TILTING MODULES FOR LIE SUPERALGEBRAS

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1. Introduction

The notion of a tilting module first emerged in Lie theory in the 1980s, see for instance [CI] where Collingwood and Irving classified the self-dual modules with a Verma flag in category $\mathcal{O}$ for a semisimple Lie algebra, generalizing earlier work of Enright and Shelton [ES]. Similar looking objects were also considered by Donkin [D1] in the representation theory of reductive algebraic groups in positive characteristic. The terminology "tilting module" comes instead from the representation theory of finite dimensional algebras, via an article of Ringel [R] which gives an elegant construction of tilting modules in the setting of quasi-hereditary algebras [CPS, DR]. Ringel's argument was subsequently adapted to algebraic groups by Donkin [D2] and to Lie algebras by Soergel [S].

The goal of the present article is to extend Soergel's framework to Lie superalgebras. Our interest in doing this arose from the papers [B1, B2] in which we conjectured that the coefficients of certain canonical bases should compute multiplicities in Δ-flags of indecomposable tilting modules over the Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{q}(n)$ respectively. Thus the present article should be viewed as a companion to [B1, B2], since we provide the general theory needed to construct the tilting modules in the first place.

We stress that the development here is very similar to Soergel's work: most of the proofs carry over unchanged to the Lie superalgebra setting. Like in [S], we have also included in the first few sections some other well-known generalities, most of which have their origins in the classic work of Bernstein, Gelfand and Gelfand [BGG]. The main result of the article is best understood from Corollary 5.7, which roughly speaking gives a duality between indecomposable projective and indecomposable tilting modules. The proof of this involves the construction of the "semi-regular bimodule", see Lemma 5.3.

At the end of the article, we have given several examples involving the Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{q}(n)$ to illustrate the usefulness of the theory. The results may also prove useful in studying the representation theory of the other classical Lie superalgebras and affine Lie superalgebras.

Notation. Throughout the article, we will work over the ground field $\mathbb{C}$. Suppose $V = \bigoplus_{d\in\mathbb{Z}} V_d = \bigoplus_{d\in\mathbb{Z}} V_{d,\bar{0}} \oplus V_{d,\bar{1}}$ is a *graded vector superspace*, i.e. a $\mathbb{Z}\times\mathbb{Z}_2$-graded vector space. To avoid confusion between the two different gradings, we use the word *degree* to refer to the $\mathbb{Z}$-grading, and *parity* to refer to the $\mathbb{Z}_2$-grading. Write $\deg(v) \in \mathbb{Z}$ (resp. $\bar{v} \in \mathbb{Z}_2$) for the degree (resp. the parity) of a

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homogeneous vector. Given two graded vector superspaces V, W, $\mathbf{Hom}_{\mathbb{C}}(V,W)$ denotes the graded vector superspace with

$$\mathbf{Hom}_{\mathbb{C}}(V,W)_{d,p} = \{f : V \to W \mid f(V_{d',p'}) \subseteq W_{d+d',p+p'} \text{ for all } (d',p') \in \mathbb{Z}\times\mathbb{Z}_2\}$$

for each $(d,p) \in \mathbb{Z}\times\mathbb{Z}_2$.

2. Graded category $\mathcal{O}$

For basic notions regarding Lie superalgebras, see [K1]. Let us recall in particular that for a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ and $\mathfrak{g}$-supermodules M, N, a homomorphism $f : M \to N$ means a (not necessarily even) linear map such that $f(Xm) = (-1)^{\bar f \bar X} X f(m)$ for all $X \in \mathfrak{g}, m \in M$. This formula needs to be interpreted additively in the case that f, X are not homogeneous! We will use the notation $M \simeq N$ as opposed to the usual $M \cong N$ to indicate that there is an *even* isomorphism between M and N.

The category of all $\mathfrak{g}$-supermodules is not an abelian category, but the *underlying even category* consisting of the same objects and only even morphisms is abelian. This, and the existence of the parity change functor Π, allows us to appeal to all the usual notions of homological algebra. Similar remarks apply to the various other categories of $\mathfrak{g}$-supermodules that we shall meet.

We will be concerned here instead with a graded Lie superalgebra, i.e. a Lie superalgebra $\mathfrak{g}$ with an additional $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{d\in\mathbb{Z}} \mathfrak{g}_d = \bigoplus_{d\in\mathbb{Z}} \mathfrak{g}_{d,\bar 0} \oplus \mathfrak{g}_{d,\bar 1}$ such that $[\mathfrak{g}_d, \mathfrak{g}_e] \subseteq \mathfrak{g}_{d+e}$ for all $d, e \in \mathbb{Z}$. A *graded* $\mathfrak{g}$-supermodule means a $\mathfrak{g}$-supermodule M with an additional $\mathbb{Z}$-grading $M = \bigoplus_{d\in\mathbb{Z}} M_d = \bigoplus_{d\in\mathbb{Z}} M_{d,\bar 0} \oplus M_{d,\bar 1}$ such that $\mathfrak{g}_d M_e \subseteq M_{d+e}$ for all $d, e \in \mathbb{Z}$. Homomorphisms $f : M \to N$ between graded $\mathfrak{g}$-supermodules are always assumed to satisfy $f(M_d) \subseteq N_d$ for each $d \in \mathbb{Z}$.

Assume from now on that we are given a graded Lie superalgebra $\mathfrak{g}$. Let $\mathfrak{h} = \mathfrak{g}_0, \mathfrak{b} = \mathfrak{g}_{\geq 0} = \bigoplus_{d\geq 0} \mathfrak{g}_d$, and $\mathfrak{n} = \mathfrak{g}_{<0} = \bigoplus_{d<0} \mathfrak{g}_d$. We write $U(\mathfrak{g}), U(\mathfrak{b})$ and $U(\mathfrak{n})$ for the corresponding universal enveloping superalgebras, all of which inherit a $\mathbb{Z}$-grading from $\mathfrak{g}$. We assume:

(A1) $\dim \mathfrak{g}_d < \infty$ for each $d \in \mathbb{Z}$;

(A2) $\mathfrak{h}_{\bar 0}$ is a reductive Lie algebra.

Fix in addition a maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{h}_{\bar 0}$ and an abelian subgroup X of $\mathfrak{t}^*$. By an *admissible* representation of $\mathfrak{h}_{\bar 0}$, we mean a locally finite dimensional $\mathfrak{h}_{\bar 0}$-supermodule such that $M = \bigoplus_{\lambda\in X} M_\lambda$, where

$$M_\lambda = \{m \in M \mid tm = \lambda(t)m \text{ for all } t \in \mathfrak{t}\}.$$

More generally, for any graded subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ containing $\mathfrak{h}_{\bar 0}$, we will say that an $\mathfrak{m}$-supermodule is admissible if it is admissible on restriction to $\mathfrak{h}_{\bar 0}$. We must also assume:

(A3) the adjoint representation $\mathfrak{g}$ is admissible.

For any graded subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ containing $\mathfrak{h}_{\bar 0}$, let $\mathcal{C}_{\mathfrak{m}}$ denote the category of all admissible graded $\mathfrak{m}$-supermodules. Finally let $\mathcal{O}$ be the category of all admissible graded $\mathfrak{g}$-supermodules that are locally finite dimensional over $\mathfrak{b}$. This is a graded analogue of the category $\mathcal{O}$ of [BGG].

