

State classification for a class of measure-valued branching diffusions in a Brownian medium[★]

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Summary. The spatial structure of a new class of measure-valued diffusions which arise as limits in distribution of a sequence of interacting branching particle systems is investigated. We obtain the following criterion of state classification for these superprocesses: their effective state space is contained in the set of purely atomic measures or the set of absolutely continuous measures according as $\epsilon = 0$ or $\epsilon \neq 0$, when the coefficient of the motion generator is a smooth function.

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1 Introduction

In this paper, we intend to investigate the spatially structural properties of a class of measure-valued branching diffusions in a Brownian medium (MBDBs) constructed and characterized in Wang [12]. Given a finite measure Z_0 with compact support on \mathbb{R} , the MBDB with initial state Z_0 is the unique solution to the $(\mathcal{L}, \delta_{Z_0})$ -martingale problem (MP), where

$$\mathcal{L}F(\mu) := \mathcal{A}F(\mu) + \mathcal{B}F(\mu) , \quad (1.1)$$

$$\mathcal{B}F(\mu) := \frac{1}{2} \gamma(m_2 - 1) \int_{\mathbb{R}} \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx) , \quad (1.2)$$

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$$\begin{aligned} \mathcal{A}F(\mu) &:= \frac{1}{2} \int_{\mathbb{R}} \rho_\epsilon \left(\frac{d^2}{dx^2} \right) \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(x-y) \left(\frac{d}{dx} \right) \left(\frac{d}{dy} \right) \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} \mu(dx) \mu(dy) \end{aligned} \quad (1.3)$$

for any bounded continuous function $F(\mu)$ that belongs to the domain $\mathcal{D}(\mathcal{L})$ of operator \mathcal{L} , the variational derivative is defined by

$$\begin{aligned} \frac{\delta F(\mu)}{\delta \mu(x)} &:= \lim_{h \downarrow 0} \frac{F(\mu + h\delta_x) - F(\mu)}{h}, \\ \rho(z) &:= \int_{\mathbb{R}} g(z-y)g(y) dy, \quad \rho_\epsilon := \rho(0) + \epsilon^2, \quad \epsilon \in \mathbb{R}, \end{aligned}$$

m_2 is the finite second moment of the offspring distribution for the branching mechanism giving rise to operator \mathcal{B} and finally g is a square-integrable, symmetric function on $\bar{\mathbb{R}} = \mathbb{R} \cup \partial$, the one point compactification of \mathbb{R} , satisfying $g(\partial) := \lim_{x \rightarrow \partial} g(x) = 0$. For the measure Z_0 , Theorems 6.4 and Theorem 7.2 of Wang [12] together show that the $(\mathcal{L}, \delta_{Z_0})$ -MP is well-posed and its solution is a measure-valued diffusion. Let us write $\{Z_t : t \geq 0\}$ for the canonical process on the space of continuous trajectories valued in the Polish space $M_F(\bar{\mathbb{R}})$ of finite, positive Borel measures on $\bar{\mathbb{R}}$, equipped with the weak topology defined by

$$\mu^n \Rightarrow \mu \quad \text{if and only if} \quad \langle f, \mu^n \rangle \rightarrow \langle f, \mu \rangle \quad \text{for} \quad \forall f \in C(\bar{\mathbb{R}}).$$

The main results of the present paper are as follows.

Under the assumption that g is a square-integrable, smooth, and symmetric function on $\bar{\mathbb{R}}$ with $g(\partial) = 0$, and that $\epsilon = 0$ holds – these conditions will be referred to as the smooth, degenerate case – then, for all times $t > 0$, the effective state space of Z_t is contained in the set of purely atomic measures.

On the other hand, if g is a square-integrable, smooth, and symmetric function on $\bar{\mathbb{R}}$ with $g(\partial) = 0$, but now $\epsilon \neq 0$ holds – these conditions will be referred to as the smooth, uniformly elliptic case – then, for all times $t > 0$, the effective state space of Z_t is contained in the set of measures which are absolutely continuous with respect to one dimensional Lebesgue measure.

In the smooth, degenerate case, if the initial state is a purely atomic measure, then by virtue of the uniqueness of solution to the martingale problem for \mathcal{L} , the conclusion for this case follows easily from the direct construction of a purely atomic measure-valued solution to the martingale problem for \mathcal{L} . The difficulty arises if the initial state is not a purely atomic measure. In order to overcome this difficulty, the following ideas are developed.

In Subsection 1.1, we prove that for any finite measure ν with compact support, the unique solution $\{\nu_t\}$ to the $(\mathcal{B}, \delta_\nu)$ -MP has the form

$$\nu_t = \sum_{i \in I(t)} a_i(t) \delta_{x_i(t)}, \quad t > 0,$$

where $I(t)$ is a random subset of \mathbb{N} , the set of natural numbers, $x_i(t)$ and $a_i(t) \geq 0$ represent the position and the mass of the i^{th} particle at time t , respectively. $\{a_i(t) : i \geq 1\}$ are independent and $a_i(t)$ is the unique solution to the $(\frac{1}{2}\gamma(m_2 - 1)x \frac{d^2}{dx^2}, \delta_{a_i(0)})$ -MP. This explicitly shows that ν_t immediately enters the set of purely atomic measures even though the initial state is not an atomic measure and the action of operator \mathcal{B} only changes the masses of atoms.

Subsection 1.2 is devoted to proving that in the smooth, degenerate case, then for any finite, atomic measure

$$\mu = \sum_{i \in J} a_i(0) \delta_{x_i(0)} ,$$

where $J \subset \mathbb{N}$, $a_i(0) \geq 0$, $x_i(0) \in \mathbb{R}$ and $x_i(0) \neq x_j(0)$ for any $i \neq j$, $i, j \in J$, the unique solution to the $(\mathcal{A}, \delta_\mu)$ -MP has the form

$$\mu_t = \sum_{i \in J} a_i(0) \delta_{x_i(t)} , \quad t \geq 0 ,$$

where $x_i(t) \neq x_j(t)$ for any $i \neq j$, $i, j \in J$, $t \geq 0$. This reflects the fact that if $\epsilon = 0$, then the action of operator \mathcal{A} only changes the positions of atoms.

Let $\{U_t : t \geq 0\}$, $\{T_t : t \geq 0\}$ and $\{S_t : t \geq 0\}$ be the semigroups corresponding to the measure-valued Markov processes which are solutions to the \mathcal{L} -MP, the \mathcal{A} -MP and the \mathcal{B} -MP, respectively. Generally, the semigroup $\{T_t\}$ does not commute with the semigroup $\{S_t\}$. However, in the smooth, degenerate case, it does indeed. In fact, in Subsection 1.3 we establish the commutativity of $\{T_t\}$ and $\{S_t\}$ in this case. Once this step is completed, the results of Subsections 1.1, 1.2 and 1.3 together yield that in the smooth, degenerate case, for any finite measure ν with compact support (ν may be nonatomic), the unique solution to the $(\mathcal{L}, \delta_\nu)$ -MP is a purely atomic measure at any time $t > 0$.

