Glauber Dynamics for Ising Model I
AMS Short Course

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January 2010

Ising model

Let $G_n = (V_n, E_n)$ be a graph with $N = |V_n| < \infty$ vertices.

The nearest-neighbor *Ising model* on G_n is the probability distribution on $\{-1, 1\}^{V_n}$ given by

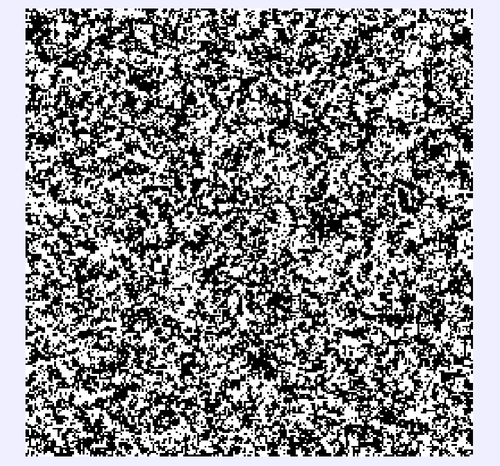
$$\mu(\sigma) = Z(\beta)^{-1} \exp\left(\beta \sum_{(u,v)\in E_n} \sigma(u)\sigma(v)\right),$$

where $\sigma \in \{-1, 1\}^{V_n}$.

The interaction strength β is a parameter which has physical interpretation as $1/$temperature.

Three regimes

High temperature ($\beta < \beta_c$):



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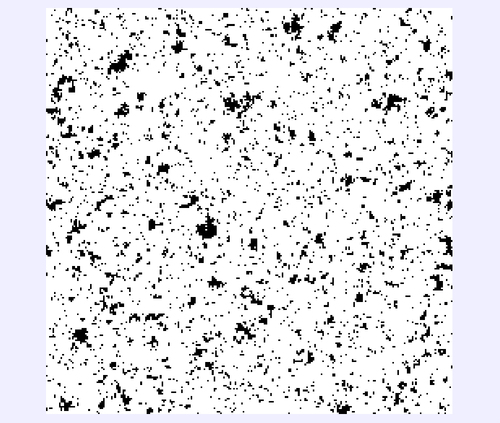
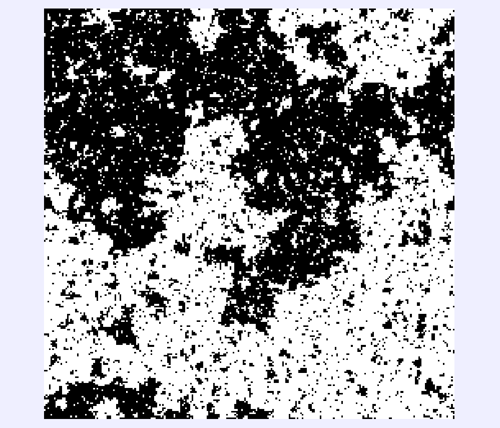


Three regimes

critical temperature ($\beta = \beta_c$),



Glauber dynamics

The (single-site) *Glauber dynamics* for μ is a Markov chain (X_t) having μ as its stationary distribution.

Transitions are made from state σ as follows:

1. a vertex v is chosen uniformly at random from V_n.
2. The new state σ' agrees with σ everywhere except possibly at v, where $\sigma'(v) = 1$ with probability

$$\frac{e^{\beta S(\sigma,v)}}{e^{\beta S(\sigma,v)} + e^{-\beta S(\sigma,v)}}$$

where

$$S(\sigma, v) := \sum_{w : w \sim v} \sigma(w).$$

Note the probability above equals the μ-conditional probability of a positive spin at v, given that that all spins agree with σ at vertices different from v.

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Cut-off

Consider a *sequence* of Markov chains, (X_t^n), and write $d_n(t)$ and $t^n_{\mathrm{mix}}(\epsilon)$ for the distance to stationary and mixing time, respectively, of the nth chain.

A sequence of Markov chains has a *cutoff* if

$$\frac{t^n_{\mathrm{mix}}(\epsilon)}{t^n_{\mathrm{mix}}(1-\epsilon)} \to 1 \quad \text{as} \quad n \to \infty.$$

A cutoff has window size $\{w_n\}$ if $w_n = o(t^n_{\mathrm{mix}})$ and

$$\lim_{\alpha \to -\infty} \liminf_n d_n(t^n_{\mathrm{mix}} + \alpha w_n) = 1$$

$$\lim_{\alpha \to \infty} \limsup_n d_n(t^n_{\mathrm{mix}} + \alpha w_n) = 0\,.$$

A conjecture

For the Glauber dynamics on graph sequences with bounded degree, $t^n_{\text{mix}} = \Omega(n \log n)$.
(T. Hayes and A. Sinclair)

Conjecture (Y.P.): If the Glauber dynamics for a sequence of transitive graphs satisfies $t^n_{\text{mix}} = O(n \log n)$, then there is a cut-off.