Lemma 2.1. *Category $\mathcal{O}$ and all the categories $\mathcal{C}_{\mathfrak{m}}$ have enough injectives.*

Proof. We explain the argument for $\mathcal{O}$; the same argument works for each $\mathcal{C}_{\mathfrak{m}}$. Let Fin be the functor from the category of all graded $\mathfrak{g}$-supermodules to $\mathcal{O}$ sending an object to its largest graded submodule belonging to $\mathcal{O}$. This is right adjoint to an exact functor, so sends injectives to injectives. Moreover, the category of all graded $\mathfrak{g}$-supermodules has enough injectives since it is isomorphic to the category of graded supermodules over the universal enveloping superalgebra $U(\mathfrak{g})$. Now given any $M \in \mathcal{O}$, we embed M into an injective graded $\mathfrak{g}$-supermodule, then apply the functor Fin. □

In view of Lemma 2.1, we can compute $\mathrm{Ext}^i(M, N)$ in category $\mathcal{O}$ or any of the categories $\mathcal{C}_{\mathfrak{m}}$ using an injective resolution of N. In the sequel, we are often going to make use of the functors $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} ?$ and $\mathbf{Hom}_{\mathfrak{g}_{\leq 0}}(U(\mathfrak{g}), ?)$. In the latter case, for a graded $\mathfrak{g}_{\leq 0}$-supermodule M, $\mathbf{Hom}_{\mathfrak{g}_{\leq 0}}(U(\mathfrak{g}), M)$ is viewed as a graded $\mathfrak{g}$-supermodule with action $(uf)(u') = (-1)^{\bar{u}\bar{f} + \bar{u}\bar{u}'} f(u'u)$, for $u, u' \in U(\mathfrak{g}), f : U(\mathfrak{g}) \to M$. The next lemma is a consequence of the PBW theorem.

Lemma 2.2. *For graded $\mathfrak{b}$-, $\mathfrak{g}_{\leq 0}$- and $\mathfrak{h}$-supermodules L, M and N,*

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} L \simeq U(\mathfrak{g}_{\leq 0}) \otimes_{U(\mathfrak{h})} L,$$
$$U(\mathfrak{g}_{\leq 0}) \otimes_{U(\mathfrak{h})} N \simeq S(\mathfrak{n}) \otimes N,$$

as graded $\mathfrak{g}_{\leq 0}$- resp. $\mathfrak{h}$-supermodules, and

$$\mathbf{Hom}_{\mathfrak{g}_{\leq 0}}(U(\mathfrak{g}), M) \simeq \mathbf{Hom}_{\mathfrak{h}}(U(\mathfrak{b}), M)$$
$$\mathbf{Hom}_{\mathfrak{h}}(U(\mathfrak{b}), N) \simeq \mathbf{Hom}_{\mathbb{C}}(S(\mathfrak{g}_{>0}), N)$$

as graded $\mathfrak{b}$- resp. $\mathfrak{h}$-supermodules. (Here $S(\mathfrak{n}), S(\mathfrak{g}_{>0})$ denote the symmetric superalgebras viewed as modules via ad*).*

Applying the lemma and (A3), $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} ?$ (resp. $U(\mathfrak{g}_{\leq 0}) \otimes_{U(\mathfrak{h})} ?$) is an exact functor from $\mathcal{C}_{\mathfrak{b}}$ to $\mathcal{C}_{\mathfrak{g}}$ (resp. from $\mathcal{C}_{\mathfrak{h}}$ to $\mathcal{C}_{\mathfrak{g}_{\leq 0}}$), which is obviously left adjoint to the natural restriction functor. Similarly, $\mathbf{Hom}_{\mathfrak{g}_{\leq 0}}(U(\mathfrak{g}), ?)$ is an exact functor from $\mathcal{C}_{\mathfrak{g}_{\leq 0}}$ to $\mathcal{C}_{\mathfrak{g}}$ that is right adjoint to restriction.

Lemma 2.3. *For $i \geq 0$, $L \in \mathcal{C}_{\mathfrak{g}}$, $M \in \mathcal{C}_{\mathfrak{h}}$ and $N \in \mathcal{C}_{\mathfrak{g}_{\leq 0}}$, we have that*

$$\mathrm{Ext}^i_{\mathcal{C}_{\mathfrak{g}}}(L, \mathbf{Hom}_{\mathfrak{g}_{\leq 0}}(U(\mathfrak{g}), N)) \simeq \mathrm{Ext}^i_{\mathcal{C}_{\mathfrak{g}_{\leq 0}}}(L, N),$$
$$\mathrm{Ext}^i_{\mathcal{C}_{\mathfrak{g}_{\leq 0}}}(U(\mathfrak{g}_{\leq 0}) \otimes_{U(\mathfrak{h})} M, N) \simeq \mathrm{Ext}^i_{\mathcal{C}_{\mathfrak{h}}}(M, N).$$

Proof. Argue by induction on i using the long exact sequence. □

3. Standard and costandard modules

Let Λ be a complete set of pairwise non-isomorphic irreducible admissible graded $\mathfrak{h}$-supermodules. Each $E \in \Lambda$ is necessarily concentrated in a single degree, denoted $|E| \in \mathbb{Z}$. Moreover, by the superalgebra analogue of Schur's lemma, the number

$$d_E := \dim \mathrm{End}_{\mathcal{C}_{\mathfrak{h}}}(E) \tag{3.1}$$

is either 1 or 2.

Lemma 3.2. *Every $E \in \Lambda$ has a finite dimensional projective cover $\widehat{E}$ in $\mathcal{C}_{\mathfrak{h}}$, with $\mathrm{cosoc}_{\mathfrak{h}} \widehat{E} \simeq E$. Moreover, given objects $M, P \in \mathcal{C}_{\mathfrak{h}}$ with P projective, $M \otimes P$ is also projective.*

Proof. For a graded $\mathfrak{h}_{\bar{0}}$-supermodule M, we observe that

$$U(\mathfrak{h}) \otimes_{U(\mathfrak{h}_{\bar{0}})} M \simeq S(\mathfrak{h}_{\bar{1}}) \otimes M$$

as graded $\mathfrak{h}_{\bar{0}}$-supermodules. Combining this with (A3) shows that the functor $U(\mathfrak{h}) \otimes_{U(\mathfrak{h}_{\bar{0}})} ?$ maps $\mathcal{C}_{\mathfrak{h}_{\bar{0}}}$ to $\mathcal{C}_{\mathfrak{h}}$. Since it is left adjoint to an exact functor, it maps projectives to projectives. By (A2) and Weyl's theorem on complete reducibility, every object in $\mathcal{C}(\mathfrak{h}_{\bar{0}})$ is projective.

Now take $E \in \Lambda$. Let $\widehat{E}$ be any indecomposable summand of $U(\mathfrak{h}) \otimes_{U(\mathfrak{h}_{\bar{0}})} E$ that maps surjectively onto E under the natural multiplication map. By the preceeding paragraph, $\widehat{E}$ is a finite dimensional indecomposable projective object in $\mathcal{C}_{\mathfrak{h}}$ mapping surjectively onto E. Now the usual arguments via Fitting's lemma show that $\widehat{E}$ is actually a projective cover of E in the category $\mathcal{C}_{\mathfrak{h}}$ and that $\mathrm{cosoc}_{\mathfrak{h}} \widehat{E} \simeq E$.

Finally let $P \in \mathcal{C}_{\mathfrak{h}}$ be an arbitrary projective object. Then, we can find $Q \in \mathcal{C}_{\mathfrak{h}}$ and $R \in \mathcal{C}_{\mathfrak{h}_{\bar{0}}}$ such that $P \oplus Q \cong U(\mathfrak{h}) \otimes_{U(\mathfrak{h}_{\bar{0}})} R$. By the tensor identity, $(P \oplus Q) \otimes M \cong U(\mathfrak{h}) \otimes_{U(\mathfrak{h}_{\bar{0}})} (R \otimes M)$. The latter is projective and $P \otimes M$ is isomorphic to a summand of it, so $P \otimes M$ is projective too. □

Define the *standard* and *costandard* $\mathfrak{g}$*-supermodules* corresponding to $E \in \Lambda$:

$$\Delta(E) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \widehat{E}, \qquad \nabla(E) := \mathbf{Hom}_{\mathfrak{g}_{\leq 0}}(U(\mathfrak{g}), E). \tag{3.3}$$

By Lemma 2.2, both $\Delta(E)$ and $\nabla(E)$ are admissible, and clearly they are locally finite dimensional over $\mathfrak{b}$, hence they belong to $\mathcal{O}$. Indeed, letting $\mathcal{O}_{\leq d}$ denote the full subcategory of $\mathcal{O}$ consisting of all objects that are zero in degrees $> d$, both $\Delta(E)$ and $\nabla(E)$ belong to $\mathcal{O}_{\leq |E|}$, with $\Delta(E)_{|E|} \simeq \widehat{E}$, $\nabla(E)_{|E|} \simeq E$. We define

$$L(E) := \mathrm{cosoc}_{\mathfrak{g}} \Delta(E) \tag{3.4}$$

for each $E \in \Lambda$. The following well-known lemma shows in particular that these are irreducible.