In the smooth, uniformly elliptic case, if the particle's motions are interaction-free or independent ($g \equiv 0$), then just as in Konno-Shiga [9], we can use the evolution equation technique to estimate the moments of the measure-valued solution μ_t to the martingale problem for \mathcal{L} , and reach the conclusion for the smooth, uniformly elliptic case. However, as pointed out in Wang [12], the interaction destroys the multiplicative property; in consequence, the log-Laplace functional and the evolution equation technique can not be applied for our dependent case. This forces us to find a new way to approach the problem. In Section 2, at first we use the duality to change the estimation of the moments of the solution to the martingale problem for \mathcal{L} to the estimation of the moments of the dual process which is a finite dimensional diffusion process. Then, by making use of the estimation of the fundamental solution for the parabolic equation of second order without the minor terms (see Ladyzenskaja et al. [10] Chapter IV or Friedman [7]), we reach the desired conclusion.

We will occasionally use the following notations: for any Polish space E , $B(E)$ will be the space of all bounded real-valued measurable functions on

E ; $C(E) \subset B(E)$ will be its subspace of all bounded continuous functions on E ; for any E -valued process $\{Y(t)\}$, we will write

$$\mathcal{F}_t^Y := \sigma(Y(s), s \leq t) \vee \sigma\left(\int_0^s h(Y(u)) du, s \leq t, \forall h \in B(E)\right).$$

Let $C_b^2(\mathbb{R}^N)$ be the space of all bounded, twice continuously differentiable functions on \mathbb{R}^N with all first and second derivatives bounded; $C_c^\infty(\mathbb{R}^N)$, the space of all infinitely differentiable functions on \mathbb{R}^N with compact support; and $C^\infty(\bar{\mathbb{R}}^N)$, the space of all infinitely differentiable functions on \mathbb{R}^N with all derivatives vanishing at infinity.

1.1 Behavior of the generator \mathcal{B}

The content of this subsection is summarized in the following theorem.

Theorem 1.1 *Let $\nu = \nu_0$ be a finite measure on \mathbb{R} with compact support. Then on a probability space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$ the $(\mathcal{B}, \delta_\nu)$ -MP has a unique solution $\{\nu_t : t \geq 0\}$ with distribution \mathbb{P}_ν^1 , which is a measure-valued diffusion process. Furthermore, for any $t > 0$, ν_t is almost surely a purely atomic measure; in fact, there exists a countable set of $\mathcal{F}_0^{\nu_t}$ -measurable real random variables $\{x_i(0) : i \geq 1\}$ such that there holds for $i \neq j$, $x_i(0) \neq x_j(0)$ a.s. (\mathbb{P}^1) and the discrete measure ν_t has the form*

$$\nu_t = \sum_{i \in I(t)} a_i(t) \delta_{x_i(0)} \quad \text{for } t > 0, \quad (1.4)$$

where $I(t)$ is a random subset of \mathbb{N} such that, given ω , $I(t, \omega)$ is decreasing in t in terms of set inclusion order, and $\{a_i(\cdot) : i \in \cup_{t>0} I(t)\}$ is a collection of independent, one dimensional diffusion processes with state space $\mathbb{R}^+ := \{x : x \geq 0\}$ and an absorbing barrier at the origin.

Proof: For the argument leading to the fact that the MP is well-posed and its unique solution is a diffusion, see Wang [12].

In order to prove that $\{\nu_t : t \geq 0\}$ takes values in the set of purely atomic measures, we will transform $\{\nu_t : t \geq 0\}$ into a probability measure-valued process and use a random time change introduced by Konno-Shiga [9] and Shiga [11].

Let $\tau_\infty := \inf\{t : \langle \nu_t, 1 \rangle = 0\}$ and define $C_t := \int_0^t \langle \nu_s, 1 \rangle^{-1} ds$, then C_t is a homeomorphism between $[0, \tau_\infty)$ and $[0, \infty)$. Let $D_t : [0, \infty) \rightarrow [0, \tau_\infty)$ be the continuous, strictly increasing inverse to C_t .

Now define $\tilde{\nu}_t := \nu_{D_t}$, $\hat{\nu}_t := \tilde{\nu}_{nor}(t) := \tilde{\nu}_t / \langle \tilde{\nu}_t, 1 \rangle$ and $\mathcal{G}_t := \mathcal{F}_{D_t}^{\nu}$, then $\{\hat{\nu}_t\}$ is a probability measure-valued process.

Since in the present case the particles are motion-free, by (1.2) we know that, for every $\phi \in C_b^2(\mathbb{R})$,

$$M_t(\phi) := \langle \nu_t, \phi \rangle - \langle \nu_0, \phi \rangle \quad (1.5)$$

is a $\{\mathcal{F}_t^v\}$ -martingale with respect to \mathbb{P}_v^1 . By Dellacherie-Meyer ([2] XV22. XV26) or Fitzsimmons ([6] p. 355 corollary (4.3)) and noting that there holds

$$\mathcal{B}(\langle v_t, \phi \rangle \langle v_t, \psi \rangle) - \langle v_t, \phi \rangle \mathcal{B} \langle v_t, \psi \rangle - \langle v_t, \psi \rangle \mathcal{B} \langle v_t, \phi \rangle = \langle v_t, c\phi\psi \rangle, \quad (1.6)$$

where $c = \frac{1}{2}\gamma(m_2 - 1)$, we obtain

$$\langle M(\phi) \rangle_t = c \int_0^t \langle v_s, \phi^2 \rangle ds \quad (1.7)$$

and

$$\langle M(\phi), M(\psi) \rangle_t = c \int_0^t \langle v_s, \phi\psi \rangle ds, \quad (1.8)$$

for every choice of $\phi, \psi \in C_b^2(\mathbb{R})$. By a random time change, the optional sampling theorem (see Ikeda-Watanabe [8] p. 34) implies that

$$\tilde{M}_t(\phi) := \langle \tilde{v}_t, \phi \rangle - \langle \tilde{v}_0, \phi \rangle \quad (1.9)$$

is a \mathcal{G}_t -martingale,

$$\begin{aligned} \langle \tilde{M}(\phi) \rangle_t &= \langle M(\phi) \rangle_{D_t} = c \int_0^{D_t} \langle v_s, \phi^2 \rangle ds \\ &= c \int_0^{D_t} \langle v_s, \phi^2 \rangle \langle v_s, 1 \rangle \frac{1}{\langle v_s, 1 \rangle} ds \\ &= c \int_0^{D_t} \langle v_{D_{C_s}}, \phi^2 \rangle \langle v_{D_{C_s}}, 1 \rangle dC_s \\ &= c \int_0^t \langle v_{D_u}, \phi^2 \rangle \langle v_{D_u}, 1 \rangle du = c \int_0^t \langle \tilde{v}_u, \phi^2 \rangle \langle \tilde{v}_u, 1 \rangle du, \end{aligned} \quad (1.10)$$

and

$$\langle \tilde{M}(\phi), \tilde{M}(\psi) \rangle_t = c \int_0^t \langle \tilde{v}_u, \phi\psi \rangle \langle \tilde{v}_u, 1 \rangle du. \quad (1.11)$$

Note that $\langle \tilde{v}_t, 1 \rangle \neq 0$ for any $t \geq 0$. Also, from the results in Wang [12], we know that $\langle \tilde{v}_t, 1 \rangle$ possesses a finite moment generating function in a neighbourhood of the origin.

