The Residue of the Global $\eta$ Function at the Origin

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INTRODUCTION

Let $M$ be a compact Riemannian manifold of dimension $m$ without boundary and let $V$ be a smooth vector bundle over $M$ which is equipped with a smooth pointwise fibre metric $(, )$. Let $P: C^\infty(V) \to C^\infty(V)$ be an elliptic self-adjoint pseudo-differential operator of order $u \geq 0$. Let $\{\lambda_i, \phi_i\}$ be a spectral resolution of $P$. For $x \in M$ and for $s > m/u$, we define:

$$\eta(s, x, P) = \sum_{\lambda_i \neq 0} \text{sign}(\lambda_i) |\lambda_i|^{-s}(\phi_i, \phi_i)(x),$$

$$\eta(s, P) = \sum_{\lambda_i \neq 0} \text{sign}(\lambda_i) |\lambda_i|^{-s} = \int_M \eta(s, x, P) |\text{dvol}(x)|,$$

where $|\text{dvol}(x)|$ denotes the Riemannian measure on $M$. Using the calculus of pseudo-differential operators depending upon a complex parameter, which was developed by Seeley [9], one can show that both $\eta(s, x, P)$ and $\eta(s, P)$ can be meromorphically extended to the complex plane with simple poles at $s = (m-n)/u$ for $n = 0, \ldots, m$. The residues of $\eta(s, x, P)$ at the poles for $n = 0, \ldots, m$ are given by local formulas in the jets of the total symbol of $P$. We define

$$R_0(x, P) = u \cdot \text{Res}_{s=0} \eta(s, x, P),$$

$$R_0(P) = u \cdot \text{Res}_{s=0} \eta(s, P) = \int_M R_0(x, P) |\text{dvol}(x)|,$$

We gave examples in [6] of differential operators in dimensions $m = 2$ and $m = 3$ such that the local residue $R_0(x, P)$ does not vanish identically. Thus a priori, there is no reason that the global integrated residue $R_0(P)$ should vanish (and in fact one can show that in general the residue of $\eta(s, P)$ need not vanish at the other poles $s = (m-n)/u \neq 0$). Atiyah et al. [4] showed that $R_0(P) = 0$ if $m$ is odd so $\eta(s, P)$ is regular at $s = 0$. The value

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\( \eta(P) = \eta(0, P) \) is a measure of the spectral asymmetry of \( P \) and plays an important role in the index theorem for manifolds with boundary.

The case of \( m \) even was left open in \([1-4]\). We shall first prove that:

**Theorem 0.1.** \( R_0(P) = 0 \) so \( \eta(s, P) \) is regular at \( s = 0 \) for any dimension \( m \).

The second half of \([4]\) was concerned with the effect of twisting the \( \eta \)-function by a unitary representation of the fundamental group. If \( \alpha : \pi_1(M) \to U(k) \) is a representation of the fundamental group, let \( W_{\alpha} \) be the locally flat bundle defined by \( \alpha \). Since the transition functions of \( W_{\alpha} \) are locally constant, we can extend to an operator \( P_{\alpha} : C^\infty(V \otimes W_{\alpha}) \to C^\infty(V = W_{\alpha}) \). If \( P \) is a differential operator, then \( P_{\alpha} \) is unique; if \( P \) is a pseudo-differential operator, then \( P_{\alpha} \) is unique modulo infinitely smoothing operators. Let \( P_k : C^\infty(V \otimes C^k) \to C^\infty(V \otimes C^k) \) be the operator which corresponds to the trivial representation of \( \pi_1(M) \to U(k) \). Let

\[
\text{ind}(\alpha, P) = \eta(0, P_{\alpha}) - \eta(0, P_k) \mod \mathbb{Z} \quad \text{in } \mathbb{R}/\mathbb{Z}.
\]

This is a homotopy invariant of the operator \( P \). If \( W_{\alpha} \) admits a global frame \( s_0 \), we can lift \( \text{ind}(\alpha, P) \) to define an invariant \( \text{ind}(\alpha, s_0, P) \in \mathbb{R} \) whose mod \( \mathbb{Z} \) reduction is \( \text{ind}(\alpha, P) \). \( \text{Ind}(\alpha, s_0, P) \) is a homotopy invariant of \( (s_0, P) \) and is related to spectral flow.

If \( m \) is odd, then Atiyah et al. gave formulas for \( \text{ind}(\alpha, P) \) and \( \text{ind}(\alpha, s_0, P) \) in terms of characteristic classes in \([4]\), but again had to leave open the case of even \( m \). We shall prove that:

**Theorem 0.2.** The formulas given in \([4]\) for \( \text{ind}(\alpha, P) \) and \( \text{ind}(\alpha, s_0, P) \) are true in any dimension \( m \).

The first three sections of this paper will be devoted to the proof of Theorem 0.1. In the first section, we review the construction given in \([4]\) interpreting \( R_0 \) as a map in \( K \)-theory. In the second section, we use Clifford algebras to construct non-trivial bundles over even dimensional spheres and to prove \( R_0(P) = 0 \) if \( m \) is odd. In the third section, we complete the proof of Theorem 0.1 by showing \( R_0(P) = 0 \) if \( m \) is even.

We can also interpret \( \text{ind}(\alpha, P) \) and \( \text{ind}(\alpha, s_0, P) \) as maps in \( K \)-theory. Atiyah et al. used the assumption that \( m \) was odd to deduce a formula for \( \text{ind}(\alpha, s_0, P) \) from the index theorem for manifolds with boundary. They then broke the computation of \( \text{ind}(\alpha, P) \in \mathbb{R}/\mathbb{Z} \) into two parts. In Section 6 of \([4]\), they computed the torsion-free part in \( \mathbb{R}/\mathbb{Q} \) using \( \text{ind}(\alpha, s_0, P) \). The torsion part in \( \mathbb{Q}/\mathbb{Z} \) was computed in Section 8 of \([4]\) and involved no restriction on the dimension. Thus to compute \( \text{ind}(\alpha, P) \) for even dimensions, it suffices to compute \( \text{ind}(\alpha, s_0, P) \).
The advantage of working with \( \text{ind}(\alpha, s_0, P) \in R \) is that there is no torsion whereas \( \text{ind}(\alpha, P) \in R/Z \) involves torsion. We will use the results of the first three sections to show that differential operators give rational generators of the \( K \)-theory groups we shall be considering and consequently it suffices to prove Theorem 0.2 for differential operators. In the fourth section, we will use a construction suggested by a remark in [4] to use the results of [4] for pseudo-differential operators on odd dimensional manifolds to deduce the corresponding results for differential operators on even dimensional manifolds. This will complete the proof of Theorem 0.2. The reader who is only interested in Theorems 0.1 and 0.2 for differential operators can read only Section four as the treatment of differential operators given in this Section four is independent of the other sections.

**SECTION ONE**

In this section we shall review briefly the results we will need concerning \( R_0(x, P) \) and \( R_0(P) \). These are proved in [1-4] and are based on the analysis in [9]. If \( P \) is a pseudo-differential operator of order \( u \), then the total symbol of \( P \) can be presented by a formal power series \( \sigma(P)(x, \xi) \approx \sum p_{u-j}(x, \xi) \). Let \( \sigma_1(P) = p_u \) be the leading symbol of \( P \). For example, if \( \nu A \) denotes the positive square root of the Laplacian, then this is a pseudo-differential operator of order 1 with leading symbol given by \( |\xi| \), the length of the covector \( \xi \). Of course, the total symbol of \( \sqrt{A} \) is more complicated.

\( R_0(x, P) \) is a smooth invariant of the jets up to order \( m \) of \( \{ p_{u-m} \} \).

Thus, although \( R_0(x, P) \) is a local invariant of the symbol of \( P \), it does not depend only of the leading symbol.

**Lemma 1.1.**

(a) If the leading symbol of \( P \) is positive (or negative) definite for \( \xi \neq 0 \), then \( R_0(x, P) = 0 \).