Lemma 3.5. *The $\{L(E)\}_{E \in \Lambda}$ form a complete set of pairwise non-isomorphic irreducibles in $\mathcal{O}$. Moreover, $L(E) \simeq \mathrm{soc}_{\mathfrak{g}} \nabla(E)$.*

Proof. Over $\mathfrak{g}_{\leq 0}$, $\Delta(E) \simeq U(\mathfrak{g}_{\leq 0}) \otimes_{U(\mathfrak{h})} \widehat{E}$, hence $\mathrm{cosoc}_{\mathfrak{g}_{\leq 0}} \Delta(E) \simeq \mathrm{cosoc}_{\mathfrak{h}} \widehat{E} \simeq E$. This immediately implies that $L(E)$ is irreducible in $\mathcal{O}$ and $\mathrm{cosoc}_{\mathfrak{g}_{\leq 0}} L(E) \simeq E$. Hence the $\{L(E)\}_{E \in \Lambda}$ are pairwise non-isomorphic irreducibles. Now take any irreducible $M \in \mathcal{O}$. There exists a non-zero $\mathfrak{b}$-homomorphism $E \to M$ for some $E \in \Lambda$. This induces by Frobenius reciprocity a non-zero $\mathfrak{g}$-homomorphism $\Delta(E) \to M$, hence $M \cong L(E)$. The same argument shows that $\mathrm{soc}_{\mathfrak{b}} L(E) \simeq E$. Finally, over $\mathfrak{b}$, $\nabla(E) \simeq \mathbf{Hom}_{\mathfrak{h}}(U(\mathfrak{b}), E)$, so $\mathrm{soc}_{\mathfrak{b}} \nabla(E) \simeq E$. Hence $\mathrm{soc}_{\mathfrak{g}} \nabla(E) \simeq L(E)$ too. □

Lemma 3.6. *Let $E, F \in \Lambda$.*

(i) $\Delta(E)$ *is the projective cover of* $L(E)$ *in* $\mathcal{O}_{\leq|E|}$.
(ii) $\dim \operatorname{Hom}_{\mathcal{O}}(\Delta(E), \nabla(F)) = 0$ *if* $E \neq F$, d_E *if* $E = F$.
(iii) $\operatorname{Ext}^1_{\mathcal{O}}(\Delta(E), \nabla(F)) = 0$.

Proof. For (i), take $M \in \mathcal{O}_{\leq|E|}$. We have the following sequence of isomorphisms natural in M:

$$\operatorname{Hom}_{\mathcal{O}_{\leq|E|}}(\Delta(E), M) \simeq \operatorname{Hom}_{\mathcal{C}_{\mathfrak{g}}}(\Delta(E), M) \simeq \operatorname{Hom}_{\mathcal{C}_{\mathfrak{b}}}(\widehat{E}, M) \simeq \operatorname{Hom}_{\mathcal{C}_{\mathfrak{h}}}(\widehat{E}, M).$$

Since $\widehat{E}$ is projective in $\mathcal{C}_{\mathfrak{h}}$, this shows that $\Delta(E)$ is projective in $\mathcal{O}_{\leq|E|}$. The same argument with $M = \Delta(E)$ shows that $\dim \operatorname{End}_{\mathcal{O}_{\leq|E|}}(\Delta(E))$ is finite dimensional, so we get that $\Delta(E)$ is actually the projective cover of $L(E)$ in $\mathcal{O}_{\leq|E|}$ from Fitting's lemma. For (ii), (iii), Lemma 2.3 implies for every $i \geq 0$ that

$$\operatorname{Ext}^i_{\mathcal{C}_{\mathfrak{g}}}(\Delta(E), \nabla(F)) \simeq \operatorname{Ext}^i_{\mathcal{C}_{\mathfrak{h}}}(\widehat{E}, F).$$

Since $\widehat{E}$ is projective with $\operatorname{cosoc}_{\mathfrak{h}} \widehat{E} \simeq E$, the right hand side is zero if $i > 0$ or if $E \neq F$, and is of dimension d_E otherwise. Now we are done since $\mathcal{O}$ is a full subcategory of $\mathcal{C}_{\mathfrak{g}}$. □

4. Projective modules and blocks

Let $M \in \mathcal{O}$. A Δ-*flag* of M means a filtration

$$0 = M_0 \subseteq M_1 \subseteq M_2 \dots$$

such that $M = \bigcup_{i\geq 0} M_i$ and each factor M_i/M_{i-1} is either zero or $\cong \Delta(E_i)$ for $E_i \in \Lambda$. If the filtration stabilizes after finitely many terms we will call it a *finite* Δ-*flag*. Arguing as in [S, Lemma 5.10], one shows:

Lemma 4.1. *Suppose we have that* $\operatorname{Ext}^1_{\mathcal{O}}(\Delta(F), N) = 0$ *for all* $F \in \Lambda$. *Then,* $\operatorname{Ext}^1_{\mathcal{O}}(M, N) = 0$ *for every* $M \in \mathcal{O}$ *admitting a* Δ*-flag.*

Applying the lemma to $N = \nabla(E)$, one easily deduces that the multiplicity of $\Delta(E)$ as a subquotient of a Δ-flag of M is equal to $\dim \operatorname{Hom}_{\mathcal{O}}(M, \nabla(E))/d_E$, for every $M \in \mathcal{O}$ admitting a Δ-flag. In particular, this multiplicity does not depend on the choice of the Δ-flag. We will denote it by $(M : \Delta(E))$.

Lemma 4.2. *A graded* $\mathfrak{g}$*-supermodule* M *admits a finite* Δ*-flag if and only if* M *is a graded free* $U(\mathfrak{n})$*-supermodule of finite rank and its restriction to* $\mathfrak{h}$ *is a projective object in* $\mathcal{C}_{\mathfrak{h}}$.

Proof. ($\Rightarrow$) It suffices to prove this for $M = \Delta(E)$. Obviously this is a graded free $U(\mathfrak{n})$-supermodule of rank $\dim \widehat{E}$. Moreover, over $\mathfrak{h}$, we have by Lemma 2.2 that $M \simeq S(\mathfrak{n}) \otimes \widehat{E}$. This is projective in $\mathcal{C}_{\mathfrak{h}}$ by Lemma 3.2.

($\Leftarrow$) We may assume that $M = \bigoplus_{i=1}^n U(\mathfrak{n}) \otimes V_i$ is a decomposition of M as a graded free $U(\mathfrak{n})$-supermodule, where V_i is a finite dimensional vector superspace concentrated in degree d_i with trivial action of $\mathfrak{n}$, and $d_1 > \cdots > d_n$. Note then that $1 \otimes V_1$ must be invariant under the action of $\mathfrak{b}$, and $\mathfrak{g}_{>0}$ acts trivially. Hence by the projectivity assumption it decomposes as a direct sum of finitely many $\widehat{E}$'s as a $\mathfrak{b}$-supermodule. Each $U(\mathfrak{n}) \otimes \widehat{E}$ in this decomposition is isomorphic as a graded $\mathfrak{g}$-supermodule to $\Delta(E)$, and the quotient of M by

$U(\mathfrak{n}) \otimes V_1$ is graded free of strictly smaller rank and is still projective over $\mathfrak{h}$, so we are done by induction. $\square$

Corollary 4.3. *If M admits a finite Δ-flag, so does any summand of M.*

Proof. Any summand of a graded free $U(\mathfrak{n})$-supermodule of finite rank is again graded free of finite rank, see [S, Remark 2.4(2)]. $\square$

We now come to the basic result on projective objects in category $\mathcal{O}$.

