Energy and Cutsets in Infinite Percolation Clusters

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Abstract

Grimmett, Kesten and Zhang (1993) showed that for $d \geq 3$, simple random walk on the infinite cluster $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ of supercritical percolation on $\mathbf{Z}^d$ is a.s. transient. Their result is equivalent to the existence of a nonzero flow f on the infinite cluster such that the 2–energy $\sum_e f(e)^2$ is finite. Here we sharpen this result, and show that if $d \geq 3$ and $p > p_c(\mathbf{Z}^d)$, then $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ supports a nonzero flow f such that the q–energy $\sum_e |f(e)|^q$ is finite for all $q > d/(d-1)$. As a corollary, we obtain that any sequence $\{\Pi_n\}$ of disjoint cutsets in $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ that separate a fixed vertex from infinity, must satisfy $\sum_n |\Pi_n|^{-\beta} < \infty$ for all $\beta > 1/(d-1)$. Our proofs are based on the method of "unpredictable paths", developed by Benjamini, Pemantle and Peres (1998) and refined by Häggström and Mossel (1998).

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1 Introduction

Bernoulli (bond) percolation with parameter p on an infinite graph $G = (V_G, E_G)$ is the probability measure $\mathbf{P}_p$ on $\{0,1\}^{E_G}$ where each edge in $e \in E_G$ satisfies $\mathbf{P}_p[\omega(e) = 1] = p$ and $\mathbf{P}_p[\omega(e) = 0] = 1 - p$, and the coordinate random variables $\{\omega(e)\}_{e \in E_G}$ are independent. The edge e is called *open* in ω if $\omega(e) = 1$ and *closed* if $\omega(e) = 0$. The connected components of open edges are called *clusters*. The infimum over p such that Bernoulli percolation with parameter p has an infinite cluster a.s. is called the *critical probability*, and denoted by $p_c(G)$. For $d \geq 2$, the cubical lattice $\mathbf{Z}^d$ satisfies $0 < p_c(\mathbf{Z}^d) < 1$, and for all $p > p_c(\mathbf{Z}^d)$ there is a.s. a unique infinite cluster, denoted $\mathcal{C}_\infty(\mathbf{Z}^d, p)$. For background on percolation, see Grimmett [5].

Grimmett, Kesten and Zhang [6] proved that if $d \geq 3$ and $p > p_c(\mathbf{Z}^d)$, then simple random walk on the infinite cluster $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ is a.s. transient. As shown, for instance, in Doyle and Snell [4], transience of (simple random walk on) a graph G is equivalent to the existence of a nonzero *flow* f of finite 2–*energy* $\sum_{e \in E_G} f(e)^2$. (See Section 2 for formal definitions.) Thus the main result of [6] is equivalent to the existence of nonzero flows on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ with finite 2–energy a.s.

Benjamini, Pemantle and Peres [2] gave an alternative proof of this result, and extended it to high-density oriented percolation, using certain "unpredictable" random paths that have *exponential intersection tails* to construct random flows of finite 2–energy on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$. Here we adapt this approach to show that these flows have finite q–energy a.s. for $q > d/(d-1)$.

Definition. The q–**energy** of a flow f on a graph $G = (V_G, E_G)$ is

$$\mathcal{E}_q(f) := \sum_{e \in E_G} |f(e)|^q.$$

Theorem 1.1 *Let $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ be the infinite cluster of independent (bond) percolation with parameter p on $\mathbf{Z}^d$. Then for $d \geq 3$ and $p > p_c(\mathbf{Z}^d)$, a.s.,*

$$\inf\{q : \exists \text{ a flow } f \neq 0 \text{ on } \mathcal{C}_\infty(\mathbf{Z}^d, p) \text{ with } \mathcal{E}_q(f) < \infty\} = \frac{d}{d-1}\,.$$

Remarks.

1. Contained in Maeda [12] is the same result for $\mathbf{Z}^d$ itself: for all $d \geq 2$, the infimum of q for which there is a nonzero flow of finite q–energy on $\mathbf{Z}^d$, is

$d/(d-1)$. Our arguments do not determine whether Theorem 1.1 extends to $d = 2$. Firstly, the assumption (1) is not satisfied for any $p < 1$ in dimension $d = 2$. Also, the assumption $d \geq 3$ implies that it suffices to find flows of finite q–energy when $d/(d-1) < q < 2$, whence the function $x \mapsto x^{q-1}$ is concave; this concavity is used to pass from (4) to (5) in the proof of theorem 2.3.

2. Theorem 1.1 also holds for site percolation, with the appropriate (site) p_c and with an identical proof. The proof of Theorem 1.1 also yields the same result for oriented percolation provided $p < 1$ is sufficiently large; the renormalization arguments of Hiemer [9] allow one to extend this to all p greater than the critical probability for oriented percolation.