An application of Ito's formula shows that process \hat{v}_t is in fact a drift-free Fleming-Viot process; in particular, for $f(x, y) = (x/y)$ we have

$$\begin{aligned} &f(\langle \hat{v}_t, \phi \rangle, \langle \hat{v}_t, 1 \rangle) - f(\langle \hat{v}_0, \phi \rangle, \langle \hat{v}_0, 1 \rangle) \\ &= \int_0^t \frac{1}{\langle \hat{v}_s, 1 \rangle} d\tilde{M}_s(\phi) - \int_0^t \frac{\langle \hat{v}_s, \phi \rangle}{\langle \hat{v}_s, 1 \rangle^2} d\tilde{M}_s(1) =: N_t. \end{aligned} \quad (1.12)$$

So $N_t = \langle \hat{v}_t, \phi \rangle - \langle \hat{v}_0, \phi \rangle$ is a \mathcal{G}_t -martingale with quadratic variation process

$$\langle N \rangle_t = c \int_0^t [\langle \hat{v}_s, \phi^2 \rangle - \langle \hat{v}_s, \phi \rangle^2] ds. \quad (1.13)$$

By Theorem 10.4.5 in Ethier-Kurtz ([4] p. 441), we conclude

$$\mathbb{P}_v^1\{\hat{\nu}_t \text{ is a purely atomic measure for all } t > 0\} = 1$$

as required, if we assume that the support of ν_0 is $[0,1]$ and also that $\nu_0([0,1]) = 1$. For the general case in which ν_0 is a finite measure on \mathbb{R} with compact support, the conclusion follows from a transformation argument.

Since for any $t > 0$, ν_t is a finite measure, it is obvious that $I(t)$ is at most countable. Now, let there be given a countable collection $\{a_i(t) : i \geq 1\}$ of independent solutions to the martingale problem for generator $(\frac{1}{2}\gamma(m_2 - 1)x \frac{d^2}{dx^2})$ (which exhibits absorption at the origin of the real line), each started at any fixed time $s > 0$ with initial value $a_i(s) > 0$, satisfying the condition $\sum_{i=1}^{\infty} a_i(s) < \infty$, as well as a countable collection $\{x_i(s) : i \geq 1\}$ of fixed, distinct points on the real line. Define $\nu_t^s := \sum_{i=1}^{\infty} a_i(t) \delta_{x_i(s)}$, a direct calculation shows that the measure-valued process ν^s solves the $(\mathcal{B}, \delta_{\nu^s})$ -MP on the time interval $[s, \infty)$. Since the MP is well-posed, we conclude that the index sets verify $I(t) \subset I(s)$ for every choice of $0 < s < t < \infty$ and therefore that their union $\cup_{t>0} I(t) = \cup_{k>0} I(\frac{1}{k})$ is also countable. The existence of a countable set of "initial" positions $\{x_i(0) : i \geq 1\}$ is now guaranteed by the right continuity of the σ -algebras involved. \square

1.2 Behavior of the Generator \mathcal{A}

In this subsection, we will show that if $\epsilon = 0$, $g \neq 0$ is a square-integrable, smooth, and symmetric function on $\bar{\mathbb{R}}$ with $g(\partial) = 0$ and the initial state is an atomic measure

$$\mu = \sum_{i \in J} a_i(0) \delta_{x_i(0)}, \quad (1.14)$$

where $a_i(0) \geq 0$, $x_i(0) \in \mathbb{R}$ and $x_i(0) \neq x_j(0)$ for $i \neq j$ and $J \subset \mathbb{N}$, then the unique solution to the $(\mathcal{A}, \delta_\mu)$ -MP has the form

$$\mu_t = \sum_{i \in J} a_i(0) \delta_{x_i(t)}, \quad (1.15)$$

where $x_i(t)$ is continuous in t and $x_i(t) \neq x_j(t)$ for $\forall t \geq 0$ and $i \neq j$. Let us begin by considering the case in which J has finitely many elements.

Lemma 1.2 *Suppose that in (1.3) we have $\epsilon = 0$ and $g \neq 0$ is a square-integrable, smooth, and symmetric function on $\bar{\mathbb{R}}$ with $g(\partial) = 0$. If $J = \{1, 2, \dots, n\}$, then for any $(x_1(0), x_2(0), \dots, x_n(0)) \in \mathbb{R}^n$ with $x_i(0) \neq x_j(0)$ for $i \neq j$ and*

$$\mu^n = \sum_{i=1}^n a_i(0) \delta_{x_i(0)} \quad \text{with } a_i(0) \geq 0,$$

there exists a unique \mathbb{R}^n -valued diffusion process $\{(x_1(t), x_2(t), \dots, x_n(t)) : t \geq 0\}$ such that for $\forall t \geq 0$ and $i \neq j$, $x_i(t) \neq x_j(t)$ and

$$\mu_t^n = \sum_{i=1}^n a_i(0) \delta_{x_i(t)} \quad (1.16)$$

is the unique solution to the $(\mathcal{A}, \delta_{\mu^n})$ -MP.

Proof: For $\mu_s^n = \sum_{i=1}^n a_i(0) \delta_{x_i(s)}$, just as in Section 2 of Wang [12], we can rewrite $\mathcal{A}F_f(\mu_s^n)$ as

$$\mathcal{A}F_f(\mu_s^n) = G_0^n g(x_1(s), x_2(s), \dots, x_n(s)) \quad (1.17)$$

for

$$F_f(\mu) = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} f(x_1, \dots, x_N) \mu^{\otimes N}(d\bar{x}), \quad (1.18)$$

where we define

$$G_\epsilon^n f := \frac{1}{2} \sum_{i,j=1}^n \rho(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} f + \frac{1}{2} \sum_{i=1}^n \epsilon^2 \frac{\partial^2}{\partial x_i^2} f \quad \text{for } \epsilon \in \mathbb{R}, \quad (1.19)$$

$$\mu^{\otimes N}(d\bar{x}) := \mu(dx_1) \cdots \mu(dx_N),$$

and

$$g(x_1(s), \dots, x_n(s)) = \sum_{l_1, \dots, l_n=1}^n a_{l_1}(0) \cdots a_{l_n}(0) f(x_{l_1}(s), \dots, x_{l_n}(s)). \quad (1.20)$$

So $\{(x_1(s), \dots, x_n(s)) : s \geq 0\}$ is a diffusion process with generator $G^n := G_0^n$. Let $\eta_t = x_i(t) - x_j(t)$ for $i \neq j$, then $\{\eta_t\}$ is a diffusion process with state space \mathbb{R} , absorbing state 0 and generator

$$\mathcal{G}f(y) = (\rho(0) - \rho(y))f''(y), \quad f \in C_b^\infty(\mathbb{R}).$$

From Feller's criterion of accessibility, the probability that η reaches 0 is 0 or 1 according as

$$\int_0^1 \frac{y}{(\rho(0) - \rho(y))} dy$$

is ∞ or $< \infty$. It is easy to check that $\rho(\cdot)$ is nonnegative definite, then by the Bochner-Khinchin theorem there is a probability distribution function $F(\cdot)$ such that

$$0 \leq 1 - \frac{\rho(y)}{\rho(0)} = \int_{\mathbb{R}} \{1 - \cos(xy)\} dF(x)$$

$$\leq \int_{\mathbb{R}} \frac{1}{2} (xy)^2 dF(x) = \frac{1}{2\rho(0)} y^2 |\rho''(0)|.$$

Hence we get

$$0 \leq \sup_y \frac{(\rho(0) - \rho(y))}{y^2} \leq \frac{1}{2} |\rho''(0)|. \quad (1.21)$$

Since g is smooth and $\rho''(0)$ is finite, state 0 is inaccessible. Define

$$\Phi_n: (x_1, \dots, x_n) \rightarrow \sum_{i=1}^n a_i(0) \delta_{x_i}. \quad (1.22)$$

Since G^n is symmetric (see Wang [12] Definition 2.3), the law of the corresponding diffusion process is exchangeable (see Wang [12] Definition 2.3) and the conditions of Theorem 10.13 of Dynkin [3] are satisfied, hence our conclusion follows. \square

Remark: With any initial state $x \in \mathbb{R}^n$, the explosion time of the G^n -diffusion is almost surely infinite; thus, the G^n -diffusion always lives in \mathbb{R}^n and this is why we often consider questions strictly within \mathbb{R}^n instead of $\bar{\mathbb{R}}^n$.