(b) If \( P_i: C^\infty V_i \to C^\infty V_i \) are elliptic self-adjoint pseudodifferential operators of the same order \( u \), then \( R_0(x, P_1 \oplus P_2) = R_0(x, P_1) + R_0(x, P_2) \).

(c) If \( P(\epsilon) \) is a smooth one-parameter family of elliptic self-adjoint pseudo-differential operators of order \( u \), then \( R_0(P(\epsilon)) \) is independent of \( \epsilon \).

Property (c) shows that the global integrated residue only depends on the homotopy class of the leading symbol. Let \( T(M) \) be the tangent space of \( M \); we identify \( T(M) \) with the cotangent space \( T^*M \) using the Riemannian metric. Let \( S(M) \) be the unit sphere bundle in \( T(M) \) and let \( p: S(M) \to M \) be the projection with fibre \( S^{m-1} \). If \( P: C^\infty V \to C^\infty V \) is an elliptic self-adjoint
pseudo-differential operator, then \( \sigma_L(P)(x, \xi) \) is an invertible self-adjoint endomorphism of the fibre \( V_x \) for \( \xi \neq 0 \). Let \( \rho^* V \) be the pull-back bundle over \( S(M) \). Let \( \Pi^\pm \) be the span in \( \rho^* V \) of the eigenvectors of \( \sigma_L(x, \xi) \) which correspond to positive and negative eigenvalues. This defines bundles \( \Pi^\pm \) over \( S(M) \) such that \( \rho^* V = \Pi^+_\pm \).

Let \( K(S(M); R) \) denote the \( K \)-theory group of complex vector bundles tensor \( R \) and let \( [P] = [\Pi^+] - [\Pi^-] = 2[\Pi^+] - \rho^*(V) \in K(S(M); R) \).

**Lemma 1.2.** There is a linear map \( R_0: K(S(M); R) \to R \) such that \( R_0([P]) = R_0(P) \) and such that \( R_0 = 0 \) on the image of \( \rho^* \) of \( K(M; R) \) in \( K(S(M); R) \).

This was proved in [4] and follows directly from Lemma 1.1; there is a similar interpretation which can be given to \( \text{ind}(\alpha, s_0, P) \). If \( \text{ch} \) is the Chern character, then \( \text{ch}: K(S(M); R) \to \bigoplus_k H^{2k}(S(M); R) \). Since \( R_0 = 0 \) on the image of \( \rho^* \), we can use \( \text{ch} \) to interpret \( R_0: \bigoplus_k H^{2k}(S(M); R) / \rho^* H^{2k}(M; R) \to R \) so \( R_0(\text{ch}[P]) = R_0(P) \).

It will be convenient to work in the oriented category. If \( M \) is not orientable, let \( \tau: \tilde{M} \to M \) be the oriented double cover and let \( \bar{P} = \tau \ast P: C^\infty(\tau \ast V) \to C^\infty(\tau \ast V) \) be the pull-back. Since \( R_0(x, P) \) is a local invariant, \( R_0(\bar{x}, \bar{P}) = R_0(\tau x, P) \) so \( R_0(\bar{P}) = 2R_0(P) \). This shows that it suffices to prove \( R_0(P) = 0 \) in the oriented category to establish that \( R_0(P) = 0 \) in general. Similarly, to prove Theorem 0.2, it suffices to compute \( \text{ind}(\alpha, s_0, P) \) for the oriented category. We work with \( \text{ind}(\alpha, s_0, P) \) rather than \( \text{ind}(\alpha, P) \in R/Z \) owing to 2 torsion. We suppose henceforth that \( M \) is oriented and fix \( \omega_m \in H^m(M; R) \) as the orientation class.

For oriented manifolds, we can use the Gysin sequence to compute \( H^k(S(M); R) / \rho^* H^k(M; R) \). Let \( D(M) \) be the unit disk bundle in \( T(M) \) and let \( \pi: D(M) \to M \) be the projection. If \( i: S(M) \to D(M) \) is the inclusion, then \( \rho = \pi i \). We regard \( w_M \in H^m(D(M), S(M), S(M); R) \); \( \omega_m \) restricts to generate \( H^m(D^m, S^{m-1}; R) = R \) on the fibres. The Gysin sequence is the following long exact sequence in cohomology (where \( D = D(M) \) and \( S = S(M) \)):

\[
\cdots \leftarrow H^{k+1}(D, S; R) \leftarrow \bigoplus H^k(S; R) \leftarrow i^* H^k(D; R) \leftarrow H^k(D, S; R) \leftarrow \cdots
\]

\[
\uparrow \omega_m \vee \pi^* \quad \uparrow \pi^* \quad \uparrow \omega_m \vee \pi^*
\]

\[
H^{k+1-m}(M; R) \quad H^k(M; R) \quad H^{k-m}(M; R)
\]

Since \( i^* \pi^* = \rho^* \), we can use the vertical isomorphisms to rewrite this in the form:
... \rightarrow \delta \rightarrow H^k(S; R) \rightarrow \rho^*H^k(M; R) \rightarrow \delta \rightarrow H^{k+1-m}(M; R) \rightarrow ...

Therefore, \( \delta \) induces a map \( \delta: H^k(S(M); R) / \rho^*H^k(M; R) \rightarrow H^{k+1-m}(M; R) \).

**Lemma 1.3.**

(a) \( \delta: H^k(S(M); R) / \rho^*H^k(M; R) \rightarrow H^{k+1-m}(M; R) \) is an isomorphism for \( k \neq m-1 \). If \( k = m-1 \), then this is an isomorphism if and only if the Euler-Poincaré characteristic \( \chi(M) = 0 \).

(b) \( \delta: \bigoplus_k H^{2k}(S(M); R) / \rho^*H^{2k}(M; R) \rightarrow \bigoplus_k H^{2k+1-m}(M; R) \) is an isomorphism.

**Proof.** If \( k > m \), then \( H^k(M; R) = 0 \) so the sequence becomes

\[ 0 \rightarrow H^{k+1-m}(M; R) \rightarrow \delta \rightarrow H^m(S(M); R) \rightarrow 0, \]

which proves (a) in this case. Similarly, if \( k < m-1 \), then \( H^{k+1-m}(M; R) = 0 \) so the sequence becomes

\[ 0 \rightarrow H^k(S(M); R) \rightarrow \delta \rightarrow H^k(M; R) \rightarrow 0. \]

\( \rho^* \) is an isomorphism and \( H^{k+1-m}(M; R) = H^k(S(M); R) / \rho^*H^k(M; R) = 0 \). In the middle dimensions, this sequence becomes:

\[ 0 \rightarrow H^1(M; R) \rightarrow \delta \rightarrow H^m(S; R) \]

\[ \rightarrow \rho^*H^m(M; R) \rightarrow \delta \rightarrow H^{m-1}(S; R) \rightarrow \rho^*H^{m-1}(M; R) \rightarrow 0. \]

Thus \( \delta: H^m(S(M); R) / \rho^*H^m(M; R) \rightarrow H^1(M; R) \) is an isomorphism and \( \delta: H^{m-1}(S(M); R) / \rho^*H^{m-1}(M; R) \rightarrow H^0(M; R) \) will be an isomorphism if and only if \( j = 0 \). Since \( j: R \rightarrow R \) is multiplication by \( \chi(M) \), this proves (a). If \( m \) is even, then \( 2k \neq m-1 \) so (b) follows from (a). If \( m \) is odd, then \( \chi(M) = 0 \) so again (b) follows from (a).

**Section Two**

In this section, we shall derive the facts concerning vector bundles on even dimensional spheres which we shall use in the third section. We shall also give the proof that \( R_0(P) = 0 \) if \( m \) is odd using these techniques.