3. The proof of transience in [6] actually yields a flow on $\mathcal{C}_\infty(p, \mathbf{Z}^3)$ with finite q–energy if $q > 1 + \log_4 3$. It might be possible to modify the construction in [6] to give an alternative proof of Theorem 1.1; however, the refinements described in Section 4 seem much harder to obtain in this manner.

A collection of edges Π is a **cutset separating v_0 from ∞**, if any infinite path emanating from v_0 must intersect Π. Nash-Williams [13] proved that if $\{\Pi_n\}_{n=1}^\infty$ is a sequence of disjoint cutsets separating v_0 from infinity in a connected transient graph, then $\sum_n |\Pi_n|^{-1} < \infty$. Theorem 1.1 provides finer information about the permissible growth rates of cutsets on supercritical infinite percolation clusters.

Corollary 1.2 *Let $d \geq 3$ and $p > p_c(\mathbf{Z}^d)$. With probability one, if $\{\Pi_n\}$ is a sequence of disjoint cutsets in the infinite cluster $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ that separate a fixed vertex v_0 from ∞, then $\sum_n |\Pi_n|^{-\beta} < \infty$ for all $\beta > \frac{1}{d-1}$.*

This follows from Theorem 1.1 and Lemma 2.1.

This corollary captures in an interesting way the similarity of the infinite cluster to all of $\mathbf{Z}^d$; a delicate issue, that we do not address, is how the "optimal" flows and cutsets in $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ behave as $p \downarrow p_c$.

The rest of the paper is organized as follows. In Section 2, after recalling some terminology, we state and prove a general sufficient condition for percolation clusters to support flows of finite q–energy (Theorem 2.3). The condition involves the moment generation function for the number of common edges in a certain random path and a fixed path, that share a given edge. The proof is based on a combination of ideas from [2] and [14]. The latter paper proves that certain self-similar measures have densities in L^q for almost all parameters; although the setting is quite different, the method of passing from L^2 to L^q bounds is similar. In Section 3 we prove Theorem 1.1, by verifying that the "unpredictable" random paths constructed in [2] satisfy the condition in Theorem 2.3. Finally, Section 4 contains refinements of Theorem 1.1 involving energy gauges more general than powers; these refinements are based on the paths with optimal predictability profiles, constructed by Häggström and Mossel [8]. At this level, a difference appears (in the power of the logarithm) between the energies that can presently be bounded on supercritical percolation clusters and on all of $\mathbf{Z}^d$; it is an interesting open problem (stated precisely at the end of the paper), to determine whether this difference is an artifact of the proofs, or a real property of percolation clusters.

2 Paths With Exponential Intersection Tails

Definitions.

1. Let $G = (V_G, E_G)$ be an infinite graph with all vertices of finite degree and let $v_0 \in V_G$. Denote by $\Upsilon = \Upsilon(G, v_0)$ the collection of infinite oriented paths in G which emanate from v_0. Let $\Upsilon_1 = \Upsilon_1(G, v_0) \subset \Upsilon$ be the set of **paths with unit speed**, those paths for which the n^{th} vertex is at distance n from v_0. A Υ-valued random element Φ may be identified with a G-valued process $\{\Phi_n\}_{n=0}^{\infty}$, where Φ_n is the n^{th} vertex in Φ.

2. Let $0 < \theta < 1$. A Borel probability measure μ on $\Upsilon(G, v_0)$ has **Exponential intersection tails** with parameter θ (in short, EIT(θ)) if there exists C such that
$$\mu \times \mu\Big\{(\varphi, \psi) : |\varphi \cap \psi| \geq n\Big\} \leq C\theta^n$$
for all n, where $|\varphi \cap \psi|$ is the number of edges in the intersection of φ and ψ.

3. If such a measure μ exists for some basepoint v_0 and some $\theta < 1$, then we say that G *admits random paths with* EIT(θ).

4. The percolation cluster containing a vertex v will be denoted $\mathcal{C}(v)$.

5. To define a flow on an undirected graph G, it is convenient to consider each undirected edge as two directed edges, one in each direction. Let vw be the directed edge from v to w. A **flow** f on G with source v_0 is an antisymmetric edge function ($f(vw) = -f(wv)$) such that the net flow out of any vertex $v \neq v_0$ is zero: $\sum_w f(vw) = 0$. The **strength** of a flow f with source v_0 is the amount flowing from v_0: $\sum_{v_0 w} f(v_0 w)$.

