

CHAPTER 2

Stress and Strain

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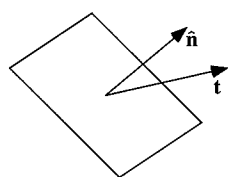
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Any quantitative description of seismic wave propagation requires the ability to characterize the internal forces and deformations in solid materials. We now begin a brief review of those parts of stress and strain theory that will be needed in subsequent chapters. Although this section is intended to be self-contained, we will not derive many equations and the reader is referred to any continuum mechanics text (e.g., Malvern, 1969) for further details.

Deformations in three-dimensional materials are termed strain; internal forces between different parts of the medium are called stress. Stress and strain do not exist independently in materials; they are linked through the constitutive relationships that describe the nature of elastic solids.

2.1 The Stress Tensor



Consider an infinitesimal plane of arbitrary orientation within a homogenous elastic medium in static equilibrium. The orientation of the plane may be specified by its unit normal vector, $\hat{n}$. The force per unit area exerted by the side in the direction of $\hat{n}$ across this plane is termed the *traction* and is represented by the vector $\mathbf{t}(\hat{n}) = (t_x, t_y, t_z)$. There is an equal and opposite force exerted by the side opposing $\hat{n}$, such that $\mathbf{t}(-\hat{n}) = -\mathbf{t}(\hat{n})$. The part of $\mathbf{t}$ that is normal to the plane is termed the *normal stress*; that parallel to it is called the *shear stress*. In the case of a fluid, there are no shear stresses and $\mathbf{t} = -P\hat{n}$, where P is the pressure.

The stress tensor, $\boldsymbol{\tau}$, in a Cartesian coordinate system (Fig. 2.1) may be defined[†] by the tractions across the yz , xz , and xy planes:

$$\boldsymbol{\tau} = \begin{bmatrix} t_x(\hat{x}) & t_x(\hat{y}) & t_x(\hat{z}) \\ t_y(\hat{x}) & t_y(\hat{y}) & t_y(\hat{z}) \\ t_z(\hat{x}) & t_z(\hat{y}) & t_z(\hat{z}) \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}. \quad (2.1)$$

[†] Often the stress tensor is defined as the transpose of (2.1) so that the first subscript of $\boldsymbol{\tau}$ represents the surface normal direction. In practice, it makes no difference as $\boldsymbol{\tau}$ is symmetric.

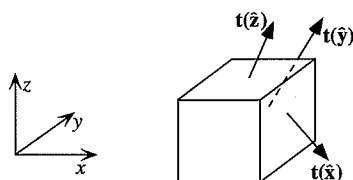
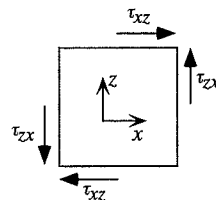


Fig. 2.1. The traction vectors $\mathbf{t}(\hat{x})$, $\mathbf{t}(\hat{y})$, and $\mathbf{t}(\hat{z})$ describe the forces on the faces of an infinitesimal cube in a Cartesian coordinate system.

Because the solid is in static equilibrium, there can be no net rotation from the shear stresses. For example, consider the shear stresses in the xz plane. To balance the torques, $\tau_{xz} = \tau_{zx}$. Similarly, $\tau_{xy} = \tau_{yx}$ and $\tau_{yz} = \tau_{zy}$, and the stress tensor $\boldsymbol{\tau}$ is symmetric, that is,

$$\boldsymbol{\tau} = \boldsymbol{\tau}^T = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \tau_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}. \quad (2.2)$$



The stress tensor $\boldsymbol{\tau}$ contains only six independent elements, and these are sufficient to completely describe the state of stress at a given point in the medium.

The traction across any arbitrary plane of orientation defined by $\hat{\mathbf{n}}$ may be obtained by multiplying the stress tensor by $\hat{\mathbf{n}}$, that is,

$$\mathbf{t}(\hat{\mathbf{n}}) = \boldsymbol{\tau} \hat{\mathbf{n}} = \begin{bmatrix} t_x(\hat{\mathbf{n}}) \\ t_y(\hat{\mathbf{n}}) \\ t_z(\hat{\mathbf{n}}) \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \tau_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} \begin{bmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{bmatrix}. \quad (2.3)$$

This can be shown by summing the forces on the surfaces of a tetrahedron (the *Cauchy tetrahedron*) bounded by the plane normal to $\hat{\mathbf{n}}$ and the xy , xz , and yz planes.

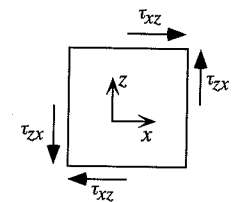
The stress tensor is simply the linear operator that produces the traction vector $\mathbf{t}$ from the normal vector $\hat{\mathbf{n}}$, and, in this sense, the stress tensor exists independent of any particular coordinate system. In seismology we almost always write the stress tensor as a 3×3 matrix in a Cartesian geometry. Note that the symmetry requirement reduces the number of independent parameters in the stress tensor to six from the nine that are present in the most general form of a second-order tensor (scalars are considered zeroth-order tensors, vectors are first order, etc.).

The stress tensor will normally vary with position in a material; it is a measure of the forces acting on infinitesimal planes at each point in the solid. Stress provides a measure only of the forces exerted across these planes and has units of force per unit area. However, other forces may be present (e.g., gravity); these are termed *body forces* and have units of force per unit volume or mass.

For any stress tensor, it is always possible to find a direction $\hat{\mathbf{n}}$ such that there are no shear stresses across the plane normal to $\hat{\mathbf{n}}$, that is, $\mathbf{t}(\hat{\mathbf{n}})$ is in the $\hat{\mathbf{n}}$ direction.