**Lemma 2.1.** Let \( \delta_{ij} \) be the Kronecker symbol and let \( n \) be even. Let \( \varepsilon_n = 1 \) if \( n \equiv 0 \) (4) and \( \varepsilon_n = \sqrt{-1} \) if \( n \equiv 2(4) \). Then there exist symmetric matrices \( \{e_0, \ldots, e_n\} \) such that: (a) \( e_ie_j + e_je_i = 2\delta_{ij} \) and (b) \( e_0 \cdots e_n = \varepsilon_n \cdot I. \)
Proof. Let \( \{v_0, \ldots, v_n\} \) be an orthonormal basis for \( R^{n+1} \) and let \( \text{CLIF}(R^{n+1}) \) denote the Clifford algebra generated by the \( \{v_i\} \) subject to the relations: \( v_iv_j + v_jv_i = -2\delta_{ij} \). \( \text{CLIF}(R^{n+1}) \) acts on the complete exterior algebra \( \Lambda(R^{n+1}) = R^{2n+1} \) by Clifford multiplication; in particular the \( \{v_i\} \) act as skew-symmetric matrices. Let \( \omega_{n+1} = v_0 \cdots v_n \in \text{CLIF}(R^{n+1}) \); \( \omega_{n+1}v_i = v_i\omega_{n+1} \). We suppose first that \( n \equiv 0(4) \); \( \omega_{n+1} \) is a skew-symmetric matrix with \( \omega_{n+1}^2 = -1 \). Let \( e_i = -\omega_{n+1}v_i \); these are symmetric and satisfy the commutation relations of (a). \( e_0 \cdots e_n = \omega_{n+1}^2 = -1 \). Next we suppose that \( n = 2(4) \); \( \omega_{n+1} \) is symmetric and \( \omega_{n+1}^2 = 1 \). Let \( e_i = -\sqrt{-1}\omega_{n+1}v_i \); these are symmetric and satisfy the commutation relations of (a). \( e_0 \cdots e_n = -(\sqrt{-1})^{n+1} = -1 \) which completes the proof.

These matrices arise in computing the symbol of \( \pm^*(d + \delta) \) as we shall see later in this section, but matrices which satisfy these two relations also arise in other contexts as we shall see in the third section.

Lemma 2.2. Let \( \{e_0, \ldots, e_n\} \) satisfy relations (a) and (b) of Lemma 2.1. Let \( x = (x_0, \ldots, x_n) \in R^{n+1} \) and let \( (x, y) = \sum_i x_i y_i \) be the usual inner product on \( R^{n+1} \). If we define \( e(x) = \sum_i x_i e_i \), then \( e(x) \) is a linear function from \( R^{n+1} \) to the symmetric matrices which satisfies the relations:

(a) \( e(x) e(y) + e(y) e(x) = 2(x, y)I \).

(b) If \( \{v_0, \ldots, v_n\} \) are an oriented orthonormal basis for \( R^{n+1} \), then \( e(v_0) \cdots e(v_n) = e_nI \).

Proof. \( e(x) e(y) + e(y) e(x) = \sum_{i,j} x_i y_j (e_i e_j + e_j e_i) = \sum_{i,j} 2x_i y_j \delta_{ij} = 2(x, y)I \) which proves (a). Let \( f(x) = \sqrt{-1} \cdot e(x) \) for \( x \in R^{n+1} \), then \( f \) extends to the Clifford algebra since \( f(x)f(y) + f(y)f(x) = -2(x, y) \). Thus \( e(v_0) \cdots e(v_n) = (-\sqrt{-1})^{n+1} f(v_0) \cdots f(v_n) = (-\sqrt{-1})^{n+1} f(v_0 \cdots v_n) = (\sqrt{-1})^{n+1} f(\omega_{n+1}) = e_0 \cdots e_n = e_nI \).

Let \( S^n = \{x \in R^{n+1} : (x, x) = 1\} \).

Lemma 2.3. Let \( e(x) \) be a linear map from \( R^{n+1} \) to the symmetric matrices which satisfies the relations of Lemma 2.2. \( e(x)^2 = 1 \) for \( x \in S^n \) is idempotent. Let \( \Pi_+(x) = \frac{1}{2}(e(x) + I) \) and \( \Pi_-(x) = I - \Pi_+(x) \) be projection on the positive and negative spectrum of \( e(x) \). Let \( \Pi_\pm \) be the bundles over \( S^n \) associating to each point \( x \in S^n \) the range of \( \Pi_\pm(x) \). Then \( \text{ch}([\Pi_+] - [\Pi_-])_n \neq 0 \) in \( H^n(S^n; R) = R \).

Proof. Since \( \Pi_+ \oplus \Pi_- = I^k \) is a trivial bundle for some \( k \) (where \( k \) is even and is the dimension of the space on which \( e(x) \) acts), \( \text{ch}([\Pi_+] - [\Pi_-])_n = 2 \text{ch}[\Pi_+] \). Let \( n = 2v \) and let \( c_v \) denote the \( v \)th Chern class. We must show \( c_v([\Pi_+]_n) \neq 0 \). Let \( \nabla \) be the flat connection on \( I^k \); we product \( \nabla \) to define connections on \( \Pi_\pm \). Fix \( x_0 \in S^n \) and let \( s_0 \) be a basis for
the range of \( \Pi_+(x_0) \). Let \( s(x) = \Pi_+(x) s_0 \); this defines a local frame for \( \Pi_+ \) near \( x_0 \). We compute:

\[
\nabla s(x) = \Pi_+(x) d\Pi_+(x) s_0 \quad \nabla^2 s(x) = \Pi_+(x) d\Pi_+(x) d\Pi_+(x) s_0.
\]

Thus at \( x_0 \) and relative to this frame, the curvature is given by

\[
\Omega(x_0) = \Pi_+(x_0) d\Pi_+(x_0) d\Pi_+(x_0).
\]

We shall show \( \text{Tr}(\Omega^n) = 2c'' \cdot \dim(\Pi_+) \omega_n \) for some universal constant \( c'' \neq 0 \). This will complete the proof of the lemma.

We choose oriented orthonormal coordinates for \( \mathbb{R}^{n+1} \) so \( x_0 = (1, 0, \ldots, 0) \). Let \( e(x) = \sum_i x_i e_i \), then the \( e_i \) satisfy the relations of Lemma 2.1. We compute:

\[
\nabla^2 v(x_0) = c(1 + e_0)^v \left( \sum_{i > 0} e_i dx^i \right)^2 = c'(1 + e_0)(e_1 \cdots e_n) dx^1 \wedge \cdots \wedge dx^n = c''(1 + e_0)(e_0) \omega_n = 2c'' \Pi_+(x_0) \omega_n \quad \text{for} \quad c'' \neq 0,
\]

which completes the proof. We note that if we replace condition (c) by \( e_0 \cdots e_n = -e_n I \), then we interchange the roles of \( \Pi_+ \) and \( \Pi_- \) so the Chern character is still non-zero. Thus the precise sign of \( e_0 \cdots e_n \) is not vital.

We use Lemma 2.3 to show that \( R_0(P) = 0 \) if \( m \) is odd. The argument we use is essentially the argument given in [4] in different notation. Let \( V = \oplus \Lambda^{k,1} \mathbb{R}^* M \) and let \( m = 2v - 1 \). Let \( P = \oplus \left( \frac{1}{2i} \right)^v (-1)^{k+1} (1 - d^*) \), where \( d \) denotes exterior differentiation and where \( * \) is the Hodge operator.

**Lemma 2.4.** If \( P \) is as defined above, then the restriction of \( \text{ch}([P])_{m-1} \) to the fibre \( \rho^{-1}(x_0) = S^{m-1} \) generates \( H^{m-1}(S^{m-1}; \mathbb{R}) \) for any \( x_0 \in M \).