In order to establish the commutativity of semigroups $\{T_t\}$ and $\{S_t\}$, we have to show that for any purely atomic measure μ , the unique solution to the $(\mathcal{A}, \delta_\mu)$ -MP is continuous in a stronger sense than that implied by the use of the weak topology. For this purpose, we have to get through two steps. The first step is to prove that all our finite branching-free particle systems can be constructed on a common probability space. The second step is to present the weak atomic topology which was first introduced by Ethier-Kurtz in [5] and to prove that the unique solution to the $(\mathcal{A}, \delta_\mu)$ -MP is continuous in the weak atomic topology.

Now let us begin our first step.

Lemma 1.3 *Assume that $g \not\equiv 0$ is a square-integrable, smooth, and symmetric function on $\bar{\mathbb{R}}$ with $g(\partial) = 0$. Let W be a cylindrical Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Then the following system of stochastic integral equations*

$$x_i(t) - x_i(0) = \int_0^t \int_{\mathbb{R}} g(y - x_i(s)) W(dy, ds), \quad i = 1, \dots, n \quad (1.23)$$

has a continuous, unique strong solution for each $n \geq 1$.

Remark: For the definition of cylindrical Brownian motion, we refer to Example 7.1.2 in Dawson [1].

Proof: We prove the existence and uniqueness by using the method of successive approximation as follows: define a sequence $x^k(t) = \{(x_1^k(t), \dots, x_n^k(t))\}$ of n -dimensional continuous processes iteratively by putting $x^0(t) = x_0$ and

$$x^k(t) = x_0 + \int_0^t \int_{\mathbb{R}} \left(g(y - x_1^{k-1}(s)), \dots, g(y - x_n^{k-1}(s)) \right) W(dy, ds),$$

where $x_0 \in \mathbb{R}^n$ is fixed. Since inequality (1.21) implies

$$\mathbb{E} \left\{ \sup_{0 \leq t \leq T} |x^{k+1}(t) - x^k(t)|^2 \right\} \leq |\rho''(0)| \int_0^T |x^k(s) - x^{k-1}(s)|^2 ds,$$

we get successively

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} |x^{k+1}(s) - x^k(s)|^2 \right] \leq (|\rho''(0)|)^k \frac{T^k}{k!} n\rho(0)T$$

and

$$\mathbb{P} \left\{ \sup_{0 \leq s \leq T} |x^{k+1}(s) - x^k(s)| > \frac{1}{2k} \right\} \leq \text{const}(2|\rho''(0)|)^k \frac{T^k}{k!}.$$

By the Borel-Cantelli lemma, we see that $x^k(s)$ converges uniformly on $[0, T]$ with probability one. Since T is arbitrary, $\lim_{k \rightarrow \infty} x^k(s) = x(s)$ determines a n -dimensional continuous process which clearly is a strong solution of (1.23). The uniqueness is obvious from above argument. \square

The second step is presented next. Let (E, r) be a complete, separable metric space. For $\mu \in M_F(E)$, the space of finite, positive, Borel measures on E with the weak topology, define χ_μ to be the purely atomic measure given by $\chi_\mu = \sum \mu(\{x\})^2 \delta_x$. Let λ denote Prohorov metric on $M_F(E)$ and $\Phi: [0, \infty) \rightarrow [0, 1]$ be a continuous and nonincreasing function with $\Phi(0) = 1$ and $\Phi(1) = 0$. Define

$$\begin{aligned} \lambda_a(\mu, \nu) = \lambda(\mu, \nu) + \sup_{0 < \epsilon \leq 1} & \left| \int_E \int_E \Phi\left(\frac{r(x, y)}{\epsilon}\right) \mu(dx) \mu(dy) \right. \\ & \left. - \int_E \int_E \Phi\left(\frac{r(x, y)}{\epsilon}\right) \nu(dx) \nu(dy) \right|. \end{aligned} \quad (1.24)$$

By Lemma 2.3 of Ethier-Kurtz [5], $(M_F(E), \lambda_a)$ is a Polish space.

Remark: In order to understand what the weak atomic topology means, let us consider an example. Let $E = \mathbb{R}$, $x(t)$ and $y(t)$ be continuous, deterministic functions on $[0, \infty)$, and $a \in (0, \infty)$ be a constant. We assume that $x(0) \neq y(0)$ and $t_0 := \inf\{t : x(t) = y(t)\} < \infty$. Define $\mu_t := a\delta_{x(t)} + a\delta_{y(t)}$. Then it is obvious that μ_t is continuous in the weak topology on $M_F(\mathbb{R})$, but μ_t is not continuous at t_0 in the weak atomic topology. In fact, this follows from the following simple calculation

$$\begin{aligned} & \sup_{0 < \epsilon \leq 1} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi\left(\frac{|x-y|}{\epsilon}\right) \mu_t(dx) \mu_t(dy) - \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi\left(\frac{|x-y|}{\epsilon}\right) \mu_{t_0}(dx) \mu_{t_0}(dy) \right| \\ & = \sup_{0 < \epsilon \leq 1} \left| 2a^2 \left\{ \Phi\left(\frac{|x(t) - y(t)|}{\epsilon}\right) - 1 \right\} \right| = 2a^2 \text{ for } 0 \leq t < t_0. \end{aligned}$$

Theorem 1.4 Suppose that $g \not\equiv 0$ is a square-integrable, smooth, and symmetric function on \mathbb{R} with $g(\partial) = 0$ and $\epsilon = 0$. Let μ be a finite, purely atomic measure with compact support of the form

$$\mu = \sum_{i \in J} a_i(0) \delta_{x_i(0)}, \quad (1.25)$$

where $x_i(0) \neq x_j(0)$ for $i \neq j$, $i, j \in J \subset \mathbb{N}$, and $a_i(0) \geq 0$. Then the unique solution to the $(\mathcal{A}, \delta_\mu)$ -MP is continuous in the weak atomic topology; it can be written in the form

$$\mu_t = \sum_{i \in J} a_i(0) \delta_{x_i(t)}, \quad x_i(t) \neq x_j(t), \quad \text{for } i \neq j, \forall t \geq 0, \quad (1.26)$$

where $x_i(t)$ is constructed in Lemma 1.3 for $i \geq 1$.