Theorem 4.4. *Every simple object $L(E) \in \mathcal{O}_{\leq n}$ admits a projective cover $P_{\leq n}(E)$ in $\mathcal{O}_{\leq n}$ with $\mathrm{cosoc}_{\mathfrak{g}} P_{\leq n}(E) \simeq L(E)$. Moreover,*

(i) *$P_{\leq n}(E)$ admits a finite Δ-flag with $\Delta(E)$ at the top;*
(ii) *for $m > n$, the kernel of any surjection $P_{\leq m}(E) \twoheadrightarrow P_{\leq n}(E)$ admits a finite Δ-flag with subquotients of the form $\Delta(F)$ for $m \geq |F| > n$;*
(iii) *$L(E)$ admits a projective cover $P(E)$ in $\mathcal{O}$ if and only if there exists $n \gg 0$ with $P_{\leq n}(E) = P_{\leq n+1}(E) = \dots$, in which case $P(E) = P_{\leq n}(E)$.*

Proof. The proof is essentially the same as [S, Theorem 3.2], so we just sketch the construction of $P_{\leq n}(E)$ and refer the reader to *loc. cit.* for everything else. For a graded $\mathfrak{b}$-supermodule M, let $\tau_{\leq n} M$ denote the quotient of M by the submodule $\bigoplus_{d>n} M_d$ of all homomogeneous parts of degree $> n$. For $E \in \Lambda$,

$$Q := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \tau_{\leq n}(U(\mathfrak{b}) \otimes_{U(\mathfrak{h})} \widehat{E})$$

is projective in $\mathcal{O}_{\leq n}$ as in the proof of [S, Theorem 3.2(1)], it is graded free over $U(\mathfrak{n})$ of finite rank, and it is projective viewed as an object of $\mathcal{C}_{\mathfrak{h}}$ by Lemma 3.2. So Lemma 4.2 shows that Q has a finite Δ-flag. Now Q clearly maps surjectively onto $L(E)$. Let $P_{\leq n}(E)$ be an indecomposable summand of Q that also maps surjectively onto $L(E)$. This has a finite Δ-flag too by Corollary 4.3, and it is a projective cover of $L(E)$ in $\mathcal{O}_{\leq n}$ by a Fitting's lemma argument, see [S, Lemma 3.3]. $\square$

For $M \in \mathcal{O}$, we write $[M : L(E)]$ for the composition multiplicity of $L(E)$ in M, i.e. the supremum of $\#\{i \mid M_i/M_{i-1} \cong L(E)\}$ over all finite filtrations $M = (M_i)_i$ of M. This multiplicity is additive on short exact sequences. Now we get "BGG reciprocity":

Corollary 4.5. $(P_{\leq n}(E) : \Delta(F)) = [\nabla(F) : L(E)]$ *for all $E, F \in \Lambda$ and $n \geq |E|, |F|$.*

Proof. In $\mathcal{O}_{\leq n}$, we have that $[\nabla(F) : L(E)] = \dim \mathrm{Hom}_{\mathcal{O}}(P_{\leq n}(E), \nabla(F))/d_E$. This equals $(P_{\leq n}(E) : \Delta(F))$ by the definition of the latter multiplicity. $\square$

Suppose finally in this section that $\sim$ is an equivalence relation on Λ with the property that

$$[\Delta(F) : L(E)] \neq 0 \text{ or } [\nabla(F) : L(E)] \neq 0 \Rightarrow F \sim E$$

for each $E, F \in \Lambda$. For an equivalence class $\theta \in \Lambda/\sim$, let $\mathcal{O}_\theta$ be the full subcategory of $\mathcal{O}$ consisting of the objects $M \in \mathcal{O}$ all of whose irreducible subquotients are of the form $L(E)$ for $E \in \theta$. We refer to $\mathcal{O}_\theta$ as a *block* of $\mathcal{O}$, in view of the following theorem which is proved exactly as in [S, Theorem 4.2].

Theorem 4.6. *The functor*

$$\prod_{\theta \in \Lambda/\sim} \mathcal{O}_\theta \to \mathcal{O}, \quad (M_\theta)_\theta \mapsto \bigoplus_{\theta \in \Lambda/\sim} M_\theta$$

is an equivalence of categories.

5. Tilting modules and Arkhipov-Soergel duality

Next, we discuss the classification of tilting modules in $\mathcal{O}$. The first main result is the analogue of [S, Theorem 5.2].

Theorem 5.1. *For any $E \in \Lambda$, there exists a unique up to isomorphism indecomposable object $T(E) \in \mathcal{O}$ such that*

(i) $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(F), T(E)) = 0$ *for all $F \in \Lambda$;*

(ii) $T(E)$ *admits a Δ-flag starting with $\Delta(E)$ at the bottom.*

We call $T(E)$ the *indecomposable tilting module* corresponding to $E \in \Lambda$. The proof given by Soergel is a variation on an argument of Ringel [R], and carries over to the present setting virtually unchanged. The main step is to show that for any $E \in \Lambda$ with $|E| \geq n$, there exists a unique up to isomorphism indecomposable object $T_{\geq n}(E)$ in $\mathcal{O}$ such that

(i)$'$ $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(F), T_{\geq n}(E)) = 0$ for all $F \in \Lambda$ with $|F| \geq n$;

(ii)$'$ $T_{\geq n}(E)$ admits a finite Δ-flag starting with $\Delta(E)$ at the bottom and with all other subquotients of the form $\Delta(F)$ for F's with $|E| > |F| \geq n$.

Moreover, given $|E| \geq m \geq n$, there exists an inclusion $T_{\geq m}(E) \hookrightarrow T_{\geq n}(E)$, and the cokernel of any such inclusion admits a finite Δ-flag with subquotients $\Delta(F)$ for $m > |F| \geq n$. Given these results, a candidate for the desired module $T(E)$ can then be constructed as a direct limit of the $T_{\geq n}(E)$'s as $n \to -\infty$. Uniqueness then needs to be established separately.

To proceed, we need to make two additional assumptions (see [S, Remark 1.2] for remarks on the first one):

(A4) $\mathfrak{g}$ is generated as a Lie superalgebra by $\mathfrak{g}_0, \mathfrak{g}_1$ and $\mathfrak{g}_{-1}$;

(A5) for $E \in \Lambda$, $(\widehat{E})^* \cong \widehat{E^\#}$ for some $E^\# \in \Lambda$.

Under the assumption (A4), an *admissible semi-infinite character* γ for $\mathfrak{g}$ is defined to be a Lie superalgebra homomorphism $\gamma : \mathfrak{h} \to \mathbb{C}$ such that $\gamma|_{\mathfrak{t}} \in X$ and

$$\gamma([X, Y]) = \mathrm{str}_{\mathfrak{h}}(\mathrm{ad}X \circ \mathrm{ad}Y) \tag{5.2}$$

for all $X \in \mathfrak{g}_1, Y \in \mathfrak{g}_{-1}$. (We recall the *supertrace* of an endomorphism $f = f_{\bar{0}} + f_{\bar{1}} : V \to V$ of a vector superspace is defined by $\mathrm{str}_V f := \mathrm{tr}_{V_{\bar{0}}} f_{\bar{0}} - \mathrm{tr}_{V_{\bar{1}}} f_{\bar{0}}$.)

In the next lemma, we write $U(\mathfrak{n})^{\circledast}$ for the graded dual $\mathbf{Hom}_{\mathbb{C}}(U(\mathfrak{n}), \mathbb{C})$ (where $\mathbb{C} = \mathbb{C}_{0,\bar{0}}$) viewed as a $U(\mathfrak{n}), U(\mathfrak{n})$-bimodule with left and right actions defined by $(nf)(n') = (-1)^{\bar{n}\bar{f} + \bar{n}\bar{n}'} f(n'n)$ and $(fn)(n') = f(nn')$ respectively, for $n, n' \in U(\mathfrak{n}), f \in U(\mathfrak{n})^{\circledast}$.

Lemma 5.3. *Let $\gamma : \mathfrak{h} \to \mathbb{C}$ be an admissible semi-infinite character for $\mathfrak{g}$. Then there exists a graded $U(\mathfrak{g}), U(\mathfrak{g})$-bimodule S_γ and an even monomorphism $\iota : U(\mathfrak{n})^{\circledast} \hookrightarrow S_\gamma$ of graded $U(\mathfrak{n}), U(\mathfrak{n})$-bimodules such that*

(i) *the map* $U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} U(\mathfrak{n})^{\circledast} \to S_\gamma, u \otimes f \mapsto u\iota(f)$ *is a bijection;*
(ii) *the map* $U(\mathfrak{n})^{\circledast} \otimes_{U(\mathfrak{n})} U(\mathfrak{g}) \to S_\gamma, f \otimes u \mapsto \iota(f)u$ *is a bijection;*
(iii) $[H, \iota(f)] = \iota(f)\gamma(H) - (-1)^{\bar{H}\bar{f}}\iota(f \circ \mathrm{ad}H)$ *for all* $H \in \mathfrak{h}$ *and* $f \in U(\mathfrak{n})^{\circledast}$.