Mean field case (Complete Graph)

Take $G_n = K_n$, the complete graph on the n vertices: $V_n = \{1, \ldots, n\}$, and E_n contains all $\binom{n}{2}$ possible edges.

The total interaction strength should be $O(1)$, so replace β by β/n. The probability of updating to a $+1$ is then

$$\frac{e^{\beta(S-\sigma(v))/n}}{e^{\beta(S-\sigma(v))/n} + e^{-\beta(S-\sigma(v))/n}}$$

where S is the *total magnetization*

$$S = \sum_{i=1}^{n} \sigma(i).$$

The statistic S is almost sufficient for determining the updating probability.

The chain (S_t) is the key to analysis of the dynamics.

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Mean field has $t_{\mathrm{mix}} = O(n \log n)$

If $\beta < 1$, a path-coupling argument can be used to show that

$$t_{\mathrm{mix}} = O(n \log n).$$

High temperature mean-field

Theorem (L.-Luczak-Peres)

Let (X_t^n) be the Glauber dynamics for the Ising model on K_n. If $\beta < 1$, then $t_{\text{mix}}(\epsilon) = (1 + o(1))\frac{n \log n}{2(1-\beta)}$ and there is a cut-off.

In fact, we show that there is *window* of size $O(n)$ centered about

$$t_n = \frac{1}{2(1-\beta)} n \log n.$$

That is,

$$\limsup_n d_n(t_n + \gamma n) \to 0 \quad \text{as} \quad \gamma \to \infty.$$

and

$$\liminf_n d_n(t_n + \gamma n) \to 1 \quad \text{as} \quad \gamma \to -\infty.$$

Critical temperature mean-field

Theorem (L.-Luczak-Peres)

Let (X_t^n) be the Glauber dynamics for the Ising model on K_n. If $\beta = 1$, then there are constants c_1 and c_2 so that

$$c_1 n^{3/2} \leq t_{\mathrm{mix}} \leq c_2 n^{3/2}.$$

Low temperature mean-field

If $\beta > 1$, then

$$t_{\text{mix}}^n > c_1 e^{c_2 n}.$$

This can be established using Cheeger constant – there is a bottleneck going between states with positive magnetization and states with negative magnetization.

Arguments for exponentially slow mixing in the low temperature regime go back at least to Griffiths, Weng and Langer (1966)

Our results show that once this barrier to mixing is removed, the mixing time is reduced to $n \log n$.

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Magnetization chain: key equation

If $S_t = \sum_{i=1}^{n} X_t(i)$, then for $S_t \geq 0$,

$$\mathbb{E}[S_{t+1} - S_t \mid \mathcal{F}_t] \approx -\left[\frac{S_t}{n} - \tanh(\beta S_t/n)\right].$$

$\beta < 1$

When $\beta < 1$, using the inequality $\tanh(x) \leq x$ for $x \geq 0$ shows that for $S_t \geq 0$,

$$\mathbb{E}[S_{t+1} \mid \mathcal{F}_t] \leq S_t \left(1 - \frac{1-\beta}{n}\right)$$

Need $[2(1-\beta)]^{-1} n \log n$ steps to drive $\mathbb{E}[S_t]$ to $\sqrt{n}$.

Additional $O(n)$ steps needed for magnetization to hit zero. (Compare with simple random walk.)

Can couple two versions of the chain so that the magnetizations agree by the time the magnetization of the top chain hits zero.

Once magnetizations agree, use a two-dimensional process to bound time until full configurations agree. Takes an additional $O(n)$ steps.

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$\beta = 1$. Why $n^{3/2}$?

Expanding $\tanh(x) = x - x^3/3 + \cdots$ in the key equation yields

$$\mathbb{E}[S_{t+1} - S_t \mid \mathcal{F}_t] \approx -\frac{1}{3}\left(\frac{S_t}{n}\right)^3 .$$

Need $t = \Theta(n^{3-2\alpha})$ steps for $\mathbb{E}[S_t] = n^\alpha$.

By comparison with nearest-neighbor random walk, need additional $n^{2\alpha}$ steps to hit zero.

Total time to hit zero is

$$O(n^{3-2\alpha}) + O(n^{2\alpha})$$

The choice $\alpha = 3/4$ makes the powers equal, and gives hitting time with expectation $n^{3/2}$.

Once the magnetizations agree, need additional $O(n \log n)$ to make the configurations agree.

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