Lemma 2.1 *Let G be a graph, and f a unit flow on G with source v_0. Then for all sequences of disjoint cutsets $\{\Pi_n\}$ separating v_0 from infinity,*

$$\sum_n |\Pi_n|^{-\beta} \leq \mathcal{E}_{1+\beta}(f).$$

PROOF. Observe first that

$$\mathcal{E}_{1+\beta}(f) = \sum_{e \in E_G} |f(e)|^{1+\beta} \geq \sum_n \sum_{e \in \Pi_n} |f(e)|^{1+\beta},$$

since the $\{\Pi_n\}$ are disjoint. By Jensen's inequality,

$$\forall n \quad \frac{1}{|\Pi_n|} \sum_{e \in \Pi_n} |f(e)|^{1+\beta} \geq \Big(\frac{1}{|\Pi_n|} \sum_{e \in \Pi_n} |f(e)| \Big)^{1+\beta} \geq |\Pi_n|^{-1-\beta}.$$

Multiplying by $|\Pi_n|$ and summing over n establishes the lemma. $\square$

Cox and Durrett [3] obtained upper bounds for the critical probability of oriented percolation using the fact that, for $d \geq 4$, oriented paths chosen uniformly in $\mathbf{Z}^d$ have EIT. In [2] the EIT property is exploited to prove transience of oriented supercritical clusters.

Proposition 2.2 ([2]) *Consider percolation with parameter p on G and let v_0 be a vertex in G. Suppose that μ is a probability measure on $\Upsilon_1 = \Upsilon_1(G, v_0)$ that satisfies*

$$\int_{\Upsilon_1} \int_{\Upsilon_1} p^{-|\varphi \cap \psi|} \, d\mu(\varphi) \, d\mu(\psi) < \infty . \tag{1}$$

Denote by φ_N the first N edges of a path φ. Then the random variables

$$Z_N := \mu\{\varphi \in \Upsilon_1 \,:\, \varphi_N \text{ is open }\} p^{-N}$$

form a nonnegative Martingale bounded in L^2, and therefore

$$\mathbf{P}_p[\mathcal{C}(v_0) \text{ is infinite }] \geq \mathbf{P}_p[\lim_N Z_N > 0] > 0 .$$

Moreover, if μ satisfies EIT(θ) for some $\theta < p$, then there is a.s. a vertex v in G such that the cluster $\mathcal{C}(v)$ is transient.

The following general theorem is used to establish Theorem 1.1 in Section 3.

Theorem 2.3 *Let μ be a probability measure on the set $\Upsilon_1(G, v_0)$ of paths with unit speed from v_0. Suppose that there exists $p \in (0,1)$, $\gamma > 1$ and $C < \infty$, so that for any fixed path ψ containing edge e_l at distance l from v_0,*

$$\int_{\Upsilon_1} p^{-|\varphi \cap \psi|} \mathbf{1}_{\{\varphi \ni e_l\}} d\mu(\varphi) \leq C l^{-\gamma}. \tag{2}$$

Then the event $|\mathcal{C}(v_0)| = \infty$ has positive probability, and on this event $\mathcal{C}(v_0)$ supports a nonzero flow f with $\mathcal{E}_{1+\beta}(f) < \infty$ for all $\beta > \gamma^{-1}$.

Proof of Theorem 2.3: It suffices to consider $\beta \in (\gamma^{-1}, 1)$. If $\Gamma \subset E_G$, let $I(\Gamma)$ be the indicator of the event that all the edges in Γ are open in the percolation and let $J_e(\Gamma)$ be the indicator of the event $\{e \in \Gamma\}$. For each $N \geq 1$ we define an edge function f_N on the ball $B(v_0, N)$ as follows. For every directed edge $e = vw$ where w is farther from v_0 than v, let

$$f_N(e) = \int_{\Upsilon_1} p^{-N} I(\varphi_N) J_e(\varphi_N)\, d\mu(\varphi)\,,$$

and define $f(wv) = -f(vw)$. If v and w are at the same distance from v_0, set $f(vw) = f(wv) = 0$. Then f_N is a flow on $\mathcal{C}(v_0) \cap B(v_0, N)$ from v_0 to the complement of $B(v_0, N-1)$, i.e., for any vertex $v \in B(v_0, N-1)$ except v_0, the incoming flow to v equals the outgoing flow from v.

The expected $(1+\beta)$–energy of f_N is

$$\mathbf{E}_p \sum_{e \in E_G} \int_{\Upsilon_1} p^{-N} I(\psi_N) J_e(\psi_N)\, d\mu(\psi_N) \left\{ \int_{\Upsilon_1} p^{-N} I(\varphi_N) J_e(\varphi_N) d\mu(\varphi) \right\}^{\beta} .$$

By Fubini's Theorem, this equals

$$\sum_{e \in E_G} \int_{\Upsilon_1} J_e(\psi_N) p^{-N} \mathbf{E}_p \left[I(\psi_N) \left\{ \int_{\Upsilon_1} p^{-N} I(\varphi_N \setminus \psi_N) J_e(\varphi_N) d\mu(\varphi) \right\}^{\beta} \right] d\mu(\psi)\,. \tag{3}$$