Proof. This is proved in almost exactly the same way as [S, Theorem 1.3]. However, the signs are rather delicate in the super case. So we describe explicitly the construction of S_γ, referring to the proof of [S, Theorem 1.3] for a fuller account of the other steps that need to be made. As a graded vector superspace, we have that

$$S_\gamma = U(\mathfrak{n})^{\circledast} \otimes_{\mathbb{C}} U(\mathfrak{b}),$$

and the map $\iota : U(\mathfrak{n})^{\circledast} \to S_\gamma$ is defined by $\iota(f) = f \otimes 1$. Note S_γ is a $U(\mathfrak{n}), U(\mathfrak{b})$-bimodule in the usual way. We now extend this structure to make S_γ into $U(\mathfrak{g}), U(\mathfrak{g})$-bimodule. First, there is a natural isomorphism of $U(\mathfrak{n}), U(\mathfrak{b})$-bimodules

$$S_\gamma = U(\mathfrak{n})^{\circledast} \otimes_{\mathbb{C}} U(\mathfrak{b}) \xrightarrow{\sim} U(\mathfrak{n})^{\circledast} \otimes_{U(\mathfrak{n})} U(\mathfrak{g})$$

mapping $u \otimes v$ to $u \otimes v$; we get the right action of $U(\mathfrak{g})$ on S_γ via this isomorphism. To obtain the left action, we use the natural isomorphisms

$$S_\gamma = U(\mathfrak{n})^{\circledast} \otimes_{\mathbb{C}} U(\mathfrak{b}) \xrightarrow{\sim} \mathbf{Hom}_{\mathbb{C}}(U(\mathfrak{n}), U(\mathfrak{b})) \xleftarrow{\sim} \mathbf{Hom}_{U(\mathfrak{b})}(U(\mathfrak{g}), \mathbb{C}_\gamma \otimes_{\mathbb{C}} U(\mathfrak{b})).$$

For the right hand space, the action of $U(\mathfrak{b})$ is the natural left action on $U(\mathfrak{g})$, and the tensor product of the action on $\mathbb{C}_\gamma = \mathbb{C}_{0,\bar{0}}$ affording the character γ and the natural left action on $U(\mathfrak{b})$. The first isomorphism maps $f \otimes b$ to the function $\widehat{f \otimes b} : n \mapsto (-1)^{\bar{b}\bar{n}} f(n)b$. The second isomorphism is given by restriction of functions from $U(\mathfrak{g})$ to $U(\mathfrak{n})$, identifying $\mathbb{C}_\gamma \otimes_{\mathbb{C}} U(\mathfrak{b})$ with $U(\mathfrak{b})$ via $1 \otimes u \mapsto u$. Now, $U(\mathfrak{g})$ acts naturally on the left on the right hand space, by $(uf)(u') = (-1)^{\bar{u}\bar{f} + \bar{u}\bar{u}'} f(u'u)$, for $u, u' \in U(\mathfrak{g})$ and $f : U(\mathfrak{g}) \to \mathbb{C}_\gamma \otimes_{\mathbb{C}} U(\mathfrak{b})$. Transferring this to S_γ via the isomorphisms gives the left $U(\mathfrak{g})$-module structure on S_γ. Now we have to check that the left and right actions of $U(\mathfrak{g})$ on S_γ just defined commute with one another, so that S_γ is a $U(\mathfrak{g}), U(\mathfrak{g})$-bimodule. This is done by brutal calculation relying on the assumption that γ is a semi-infinite character, see the proof of [S, Theorem 1.3] for the detailed argument which generalizes routinely to our setting. Once that is done, (i)–(iii) are relatively easy to check to complete the proof. □

For the remainder of the section, we fix an admissible semi-infinite character γ for $\mathfrak{g}$ and let S_γ be the *semi-regular bimodule* constructed in Lemma 5.3. Let $\mathcal{M}$ resp. $\mathcal{K}$ be the category of all admissible graded $\mathfrak{g}$-supermodules that are free resp. cofree of finite rank as graded $U(\mathfrak{n})$-supermodules, i.e. isomorphic to direct sums of maybe graded shifted copies of $U(\mathfrak{n})$ resp. $U(\mathfrak{n})^{\circledast}$. The following theorem is the super analogue of [S, Theorem 2.1], which Soergel attributes originally to Arkhipov [A].

Theorem 5.4. *The functors* $\mathcal{M} \to \mathcal{K}, M \mapsto S_\gamma \otimes_{U(\mathfrak{g})} M$ *and* $\mathcal{K} \to \mathcal{M}, M \mapsto \mathbf{Hom}_{U(\mathfrak{g})}(S_\gamma, M)$ *are mutually inverse equivalences between the categories* $\mathcal{M}$ *and* $\mathcal{K}$*, such that short exact sequences correspond to short exact sequences.*

Proof. Take $M \in \mathcal{M}$. Recalling Lemma 5.3, the map $f \otimes m \mapsto \iota(f) \otimes m$ is a $U(\mathfrak{n})$-isomorphism $U(\mathfrak{n})^{\circledast} \otimes_{U(\mathfrak{n})} M \to S_\gamma \otimes_{U(\mathfrak{g})} M$. Hence $S_\gamma \otimes_{U(\mathfrak{g})} M$ is graded cofree of finite rank, so in particular it is finite dimensional in each degree. Moreover, for $f \in U(\mathfrak{n})^{\circledast}, m \in M$ and $H \in \mathfrak{h}$, we have by Lemma 5.3(iii) that

$$H(\iota(f) \otimes m) = (-1)^{\bar{H}\bar{f}}\iota(f) \otimes (H + \gamma(H))m - (-1)^{\bar{H}\bar{f}}\iota(f \circ \mathrm{ad}H) \otimes m. \quad (5.5)$$

It follows from this and (A3) that $S_\gamma \otimes_{U(\mathfrak{g})} M$ is admissible. Hence $S_\gamma \otimes_{U(\mathfrak{g})} ?$ is a well-defined functor from $\mathcal{M}$ to $\mathcal{K}$. For the other direction, we note that $\mathbf{Hom}_{U(\mathfrak{n})}(U(\mathfrak{n})^{\circledast}, U(\mathfrak{n})^{\circledast}) \simeq U(\mathfrak{n})$ as a $U(\mathfrak{n}), U(\mathfrak{n})$-bimodule; an isomorphism maps $u \in U(\mathfrak{n})$ to $\widehat{u} \in \mathbf{Hom}_{U(\mathfrak{n})}(U(\mathfrak{n})^{\circledast}, U(\mathfrak{n})^{\circledast})$ where $(\widehat{u}f)(n) = (-1)^{\bar{u}\bar{f}} f(un)$ for each $f \in U(\mathfrak{n})^{\circledast}, n \in U(\mathfrak{n})$. So for $N \in \mathcal{K}$, we deduce that $\mathbf{Hom}_{U(\mathfrak{g})}(S_\gamma, N) \simeq \mathbf{Hom}_{U(\mathfrak{n})}(U(\mathfrak{n})^{\circledast}, N)$ is graded free of finite rank over $U(\mathfrak{n})$. Moreover, given $\theta \in \mathbf{Hom}_{U(\mathfrak{g})}(S_\gamma, N)$,

$$(H\theta)(\iota(f)) = (H - \gamma(H))\theta(\iota(f)) + (-1)^{\bar{H}\bar{\theta}+\bar{H}\bar{f}}\theta(\iota(f \circ \mathrm{ad}H)) \quad (5.6)$$

for each $H \in \mathfrak{h}$ and $f \in U(\mathfrak{n})^{\circledast}$. Using this and (A3) one can check that $\mathbf{Hom}_{U(\mathfrak{g})}(S_\gamma, N)$ is admissible. Hence, $\mathbf{Hom}_{U(\mathfrak{g})}(S_\gamma, ?)$ is a well-defined functor from $\mathcal{K}$ to $\mathcal{M}$. The remainder of the proof is exactly as in the proof of [S, Theorem 2.1]. $\square$