**Proof.** Introduce an oriented orthonormal frame \( v_0, \ldots, v_{m-1} \) for \( \mathbb{R}^* M_{x_0} \). Let \( \xi = \sum_i \xi_i v_i \in T^* M_{x_0} \) and let \( p(x_0, \xi) \) be the leading symbol of \( P \). This is linear in \( \xi \) and \( p(x_0, \xi)^2 = |\xi|^2 \). To show that \( \text{ch}([P])_{m-1} \neq 0 \) in \( H^{m-1}(S^{m-1}; \mathbb{R}) \), we must verify that \( p \) satisfies the hypothesis of Lemma 2.1. Let \( \omega_m = v_0 \cdots v_{m-1} \); modulo factors of \( \sqrt{-1} \) (which depend upon whether \( m \equiv 1(4) \) or \( m \equiv 3(4) \)) we have \( p(x_0, \xi) = \omega_m \sum_i \xi_i v_i \). This is exactly the example discussed in the proof of Lemma 2.1 so this completes the proof of Lemma 2.4.

Let \( \alpha = \text{ch}([P])_{m-1} \). Since \( \alpha \) restricts to the generator of \( H^{m-1}(S^{m-1}; \mathbb{R}) \) on each fibre, this represents a cohomology extension. The Gysin sequence simplifies considerably if \( m \) is odd; \( H^k(S(M); \mathbb{R}) = H^k(M; \mathbb{R}) \oplus \)
\( \alpha H^{k+1-m}(M; R) \) and \( \rho^* \) is injective. Let \( W \) be an auxiliary bundle over \( M \) which is equipped with a Riemannian connection. We can define \( P_w : C^\infty(V \otimes W) \to C^\infty(V \otimes W) \); this is the operator \( P \) with coefficients in \( W \); the leading symbol is \( \rho \otimes 1 \). \( \text{ch}([P_w]) = \text{ch}([P]) \wedge \rho^* \text{ch}(W) = \alpha \wedge \rho^* \text{ch}(W) \) modulo the image of \( \rho^* \). Therefore, \( \{[P_w]\} \) generates \( K(S(M); R) \rho^* K(M; R) \) as \( W \) ranges over the bundles in \( K(M; R) \). Therefore to show that \( R_0(Q) = 0 \) in general, it suffices to show that \( R_0(P_w) = 0 \) for these specific examples.

\( R_0(x, P_w) \) is a local formula in the jets of the metric on \( M \) and in the jets of the connection on \( W \). It also depends on the choice of the orientation of \( M \). \( R_0(P_w) = \int_M R_0(x, P_w) \omega_M \) is independent of the choice of the metric on \( M \) and of the connection on \( W \). By [7], the only invariants which arise from integrating such local formulas are products of characteristic classes of \( T(M) \) and \( W \) evaluated on the fundamental cycle of \( M \). These vanish since \( m \) is odd which completes the proof of Theorem 0.1 in this case.

One can in fact use the results of [8] to show the local formula \( R_0(x, P_w) = 0 \) (although this pointwise vanishing is not true for general \( P \) as noted in [6]). As this requires slightly more information about \( R_0(x, P) \) than just the knowledge that it is a smooth local invariant, we have used instead the results of [7].

In the even dimensional case, the fibres \( S^{m-1} \) will have odd dimension and \( H^{m-1}(S^{m-1}; R) \) will not lie in the range of the Chern character. We shall therefore have to look for a different set of classifying examples although we shall still rely on the results of [7] to prove the vanishing of \( R_0(P) \).

**SECTION THREE**

Let \( \dim(M) = m \) be even and let \( M \) be oriented. In the first section, we used the Chern character to interpret

\[
R_0 : \bigoplus_k H^{2k}(S(M); R) / \rho^* H^{2k}(M; R) \to R.
\]

We used the Gysin sequence to construct an isomorphism

\[
\delta : H^k(S(M); R) / \rho^* H^k(M; R) \simeq H^{k+1-m}(M; R) \quad \text{for} \quad k \neq m - 1.
\]

In particular, this shows \( R_0 = 0 \) on \( H^{2k}(S(M); R) \) for \( 2k < m \).

Let \( \omega_j \in H^2(S^j; R) \) be the orientation class of the sphere. We will use the following lemma and corollary to complete the proof of Theorem 0.1.

**Lemma 3.1.** Let \( 0 < j < m \) be given with \( j \) odd. Let \( \Psi : M \to S^j \). There exists a self-adjoint differential operator \( P(\Psi) : C^\infty(\Lambda(M)) \to C^\infty(\Lambda(M)) \) such that
\[(a) \ R_0(P(\Psi)) = 0,\]
\[(b) \ \text{ch}([P(\Psi)]) = \rho^*a_1 + a_2 + a_3 \text{ where } a_1 \in \bigoplus_k H^{2k}(M; R), \ a_2 \in \bigoplus_{2k > j+m-1} H^{2k}(S(M); R), \ a_3 \in H^{j+m-1}(S(M); R), \text{ and } \delta a_3 = c\Psi^*(\omega_j) \text{ for } c \neq 0.\]

We shall prove Lemma 3.1 later in this section. If \(j\) is odd, the set of cohomology classes in \(H^j(M; R)\) which can be represented in the form \(\Psi^*(\omega_j)\) for some map \(\Psi: M \to S^j\) generates \(H^j(M; R)\). This proves:

**Corollary 3.2.** Let \(P(\Psi)\) be as in Lemma 3.1, then \(\{\text{ch}([P(\Psi)])\}\) generates \(\bigoplus_k H^{2k}(S(M); R)/\rho^*H^{2k}(M; R)\) as \(j\) ranges over the odd integers \(0 < j < m\) and as \(\Psi\) ranges over the set of all maps \(\Psi: S^j \to M\).

Since \(R_0(P(\Psi)) = 0\) by Lemma 3.1(a), \(R_0 = 0\) on the generators so \(R_0 \equiv 0\). This completes the proof of Theorem 0.1. We will use Corollary 3.2 in the next section in the proof of Theorem 0.2 for pseudo-differential operators.

We begin the proof of Lemma 3.1. Let \(\Psi: M \to S^j\) for \(j\) odd and let \(D^j\) be the lower hemisphere of \(S^j\). By making \(\Psi\) transverse at a point and by adjusting the parametrization of \(S^j\), we may assume that \(\Psi^{-1}(D^j) = D^j \times N^{m-j} = M_0\), where \(N^{m-j}\) is a smooth submanifold of dimension \(m - j\). This replaces \(\Psi\) by a homotopic map and does not change \(\Psi^*(\omega_j)\). Fix a metric on \(N^{m-j}\) and choose a metric on \(M\) which is product near \(M_0\).

The orientations \(\omega_M\) on \(M\) and \(\omega_j\) on \(D^j\) induce an orientation \(\omega_N\) on \(N^{m-j}\) so \(\omega_M = \omega_j \omega_N\) over \(M_0\). Let \(\omega_M, \omega_j,\) and \(\omega_N\) act on \(A(M), A(D^j),\) and \(A(N^{m-j})\) by Clifford multiplication. To simply expressions which would otherwise involve complicated powers of \(\sqrt{-1}\), we adopt the convention that symbols \(e_1, \ldots, e_j\) represent appropriate powers of \(\sqrt{-1}\). Decompose \(A(M_0) = A(D^j) \otimes A(N^{m-j}) = D^j \times C^{2j} \otimes A(N^{m-j}).\)

**Lemma 3.3.** There exist self-adjoint matrices on \(C^{2j}, \{f_0, \ldots, f_{2j}\},\) such that
\[(a) \ \ f_i f_j + f_j f_i = 2\delta_{ij},\]
\[(b) \ \ f_0 \cdots f_{2j} = e_j I,\]
\[(c) \ \ \omega_M = e_j f_0 \otimes \omega_N \text{ in the decomposition } A(M_0) = D^j \times C^{2j} \otimes A(N^{m-j}).\]

**Proof.** Decompose \(A(D^j) = A^e(D^j) \oplus A^o(D^j)\) into even and odd forms. Let \(f_0 = e_j \omega_0\) on \(A^e(D)\) and \(-e_j \omega_j\) on \(A^o(D)\). \(f_0\) is a traceless idempotent self-adjoint matrix such that \(\omega_M = e_j f_0 \otimes \omega_N\). By using the spin representation, we can find self-adjoint matrices \(\{f_0', \ldots, f_{2j}'\}\) on \(C^{2j}\) which satisfy the relations (a) and (b). Any two traceless idempotent self-adjoint matrices are conjugate so \(f_0'\) is conjugate to \(f_0\). By applying this conjugation to change the basis for \(C^{2j}\) we can assume \(f_0' = f_0\) which completes the proof.
Let $\xi = (\xi_1, \ldots, \xi_m)$ represent a point of $S^{m-1}$ and let $\xi' = (\xi'_2, \ldots, \xi'_m) \in S^{m-2}$.

