MATH 243, LECTURE 11

1. Using probability rules

Let's recall the basic rules for *discrete* probability problems.

- If there are D outcomes in the sample space which are a priori equally likely, then the chance of achieving one of N of these outcomes is $\frac{N}{D}$.
- $P(A \text{ doesn't happen}) = 1 - P(A)$.
- If event A and event B have no outcomes in common,
$$P(A \text{ or } B) = P(A) + P(B).$$
- If the outcome of event X is unrelated to the outcome of event Y (they are *independent*) then
$$P(X = A \text{ and } Y = B) = P(X = A) \times P(Y = B)$$

Example 1. *Find the probability of getting exactly one four when rolling a die three times.*

Example 2. *Find the probability that you there is a pair dealt in a hand with five cards.*

2. Continuous (vs. discrete) random variables

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Let's remember how to find probabilities of flipping some number of heads in a sequence of coin flips. As the number of flips gets large, the natural of the questions change.

For example, with 80 flips, we probably won't get exactly 40 heads. The probability of that is .089 But $P(35 \leq Y \leq 45) = .781$ and $P(30 \leq Y \leq 50) = .982$, so the probability of getting "near 30 heads" is high.

As the number of trials gets very large, the probability of any one outcome gets very small. It starts being less relevant to ask for example what is the probability that you would get exactly 31 heads out of 80. The answer is just too small to matter. What works better is asking the chances that the number of heads will be in some range. This change of emphasis (from "individual" measurement to ranges) is important when we look at continuous variables.

- If a random variable X takes on a finite number of possible values, it is called a *discete* random variable. All of our examples so far have been discrete.
- If a random variable X takes on a *range* of possible values, it is called a *continuous* random variable.
- The sample space is still the collection of possible values for X.
- The probability distribution of a random variable is the assignment of probabilities to the values in the sample space.
- It no longer makes sense to ask what $P(X = a)$ is, but rather what is $P(a \leq X \leq b)$.

Example 3. *Let X be a number randomly chosen between 0 and 1. (Note that it is hard to randomly choose numbers in this fashion.)*

- *What is $P(X = .3)$*
- *If we assume all choices between 0 and 1 are equally likely, what is $P(X \geq .5)$.*
- *What is $P(X \leq .3$ or $X \geq .6)$?*

Example 4. *What is the probability that a number chosen between 0 and 3 lies between 1.5 and 2.4?*

The probabilities end up corresponding to (ratios of) lengths within these state spaces. Probabilities can also correspond to (ratios of) areas.

Example 5. *If a dartboard is 8 inches in radius and the center circle is $\frac{1}{2}$ inch in radius, what is the probability that a dart thrown at random will hit the center square? What is the probability that a dart thrown at random will hit in the 20-point wedge?*

To be systematic about probabilities in the continuous setting, we have to formalize the notion of distribution.

Definition 6.
(1) *A distribution for a continuous random variable can be any non-negative function D where the total area under the function is 1.*
(2) *If X is distributed according to a distribution D then $P(a \leq X \leq b)$ is the area between the line $x = a$, the line $x = b$, the graph of D, and the x-axis.*
(3) *The sample space for such a variable is all x where D is non-zero.*
(4) *If D is equal to either 0 or one other fixed number, we say D is a uniform distribution.*

Example 7. *Revisit a previous example.*
- *Our sample space is the closed interval $[0, 3]$.*
- *Our probability distribution is the function*

$$p(t) = \begin{cases} 1/3 & 0 \leq t \leq 3 \\ 0 & \textit{else} \end{cases}$$

Example 8. *Suppose that X is distributed according to*

$$D = \begin{cases} \frac{1}{12} & 1 \leq x \leq 3 \\ \frac{1}{2} & 3 \leq x \leq 4 \\ \frac{1}{9} & 4 \leq x \leq 7 \\ 0 & \textit{otherwise} \end{cases}$$

- *What is the state space?*
- *What is $P(2 \leq X \leq 3)$?*
- *Which is more likely, that X is between 3 and 4 or that X is between 5 and 7?*

Compare the answer to the last question to the answer you would get if X were uniformly distributed.

3. Normal distributions

Example 9. *[Main Example] $\mathbf{Z}$ is a number chosen with normal distribution $N(\mu, \sigma)$. We'll use $N(0, 1)$ as our first example.*
- *Our random event is the choice of a number.*
- *Our random variable $\mathbf{Z}$ is the value of the number.*
- *Our sample space is all possible real numbers.*
- *Our probability distribution is the function*

$$e^{-t^2/2}/\sqrt{2\pi}.$$

- *What is $P(\mathbf{Z} > 3)$? (Hint: use 68-95-99.7 rule.)*
- *What is $P(0 \leq Z \leq 2)$?*
- *$P(\mathbf{Z} > 100)$?*

Doing calculations with normal distributions works much as before. Instead of calculating percentiles, we are calculating probabilities.