Finally, let $\mathcal{O}^\Delta$ be the full subcategory of $\mathcal{O}$ consisting of all objects admitting a finite Δ-flag. We recall from Corollary 4.3 that $\mathcal{O}^\Delta$ is closed under taking direct summands. For a graded $\mathfrak{g}$-supermodule M, we let $M^\star$ denote its graded dual, namely, the space $\mathbf{Hom}_{\mathbb{C}}(M, \mathbb{C})$, where $\mathbb{C} = \mathbb{C}_{0,\bar{0}}$, with action defined by $(Xf)(m) = -(-1)^{\bar{X}\bar{f}} f(Xm)$ for each $X \in \mathfrak{g}, m \in M$ and $f : M \to \mathbb{C}$. Recalling the assumption (A5), the theorem has the following corollary:

Corollary 5.7. *The functor $M \mapsto (S_\gamma \otimes_{U(\mathfrak{g})} M)^\star$ defines a contravariant equivalence of categories $\mathcal{O}^\Delta \to \mathcal{O}^\Delta$ under which short exact sequences correspond to short exact sequences, $\Delta(\mathbb{C}_{-\gamma} \otimes E^\#)$ maps to $\Delta(E)$ and $P_{\leq -n}(\mathbb{C}_{-\gamma} \otimes E^\#)$ maps to $T_{\geq n}(E)$, for every $E \in \Lambda$ and $n \leq |E|$.*

Proof. It is easy to see using (5.5) and (A5) that the degree $-|E|$ piece of $(S_\gamma \otimes_{U(\mathfrak{g})} \Delta(E))^\star \simeq (U(\mathfrak{n})^{\circledast} \otimes \widehat{E})^\star$ is isomorphic to $\mathbb{C}_{-\gamma} \otimes \widehat{E^\#}$ as an $\mathfrak{h}$-supermodule. Moreover, this generates $(S_\gamma \otimes_{U(\mathfrak{g})} \Delta(E))^\star$ freely as a $U(\mathfrak{n})$-supermodule, hence $(S_\gamma \otimes_{U(\mathfrak{g})} \Delta(E))^\star \cong \Delta(\mathbb{C}_{-\gamma} \otimes E^\#)$. It follows from this and Theorem 5.4 that the functor $(S_\gamma \otimes_{U(\mathfrak{g})} ?)^\star$ maps $\mathcal{O}^\Delta$ to $\mathcal{O}^\Delta$ and sends short exact sequences to short exact sequences. Similarly, one shows using (5.6) that $\mathbf{Hom}_{U(\mathfrak{g})}(S_\gamma, \Delta(E)^\star) \cong \Delta(\mathbb{C}_{-\gamma} \otimes E^\#)$. Hence the functor $\mathbf{Hom}_{U(\mathfrak{g})}(S_\gamma, ?^\star)$ maps $\mathcal{O}^\Delta$ to $\mathcal{O}^\Delta$. Now it is immediate from Theorem 5.4 that our two functors are mutually inverse equivalences. It just remains to show that $(S_\gamma \otimes_{U(\mathfrak{g})} P_{\leq -n}(\mathbb{C}_{-\gamma} \otimes E^\#))^\star \cong T_{\geq n}(E)$, for $n \leq |E|$, for which one uses the characterization of $T_{\geq n}(E)$ given in (i)′, (ii)′ above. $\square$

Corollary 5.8. *For $E, F \in \Lambda$, we have that*

$$(T(E) : \Delta(F)) = [\nabla(\mathbb{C}_{-\gamma} \otimes F^\#) : L(\mathbb{C}_{-\gamma} \otimes E^\#)].$$

Proof. We have for $n \leq |E|, |F|$ that

$$\begin{aligned}(T(E):\Delta(F)) &= (T_{\geq n}(E):\Delta(F)) = (P_{\leq -n}(\mathbb{C}_{-\gamma}\otimes E^{\#}):\Delta(\mathbb{C}_{-\gamma}\otimes F^{\#}))\\ &= [\nabla(\mathbb{C}_{-\gamma}\otimes F^{\#}):L(\mathbb{C}_{-\gamma}\otimes E^{\#})],\end{aligned}$$

using Corollary 5.7 and Lemma 4.5. □

6. Some variations

We now mention some variations to the general framework considered so far. First of all, we recall from [S, §6] how to deduce results about ungraded $\mathfrak{g}$-supermodules from the graded theory above. To do this, one needs to require in addition that

(A6) there is an element $D \in \mathfrak{h}_{\bar{0}}$ such that $[D, X] = \deg(X)X$ for all homogeneous $X \in \mathfrak{g}$.

Let $\overline{\mathcal{O}}$ be the category of all admissible (but no longer graded!) $\mathfrak{g}$-supermodules that are locally finite dimensional over $\mathfrak{b}$. Since D necessarily belongs to $\mathfrak{t}$, every $M \in \mathcal{O}$ resp. $M \in \overline{\mathcal{O}}$ decomposes into eigenspaces $M = \bigoplus_{a\in\mathbb{C}} M^{(a)}$ with respect to the action of D. For $a \in \mathbb{C}$, let $\mathcal{O}_a$ denote the full subcategory of $\mathcal{O}$ consisting of all $M \in \mathcal{O}$ such that $M^{(a+i)} = M_i$ for all $i \in \mathbb{Z}$. For $\bar{a} \in \mathbb{C}/\mathbb{Z}$, let $\overline{\mathcal{O}}_{\bar{a}}$ denote the full subcategory of $\overline{\mathcal{O}}$ consisting of all $M \in \overline{\mathcal{O}}$ such that $M^{(b)} = 0$ for all $b \notin \bar{a}$. Then,

$$\mathcal{O} = \prod_{a\in\mathbb{C}} \mathcal{O}_a, \qquad \overline{\mathcal{O}} = \prod_{\bar{a}\in\mathbb{C}/\mathbb{Z}} \overline{\mathcal{O}}_{\bar{a}}.$$

Forgetting the grading gives an isomorphism of categories $\mathcal{O}_a \to \overline{\mathcal{O}}_{\bar{a}}$, the inverse functor being defined on $M \in \overline{\mathcal{O}}_{\bar{a}}$ by introducing a $\mathbb{Z}$-grading according to the rule $M_i = M^{(a+i)}$. In this way, we can transfer results from $\mathcal{O}$ to $\overline{\mathcal{O}}$.

To describe some of the things that can be obtained in this way, let $\overline{\Lambda}$ denote a set of representatives for the equivalence classes of $E \in \Lambda$ viewed up to degree shifts, so that $\overline{\Lambda}$ is a complete set of pairwise non-isomorphic irreducible admissible $\mathfrak{h}$-supermodules. Also let $\widehat{E}$ denote the projective cover of $E \in \overline{\Lambda}$ in the category of admissible $\mathfrak{h}$-supermodules. We have the objects $L(E), \Delta(E)$ and $\nabla(E) \in \overline{\mathcal{O}}$ obtained from the ones defined before by forgetting the grading. Intrinsically, $\Delta(E) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \widehat{E}$, $\nabla(E)$ is the largest submodule of $\mathrm{Hom}_{U(\mathfrak{g}_{\leq 0})}(U(\mathfrak{g}), E)$ that belongs to $\overline{\mathcal{O}}$, and $L(E) = \mathrm{cosoc}_{\mathfrak{g}}\Delta(E) \simeq \mathrm{soc}_{\mathfrak{g}}\nabla(E)$. In particular, $\{L(E)\}_{E\in\overline{\Lambda}}$ is a complete set of pairwise non-isomorphic irreducible objects in $\overline{\mathcal{O}}$.