**Lemma 3.4.** The map $h(\xi) = (\xi_1^2 - |\xi'|^2, 2\xi_1 \xi_2, \ldots, 2\xi_1 \xi_m)$ defines a map from $S^{m-1} \to S^{m-1}$ which is of degree 2 and which is $O(m-1)$ equivariant in the last $m-1$ coordinates.

*Proof.* It is clear $|h(\xi)|^2 = |\xi|^2$ so $h: S^{m-1} \to S^{m-1}$. Geometrically, $h$ is defined by wrapping each great circle thru the north pole $(1, 0, \ldots, 0)$ around itself twice. $h$ is two to one almost everywhere. Since $m$ is even, $h$ preserves the orientation and hence is degree 2 (if $m$ were odd, $h$ would be degree 0). $h(\xi_1, g\xi') = (\xi_1^2 - |\xi'|^2, 2\xi_1 g\xi')$ for $g \in O(m-1)$ so $h$ is $O(m-1)$ equivariant.

Let $O(m-1)$ act on the last $m-1$ coordinates of $S^{m+j-1}$.

**Lemma 3.5.** There is a smooth map $\tau(t, \xi): D^j \times S^{m-1} \to S^{m+j-1}$ such that

(a) $\tau$ is $O(m-1)$ equivariant.

(b) $\tau$ is the restriction of a map from $D^j \times R^m \to R^{m+j}$ which is quadratic in $\xi$.

(c) $\tau(t, \xi) = (1, 0, \ldots, 0)$ if $t \in \partial D$; $\tau$ extends to a map on $S^j \times S^{m-1} \to S^{m+j-1}$ of degree 2.

*Proof.* First let $j = 1$. Let $D^1 = [0, 2\pi]$. We define:

\[
\tau(r, \xi) = (\cos(r)|\xi|^2, \sin(r)h(\xi)) \quad r \in [0, \pi],
\]

\[
= (\cos(r)|\xi|^2, \sin(r)|\xi|^2, 0, \ldots, 0) \quad r \in [\pi, 2\pi].
\]

Geometrically, we are, after composing with $h$ on $S^{m-1}$, just pinching off $[0, \pi] \times S^{m-1}$ at the poles and then using the interval $[\pi, 2\pi]$ to drag the south pole back up to the north pole. This map is two to one almost everywhere on the image $S^m$ and defines a degree 2 map from $S^1 \times S^{m-1} \to S^m$. Conditions (a) and (b) are immediate.

We now deal with the more general case $j \geq 3$. Decompose $t \in D^j$ into polar coordinates $t = (r, \theta)$ for $\theta \in S^{j-1}$ and $r \in [0, 3\pi/2]$, $0 = (0, \theta)$ is the origin of $D^j$. Define:

\[
\tau(r, \theta, \xi) = (\sin(r)|\xi|^2\theta, \cos(r)h(\xi)) \quad r \in \left[0, \frac{\pi}{2}\right],
\]

\[
= (\sin(r)|\xi|^2\theta, \cos(r)|\xi|^2, 0, \ldots, 0) \quad r \in \left[\frac{\pi}{2}, \pi\right],
\]

\[
= (-\sin(r)|\xi|^2, 0, \ldots, 0, \cos(r)|\xi|^2, 0, \ldots, 0) \quad r \in \left[\pi, \frac{3\pi}{2}\right].
\]
This is continuous in \( r \); we choose a fixed smoothing in \( r \) to make this \( C^\infty \).

(a) and (b) are immediate. On \( S^j \times S^{m-j} \to S^{m+j-1} \), in the image space, this is two to one almost everywhere and defines a degree 2 map. This completes the proof.

Let \( x = (t, y) \) be a point of \( D^j \times N^{m-j} = M_0 \). Let \( v_i = dt_i \) for \( i = 1, \ldots, j \). Choose a local orthonormal frame \( \{v_i\} \) for \( T^*(N^{m-j}) \) for \( i = j + 1, \ldots, m \). We define

\[ e_i(y) = \epsilon_4 f^i_k \otimes \omega_N \quad \text{for} \quad i = 0, \ldots, 2j - 1, \]

\[ = \epsilon_5 f^j_k \otimes \omega_N v_{i+1-j}(y) \quad \text{for} \quad i = 2j, \ldots, m + j - 1. \]

These are self-adjoint matrices acting on \( A(M_0) \) by Clifford multiplication which satisfy the relations: \( e_i e_j + e_j e_i = 2\delta_{ij} \) and \( e_0 \cdots e_{m+j-1} = \epsilon I \). \( e_0 = \omega_M \) is globally defined on \( M \). \( \{e_1, \ldots, e_{2j-1}\} \) are globally defined on \( M_0 \).

\( \{e_2j, \ldots, e_{m+j-1}\}(y) \) depend on the choice of a local frame for \( T^*(N) \).

Let \( e(\xi, y) = \sum_{i} \xi_i e_i(y) \) for \( \xi \in S^{m+j-1} \) and let \( p(t, y, \xi) = e(\tau(t, \xi), y) \). Since \( \tau \) is \( O(m-1) \) equivariant, this defines an invariant symbol on \( S(M_0) \) which is the restriction of a pure quadratic polynomial on \( T^*M_0 \). Since \( p(t, y, \xi) = |\xi|^2 e_0 = \epsilon_0 |\xi|^2 \omega_M \) on \( \delta M_0 \), we can extend \( p \) to all of \( T^*M \) by defining

\[ p(x, \xi) = \epsilon_0 |\xi|^2 \omega_M \quad \text{if} \quad x \in M_0, \]

\[ = \epsilon(\tau(t, \xi), y) \quad \text{if} \quad x = (t, y) \in D^j \times N^{m-j}. \]

\( p(x, \xi) \) is a self-adjoint matrix acting on \( A(M) \) by Clifford multiplication which is a pure-quadratic in the dual variable \( \xi \). We use the Levi–Civita connection to find a canonical operator \( P \) whose leading symbol is \( p \). This operator will be denoted by \( P = P(\Psi) \).