Proof: We only need to consider the case in which J is a countable set. Without loss of generality, we can suppose that J is the set of all natural numbers. Let $\mu^n := \sum_{i=1}^n a_i(0) \delta_{x_i(0)}$. By Lemma 1.3, let $(x_1(t), \dots, x_n(t))$ be the unique solution for (1.23) with initial state $(x_1(0), \dots, x_n(0))$. Set $\mu_t^n := \sum_{i=1}^n a_i(0) \delta_{x_i(t)}$. Then by Theorem 10.13 of Dynkin [3], μ_t^n is the unique solution to the $(\mathcal{A}, \delta_{\mu^n})$ -MP and all $\{\mu_t^n\}$ are defined on a common probability space. Note that for any $n \geq 1$,

$$\mu_t^n \in C([0, \infty), (M_F(\mathbb{R}), \lambda_a)).$$

If we can prove that for any $\varsigma > 0$, there exists N_0 such that

$$\sup_{0 \leq t < \infty} \lambda_a(\mu_t^n, \mu_t^m) \leq \varsigma \quad \text{for any } n, m \geq N_0 \quad \text{a.s.} \quad (1.27)$$

then

$$\mu_t := \lim_{n \rightarrow \infty} \mu_t^n \in C([0, \infty), (M_F(\mathbb{R}), \lambda_a))$$

and this implies our conclusion.

Indeed, since μ is a finite measure, there exists N_0 such that

$$\sum_{i \geq N_0} a_i(0) < \frac{\varsigma}{2} \quad \text{and} \quad \sum_{i \geq N_0} a_i^2(0) < \frac{\varsigma}{2}. \quad (1.28)$$

From these, we obtain

$$\sup_{0 \leq t < \infty} |\chi_{\mu_t^n}(\mathbb{R}) - \chi_{\mu_t^m}(\mathbb{R})| \leq \varsigma \quad \text{for } n, m \geq N_0 \quad \text{a.s.}$$

By Lemma 2.1 and Lemma 2.2 of Ethier-Kurtz [5], (1.27) follows and we are done. \square

1.3 Commutativity of semigroups

For any finite measures ν, μ on \mathbb{R} with compact supports, let $\{Z_t^1 : t \geq 0\}$, $(\{Z_t^2 : t \geq 0\})$ be the unique solution to the $(\mathcal{B}, \delta_\nu)$ -MP, (the $(\mathcal{A}, \delta_\mu)$ -MP) with distribution $\mathbb{P}_\nu^1, (\mathbb{P}_\mu^2)$ on a probability space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1), ((\Omega^2, \mathcal{F}^2, \mathbb{P}^2))$, respectively. Define

$$S_t F(\nu) := \mathbb{E}_\nu^1 F(Z_t^1) := \int F(\omega^1(t)) \mathbb{P}_\nu^1(d\omega^1), \quad (1.29)$$

$$T_t F(\mu) := \mathbb{E}_\mu^2 F(Z_t^2) := \int F(\omega^2(t)) \mathbb{P}_\mu^2(d\omega^2)$$

for $F \in C(M_F(\bar{\mathbb{R}}))$.

Lemma 1.5 *Above defined $\{S_t\}$ and $\{T_t\}$ are Feller semigroups on $C(M_F(\bar{\mathbb{R}}))$. If $\epsilon = 0$ and g is a square-integrable, smooth, and symmetric function on $\bar{\mathbb{R}}$ with $g(\partial) = 0$, then $\{T_t\}$ commutes with $\{S_t\}$. Furthermore, for any finite measure ξ on \mathbb{R} with compact support, let $\{Z_t : t \geq 0\}$ be the unique solution to the $(\mathcal{A} + \mathcal{B}, \delta_\xi)$ -MP with distribution \mathbb{P}_ξ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define*

$$U_t F := T_t S_t F \quad \text{for } F \in C(M_F(\bar{\mathbb{R}})) .$$

Then there comes

$$U_s F(Z_t) = \mathbf{E}_\xi [F(Z_{t+s}) | \mathcal{F}_t^Z] \quad (1.30)$$

for all $s, t \geq 0$ and $F \in C(M_F(\bar{\mathbb{R}}))$.

Proof: The proof of the assertion that $\{S_t\}$ and $\{T_t\}$ are strongly continuous, positive, contraction semigroups is elementary and omitted here. The assertion of Feller semigroups follows from Ethier-Kurtz ([4] p. 166. Lemma 2.3). Let us turn to the proof of the commutativity.

First, let us consider the case in which the initial state is a purely atomic measure given by $\mu = \sum_{i \in J} a_i(0) \delta_{x_i(0)}$, where $J \subset \mathbb{N}$. By Theorem 1.1, let $v_t = \sum_{i \in J} a_i(t) \delta_{x_i(t)}$ be the unique solution to the $(\mathcal{B}, \delta_\mu)$ -MP. Writing the respective finite approximations as $\mu^n := \sum_{i=1}^n a_i(0) \delta_{x_i(0)}$ and $v_t^n := \sum_{i=1}^n a_i(t) \delta_{x_i(t)}$, we see that all the measures $\{v_t^n : n \geq 1\}$ are defined on a common probability space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$. If we take $a_i(0) = 0$ for $i \notin \{1, \dots, n\}$, then by Theorem 1.1, v_t^n is the unique solution to the $(\mathcal{B}, \delta_{\mu^n})$ -MP. Let $\mu_t^n = \sum_{i=1}^n a_i(0) \delta_{x_i(t)}$ be the unique solution to the $(\mathcal{A}, \delta_{\mu^n})$ -MP constructed in Lemma 1.3. Then $\{\mu_t^n\}$ are defined on a common probability space $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$. Define

$$\tilde{\Omega} := \Omega^1 \times \Omega^2, \quad \tilde{\mathcal{F}} := \mathcal{F}^1 \times \mathcal{F}^2, \quad \tilde{\mathbb{P}} := \mathbb{P}^1 \times \mathbb{P}^2, \quad \mathcal{F}_t^{\mu, v} := \mathcal{F}_t^\mu \vee \mathcal{F}_t^v .$$

We claim that $Z_t^n = \sum_{i=1}^n a_i(t) \delta_{x_i(t)}$ is the unique solution to the $(\mathcal{A} + \mathcal{B}, \delta_{\mu^n})$ -MP on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

To prove the claim, we only need to show that for any $f \in C^\infty(\bar{\mathbb{R}}^l)$ and $\phi_i \in C^\infty(\bar{\mathbb{R}})$, $i = 1, 2, \dots, l$,

$$\begin{aligned} & f(\langle Z_t^n, \phi_1 \rangle, \dots, \langle Z_t^n, \phi_l \rangle) - f(\langle Z_0^n, \phi_1 \rangle, \dots, \langle Z_0^n, \phi_l \rangle) \\ & - \int_0^t (\mathcal{A} + \mathcal{B}) f(\langle Z_s^n, \phi_1 \rangle, \dots, \langle Z_s^n, \phi_l \rangle) ds \end{aligned} \quad (1.31)$$

is a $\mathcal{F}_t^{\mu^n, v^n}$ -martingale with respect to $\tilde{\mathbb{P}}$. Let

$$g(t, s) := f \left(\left\langle \sum_{i=1}^n a_i(t) \delta_{x_i(s)}, \phi_1 \right\rangle, \dots, \left\langle \sum_{i=1}^n a_i(t) \delta_{x_i(s)}, \phi_l \right\rangle \right) . \quad (1.32)$$

Then (1.31) is equivalent to for any $t > s \geq 0$,

$$\tilde{\mathbf{E}} \left[g(t, t) - g(s, s) - \int_s^t (\mathcal{A} + \mathcal{B}) g(u, u) du \middle| \mathcal{F}_s^{\mu^n, v^n} \right] = 0 . \quad (1.33)$$