The two factors appearing in the expectation above depend on disjoint edges, hence they are independent and the expectation of the product can be replaced by a product of expectations. Consequently (3) is equal to

$$\sum_{e \in E_G} \int_{\Upsilon_1} J_e(\psi_N) p^{-N} p^N \mathbf{E}_p \left[\left\{ \int_{\Upsilon_1} p^{-N} I(\varphi_N \setminus \psi_N) J_e(\varphi_N) d\mu(\varphi) \right\}^{\beta} \right] d\mu(\psi)\,. \tag{4}$$

An application of Jensen's inequality to the (concave) function $x \mapsto x^\beta, \beta < 1$ then yields that (4) is bounded by

$$\begin{aligned} &\sum_{e \in E_G} \int_{\Upsilon_1} J_e(\psi_N) \left\{ \mathbf{E}_p \left[\int_{\Upsilon_1} p^{-N} I(\varphi_N \setminus \psi_N) J_e(\varphi_N) d\mu(\varphi) \right] \right\}^\beta \qquad (5) \\ = &\sum_{l=1}^{\infty} \int_{\Upsilon_1} \sum_{|e|=l} J_e(\psi_N) \left\{ \int_{\Upsilon_1} p^{-|\varphi_N \cap \psi_N|} J_e(\varphi_N) d\mu(\varphi) \right\}^\beta d\mu(\psi), \end{aligned}$$

where, for directed $e = vw$, $|e|$ is the distance from v to v_0. The above is not larger than

$$\sum_{l=1}^{\infty} \int_{\Upsilon_1} \left\{ \int_{\Upsilon_1} p^{-|\varphi \cap \psi|} J_{e(l)}(\varphi) d\mu(\varphi) \right\}^\beta d\mu(\psi) \le C \sum_{l=1}^{\infty} l^{-\beta\gamma}, \qquad (6)$$

where $e(l) = e(l, \psi)$ is the unique edge in ψ at distance l from v_0, and we have used the hypothesis (2).

For each directed edge e, the sequence $\{f_N(e)\}_{N>|e|}$ is a nonnegative martingale, so it converges a.s. to a limit denoted $f(e)$. Clearly, f is a flow from v_0 to infinity. The strength of f_N is precisely the random variable Z_N that appears in Proposition 2.2. The assumption (2) implies the condition (1) of Proposition 2.2, and hence $\mathcal{C}(v_0)$ is infinite and the strength of f, $\lim_N Z_N$, is positive with positive probability. Finally, $\mathbf{E}_p[\mathcal{E}_{1+\beta}(f)] \le \sup_N \mathbf{E}_p[\mathcal{E}_{1+\beta}(f_N)] < \infty$, since the right-hand side of (6) is finite for any $\beta > 1/\gamma$.

□

3 Unpredictable paths and percolation in $\mathbf{Z}^d$

Definition. For a sequence of random variables $S = \{S_n\}_{n \ge 0}$ taking values in a countable set V, we define its **predictability profile** $\{\mathsf{PRE}_S(k)\}_{k \ge 1}$ by

$$\mathsf{PRE}_S(k) = \sup \mathbf{P}[S_{n+k} = x \mid S_0, \ldots, S_n], \qquad (7)$$

where the supremum is over all $x \in V$, all $n \ge 0$ and all histories $S_0, \ldots, S_n$.

The following was used in [2] along with Proposition 2.2 to prove the theorem of Grimmett, Kesten and Zhang. We use it in the proof of Theorem 1.1 below.

Lemma 3.1 ([2]) *Let $\{\Gamma_n\}$ be a sequence of random variables taking values in a countable set V. If the predictability profile of Γ satisfies $\sum_{k=1}^{\infty} \mathsf{PRE}_\Gamma(k) < \infty$, then* *there exist* $C < \infty$ *and* $0 < \theta < 1$*, such that for any sequence* $\{v_n\}_{n\geq 0}$ *in* V *and all* $m \geq 1$*,*

$$\mathbf{P}[\#\{n \geq 0 \,:\, \Gamma_n = v_n\} \geq m] \leq C\theta^m\,.$$

We now specialize to the case where $G = \mathbf{Z}^d$ for $d \geq 3$. We shall need paths whose predictability profiles are controlled. The basic building block for such paths in $\mathbf{Z}^d$ is an integer–valued nearest neighbor process:

Theorem 3.2 (Benjamini, Pemantle, Peres [2]) *For any* $\alpha < 1$ *there exists an integer-valued stochastic process* $\{S_n\}_{n\geq 0}$ *such that* $|S_n - S_{n-1}| = 1$ *a.s. for all* $n \geq 1$ *and*

$$\mathsf{PRE}_S(k) \leq C_\alpha k^{-\alpha} \qquad \text{for some } C_\alpha < \infty, \text{ for all } k \geq 1\,.$$

Remark: Let $\mathbf{T}_b(M)$ be the tree of depth M where each vertex not at the deepest level has b children. The construction of S in Theorem 3.2 uses a random element σ with values in $\{-1,1\}^{\mathbf{T}_b(M)}$, which can be obtained from a variant of the Ising model at low temperature. Order the vertices on the boundary from left to right as $w_1,\ldots,w_{b^M}$. Processes $S^M, M > 1$, are defined for $n \leq b^M$ by $S^M_n = \sum_{k=1}^n \sigma(w_k)$, and S is then defined for all $n \geq 0$ using the consistency of the laws of the S^M.

