m^2 \tag{3.5}$$

holds. By a limit argument similar to the proof of Rozovskii ([5] p111, Theorem 2), we can get that $T_{r,s}(x)$, the solution of (3.2), satisfies

$$\mathbb{E} \sup_{s \in [r,T]} \|T_{r,s}\|_m^2 \leq K \mathbb{E} \|\phi\|_m^2, \tag{3.6}$$

which implies the uniqueness of the solution of (3.2). Now we only need to prove the semigroup property. Let $T_{r,t}$ be the unique strong solution of (3.2). Then, for any $s \geq r$ and $t \geq 0$, we have

$$\begin{aligned} T_{r,s+t}(x) = \phi(x) + \int_r^{s+t} \left[\frac{1}{2} a(x)\partial_{xx}^2 T_{r,u}(x)\right] du \\ + \int_r^{s+t} \int_{\mathbb{R}} h(y-x)\partial_x T_{r,u}(x) W(du,dy), \qquad s \geq r, \qquad t \geq 0. \end{aligned} \tag{3.7}$$

We subtract each side of (3.2) from the corresponding side of (3.7), respectively. Then, we get

$$\begin{aligned} T_{r,s+t}(x) - T_{r,s}(x) = &\int_s^{s+t} \left[\frac{1}{2}a(x)\partial_{xx}^2 T_{r,u}(x)\right] du \\ &+ \int_s^{s+t}\int_{\mathbb{R}} h(y-x)\partial_x T_{r,u}(x) W(du,dy), \quad s \geq r, \quad t \geq 0. \end{aligned} \tag{3.8}$$

If we look $T_{r,s}(x)$ as the initial data in (3.8), since $a(x)$ and $h(x)$ are time homogeneous and the lower limits of the integrals on the right hand side are s, we can reform (3.8) to get

$$\begin{aligned} T_{s,s+t}(x) = T_{r,s}(x) + &\int_s^{s+t} \left[\frac{1}{2}a(x)\partial_{xx}^2 T_{s,u}(x)\right] du \\ &+ \int_s^{s+t}\int_{\mathbb{R}} h(y-x)\partial_x T_{s,u}(x) W(du,dy), \qquad s \geq r, \qquad t \geq 0. \end{aligned} \tag{3.9}$$

By the uniqueness of the strong solution of (3.2), for any fixed $s \geq r$, $T_{r,s+t}(x,\phi)$ is the unique strong solution of (3.8). On the other hand, for the same fixed $s \geq r$, just following the same idea to prove Theorem 2.1 we can prove that (3.9) has a unique strong solution, which is just $T_{s,s+t}(x, T_{r,s}(\cdot,\phi))$ since the initial value is $T_{r,s}(\cdot,\phi)$. This obviously gives that for any fixed $s \geq r$,

$$T_{r,s+t}(x,\phi) = T_{s,s+t}(x, T_{r,s}(\cdot,\phi)), \qquad t \geq 0, \tag{3.10}$$

holds for all $\omega \notin N$ with $\mathbb{P}(N) = 0$. The existence of the set N comes from the continuity of the unique strong solution of (3.2) and (3.9). $\square$

In the following we derive an equivalent SPDE form of the equation (1.1). Based on this new SPDE form, we can use the existing results of the linear SPDE, which are discussed above, to construct a unique, nonnegative, smooth classical solution of (1.1).

Let $T_{r,t}(x) = T_{r,t}(x,\phi)$ and $\psi_{r,t}(x) = \psi_{r,t}(x,\phi)$ denote the unique solution of (3.2) and (1.1), respectively. For any $\phi \in C_b^m(\mathbb{R}) \bigcap \mathbb{H}^m(\mathbb{R})$ we consider the following stochastic equation:

$$\Psi_{r,t}(x) = T_{r,t}\phi(x) - \frac{1}{2}\int_r^t T_{s,t}[\sigma(x)(\Psi_{r,s}(x))^2] ds, \tag{3.11}$$

where $T_{r,t}\phi(x)$ is the unique strong solution of (3.2). From the inequality (3.6), we can prove that the equation (3.11) has a unique solution by the Picard iterative scheme. Then, we have following theorem.

THEOREM 3.2. *Suppose that the basic condition holds for* $m \geq 1$. *Then, for any* $\phi \in C_b^m(\mathbb{R}) \bigcap \mathbb{H}^m(\mathbb{R})$, (3.11) *has a unique strong solution* $\psi_{r,t} \in C_b^m(\mathbb{R}) \bigcap \mathbb{H}^m(\mathbb{R}), t \in [r,T]$, *which defines a* ψ*-semigroup for all* $\omega \notin N$ *with* $\mathbb{P}(N) = 0$ *and* (3.11) *is equivalent to* (1.1). *Thus,* $\{\psi_{r,t} : r \leq t\}$ *is also the unique strong solution of* (1.1).

Proof. Let $T_{r,t}(x) = T_{r,t}(x,\phi)$ be the unique strong solution of (3.2). By Theorem 3.1, we know that $\{T_{r,t} : t \geq r\}$ is a ψ- semigroup.