Given a partition $s = t_0 < t_1 < \dots < t_m = t$ of the interval $[s, t]$, we can rewrite the left hand side of (1.33) as follows:

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[g(t, t) - g(s, s) - \int_s^t (\mathcal{A} + \mathcal{B})g(u, u) du \middle| \mathcal{F}_s^{\mu^n, \nu^n} \right] \\
&= \tilde{\mathbb{E}} \left[\sum_{i=1}^m \left\{ g(t_i, t_i) - g(t_{i-1}, t_{i-1}) - \int_{t_{i-1}}^{t_i} (\mathcal{A} + \mathcal{B})g(u, u) du \right\} \middle| \mathcal{F}_s^{\mu^n, \nu^n} \right] \\
&= \tilde{\mathbb{E}} \left[\sum_{i=1}^m \left\{ g(t_i, t_i) - g(t_{i-1}, t_{i-1}) - \int_{t_{i-1}}^{t_i} (\mathcal{A} + \mathcal{B})g(u, u) du - g(t_i, t_{i-1}) \right. \right. \\
&\quad \left. \left. + \int_{t_{i-1}}^{t_i} \mathcal{A}g(t_i, u) du + g(t_i, t_{i-1}) - \int_{t_{i-1}}^{t_i} \mathcal{A}g(t_i, u) du - \int_{t_{i-1}}^{t_i} \mathcal{B}g(u, t_{i-1}) du \right. \right. \\
&\quad \left. \left. + \int_{t_{i-1}}^{t_i} \mathcal{B}g(u, t_{i-1}) du \right\} \middle| \mathcal{F}_s^{\mu^n, \nu^n} \right] \\
&= \tilde{\mathbb{E}} \left[\sum_{i=1}^m \left\{ \left(g(t_i, t_i) - g(t_i, t_{i-1}) - \int_{t_{i-1}}^{t_i} \mathcal{A}g(t_i, u) du \right) \right. \right. \\
&\quad \left. \left. + \left(g(t_i, t_{i-1}) - g(t_{i-1}, t_{i-1}) - \int_{t_{i-1}}^{t_i} \mathcal{B}g(u, t_{i-1}) du \right) \right. \right. \\
&\quad \left. \left. + \int_{t_{i-1}}^{t_i} (\mathcal{A}g(t_i, u) - \mathcal{A}g(u, u)) du + \int_{t_{i-1}}^{t_i} (\mathcal{B}g(u, t_{i-1}) - \mathcal{B}g(u, u)) du \right\} \middle| \mathcal{F}_s^{\mu^n, \nu^n} \right].
\end{aligned} \tag{1.34}$$

Note that for $i = 1, \dots, m$,

$$\mathbb{E}_{\mu^n}^2 \left[g(t_i, t_i) - g(t_i, t_{i-1}) - \int_{t_{i-1}}^{t_i} \mathcal{A}g(t_i, u) du \middle| \mathcal{F}_{t_{i-1}}^{\mu^n, \nu^n} \right] = 0 \tag{1.35}$$

and

$$\mathbb{E}_{\mu^n}^1 \left[g(t_i, t_{i-1}) - g(t_{i-1}, t_{i-1}) - \int_{t_{i-1}}^{t_i} \mathcal{B}g(u, t_{i-1}) du \middle| \mathcal{F}_{t_{i-1}}^{\mu^n, \nu^n} \right] = 0. \tag{1.36}$$

Therefore, to prove (1.33), it suffices to prove

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[\int_{t_{i-1}}^{t_i} (\mathcal{A}g(t_i, u) - \mathcal{A}g(u, u)) du \right. \\
&\quad \left. + \int_{t_{i-1}}^{t_i} (\mathcal{B}g(u, t_{i-1}) - \mathcal{B}g(u, u)) du \middle| \mathcal{F}_{t_{i-1}}^{\mu^n, \nu^n} \right] = o(|t_i - t_{i-1}|).
\end{aligned} \tag{1.37}$$

But (1.37) follows from the continuity of $\{a_i(t)\}$, $\{x_i(t)\}$, $\mathcal{A}g(s, t)$, and $\mathcal{B}g(s, t)$. So the claim is proved. Combining this result with Theorem 1.1 and Theorem 1.4, we obtain

$$T_i S_i F(\mu^n) = S_i T_i F(\mu^n) \quad \text{for } F \in C(M_F(\bar{\mathbb{R}})). \tag{1.38}$$

It follows from Lemma 3.2 of Wang [12] and the above limiting argument that $Z_t := \sum_{i=1}^{\infty} a_i(t) \delta_{x_i(t)}$ is the unique solution to the $(\mathcal{A} + \mathcal{B}, \delta_\mu)$ -MP. Since $\{S_t : t \geq 0\}$ and $\{T_t : t \geq 0\}$ are Feller semigroups, we obtain

$$T_t S_t F(\mu) = S_t T_t F(\mu) \quad \text{for } F \in C(M_F(\bar{\mathbb{R}})). \tag{1.39}$$

As for the general case in which the initial state is a finite measure with compact support, since purely atomic measures are dense in $M_F(\bar{\mathbb{R}})$ and (1.39) holds for any purely atomic measure, it is easy to conclude that (1.39) holds for any $\mu \in M_F(\bar{\mathbb{R}})$. (1.30) follows from the uniqueness of solution to the $(\mathcal{A} + \mathcal{B}, \delta_\mu)$ -MP for any finite measure μ with compact support. \square

The following theorem is the summation of above results.

Theorem 1.6 *Suppose that $g \not\equiv 0$ is a square-integrable, smooth, and symmetric function on $\bar{\mathbb{R}}$ with $g(\partial) = 0$ and $\epsilon = 0$. Let ν be a finite measure on $\bar{\mathbb{R}}$ with compact support. Then on a probability space $(\Omega, \mathcal{F}, \mathbf{P}_\nu)$ the unique solution $\{\nu_t\}$ to the $(\mathcal{A} + \mathcal{B}, \delta_\nu)$ -MP has the form*

$$\nu_t = \sum_{i \in I(t)} a_i(t) \delta_{x_i(t)},$$

where $\{a_i(t) \geq 0 : i \in \cup_{t>0} I(t)\}$ is a collection of independent, one dimensional diffusion processes with state space \mathbb{R}^+ and an absorbing barrier at the origin, $x_i(t)$ is constructed in Lemma (1.3) for $i \in \cup_{t>0} I(t)$, $x_i(t) \neq x_j(t)$ for $t > 0$, $i \neq j$, and $I(t)$ is a random subset of \mathbb{N} such that, given $\omega \in \Omega$, $I(t, \omega)$ is decreasing in t in terms of set inclusion order.

Proof: Lemma 1.5, Theorem 1.1, and Theorem 1.4 together imply our conclusion.