Given a $\mathbf{Z}^d$-valued process Y up to time T, we define the *time-reversal* $\overleftarrow{Y}$ of Y up to time T, started at $z \in \mathbf{Z}^d$, by

$$\overleftarrow{Y}_k := z + Y_{T-k} - Y_T \text{ for } k \in [0,T]\,.$$

Since the processes S^M used in Theorem 3.2 to construct S are defined by summing the spins $\sigma(v)$ over v in the deepest level of $\mathbf{T}_b$, the process S has the property that

$$\mathsf{PRE}_{\overleftarrow{S}}(k) \leq C_\alpha k^{-\alpha}. \tag{8}$$

Corollary 3.3 *For each* $\frac{1}{2} < \alpha < 1$*, there is a* $\mathbf{Z}^d$*–valued process* $\Phi = \Phi^{\alpha,d}$ *so that*

$$\mathsf{PRE}_\Phi(k) \leq C(\alpha,d)k^{-(d-1)\alpha}, \tag{9}$$

and so the random edge sequence $\{\Phi_{n-1}\Phi_n\}_{n\geq 1}$ *is supported on* Υ_1*. Moreover, its time-reversal* $\overleftarrow{\Phi}$*, started at* $z \in \mathbf{Z}^d$ *and defined for times* $k \leq M$*, also satisfies for* $k \leq M$

$$\mathsf{PRE}_{\overleftarrow{\Phi}}(k) \leq C(\alpha,d)k^{-(d-1)\alpha}. \tag{10}$$

PROOF. Let $W_k^r = (S_k^{(r)} + k)/2$ for $r = 1, \ldots, d-1$, where $S^{(r)}$ are independent copies of the process described in Theorem 3.2. For $r = 1, \ldots, d-1$, define clocks

$$t_r(n) := \lfloor \frac{n+d-1-r}{d-1} \rfloor ,$$

and let $D(n) := n - \sum_{r=1}^{d-1} W^r_{t_r(n)}$.

Write $\Phi_n = (W^1_{t_1(n)}, \ldots, W^{d-1}_{t_{d-1}(n)}, D(n))$. It is then easy to see that

$$\mathsf{PRE}_\Phi(k) \le \left[\mathsf{PRE}_S(\lfloor \frac{k}{d-1} \rfloor)\right]^{d-1} \le \left(\frac{C_\alpha k}{d-1}\right)^{-\alpha(d-1)} \le C(\alpha,d)k^{-\alpha(d-1)} .$$

The same bound for $\mathsf{PRE}_{\overleftarrow{\Phi}}(k)$ is obtained similarly, using (8). □

Proof of Theorem 1.1. Since a flow on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ is also a flow on $\mathbf{Z}^d$, there can be no flows of finite q–energy on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ for $q \le d/(d-1)$.

For the remainder of the proof let $q > d/(d-1)$ and denote $\beta = q-1$. We want to show that for $p > p_c$, a.s. a flow f on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ with finite $(1+\beta)$–energy exists. Since $\beta > 1/(d-1)$, we may choose $\alpha \in (1/2, 1)$ so that $\beta\alpha > 1/(d-1)$.

We first verify the hypotheses of Theorem 2.3 for $\gamma = \alpha(d-1)$. Fix a path $\psi \in \Upsilon_1$, and let $(e_0, e_1, e_2, \ldots)$ be its constituent edges. If $e = vw$, write $\underline{e}$ for v and $\overline{e}$ for w.

For any path φ, thought of as a sequence of edges, denote by φ_l the first l edges of φ and write $U(\varphi, \psi, l) := |\varphi \cap \psi| - |\varphi_l \cap \psi|$. Let Φ be the process constructed in Corollary 3.3 and let μ denote the distribution of the random edge sequence $\{\Phi_{n-1}\Phi_n\}_{n \ge 1}$. By Lemma 3.1, the process Φ constructed in Corollary 3.3 has the property that, given the history of the first l steps, the number of subsequent intersections with a fixed trajectory has an exponential tail:

$$\mu[\varphi : U(\varphi, \psi, l) > n \mid \mathcal{F}_l] \le C_1 \theta^n , \tag{11}$$

where $\mathcal{F}_l$ is the σ-field generated by the random variables $\{\mathbf{1}_{\{e \in \varphi\}} : |e| \le l\}$.