Since \( p(x, \xi)^2 = |\xi|^4 \), \( P(\Psi) \) is elliptic. Let \( A^\pm(M) \) denote the decomposition of \( A(M) \) into \( \pm 1 \) eigenspaces under Clifford multiplication by \( \epsilon_0 \omega_M = e_0 \). Up to a possible sign convention, these are just the bundles of the signature complex. On \( M - M_0 \), \( p(x, \xi) = \pm |\xi|^2 \) on \( A^\pm(M) \) so \( P = \pm A \) on \( C^\infty(A^\pm(M - M_0)) \). This implies that the local invariant \( R_0(x, P) = 0 \) on \( M - M_0 \) so \( R_0(x, P) \) has compact support in \( M_0 \). Let \( R_0(y) = \int_{D^j} R_0(t, y, P) dt \), then \( R_0(P) = \int_{N} R_0(y) dvol(y) \). The total symbol of \( P \) depends canonically on the metric chosen for \( M \) so \( R_0(t, y, P) \) and \( R_0(y) \) are smooth local invariants of the metric on \( N^{m-j} \). \( R_0(y) \) depends on the orientation of \( M \) or equivalently on the orientation of \( N^{m-j} \). Since \( R_0(P) \) is a homotopy invariant, \( R_0(P) = \int_{N^{m-j}} R_0(y, P) dvol(y) \) is independent of the metric. We apply the results of [7] to show \( R_0(P) = c\chi(N^{m-j}) - Q(N^{m-j}) \), where \( Q \) is a Pontrjagin characteristic number. Since \( m-j \) is odd, \( \chi(N^{m-j}) = Q(N^{m-j}) = 0 \) so \( R_0(P) = 0 \) which completes the proof of Lemma 3.1(a).
On $M - M_0$, $[P] = \rho^*(A^+ - A^-)$ so $\text{ch}([P]) - \rho^* \text{ch}(A^+ - A^-)$ has compact support in $M_0$. We work modulo the image of $\rho^*$ in Lemma 3.1(b) so it suffices to prove (b) for the special case $M = S^j \times N^{m-j}$ and $\Psi(t, y) = t$.

In this case, $A^+ \simeq A^-$ so $\text{ch}(A^+ - A^-) = 0$.

Since $\chi(S^j \times N^{m-j}) = 0$, there is a global cohomology class $\theta_{m-1} \in H^{m-1}(S(M); R)$ which restricts to generate $H^{m-1}(S^{m-1}; R)$ on the fibres of $\rho$. $\rho^* : H^k(M; R) \rightarrow H^k(S(M); R)$ is injective and the map $\delta$ is given by $\delta(a) = \theta_{m-1} \wedge \rho^*(a)$. We decompose:

$$H^k(S(M); R) = \rho^*H^k(M; R) \oplus \theta_{m-1} \wedge \rho^*H^{k+1-m}(M; R).$$

Let $\omega_j \in H^j(S^j; R)$ be the generator. We identify $\omega_j = \Psi^*\omega_j$ and $H^*(N^{m-j}; R)$ with its image in $H^*(M; R)$ for notational convenience. Then:

$$H^k(M; R) = H^k(N^{m-j}; R) \oplus \omega_j \wedge H^j(N^{m-j}; R).$$

Decompose $\text{ch}([P])$ in the form

$$\text{ch}([P]) = \rho^*\alpha_1 + \rho^*\omega_j \wedge \alpha_2 + \theta_{m-1} \wedge \rho^*\alpha_3 + \theta_{m-1} \wedge \rho^*(\omega_j \wedge \alpha_4)$$

for $\alpha_i \in H^*(N^{m-j}; R)$. Fix $t_0 \in S^j - D^j$ and let $r$ be the restriction to $\rho^{-1}(t_0 \times N^{m-j})$. Then:

$$r^*\text{ch}([P]) = \rho^*\alpha_1 + \theta_{m-1} \wedge \rho^*\alpha_3.$$

However, over $t_0 \in S^j - D^j$, $[P] = \rho^*(A^+ - A^-)$ so $r^*\text{ch}([P]) = 0$. This implies $\alpha_1 = \alpha_3 = 0$. If we decompose $\alpha_4 = c + \alpha_4'$ for $\alpha_4' \in \bigoplus_{k > 0} H^k(N^{m-j}; R)$, then

$$\text{ch}([P]) = \rho^*(\omega_j \wedge \alpha_2) + \theta_{m-1} \wedge \rho^*(\omega_j \wedge \alpha_4) + \cdots$$

$$= \rho^*(\omega_j \wedge \alpha_2) + c\delta(\Psi^*\omega_j) + \cdots,$$

where we have omitted terms on $\bigoplus_{k > m-j+1} H^k(S(M); R)$. To complete the proof of Lemma 3.1(b), it suffices to check that $c \neq 0$.

Fix $y_0 \in N^{m-1}$ and restrict $\text{ch}([P])$ to $\rho^{-1}(S^j \times y_0) = S^j \times S^{m-1}$. Under this restriction, $\text{ch}([P]) = c\theta_{m-1} \wedge \omega_j$. Let $\{e_0, \ldots, e_{m-j-1}\}$ be the matrices defined earlier and let $e(\zeta) = \sum e_i(\zeta)\epsilon_i$ for $\zeta \in S^{m+j-1}$. The $\{e_i\}$ satisfy the hypothesis of Lemma 2.4 so if $\Pi_\pm$ are the bundles corresponding to the $\pm 1$ eigenvalues of $e$, then $\text{ch}(\Pi_+) - \text{ch}(\Pi_-)$ represents a non-zero element of $H^{m+j-1}(S^j \times S^{m-1}; R)$. Since $\tau$ is a degree 2 map, $\tau^*\text{ch}(\Pi_+ - \Pi_-)$ represents a non-zero element of $H^{m+j-1}(S^j \times S^{m-1}; R)$. We defined $P$ so $\sigma_1(P) = e(\tau)$ so
\[ [P] = \tau^* (\Pi_+ - \Pi_-) \] and \( \text{ch}([P]) = \tau^* \text{ch}(\Pi_+ - \Pi_-) \) represents a non-zero element of \( H^{m+j-1}(S^j \times S^{m-1}; R) \), which completes the proof of Lemma 3.1(b) by showing that \( c \neq 0 \).

SECTION FOUR

Atiyah et al. [4] proved the regularity of \( \eta(s, P) \) at \( s = 0 \) for odd dimensional manifolds. They made the following remark about the even dimensional case which we paraphrase slightly for notational consistency.

In [1] this Theorem was asserted for all \( m \). Unfortunately, we do not yet know to deal with even dimensional manifolds. The usual device, to change parity, is to replace \( M \) by \( M \times S^1 \). Now it is true that if \( A \) is self-adjoint and elliptic on \( M \) and \( B \) is elliptic on \( N \) with \( \text{order } B = \text{order } A \), then

\[
P = \begin{pmatrix} A \otimes 1 & 1 \otimes B^* \\ 1 \otimes B & -A \otimes 1 \end{pmatrix}
\]

is self-adjoint and elliptic on \( M \times N \) and \( \eta(s, P) = \text{index}(B) \eta(s, A) \). If \( A, B \) are differential operators then so is \( P \) but unfortunately if \( A \) or \( B \) is pseudodifferential \( P \) is not pseudodifferential. On the other hand, we cannot use only differential operators because \( \text{index}(B) = 0 \) if \( B \) is differential and \( \text{dim } N \) is odd—for example if \( N \) is the circle... In studying the index, this problem is not serious because \( P \) can be approximated (with respect to suitable Sobolev-norms) by pseudo-differential operators and the index is norm-continuous. The residue \( R_0(A) \) is a more sophisticated invariant, depending (in its explicit integral formula) on much of the complete symbol of \( A \) and so is not obviously continuous under a crude norm approximation. Atiyah et al. [4, pp. 84–86].

In this section, we give a slightly different version of the argument outlined above to give a proof of Theorem 0.1 for differential operators which is independent of Sections one through three. We will also use this argument to compute \( \text{ind}(a, P) \) and \( \text{ind}(a, s_0, P) \) for differential operators.

We must first review some results concerning pseudo-differential operators on the circle. Let \( S^1 = [0, 2\pi] \) with the endsections identified and let \( t \) be the usual periodic parameter. The functions \( \{\exp(int)\} \) for \( n \in \mathbb{Z} \) are an orthonormal basis for \( L^2(S^1) \).