2 Absolutely continuous measure states

The objective of this section is to prove that in the smooth, uniformly elliptic case, for time $t > 0$, the effective state space of the MBDBs is contained in the set of measures absolutely continuous with respect to one dimensional Lebesgue measure. We will use the duality method to establish the existence of density process. To this end, first let us recall the construction of the dual process given in Wang [12]. Let $\mathbf{M} = \{M(t) : t \geq 0\}$ be a pure jump Markov process on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbf{P}})$ with state space \mathbb{N} and transition intensities $q_{m,m-1} = \sigma m(m-1)$ for $m \in \mathbb{N}$, and $q_{i,j} = 0$ for all other pair (i, j) of natural numbers. Let $\{\tau_k\}$ be the sequence of jump times of \mathbf{M} (take $\tau_0 = 0$) and $\{\Gamma_k\}$ be a sequence of random operators which are conditionally independent given \mathbf{M} and satisfy

$$\hat{\mathbf{P}}(\Gamma_k = \Phi_{i,j} | M) = \frac{1}{m(m-1)} \mathbf{1}_{\{M(\tau_k^-) = m, M(\tau_k) = m-1\}} \quad (2.40)$$

for $1 \leq i, j \leq m$ and $j \neq i$, where $\Phi_{i,j}$ is defined by

$$\Phi_{i,j} f(x_1, \dots, x_N) := \begin{cases} f(x_1, \dots, x_j, \dots, x_{i-1}, x_j, x_{i+1}, \dots, x_N) & \text{if } j < i \\ f(x_1, \dots, x_{i-1}, x_j, x_{i+1}, \dots, x_j, \dots, x_N) & \text{if } i < j. \end{cases}$$

Then for the generator \mathcal{L} , the dual process is given by

$$Y(t) = T_{M(\tau_k)}(t - \tau_k) \Gamma_k T_{M(\tau_{k-1})}(\tau_k - \tau_{k-1}) \Gamma_{k-1} \dots \Gamma_1 T_{M(0)}(\tau_1) Y(0), \quad (2.41)$$

for $\tau_k \leq t < \tau_{k+1} \quad k = 0, 1, 2, \dots$,

where $Y(0) \in D(\bar{\mathbb{R}}^{M(0)}) = C^\infty(\bar{\mathbb{R}}^{M(0)})$.

For any $f \in D(\bar{\mathbb{R}}^m)$, take $Y(0) = f, M(0) = m$, let $\{\mu_t\}$ be the solution to the $(\mathcal{L}, \delta_\mu)$ -MP defined on $(\Omega, \mathcal{F}, \mathcal{F}_t^\mu, \mathbb{P}_\mu)$. In Wang [12], we have proved that the following duality

$$\mathbb{E}_\mu[\langle f, \mu_t^m \rangle] = \hat{\mathbb{E}} \left[\langle Y(t), \mu^{M(t)} \rangle \exp \left\{ \sigma \int_0^t M(u)(M(u) - 1) du \right\} \right] \quad (2.42)$$

holds for the solution $\{\mu_t\}$ to the $(\mathcal{L}, \delta_\mu)$ -MP.

We can now state the central result of this section.

Theorem 2.1 *Let μ be a finite measure on \mathbb{R} with compact support. We denote by \mathbb{P}_μ the solution to the $(\mathcal{L}, \delta_\mu)$ -MP with canonical process μ_t on the space $D([0, \infty), M_F(\bar{\mathbb{R}}))$. If $\epsilon \neq 0$ and $g(\cdot)$ is a square-integrable, smooth, and symmetric function on $\bar{\mathbb{R}}$ with $g(\partial) = 0$ in the definition of \mathcal{L} , then μ_t is absolutely continuous with respect to one dimensional Lebesgue measure for almost all $t > 0$, \mathbb{P}_μ -almost surely.*

Proof: Put $p_h(x, y) \equiv p(h, x, y) := (2\pi h)^{-1/2} \exp[-(x - y)^2/2h]$ for any $h > 0$ and any $x, y \in \mathbb{R}$. First we claim that

$$\int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu \langle \mu_t, p_h(x, \cdot) \rangle^2 dx dt < \infty \quad (2.43)$$

and

$$\lim_{h \downarrow 0} \lim_{\tilde{h} \downarrow 0} \int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu (\langle \mu_t, p_h(x, \cdot) \rangle - \langle \mu_t, p_{\tilde{h}}(x, \cdot) \rangle)^2 dx dt = 0 \quad (2.44)$$

hold for every $T > 0$. Once this claim is proved, we can find a subsequence of values of h (if necessary) such that there exists a jointly measurable function $\ell_t(x, \omega): (0, \infty) \times \mathbb{R} \times \Omega \rightarrow [0, \infty)$ satisfying both $\int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu (\ell_t(x))^2 dx dt < \infty$ and

$$\lim_{h \downarrow 0} \int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu (\langle \mu_t, p_h(x, \cdot) \rangle - \ell_t(x))^2 dx dt = 0 \quad (2.45)$$

for every $T > 0$. Moreover, for every $\phi \in C_c^\infty(\mathbb{R})$, we get

$$\begin{aligned} & \mathbb{E}_\mu \left| \langle \mu_t, \phi \rangle - \int_{\mathbb{R}} \ell_t(x) \phi(x) dx \right|^2 \\ & \leq \lim_{h \downarrow 0} \mathbb{E}_\mu \left| \langle \mu_t, \phi \rangle - \int_{\mathbb{R}} \left[\int_{\mathbb{R}} p_h(x, y) \phi(x) dx \right] \mu_t(dy) \right|^2 \\ & \quad + \lim_{h \downarrow 0} \mathbb{E}_\mu \left| \int_{\mathbb{R}} \langle \mu_t, p_h(x, \cdot) \rangle \phi(x) dx - \int_{\mathbb{R}} \ell_t(x) \phi(x) dx \right|^2 \end{aligned} \quad (2.46)$$

$$\begin{aligned}
&= \lim_{h \downarrow 0} \mathbb{E}_\mu \left| \int_{\mathbb{R}} \langle \mu_t, p_h(x, \cdot) \rangle \phi(x) dx - \int_{\mathbb{R}} \ell_t(x) \phi(x) dx \right|^2 \\
&\leq \lim_{h \downarrow 0} \int_{\mathbb{R}} \mathbb{E}_\mu \left| \langle \mu_t, p_h(x, \cdot) \rangle - \ell_t(x) \right|^2 dx \int_{\mathbb{R}} \phi^2(x) dx.
\end{aligned}$$

By (2.45), we conclude that μ_t has a Lebesgue density:

$$\mathbb{P}_\mu(\mu_t(dx) \ll dx, \text{ with density } \ell_t(x), \text{ for almost all } t > 0) = 1.$$

We prove our claim next. With G_ϵ^n defined by (1.19), let $\tilde{\mathbb{P}}^{(x_1, x_2)}$ ($\tilde{\mathbb{P}}^{(x_1)}$) be the solutions to $(G_\epsilon^2, \delta_{(x_1, x_2)})$ -MP ($(G_\epsilon^1, \delta_{(x_1)})$ -MP) with canonical processes $\{X_t^1, X_t^2\}$ ($\{X_t^1\}$) on $C([0, \infty), \mathbb{R}^2)$ ($C([0, \infty), \mathbb{R}^1)$), respectively, and $f_s^{(x_1, x_2)}(\cdot, \cdot)$ ($f_s^{(x_1)}(\cdot)$) be the transition densities corresponding to $\tilde{\mathbb{P}}^{(x_1, x_2)}$ ($\tilde{\mathbb{P}}^{(x_1)}$), respectively.