Our next goal is to verify

$$\int_{\Upsilon_1} p^{-|\varphi \cap \psi|} \mathbf{1}_{\{\varphi \in e_l\}} d\mu(\varphi) \le C l^{-\gamma}, \tag{12}$$

for p sufficiently close to 1. The left hand side of (12) equals

$$E^\mu \left[p^{-|\varphi_l \cap \psi|} \mathbf{1}_{\{\varphi \ni e_l\}} E^\mu[p^{-U(\varphi,\psi,l)} \mid \mathcal{F}_l]\right] . \tag{13}$$

By (11), this is bounded by

$$\frac{C_1}{1-p^{-1}\theta}\sum_{m=1}^{\infty} p^{-m}\mu[\varphi \ni e_l \text{ and } |\varphi_l \cap \psi| = m]. \tag{14}$$

Let $A := \{|\varphi_{l/2} \cap \psi| \geq m/2\}$ and $B := \{|(\varphi_l \setminus \varphi_{l/2}) \cap \psi| \geq m/2\}$. We have

$$\mu[|\varphi_l \cap \psi| = m \text{ and } \varphi \ni e_l] \leq \mu\left[A \cap \{\varphi \ni e_l\}\right] + \mu\left[B \cap \{\varphi \ni e_l\}\right]. \tag{15}$$

By (9), $\mu[\varphi \ni e_l \mid A] \leq C_2 l^{-\gamma}$, and by Lemma 3.1, $\mu[A] \leq C_1\theta^{m/2}$. Thus

$$\mu\left[A \cap \{\varphi \ni e_l\}\right] = \mu[\varphi \ni e_l \mid A] \cdot \mu[A] \leq C_3\theta^{m/2} l^{-\gamma}. \tag{16}$$

Let $\overleftarrow{\Phi}$ be the time–reversal of Φ started at $\overline{e_l}$, and let $\overleftarrow{B}$ be the event that $\{\overleftarrow{\Phi}_n\}_{n \leq l/2}$ intersects the vertices determined by ψ at least $m/2$ times. Then

$$\mu[B \cap \{\varphi \ni e_l\}] \leq \mathbf{P}[\overleftarrow{B} \cap \{\overleftarrow{\Phi} \ni 0\}],$$

because the number of edge intersections of two paths is bounded by the number of vertex intersections. By Lemma 3.1, $\mathbf{P}[\overleftarrow{B}] \leq C_1\theta^{m/2}$, and (9) implies that $\mathbf{P}[\overleftarrow{\Phi} \ni 0 \mid \overleftarrow{B}] < C_2 l^{-\gamma}$. Thus

$$\mu[B \cap \{\varphi \ni e_l\}] \leq \mathbf{P}[\overleftarrow{B} \cap \{\overleftarrow{\Phi} \ni 0\}] = \mathbf{P}[\overleftarrow{B}] \cdot \mathbf{P}[\overleftarrow{\Phi} \ni 0 \mid \overleftarrow{B}] \leq C_3\theta^{m/2} l^{-\gamma}.$$

We conclude that the right-hand side of (15) is bounded by $2C_3\theta^{m/2}l^{-\gamma}$. Thus for $p > \sqrt{\theta}$, the sum (14) is bounded by $Cl^{-\gamma}$, and (12) follows. Since $\beta > [\alpha(d-1)]^{-1}$, by Theorem 2.3, $\mathbf{P}[I(0)] > 0$, where $I(0)$ is the event that $\mathcal{C}(0)$ is infinite and supports a flow of finite $(1+\beta)$–energy. The event $\bigcup_{v \in \mathbf{Z}^d} I(v)$ does not depend on the status of any finite collection of edges, and hence by Kolmogorov's zero–one law, has probability one.

This concludes the proof for p near 1; The general case $p > p_c$ is reduced to this by the renormalization argument used in Corollary 2.1 of [2], which relies on techniques of [7],[1] and [15]; a result of Soardi and Yamasaki [17], that the existence of a flow of finite q–energy is invariant under rough isometries, is also needed. □

4 A Refinement

The concept of energy can be further generalized by defining the H–energy of a flow f as $\mathcal{E}_H(f) := \sum_e H(|f(e)|)$, where $H : [0,\infty) \to [0,\infty)$ is nondecreasing. For the rest of this section, we fix $d \geq 3$ and compare H–energy of flows on $\mathbf{Z}^d$ and on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$. As we shall see, in both cases the "critical" gauges are obtained by logarithmic corrections to the power law $u \mapsto u^{d/(d-1)}$.

Notation. For any positive decreasing function h, let $H_h(u) := u^{d/(d-1)}/h(u)$ for $u > 0$ and $H_h(0) = 0$. If $h(u) = [\log(1 + u^{-1})]^\alpha$, then we abbreviate H_h by H_α. We let C, C_i denote positive finite constants whose value is unimportant.