The notion of a Δ-flag of an object of $\overline{\mathcal{O}}$ is defined as before. The multiplicity $(M : \Delta(E))$ of $\Delta(E)$ as a subquotient of a Δ-flag of an object $M \in \overline{\mathcal{O}}$ is independent of the choice of flag. We have that

$$(M : \Delta(E)) = \dim \mathrm{Hom}_{\overline{\mathcal{O}}}(M, \nabla(E))/d_E. \tag{6.1}$$

We also note from Corollary 4.3 that summands of objects with finite Δ-flags have finite Δ-flags. We can always choose a partial ordering $\preceq$ on $\overline{\Lambda}$ such that

$$[\Delta(F) : L(E)] \neq 0 \text{ or } [\nabla(F) : L(E)] \neq 0 \Rightarrow E \preceq F.$$

Let $\sim$ be the equivalence relation on $\overline{\Lambda}$ generated by the partial order $\preceq$. For $\overline{\theta} \in \overline{\mathcal{O}}/\sim$, let $\overline{\mathcal{O}}_{\overline{\theta}}$ be the full subcategory of $\overline{\mathcal{O}}$ consisting of the objects $M \in \overline{\mathcal{O}}$ all of whose irreducible subquotients are of the form $L(E)$ for $E \in \overline{\theta}$. Then Theorem 4.6 gives us the block decomposition of $\overline{\mathcal{O}}$:

Theorem 6.2. *The functor*

$$\prod_{\overline{\theta} \in \overline{\Lambda}/\sim} \overline{\mathcal{O}}_{\overline{\theta}} \to \overline{\mathcal{O}}, \quad (M_{\overline{\theta}})_{\overline{\theta}} \mapsto \bigoplus_{\overline{\theta} \in \overline{\Lambda}/\sim} M_{\overline{\theta}}$$

is an equivalence of categories.

Next, we use Theorem 5.1 to define the *indecomposable tilting module* $T(E) \in \overline{\mathcal{O}}$ for each $E \in \overline{\Lambda}$:

Theorem 6.3. *For each $E \in \overline{\Lambda}$ there exists a unique up to isomorphism indecomposable object $T(E) \in \overline{\mathcal{O}}$ such that*

(i) $\mathrm{Ext}^1_{\overline{\mathcal{O}}}(\Delta(F), T(E)) = 0$ *for all $F \in \overline{\Lambda}$;*
(ii) $T(E)$ *admits a Δ-flag starting with $\Delta(E)$ at the bottom.*

Let γ be an admissible semi-infinite character for $\mathfrak{g}$ and construct the semi-regular bimodule S_γ as in Lemma 5.3. Let $\overline{\mathcal{O}}^\Delta$ be the full subcategory of $\overline{\mathcal{O}}$ consisting of the objects that admit a finite Δ-flag. Then Corollaries 5.7 and 5.8 give us:

Theorem 6.4. *The functor $M \mapsto (S_\gamma \otimes_{U(\mathfrak{g})} M)^\star$ defines a contravariant equivalence of categories $\overline{\mathcal{O}}^\Delta \to \overline{\mathcal{O}}^\Delta$ under which short exact sequences correspond to short exact sequences and $\Delta(E)$ maps to $\Delta(\mathbb{C}_{-\gamma} \otimes E^\#)$ for every $E \in \overline{\Lambda}$. Moreover,*

$$(T(E) : \Delta(F)) = [\nabla(\mathbb{C}_{-\gamma} \otimes F^\#) : L(\mathbb{C}_{-\gamma} \otimes E^\#)] \tag{6.5}$$

for all $E, F \in \overline{\Lambda}$.

Still assuming that (A6) holds, we now impose some finiteness conditions. First, assume

(A7) for each $E \in \overline{\Lambda}$, $\nabla(E)$ has a composition series.

Given (A7), it is not hard to show that every object M in the category $\overline{\mathcal{O}}^{\mathrm{fin}}$ of all *finitely generated* admissible $\mathfrak{g}$-supermodules that are locally finite dimensional over $\mathfrak{b}$ has a composition series. We remark that (A7) holds automatically if the partial ordering $\preceq$ chosen above has the property that for each $E \in \overline{\Lambda}$, there are only finitely many $F \in \overline{\Lambda}$ with $F \preceq E$. Next assume

(A8) the category $\overline{\mathcal{O}}^{\mathrm{fin}}$ has enough projectives.

By Theorem 4.4(iii), (A8) holds automatically if the partial ordering $\preceq$ has the property that for each $E \in \overline{\Lambda}$, there are only finitely many $F \in \overline{\Lambda}$ with $E \preceq F$.

Using (A7), (A8) and Fitting's lemma, one deduces that each $L(E)$ has a projective cover denoted $P(E)$ in the category $\overline{\mathcal{O}}^{\mathrm{fin}}$. Moreover, Theorem 4.4

and Corollary 4.5 imply in the present setting that $P(E)$ has a finite Δ-flag satisfying BGG reciprocity

$$(P(E) : \Delta(F)) = [\nabla(F) : L(E)] \tag{6.6}$$

for all $E, F \in \overline{\Lambda}$. Under the equivalence of categories from Theorem 6.4, $P(E)$ gets mapped to $T(\mathbb{C}_{-\gamma} \otimes E^{\#})$, so the tilting modules $T(E)$ also all have *finite* Δ-flags, i.e. they belong to the category $\overline{\mathcal{O}}^{\text{fin}}$ too.

7. Examples

We now give some examples, beginning with the classical ones to set the scene.

Example 7.1. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal toral subalgebra, and $\Delta \subset \mathfrak{t}^*$ be a choice of simple roots. Let $\rho \in \mathfrak{t}^*$ be half the sum of the corresponding positive roots. We take the $\mathbb{Z}$-grading on $\mathfrak{g}$ defined so that $\mathfrak{g}_\alpha$ is in degree 1 and $\mathfrak{g}_{-\alpha}$ is in degree -1 for each $\alpha \in \Delta$. Clearly this grading is induced by the adjoint action of some $D \in \mathfrak{t}$, and $\mathfrak{h} := \mathfrak{g}_0 = \mathfrak{t}$. Taking the group X of admissible weights to be all of $\mathfrak{t}^*$, the category $\overline{\mathcal{O}}^{\text{fin}}$ is exactly the category introduced in [BGG].

It is easy to see that our assumptions (A1)–(A6) are all satisfied. Moreover, by Harish-Chandra's theorem on central characters, we can choose the equivalence relation $\sim$ so that the equivalence classes are the orbits of the finite Weyl group W under the dot action. Hence the equivalence classes are finite, so (A7) and (A8) automatically hold too. We also note that the usual Verma modules $M(\lambda)$ for $\lambda \in \mathfrak{h}^*$ are the standard modules here, and their duals under the duality of [BGG, §4, Remark] are the costandard modules. The indecomposable tilting modules $T(\lambda)$ are the modules defined originally by Collingwood and Irving in [CI].

This setup is generalized to an arbitrary symmetrizable Kac-Moody algebra in [S, §7], see also [DGK, RCW]. In general, (A7) and (A8) do not hold, so it becomes important to work in category $\overline{\mathcal{O}}$ rather than $\overline{\mathcal{O}}^{\text{fin}}$. Soergel also discusses certain parabolic analogues.

Example 7.2. In the next two examples, we take $\mathfrak{g}$ to be the Lie superalgebra $\mathfrak{gl}(m|n)$. We recall that $\mathfrak{g}$ consists of $(m+n) \times (m+n)$ matrices over $\mathbb{C}$, where we label rows and columns of such matrices by the ordered index set $\{-m, \dots, -1, 1, \dots, n\}$. Writing $\bar{i} = \bar{0}$ if $i > 0$ and $\bar{1}$ if $i < 0$, the parity of the ij-matrix unit $e_{i,j} \in \mathfrak{g}$ is $\bar{i} + \bar{j}$, and the superbracket satisfies $[e_{i,j}, e_{k,l}] = \delta_{j,k} e_{i,l} - (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})} \delta_{i,l} e_{k,j}$. The subalgebra $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ is isomorphic to $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$. We will always take the maximal toral subalgebra $\mathfrak{t}$ to be the subalgebra consisting of all diagonal matrices, and the group X of admissible weights to be all of $\mathfrak{t}^*$. Let $\delta_{-m}, \dots, \delta_{-1}, \delta_1, \dots, \delta_n$ be the basis for $\mathfrak{t}^*$ dual to the basis $e_{-m,-m}, \dots, e_{-1,-1}, e_{1,1}, \dots, e_{n,n}$ of $\mathfrak{t}$.