First let us consider (2.43). Recall that τ_1 is the first jump time of \mathbf{M} . By duality, we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu \langle \mu_t, p_h(x, \cdot) \rangle^2 dx dt \tag{2.47} \\
&= \int_0^T \int_{\mathbb{R}} \hat{\mathbb{E}} \mathbf{1}_{(t < \tau_1)} \left[\left\langle \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi h} \exp \left\{ -\frac{(x-y_1)^2}{2h} - \frac{(x-y_2)^2}{2h} \right\} \right. \right. \\
&\quad \left. \left. \times f_t^{(x_1, x_2)}(y_1, y_2) dy_1 dy_2, \mu^{M(t)}(dx_1, dx_2) \right\rangle \exp\{\sigma 2t\} \right] dx dt \\
&\quad + \int_0^T \int_{\mathbb{R}} \hat{\mathbb{E}} \mathbf{1}_{(t \geq \tau_1)} \left[\left\langle \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi h} \exp \left\{ -\frac{(x-y_1)^2}{2h} - \frac{(x-y_2)^2}{2h} \right\} \right. \right. \\
&\quad \left. \left. \times f_{\tau_1}^{(z, z)}(y_1, y_2) dy_1 dy_2 \times f_{t-\tau_1}^{(x_1)}(z) dz, \mu(dx_1) \right\rangle \right] \exp\{2\sigma\tau_1\} dx dt \\
&\rightarrow \int_0^T \int_{\mathbb{R}} \hat{\mathbb{E}} \mathbf{1}_{(t < \tau_1)} \left[\left\langle f_t^{(x_1, x_2)}(x, x), \mu(dx_1) \mu(dx_2) \right\rangle \exp\{\sigma 2t\} \right] dx dt \\
&\quad + \int_0^T \int_{\mathbb{R}} \hat{\mathbb{E}} \mathbf{1}_{(t \geq \tau_1)} \exp\{2\sigma\tau_1\} \left[\left\langle \int_{\mathbb{R}} f_{\tau_1}^{(z, z)}(x, x) f_{t-\tau_1}^{(x_1)}(z) dz, \mu(dx_1) \right\rangle \right] dx dt
\end{aligned}$$

as $h \rightarrow 0$

$$\begin{aligned}
&= \left\langle \int_0^T \int_{\mathbb{R}} f_t^{(x_1, x_2)}(x, x) dx dt, \mu(dx_1) \mu(dx_2) \right\rangle \\
&\quad + \int_0^T \left[\left\langle \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} 2\sigma f_s^{(z, z)}(x, x) dx f_{t-s}^{(x_1)}(z) ds dz, \mu(dx_1) \right\rangle \right] dt.
\end{aligned}$$

Since $f_t^{(\xi_1, \xi_2)}(x_1, x_2)$ is the fundamental solution of the following parabolic equation

$$\frac{\partial u}{\partial t} = G_\epsilon^2 u,$$

(2.43) follows from (2.47) and the inequality (6.12) of Friedman ([7] p. 24).

To prove (2.44), it suffices to show that

$$\lim_{h \downarrow 0} \lim_{\hbar \downarrow 0} \int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu(\langle \mu_t, p_h(x, \cdot) \rangle \langle \mu_t, p_{\hbar}(x, \cdot) \rangle) dx dt = N_T \quad (2.48)$$

and N_T is finite and independent of the ways of h and \hbar approaching to zero. In a similar way as (2.47), we can get

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \mathbb{E}_\mu(\langle \mu_t, p_h(x, \cdot) \rangle \langle \mu_t, p_{\hbar}(x, \cdot) \rangle) dx dt \quad (2.49) \\ &= \int_0^T \int_{\mathbb{R}} \hat{\mathbb{E}}1_{(t < \tau_1)} \left[\left\langle \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi(h\hbar)^{1/2}} \exp \left\{ -\frac{(x-y_1)^2}{2h} - \frac{(x-y_2)^2}{2\hbar} \right\} \right. \right. \\ & \quad \left. \left. \times f_t^{(x_1, x_2)}(y_1, y_2) dy_1 dy_2, \mu^{M(t)}(dx_1, dx_2) \right\rangle \exp\{\sigma 2t\} \right] dx dt \\ & \quad + \int_0^T \int_{\mathbb{R}} \hat{\mathbb{E}}1_{(t \geq \tau_1)} \left[\left\langle \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi(h\hbar)^{1/2}} \exp \left\{ -\frac{(x-y_1)^2}{2h} - \frac{(x-y_2)^2}{2\hbar} \right\} \right. \right. \\ & \quad \left. \left. \times f_{\tau_1}^{(z, z)}(y_1, y_2) dy_1 dy_2 f_{t-\tau_1}^{(x_1)}(z) dz, \mu(dx_1) \right\rangle \right] \exp\{2\sigma\tau_1\} dx dt \\ & \rightarrow \int_0^T \int_{\mathbb{R}} \hat{\mathbb{E}}1_{(t < \tau_1)} \left[\left\langle f_t^{(x_1, x_2)}(x, x), \mu(dx_1)\mu(dx_2) \right\rangle \exp\{\sigma 2t\} \right] dx dt \\ & \quad + \int_0^T \int_{\mathbb{R}} \hat{\mathbb{E}}1_{(t \geq \tau_1)} \left[\left\langle \int_{\mathbb{R}} f_{\tau_1}^{(z, z)}(x, x) f_{t-\tau_1}^{(x_1)}(z) dz, \mu(dx_1) \right\rangle \right] \exp\{2\sigma\tau_1\} dx dt. \end{aligned}$$

By the inequality (6.12) of Friedman ([7] p.24), clearly the above limit is finite and independent of the ways of h and \hbar approaching to zero. Therefore, (2.44) holds, as desired. \square

Remark: In the smooth, degenerate case, if two particles already met, then they never separate. So

$$\begin{aligned} & \hat{\mathbb{E}}^{(x_1, x_1)} \frac{1}{2\pi\hbar} e^{-\frac{(x-x_1)^2}{2\hbar}} e^{-\frac{(x-x_1)^2}{2h}} \quad (2.50) \\ &= \int_{\mathbb{R}} \frac{1}{2\pi\hbar} e^{-\frac{(x-y)^2}{\hbar}} f_t^{(x_1)}(y) dy, \quad x \in \mathbb{R}, \end{aligned}$$

where $f_t^{(x_1)}(y)$ is the fundamental solution for the equation

$$\frac{\partial u}{\partial t} = 2\rho(0) \frac{\partial^2 u}{\partial y^2}.$$

If $f_t^{(x_1)}(y_0) > 0$, since

$$\lim_{h \downarrow 0} \int_{\mathbb{R}} \frac{1}{(\pi h)^{1/2}} e^{-\frac{(y_0-y)^2}{h}} f_t^{(x_1)}(y) dy = f_t^{(x_1)}(y_0) > 0, \quad (2.51)$$

we have

$$\begin{aligned} & \lim_{h \downarrow 0} \int_{\mathbb{R}} \frac{1}{2\pi\hbar} e^{-\frac{(y_0-y)^2}{\hbar}} f_t^{(x_1)}(y) dy \quad (2.52) \\ &= \lim_{h \downarrow 0} \frac{1}{2(\pi h)^{1/2}} \int_{\mathbb{R}} \frac{1}{(\pi h)^{1/2}} e^{-\frac{(y_0-y)^2}{h}} f_t^{(x_1)}(y) dy = \infty. \end{aligned}$$

This suggests that in the smooth, degenerate case it is impossible to prove that $\mu_t(dx)$ is absolutely continuous with respect to dx —the result which we proved above.

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