First we consider the case of $\mathbf{Z}^d$ itself. Let $D(l)$ be the collection of edges at distance l from the origin. T. Lyons [11] constructed a nonzero flow f_* on $\mathbf{Z}^d$ that satisfies $|f_*(e)| \leq Cl^{1-d}$ for any edge $e \in D(l)$. Thus

$$\sum_e H_h(|f_*(e)|) \leq C_1 \sum_{l=1}^\infty l^{d-1} \frac{(Cl^{1-d})^{d/(d-1)}}{h(Cl^{1-d})} \leq C_2 \sum_l \frac{1}{lh(l^{1-d})} \asymp \int_1^\infty \frac{dx}{xh(x^{-1})} \, ,$$

where $y \asymp z$ means that the ratio y/z is bounded above and below by positive constants.

Let f be a unit flow from 0. If H_h is convex, then

$$|D(l)|^{-1} \sum_{e \in D(l)} H_h(|f(e)|) \ \geq \ H_h \left(|D(l)|^{-1} \sum_{e \in D(l)} |f(e)| \right) ,$$

by Jensen's inequality. Since f is a unit flow, $\sum_{e \in D(l)} |f(e)| \geq 1$, so

$$\sum_{e \in D(l)} H_h(|f(e)|) \geq |D(l)| \ H_h(|D(l)|^{-1}) \geq \frac{C_3}{lh(C_4 l^{1-d})} \, .$$

Thus the H_h–energy of any unit flow f is at least

$$\sum_l \sum_{e \in D(l)} H_h(|f(e)|) \geq \sum_l \frac{C_3}{lh(C_4 l^{1-d})} \asymp \int_1^\infty \frac{dx}{xh(x^{-1})} \, .$$

In particular, $\mathbf{Z}^d$ supports a flow of finite H_α–energy iff $\alpha > 1$.

Proposition 4.1 *Let h be a decreasing function satisfying*

$$\sum_j^\infty \frac{1}{jh(j^{-1})} < \infty \tag{17}$$

and $h(x^2) \leq \kappa h(x)$ for all $x > 0$. Define $G_h(u) := H_{h^2}(u)/u$, and assume that G_h is concave. Then for $p > p_c$, there is a.s. a flow $f \neq 0$ on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ with finite H_{h^2}–energy,i.e.,

$$\sum_{e \in E(\mathbf{Z}^d)} \frac{|f(e)|^{d/(d-1)}}{h(|f(e)|)^2} < \infty.$$

In particular, $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ supports a flow of finite H_α–energy for $\alpha > 2$.

We proceed as in the proof of Theorems 1.1 and 2.3, which may be consulted for notation. Let $g(y) = h(y^{1-d})$. Convergence of the sum in (17) implies that $\sum_j (jg(j))^{-1} < \infty$, and hence by Theorem 1.4 in Häggström and Mossel [8], there is an integer–valued process S with $\mathsf{PRE}_S(k) \leq Cg(k)/k$; their constructions also yield the same bound for the time-reversal of S. As in Corollary 3.3, we can define a process Φ supported on $\Upsilon_1(0, \mathbf{Z}^d)$ such that $\mathsf{PRE}_\Phi(k) \leq C(g(k)/k)^{d-1}$ and $\mathsf{PRE}_{\overleftarrow{\Phi}}(k) \leq C(g(k)/k)^{d-1}$. Let μ be the distribution of the edge sequence φ determined by Φ, and write

$$f_N(e) = \int_{\Upsilon_1} p^{-N} I(\varphi_N) J_e(\varphi_N) d\mu(\varphi)$$

for edges directed away from 0. As before it is enough to show that

$$\mathbf{E}_p \sum_{e \in E(\mathbf{Z}^d)} H_{h^2}(|f_N(e)|) = \mathbf{E}_p \sum_{e \in E(\mathbf{Z}^d)} G_h(|f_N(e)|)|f_N(e)| \tag{18}$$

is bounded uniformly in N. By Fubini's Theorem, (18) equals

$$\sum_e \int_{\Upsilon_1} \mathbf{E}_p \left[p^{-N} I(\psi_N) J_e(\psi_N) G_h \left(\int_{\Upsilon_1} p^{-N} I(\varphi_n \setminus \psi_N) J_e(\varphi_N) d\mu(\varphi) \right) \right] d\mu(\psi).$$

By independence of the status of different edges, this can be rewritten as

$$\int_{\Upsilon_1} \sum_e \mathbf{E}_p \left[p^{-N} I(\psi_N) J_e(\psi_N) \right] \mathbf{E}_p \left[G_h \left(\int_{\Upsilon_1} p^{-N} I(\varphi_n \setminus \psi_N) J_e(\varphi_N) d\mu(\varphi) \right) \right] d\mu(\psi).$$