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$$\begin{aligned} \mathbf{t}(\hat{\mathbf{n}}) &= \lambda \hat{\mathbf{n}} = \boldsymbol{\tau} \hat{\mathbf{n}}, \\ \boldsymbol{\tau} \hat{\mathbf{n}} - \lambda \hat{\mathbf{n}} &= 0, \\ (\boldsymbol{\tau} - \mathbf{I}\lambda) \hat{\mathbf{n}} &= 0, \end{aligned} \quad (2.4)$$

where $\mathbf{I}$ is the identity matrix and λ is a scalar. This is an eigenvalue problem that has a nontrivial solution only when

$$\det[\boldsymbol{\tau} - \mathbf{I}\lambda] = 0. \quad (2.5)$$

This is a cubic equation with three solutions, the eigenvalues λ_1 , λ_2 , and λ_3 (do not confuse these with the Lamé parameter λ that we will discuss later). Since $\boldsymbol{\tau}$ is symmetric and real, the eigenvalues are real. Corresponding to the eigenvalues are the eigenvectors $\hat{\mathbf{n}}^{(1)}$, $\hat{\mathbf{n}}^{(2)}$, and $\hat{\mathbf{n}}^{(3)}$. The eigenvectors are orthogonal and define the *principal axes* of stress. The planes perpendicular to these axes are termed the *principal planes*. We can rotate $\boldsymbol{\tau}$ into the $\hat{\mathbf{n}}^{(1)}$, $\hat{\mathbf{n}}^{(2)}$, $\hat{\mathbf{n}}^{(3)}$ coordinate system by applying a similarity transformation:

$$\boldsymbol{\tau}^R = \mathbf{N}^T \boldsymbol{\tau} \mathbf{N} = \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix}, \quad (2.6)$$

where $\boldsymbol{\tau}^R$ is the rotated stress tensor and τ_1 , τ_2 , and τ_3 are the *principal stresses* (identical to the eigenvalues λ_1 , λ_2 , and λ_3). Here $\mathbf{N}$ is the matrix of eigenvectors

$$\mathbf{N} = \begin{bmatrix} n_x^{(1)} & n_x^{(2)} & n_x^{(3)} \\ n_y^{(1)} & n_y^{(2)} & n_y^{(3)} \\ n_z^{(1)} & n_z^{(2)} & n_z^{(3)} \end{bmatrix}, \quad (2.7)$$

with $\mathbf{N}^T = \mathbf{N}^{-1}$ for orthogonal eigenvectors normalized to unit length.

If $\tau_1 = \tau_2 = \tau_3$, then the stress field is called *hydrostatic* and there are no planes of any orientation in which shear stress exists. In a fluid the stress tensor can be written

$$\boldsymbol{\tau} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix}, \quad (2.8)$$

where P is the pressure.

2.1.1 Values for Stress

Stress has units of force per unit area. In SI units

$$1 \text{ pascal (Pa)} = 1 \text{ N m}^{-2}.$$

Table 2.1. Pressure Versus Depth Inside Earth

Depth (km)	Region	Pressure (GPa)
0–24	Crust	0–0.6
24–400	Upper Mantle	0.6–13.4
400–670	Transition Zone	13.4–23.8
670–2891	Lower Mantle	23.8–135.8
2891–5150	Outer Core	135.8–328.9
5150–6371	Inner Core	328.9–363.9

Recall that 1 newton (N) = $1 \text{ kg m s}^{-2} = 10^5 \text{ dyne}$. Another commonly used unit for stress is the *bar*:

$$1 \text{ bar} = 10^5 \text{ Pa},$$

$$1 \text{ kbar} = 10^8 \text{ Pa} = 100 \text{ MPa},$$

$$1 \text{ Mbar} = 10^{11} \text{ Pa} = 100 \text{ GPa}.$$

Pressure increases rapidly with depth in Earth, as shown in Table 2.1 using values taken from the reference model PREM (Dziewonski and Anderson, 1981). Pressures reach 13.4 GPa at 400 km depth, 136 GPa at the core–mantle boundary, and 329 GPa at the inner-core boundary. In contrast, the pressure at the center of the Moon is only about 4.8 GPa, a value reached in Earth at 150 km depth (Latham et al., 1969). This is a result of the much smaller mass of the Moon.

These are the hydrostatic pressures inside Earth; shear stresses at depth are much smaller in magnitude and include stresses associated with mantle convection and the dynamic stresses caused by seismic wave propagation. Static shear stresses can be maintained in the upper, brittle part of the crust. Measuring shear stress in the crust is a topic of current research and the magnitude of the stress is a subject of some controversy. Crustal shear stress is probably between about 100 and 1,000 bars (10 to 100 MPa), with a tendency for lower stresses to occur close to active faults (which act to relieve the stress).

2.2 The Strain Tensor

Now let us consider how to describe changes in the positions of points within a continuum. The location of every point relative to its position at a reference time t_0 can be expressed as a vector field, that is, the displacement field $\mathbf{u}$ is given by

$$\mathbf{u}(\mathbf{r}_0) = \mathbf{r} - \mathbf{r}_0, \quad (2.9)$$

where $\mathbf{r}$ is the current position of the point and $\mathbf{r}_0$ is the reference location of the point. The displacement field is an important concept and we will refer to it often in this book. It is an absolute measure of position changes. In contrast, *strain* is a local measure of relative changes in the displacement field, that is, the spatial gradients in the displacement field. Strain is related to the deformation, or change in shape, of a material rather than any absolute change in position. For example,