To complete the proof of the theorem, It suffices to prove that (3.11) is equivalent to (1.1). To this end, in the following we prove that given a solution of (3.11), we can change the form of (3.11) into that of (1.1) by a stochastic Fubini theorem (see Theorem 2.6 of Walsh [6]) as follows: For any $\phi \in C_b^m(\mathbb{R}) \bigcap \mathbb{H}^m(\mathbb{R})$, let $\psi_{r,t}(x)$ be a solution of (3.11). Thus, we have

$$
\begin{aligned}
\psi_{r,t}(x) =\ & T_{r,t}(x) - \frac{1}{2}\int_r^t T_{s,t}[\sigma(x)(\psi_{r,s}(x))^2]ds \\
=\ & \phi(x) + \int_r^t [\frac{1}{2}a(x)\partial^2_{xx}T_{r,s}(x)]ds + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x T_{r,s}(x)W(ds,dy) \\
& -\frac{1}{2}\int_r^t [\sigma(x)(\psi_{r,s}(x))^2]ds - \frac{1}{2}\int_r^t \left\{ \int_s^t \frac{1}{2}a(x)\partial^2_{xx}T_{s,u}[\sigma(x)(\psi_{r,s}(x))^2]du \right. \\
& \left. + \int_s^t \int_{\mathbb{R}} h(y-x)\partial_x T_{s,u}[\sigma(x)(\psi_{r,s}(x))^2]W(du,dy) \right\} ds \\
=\ & \phi(x) + \int_r^t [\frac{1}{2}a(x)\partial^2_{xx}T_{r,u}(x)]du \\
& -\frac{1}{2}\int_r^t \left\{ \int_r^u \frac{1}{2}a(x)\partial^2_{xx}T_{s,u}[\sigma(x)(\psi_{r,s}(x))^2]ds \right\} du \\
& -\frac{1}{2}\int_r^t [\sigma(x)(\psi_{r,s}(x))^2]ds + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x T_{r,u}(x)W(du,dy) \\
& -\frac{1}{2}\int_r^t \int_{\mathbb{R}} \left\{ \int_r^u h(y-x)\partial_x T_{s,u}[\sigma(x)(\psi_{r,s}(x))^2]ds \right\} W(du,dy) \\
=\ & \phi(x) + \int_r^t [\frac{1}{2}a(x)\partial^2_{xx}\psi_{r,u}(x)]du - \frac{1}{2}\int_r^t [\sigma(x)(\psi_{r,s}(x))^2]ds \\
& + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x \psi_{r,u}(x)W(du,dy).
\end{aligned}
\tag{3.12}
$$

This completes the proof. □

Theorem 3.3. *Suppose that the basic condition and $\phi \in C_b^m(\mathbb{R})^+ \bigcap \mathbb{H}^m(\mathbb{R})$ hold for any $m \in \mathbb{N}$. Then, (1.1) has a version of unique, nonnegative, strong solution $\{\psi_{r,t} : r \le t\}$, which is continuous in t and infinitely differentiable with respect to x, and defines a ψ-semigroup for all $\omega \notin N$ with $\mathbb{P}(N) = 0$.*

Proof. Based on Theorem 3.2, we only need to prove that the conclusion is true for equation (3.11). Since now for any $m \in \mathbb{N}$ we have $c \in C_b^{m+1}(\mathbb{R})$, $c^2(x) \ge \epsilon > 0$, $h \in C_b^{m+1}(\mathbb{R}) \bigcap \mathbb{H}^{m+1}(\mathbb{R})$. For any $\phi \in \{C_b^m(\mathbb{R})^+ \bigcap\ \mathbb{H}^m(\mathbb{R})\}$, by Theorem 3.1, the equation (3.2) has a unique solution $T_{r,t}(x) \in \mathbb{L}_2([r,T],\mathcal{P},\mathbb{H}^{m+1}) \bigcap C([r,T],\mathcal{P},\mathbb{H}^m)$ and the following inequality

$$
\mathbb{E} \sup_{s\in[r,T]} \|T_{r,s}\|_m^2 \le K(m)\mathbb{E}\|\phi\|_m^2 \tag{3.13}
$$

holds. Then, by the Picard iterative scheme, the equation (3.11) has a unique, strong solution $\psi_{r,t}$, which satisfies

$$
\mathbb{E} \sup_{s\in[r,T]} \|\psi_{r,s}\|_m^2 \le K(m)\mathbb{E}\|\phi\|_m^2. \tag{3.14}
$$

Since under the basic condition and $\phi \in C_b^m(\mathbb{R})^+ \bigcap \mathbb{H}^m(\mathbb{R})$ with $m \ge 1$, Theorem 1.1 can guarantee the existence of a nonnegative solution and the solution of equation (1.1) has uniqueness, above $\psi_{r,t}$, thus, is just the nonnegative solution of equation (1.1). The remaining parts of the conclusion follow from an argument similar to the proofs of Proposition 3 and Theorem 3 on page 139 of Rozovskii [5]. □

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