Let \( k \) be a positive integer and let \( \lambda \in \mathbb{R} \). Let

\[
B_{1,k}(\lambda) = \left\{ \left(-\sqrt{-1} \frac{d}{dt}\right)^{2k} + \lambda^2 \right\}^{1/2k} \quad B_2 = -\sqrt{-1} \frac{d}{dt}
\]

be first order pseudo-differential operators on \( S^1 \). Their symbols are

\[
\sigma_L(B_{1,k}(\lambda))(t, \xi_1) = |\xi_1|^{2k} \quad \sigma_L(B_2)(t, \xi_1) = \xi_1.
\]
Let
\[ B_k(\lambda) = \exp(-\sqrt{-1}t)(B_1(\lambda)^k + B_2B_4(\lambda)^{k-1}) + (B_1(\lambda)^k - B_2B_1(\lambda)^{k-1}), \]
\[ \sigma(B_k(\lambda))(t, \xi_1) = \exp(-\sqrt{-1}t) |\xi_1|^k (1 + \text{sign}(\xi_1)) + |\xi_1|^k (1 - \text{sign}(\xi_1)) \]
\[ = \begin{cases} 
\exp(-\sqrt{-1}t) 2|\xi_1|^k & \text{if } \xi_1 > 0 \\
2|\xi_1|^k & \text{if } \xi_1 < 0
\end{cases}. \]

This is an elliptic pseudo-differential operator of order \(k\).

**Lemma 4.1.** Index \(B_k(\lambda) = 1\).

**Proof.** The index is a continuous integer valued function of \(\lambda\) so it suffices to check this for \(\lambda = 0\). We compute:
\[ B_{1,k}(0)(\exp(\sqrt{-1}nt)) = |n| \exp(\sqrt{-1}nt), \]
\[ B_2(\exp(\sqrt{-1}nt)) = n \exp(\sqrt{-1}nt), \]
\[ B_k(0)(\exp(\sqrt{-1}nt)) = |n|^k \exp(\sqrt{-1}(n-1)t)(1 + \text{sign}(n)) + |n|^k \exp(\sqrt{-1}nt)(1 - \text{sign}(n)) \]
\[ = \begin{cases} 
2|n|^k \exp(\sqrt{-1}(n-1)t) & \text{if } n \geq 0 \\
2|n|^k \exp(\sqrt{-1}nt) & \text{if } n \leq 0
\end{cases}. \]

This shows \(\dim(\ker(B_k(0))) = 1\). Since \(B_k(0)\) is surjective from \(L^2(S^1)\) to \(L^2(S^1)\), \(\dim(\ker(B_k^*(0))) = 0\) so the index is 1.

Let \(P: C^\infty(V) \to C^\infty(V)\) be a self-adjoint elliptic differential operator of order \(k\) on a manifold \(M\). Let \(\mathcal{M} = M \times S^1\) and let \(\mathcal{Y} = V \oplus V\) over \(\mathcal{M}\). We define \(B_k(\epsilon P)\) by replacing \(\lambda\) by \(\epsilon P\).
\[ B_{1,k}(\epsilon P) = \left( -\sqrt{-1} \frac{d}{dt} \right)^{2k} + \epsilon^2 P^2 \]
\[ B_k(\epsilon P) = \exp(-\sqrt{-1}t)(B_1(\epsilon P)^k + B_2B_1(\epsilon P)^{k-1}) + B_1(\epsilon P)^k - B_2B_1(\epsilon P)^{k-1}. \]

Let
\[ \mathcal{P}(\epsilon) = \begin{pmatrix} \epsilon P & B_k(\epsilon P)^* \\ B_k(\epsilon P) & -\epsilon P \end{pmatrix}: C^\infty(\mathcal{Y}) \to C^\infty(\mathcal{Y}) \text{ over } \mathcal{M}. \]

If \(\epsilon > 0\), this is a pseudo-differential operator of order \(k\) on \(M \times S^1\). If \(\epsilon = 0\), this is the operator defined by Atiyah et al. earlier.
LEMMA 4.2. \( P(\varepsilon) \) is elliptic for all \( \varepsilon \).

Proof. Since \( P \) and \( B_k(\varepsilon P) \) commute,

\[
\mathcal{P}(\varepsilon)^2 = \begin{pmatrix} B^2 + B_k(\varepsilon P) B_k(\varepsilon P) & 0 \\ 0 & P^2 + B_k(\varepsilon P) B_k(\varepsilon P)^* \end{pmatrix}
\]

so it suffices to check \( P^2 + B_k(\varepsilon P)^* B_k(\varepsilon P) \) and \( B_k(\varepsilon P) B_k(\varepsilon P)^* \) are elliptic. These two cases are similar so we shall only check the first. We compute:

\[
\sigma_L(P^2 + B_k(\varepsilon P)^* B_k(\varepsilon P)) = \sigma_L(P)^2 + \sigma_L(B_k(\varepsilon P)^*) \sigma_L(B_k(\varepsilon P)).
\]

Let \( \xi' = (\xi_1, \ldots, \xi_{m-1}) \) and \( \xi = (\xi', \xi_m) \). We suppose that \( \sigma_L(\cdots)(\xi) = 0 \). Since this symbol is the sum of two positive definite symbols, this implies the two equations:

\[
\sigma_L(P)(\xi') = 0 \quad \sigma_L(B_k(\varepsilon P))(\xi', \xi_m) = 0.
\]

By hypothesis \( P \) is elliptic so the first equation implies \( \xi' = 0 \). Therefore,

\[
\sigma_L(B_k(\varepsilon P))(0, \xi_m) = \sigma_L(B_k(0))(\xi_m) = 0,
\]

which implies \( \xi_m = 0 \) since \( B_k(0) \) is elliptic. This proves \( \xi = 0 \) and completes the proof.

LEMMA 4.3. \( \eta(s, P) = \eta(s, \mathcal{P}(\varepsilon)) \) for all \( \varepsilon \).

Proof. Let \( \{\lambda, \Phi_v\} \) be a spectral resolution of \( P \). We decompose

\[
L^2(\mathcal{Y}) = \bigoplus_v \Phi_v \otimes (L^2(S^1) \oplus L^2(S^1)).
\]

The operator \( \mathcal{P}(\varepsilon) \) respects this orthogonal decomposition. Let \( \mathcal{P}_v(\varepsilon) \) be the restriction of \( \mathcal{P}(\varepsilon) \) to the subspace \( \Phi_v \otimes (L^2(S^1) \oplus L^2(S^1)) \). \( B_k(\varepsilon P) = B_k(\varepsilon \lambda_v) \) on this subspace.

Let \( A^0_k(\lambda) = B_k(\lambda)^* B_k(\lambda) \) and let \( A^1_k(\lambda) = B_k(\lambda) B_k(\lambda)^* \). Let \( N(A^0_k(\lambda)) = N(B_k(\lambda)) \) and let \( N(A^1_k(\lambda)) = N(B_k(\lambda)^*) \) be the corresponding harmonic spaces; index \( B_k(\lambda) = \dim(N(A^0_k(\lambda)) - \dim N(A^1_k(\lambda)) = 1 \) by Lemma 4.1. Let \( \{\mu_i(\lambda), \Psi_i(\lambda)\} \) be a spectral resolution of \( A^0_k(\lambda) \) on \( N(A^0_k(\lambda)) \) so the \( \mu_i(\lambda) > 0 \). Then \( \{\mu_i(\lambda), B_k(\lambda) \Psi_i(\lambda)/\sqrt{\mu_i(\lambda)}\} \) is a spectral resolution of \( A^1_k(\lambda) \) on \( N(A^1_k(\lambda)) \). The two-dimensional subspace spanned by

\[
\Phi_v \otimes (\Psi_i(\varepsilon \lambda_v), 0), \quad \Phi_v \otimes (0, B_k(\varepsilon \lambda_v) \Psi_i(\varepsilon \lambda_v)/\sqrt{\mu_i(\varepsilon \lambda_v)})
\]
is invariant under $\mathcal{P}_\epsilon(e)$. On this subspace, $\mathcal{P}_\epsilon(e)$ is given by the matrix:

$$
\begin{pmatrix}
\lambda_\epsilon & \sqrt{\mu_\epsilon(e\lambda_\epsilon)} \\
\sqrt{\mu_\epsilon(e\lambda_\epsilon)} & -\lambda_\epsilon
\end{pmatrix}.
$$

This matrix has eigenvalues $\pm \sqrt{(\lambda_\epsilon^2 + \mu_\epsilon(e\lambda_\epsilon))}$. This eigenvalues have opposite signs and cancel off in the computation of $\eta$.