Applying Jensen's inequality to the second expectation bounds the preceding formula by

$$\int_{\Upsilon_1} \sum_e J_e(\psi) G_h \left(\mathbf{E}_p \left[\int_{\Upsilon_1} p^{-N} I(\varphi_N \setminus \psi_N) J_e(\varphi_N) \right] \right) d\mu(\psi). \tag{19}$$

Since ψ contains one edge $e(l)$ in $D(l)$, (19) equals

$$\int_{\Upsilon_1} \sum_l G_h \left(\int_{\Upsilon_1} p^{-|\varphi_N \cap \psi_N|} J_{e(l)}(\varphi_N) d\mu(\varphi) \right) d\mu(\psi). \tag{20}$$

Arguing as in Theorem 1.1, we obtain that

$$\int_{\Upsilon_1} p^{-|\varphi_N \cap \psi_N|} J_{e(l)}(\varphi_N) d\mu(\varphi) \leq \left(C_1 \frac{g(l)}{l} \right)^{d-1}.$$

Thus (20) is bounded by

$$\sum_l G_h \left(C_1 \left\{ \frac{g(l)}{l} \right\}^{d-1} \right) = \sum_l \frac{(C_2 g(l)/l)}{h\left((C_2 g(l)/l)^{d-1} \right)^2}. \tag{21}$$

Since $h((l/C_2 g(l))^{1-d}) = g(C_3 l/g(l))$, (21) is bounded by

$$\sum_l C_2 \frac{g(l)}{l} \frac{1}{g(C_3 l/g(l))^2}. \tag{22}$$

The assumption that $h(x^2) \leq \kappa h(x)$ implies that

$$\forall\, y > 0, \;\; g(y^2) \leq \kappa g(y). \tag{23}$$

Therefore $g(y)^2 \leq C_4 y$ for all y. Consequently,

$$\frac{1}{g(C_3 l/g(l))} \leq \frac{\kappa}{g(C_5 l^2/g(l)^2)} \leq \frac{\kappa}{g(C_6 l)},$$

where the last inequality follows since g is increasing. Thus (22) is bounded by

$$C_7 \sum_l \frac{g(l)}{l g(C_6 l)^2} \asymp \sum_l \frac{1}{l g(l)},$$

because (23) implies that $g(C_6 l) \asymp g(l)$. Hence convergence of (22) follows from convergence of $\sum_l (g(l)l)^{-1}$. □

The preceding proposition has implications for the permissible growth rate of cutsets on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$. Let $\{\Pi_n\}$ be a sequence of disjoint cutsets in the percolation cluster. Assume that h satisfies the hypothesis of that proposition, and also that H_{h^2} is convex. Let f be a unit flow of finite H_{h^2}–energy on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$. Then

$$\infty > \sum_e H_{h^2}(|f(e)|) \geq \sum_n \sum_{e \in \Pi_n} H_{h^2}(|f(e)|) \geq \sum_n \frac{|\Pi_n|^{-1/(d-1)}}{h^2(|\Pi_n|^{-1})}.$$

In particular, taking $h(u) = (\log(1 + u^{-1}))^{2+\epsilon}$ shows that

$$\sum_n |\Pi_n|^{-1/(d-1)} (\log(|\Pi_n|))^{-2-\epsilon} < \infty.$$

While we know that $\mathbf{Z}^d$ itself will support flows of finite H_h energy iff h satisfies the summability condition (17), Proposition 4.1 only gives a *sufficient* condition for finiteness of energy on $\mathcal{C}_\infty(\mathbf{Z}^d, p)$. The proof above used Theorem 1.4 of Häggström and Mossel [8], which states that for any increasing function g satisfying $\sum_j (jg(j))^{-1} < \infty$, there is a $\mathbf{Z}$–valued nearest–neighbor process with predictability profile at k bounded by $Cg(k)/k$. Hoffman [10] proved that if g does not satisfy this summability condition, then such a predictability profile cannot be attained for any nearest–neighbor process on $\mathbf{Z}$. In a previous version of this paper, the following **conjecture** was made:

For $d \geq 3$ and $p > p_c(\mathbf{Z}^d)$, any nonzero flow f on the infinite cluster $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ must satisfy

$$\mathcal{E}_{H_2}(f) = \sum_{e: f(e) \neq 0} \frac{|f(e)|^{d/(d-1)}}{\log^2(1 + |f(e)|^{-1})} = \infty \,. \tag{24}$$

This conjecture motivated E. Mossel and C. Hoffman to find a different construction of low energy flows on percolation clusters. By combining their new ideas with the methods of the present paper, they showed in a recent preprint (entitled "Energy of flows on percolation clusters") that $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ can support flows that do not satisfy (24). Moreover, under a mild regularity hypothesis on h, they proved the remarkable result that for all $d \geq 3$ and $p > p_c(\mathbf{Z}^d)$, the infinite cluster $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ a.s. supports a nonzero flow of finite H_h energy iff $\mathbf{Z}^d$ supports such a flow.

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