

## MATH 241, LECTURE 2

### 1. FUNCTIONS: THE BASIC OBJECTS OF STUDY IN CALCULUS

Functions are means for producing one quantity (or set of quantities) from another. More formally we have the following.

**Definition 1.** *A function assigns to each input, from some set called the domain of the function, a unique output, in a set called the range of the function.*

Functions can be about measurements of a quantity, such as a cost function which measures for each time the amount it costs to produce a given good. Functions can also be algebraic, such as the function which takes a number, squares it, subtracts  $\pi$  from the answer, and then takes the reciprocal of that quantity.

Even if you are more interested in functions which measure real-world quantities, part of the power of calculus, and of mathematics in general, is connecting those functions with algebraic functions. For example, both  $F = ma$  (Newton) and  $E = mc^2$  (Einstein) changed the world, relating fundamental physical quantities through simple functions. Also, with algebraic functions we can compute explicitly, which helps develop understanding we can carry with us when we do applications.

### 2. BASIC NUMERICAL FUNCTIONS

There are three kinds of basic functions we use in mathematics.

- Polynomials, ratios of polynomials, fractional powers.  
For example  $f(x) = x^2$  or  $\frac{x-1}{x+1}$  or  $\sqrt{x^5 - 3}$ .
- Exponential and logarithmic functions.  
For example  $f(x) = 2^x$  or  $\log_{10}(x)$ , which we will define later for those of you who haven't seen them.
- Trigonometric functions such as  $\sin(x)$ , which we will not be studying.

Understanding basic functions better will be an important outcome of our study of calculus.

**Question 2.** *What is the better deal: getting one million dollars a day for a month? getting  $n^5$  dollars on the  $n$ th day for a month? getting one dollar the first day and then on each day getting twice what one got the previous, for a month?*

*What if a month is changed to a week? or a year?*

### 3. MAKING NEW FUNCTIONS FROM OLD

Adding, subtracting and multiplying functions is straightforward to both execute and understand.

**Example 3.** *"Profit is the difference between total revenue and total cost" is translated into taking the difference of functions.*

Taking the quotient of functions can be trickier. For algebraic functions, you have to be careful because the domain might change.

**Example 4.** *If  $f(x) = \sqrt{x}$  and  $g(x) = x^2 - 3x + 2$ , what is the domain of the function  $\frac{f(x)}{g(x)}$ ?*

For real-world functions, which may contain error, what would seem to be a small amount of error in the denominator could lead to huge error in the final answer.

**Example 5.** *To measure speed, we take distance travelled and divide it by time. What if we tried to measure the speed of a jet by using a stopwatch over 100 yards?*

#### 4. LINEAR FUNCTIONS

Linear functions are the simplest possible functions, so they are used often throughout mathematics. One of the main themes of calculus is the understanding of general functions through related linear functions. In short, linear functions are the basis of calculus.

**Definition 6.** *Linear functions are functions of the form  $f(x) = mx + b$ , where  $m$  and  $b$  are constants.*

The graph of a linear function  $y = mx + b$  is a line in the plane.

**Question 7.** *Are all lines in the plane graphs of linear functions?*

There are many ways to describe a given linear function. An important skill to develop is the ability to translate between the different descriptions. Key to many of these descriptions is the notion of slope.

**Definition 8.** *The slope of the line  $y = mx + b$  is equal to  $m$ . It measures the change in  $y$  if  $x$  is increased by one. If one is not given the slope explicitly, it can be computed by  $m = \frac{y_2 - y_1}{x_2 - x_1}$  where  $(x_1, y_1)$  and  $(x_2, y_2)$  are any two points on the line.*

**Example 9.** *Sketch some lines with slope 1,  $-1$ , 2,  $-3$ ,  $\frac{1}{2}$  and  $-\frac{2}{3}$ .*

**Example 10.** *The points  $(-1, -1)$  and  $(3, 7)$  are both on the line  $y = 2x + 1$ . We can verify the formula for  $m$  in this case, and then take two other points on the line and use them to calculate the slope.*

The different descriptions we will use are as follows, listed from simplest to most complicated. We translate each to the standard form, and give an example.

- Standard, slope-intercept form:  $y = mx + b$ .
- Linear equation form:  $ax + by = c$ .
- Point-slope form: “the line which goes through  $(x_0, y_0)$  with slope  $m$ ” namely,  $(y - y_0) = m(x - x_0)$ .
- Two-point form: “the line between  $(x_1, y_1)$  and  $(x_2, y_2)$ ,” namely  $(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ .

**Example 11.** *Find the equations - in slope-intercept form - of the following lines:*

- The line  $3x + 4y = 7$ .
- Through the points  $(-1, 4)$  and  $(2, 3)$ .
- Through the point  $(1, 2)$  with slope 3.

Two lines are perpendicular if they intersect at ninety degree angles. In equations, it means that if the slope of one line is  $m$ , the slope of the other is  $-m$ .

**Question 12.** *Why does this make sense?*

**Example 13.** *Find the equation of the line perpendicular to the line  $2x - 3y = 3$  at the point  $(3, 1)$ .*