The remaining eigenfunctions of $\mathcal{P}_\epsilon(e)$ arise from $\Phi_\epsilon \otimes N(A^0(e\lambda_\epsilon))$ and $\Phi_\epsilon \otimes N_A^1(e\lambda_\epsilon))$. On the first subspace, $\mathcal{P}_\epsilon(e) = \lambda_\epsilon$ and on the second subspace, $\mathcal{P}_\epsilon(e) = -\lambda_\epsilon$. Consequently $\eta(s, \mathcal{P}_\epsilon(e)) = \text{sign}(\lambda_\epsilon) |\lambda_\epsilon|^{-s} (\text{dim } N(A^0(e\lambda_\epsilon)) - \text{dim } N(A^1(e\lambda_\epsilon)) = \text{sign}(e\lambda_\epsilon) |\lambda_\epsilon|^{-s}$ if $\lambda_\epsilon \neq 0$ and 0 otherwise.

$$
\eta(s, \mathcal{P}_\epsilon(e)) = \sum_{\lambda_\epsilon \neq 0} \eta(s, \mathcal{P}_\epsilon(e)) = \sum_{\lambda_\epsilon \neq 0} \text{sign}(\lambda_\epsilon) |\lambda_\epsilon|^{-s} = \eta(s, P),
$$

which completes the proof of the lemma.

We can now give a second proof of Theorem 0.1 for differential operators which is independent of the first three sections. Let $m$ be even and $P: C^\infty(V) \to C^\infty(V)$ an elliptic self-adjoint differential operator. We form $\mathcal{P}(e)$ as above. This is a pseudo-differential operator on $M \times S^1$ for $e > 0$. If $P$ is a pseudo-differential operator on $M$, then $\mathcal{P}(e)$ is not a pseudo-differential operator on $M \times S^1$ for any $e$.

Since $M \times S^1$ is odd dimensional, we can apply the results of [4] to show $\eta(s, \mathcal{P}(e))$ is regular at $s = 0$ for $e > 0$ and that therefore $\eta(s, P)$ is regular at $s = 0$.

Let $\alpha: \pi_1(M) \to U(k)$ be a representation of the fundamental group. We extend $\alpha$ to be trivial on $\pi_1(S^1)$. It is clear that the construction $P \to \mathcal{P}$ commutes with tensoring over $\alpha$ so Lemma 4.3 implies

$$
\text{ind}(\alpha, P) = \text{ind}(\alpha, \mathcal{P}(e))
$$

for all $e$.

Let $\tilde{\eta}(s, P)$ be the mod $Z$ reduction. If $P(\tau)$ is a one-parameter family of such operators, $(d/d\tau)(\tilde{\eta}(0, P\tau)))$ is given by a local integral formula in the jets of the symbols of $P(\tau)$ and of $(d/d\tau)(P\tau))$. Suppose that $W_\alpha$ admits a global frame $s_0$. Let $P_\alpha(f s_0) = P(f) s_0$; this defines both $P_\alpha$ and $P_k$ on the same bundle $W_\alpha$. It is clear that $\sigma_\alpha(P_\alpha) = \sigma_\alpha(P_k)$. Let $P(\tau)$ be any one-parameter family of self-adjoint operators on $W_\alpha$ such that $\sigma_\alpha(P(\tau)) = \sigma_\alpha(P_k)$ and such that $P(0) = P_k$ and $P(1) = P_\alpha$. For example, we could define $P(\tau) = \tau P_\alpha + (1 - \tau) P_k$. We define:

$$
\text{ind}(\alpha, s_0, P) = \int_0^1 \frac{d}{d\tau}(\tilde{\eta}(0, P\tau)) d\tau.
$$
This real-valued invariant is independent of the homotopy chosen and defines a lift on \( \text{ind}(\alpha, P) \) to \( R \) which only depends on the homotopy class of \( s_0 \) and of \( P \).

We extend \( s_0 \) to be flat along \( S^1 \) on \( W_\alpha \) over \( M \times S^1 \). Let \( \tau P_\alpha + (1 - \tau) P \). We use the correspondence \( P \to \mathcal{P} \) to define \( \mathcal{P}(\varepsilon, \tau) \). Lemma 4.3 implies that \( \frac{d}{d\tau}(\mathcal{P}(\varepsilon, \tau)) = \frac{d}{d\tau}(\mathcal{P}(\varepsilon, \mathcal{P}(\varepsilon, \tau))) \) so that:

\[
\text{ind}(\alpha, s_0, P) = \text{ind}(\alpha, s_0, \mathcal{P}(\varepsilon))
\]

for any \( \varepsilon \).

Let \( V_\alpha \) be the connection which corresponds to the locally flat structure defined by \( \alpha \). Since the curvature of \( V_\alpha \) is zero, \( W_\alpha \) is a torsion element of \( K(M) \). Suppose that in fact \( W_\alpha \) is trivial and let \( s_0 \) be a global frame for \( W_\alpha \). Let \( V_\alpha(s_0) = \theta_\alpha s_0 \) be the connection 1-form. We define

\[
\varepsilon_k = k\int_0^1 (t - t^2)^{k-1} dt,
\]

\[
\text{Tch}(\alpha) = \sum_k \left( \frac{-1}{2\pi} \right)^k \frac{1}{k!} \varepsilon_k \text{Tr}(\theta^2 k^{-1}).
\]

\( \text{Tch}(\alpha) \) is called the transgression of the Chern character. We refer to [5] for a more detailed explanation of secondary characteristic classes. If \( \text{Todd}(M) \) is the Todd class of \( T^*M \), then the formula of [4] for odd dimensions is

\[
\text{ind}(\alpha, s_0, P) = \int_{S(M)} \text{ch}[P] \wedge \rho^*(\text{Tch}(\alpha) \wedge \text{Todd}(M)). \quad (*)
\]

The formula for \( \text{ind}(\alpha, P) \) is the appropriate interpretation of this formula in \( R/Z \) in \( K \)-theory.

The formulas for \( \text{ind}(\alpha, \mathcal{P}(\varepsilon)) \) and for \( \text{ind}(\varepsilon, s_0, \mathcal{P}(\varepsilon)) \) are valid for odd dimensional manifolds and only depend on the homotopy class of the leading symbol of \( \mathcal{P}(\varepsilon) \). We may therefore set \( \varepsilon = 0 \) in applying these formulas. Using the multiplicative nature of these formulas, it is not difficult to show that the formulas applied to \( \mathcal{P}(0) \) in odd dimension \( m + 1 \) are just the corresponding formulas applied to \( P \) in even dimension \( m \). This completes the proof of Theorems 0.1 and 0.2 for differential operators.

If \( P \) is pseudo-differential, we cannot use this trick since the resulting operator \( \mathcal{P}(\varepsilon) \) is not pseudo-differential on \( M \times S^1 \). We regard

\[
\text{ind}(\alpha, P) : K(S(M))/\rho^*K(M) \to R/Z,
\]

\[
\text{ind}(\alpha, s_0, P) : K(S(M))/\rho^*K(M) \to R.
\]
Since $R$ has no torsion, this second map can be viewed:

$$\text{ind}(\alpha, s_0, P) : K(S(M); R) / p*K(M; R) \rightarrow R.$$ 

In Corollary 3.2, we proved that the differential operators generate $K(S(M); R) / p*K(M; R)$. Since we have proved formula (*) for a set of generators, this establishes this formula in general. As noted in the introduction, this is the essential step in the calculation of $\text{ind}(\alpha, P)$ which was missing in even dimensions. This completes the proof of Theorem 0.2.

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**References**