Now there are two natural $\mathbb{Z}$-gradings to consider. First, we discuss the *principal grading* induced by the adjoint action of the matrix $D = \text{diag}(m+n, m+n-1, \dots, 2, 1) \in \mathfrak{h}$, so the degree of $e_{i,j}$ is defined by the equation

$[D, e_{i,j}] = \deg(e_{i,j})e_{i,j}$. For this grading, $\mathfrak{h} := \mathfrak{g}_0$ coincides with the subalgebra $\mathfrak{t}$ of diagonal matrices and $\mathfrak{b} := \mathfrak{g}_{\geq 0}$ is the subalgebra of all upper triangular matrices. Let $\overline{\mathcal{O}}^{\text{fin}}$ be the resulting category as in section 6. We should check that the assumptions (A1)–(A8) all hold, the only difficult ones being (A7) and (A8):

Lemma 7.3. *Every object $M \in \overline{\mathcal{O}}^{\text{fin}}$ has a composition series, and $\overline{\mathcal{O}}^{\text{fin}}$ has enough projectives.*

Proof. Let $\mathcal{E}$ be the category of all finitely generated $\mathfrak{g}_{\bar{0}}$-supermodules that are locally finite dimensional over $\mathfrak{b}_{\bar{0}}$ and semisimple over $\mathfrak{h}$. By the PBW theorem, $U(\mathfrak{g})$ is free of finite rank as a (left or right) $U(\mathfrak{g}_{\bar{0}})$-module. Hence, the functor $U(\mathfrak{g})\otimes_{U(\mathfrak{g}_{\bar{0}})}?$ maps objects in $\mathcal{E}$ to objects in $\overline{\mathcal{O}}^{\text{fin}}$, and it is left adjoint to the natural restriction functor from $\overline{\mathcal{O}}^{\text{fin}}$ to $\mathcal{E}$. So it sends projectives to projectives. By Example 7.1, $\mathcal{E}$ has enough projectives, so we deduce that $\overline{\mathcal{O}}^{\text{fin}}$ does too. Finally, to see that every object $M \in \overline{\mathcal{O}}^{\text{fin}}$ has a composition series, note that $U(\mathfrak{g})$ is Noetherian, so M has a descending filtration $M = M_0 \geq M_1 \geq \dots$ such that each M_i/M_{i+1} is irreducible. We just need to show that this filtration stabilizes after finitely many terms. But every object in $\mathcal{E}$ has a composition series by Example 7.1 so this follows immediately on restricting M to $\mathfrak{g}_{\bar{0}}$. $\square$

The standard modules $\Delta(\lambda)$ in this case are the *Verma modules* $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one dimensional $\mathfrak{b}$-module with character $\lambda \in \mathfrak{h}^*$. The costandard modules $\nabla(\lambda)$ are the *dual Verma modules* $M(\lambda)^\tau$, where τ is the duality defined using the "supertranspose" antiautomorphism $e_{i,j} \mapsto (-1)^{\bar{i}(\bar{i}+\bar{j})}e_{j,i}$ of $\mathfrak{g}$. Finally the indecomposable tilting modules are denoted $T(\lambda)$ and the irreducible modules are denoted $L(\lambda)$, for $\lambda \in \mathfrak{h}^*$. Like in Example 7.1, an admissible semi-infinite character for $\mathfrak{g}$ with respect to the principal grading is given by the character 2ρ, where $\rho = m\delta_{-m} + \dots + 2\delta_{-2} + \delta_{-1} - \delta_1 - 2\delta_2 - \dots - n\delta_n$. Now we get from (6.5) that

$$(T(\lambda) : M(\mu)) = [M(-\mu - 2\rho) : L(-\lambda - 2\rho)], \tag{7.4}$$

for $\lambda, \mu \in \mathfrak{h}^*$. A precise conjecture for these multiplicities in the case that λ, μ are integral linear combinations of the δ_i can be found in [B1].

It is interesting to note in this example that both (A7) and (A8) hold, despite the fact (as seen in [B1]) that the partial ordering $\preceq$ of section 6 always has infinite chains.

Example 7.5. Continuing with $\mathfrak{g} = \mathfrak{gl}(m|n)$, we now discuss the second natural $\mathbb{Z}$-grading, namely, the *compatible grading*. This is induced by the adjoint action of the matrix $D = \mathrm{diag}(1/2, 1/2, \dots, 1/2; -1/2, -1/2, \dots, -1/2)$. Note this time that $\mathfrak{h} := \mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$, and $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1 = \mathfrak{g}_{\bar{1}}$. This time, as is easy to show, the category $\overline{\mathcal{O}}^{\text{fin}}$ is precisely the category of all finite dimensional $\mathfrak{g}$-supermodules that are semisimple over $\mathfrak{t}$. The hypothesis (A1)–(A8) are all satisfied, arguing as in Lemma 7.3 for (A7) and (A8).

Recalling $\mathfrak{h} \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, the irreducible finite dimensional $\mathfrak{h}$-supermodules are parametrized by the set X^+ of dominant weights, namely, the $\lambda = \lambda_{-m}\delta_{-m}+$

$\cdots + \lambda_{-1}\delta_{-1} + \lambda_1\delta_1 + \cdots + \lambda_n\delta_n \in \mathfrak{h}^*$ with each $\lambda_{-m} - \lambda_{1-m}, \dots, \lambda_{-2} - \lambda_{-1}, \lambda_1 - \lambda_2, \dots, \lambda_{n-1} - \lambda_n$ being non-negative integers. Given $\lambda \in X^+$, we denote the corresponding standard module $\Delta(\lambda)$ instead by $K(\lambda)$ and call it the *Kac module* of highest weight λ, since it was first defined by Kac in [K2]. The costandard modules are the *dual Kac modules* $K(\lambda)^\tau$. We also write $L(\lambda)$ for the unique irreducible quotient of $K(\lambda)$, $P(\lambda)$ for its projective cover, and $U(\lambda)$ for the indecomposable tilting module of highest weight λ in this finite dimensional setting. By (6.6), $P(\lambda)$ has a finite Kac flag with $K(\lambda)$ at the top, satisfying the BGG reciprocity

$$(P(\lambda) : K(\mu)) = [K(\mu) : L(\lambda)], \tag{7.6}$$

as was also proved in [Z, Proposition 2.5].

Now let $\beta = n(\delta_{-m} + \cdots + \delta_{-1}) - m(\delta_1 + \cdots + \delta_n)$ be the sum of the positive odd roots. It is easy to check that the unique 1-dimensional representation $\gamma : \mathfrak{h} \to \mathbb{C}$ of weight $-\beta$ is an admissible semi-infinite character for $\mathfrak{g}$ with respect to the compatible grading. In fact in this case, there is an even isomorphism of $U(\mathfrak{g}), U(\mathfrak{g})$-bimodules between the semi-regular bimodule S_γ from Lemma 5.3 and the regular bimodule $\Pi^{mn}U(\mathfrak{g})$. In the notation of Lemma 5.3, an isomorphism maps $1 \in U(\mathfrak{g})$ to the element $\iota(\delta) \in S_\gamma$, where $\delta \in U(\mathfrak{n})^{\circledast}$ is the function mapping $\prod_{-m \le i < 0 < j \le n} e_{j,i}$ to 1 (product taken in some fixed order) and all other monomials in the $e_{j,i}$ of strictly smaller length to 0. So in this case the duality in Theorem 6.4 is (up to parity change and degree shift) just the usual duality $*$ on finite dimensional $\mathfrak{g}$-supermodules. In particular,

$$K(\beta - w_0\lambda)^* \cong K(\lambda), \tag{7.7}$$

$$P(\beta - w_0\lambda)^* \cong U(\lambda), \tag{7.8}$$

where w_0 denotes the longest element of the Weyl group $W \cong S_m \times S_n$ of $\mathfrak{h}$ acting on $\mathfrak{t}^*$ in the obvious way. The statement (6.5) says

$$(U(\lambda) : K(\mu)) = [K(\beta - w_0\mu) : L(\beta - w_0\lambda)], \tag{7.9}$$

for $\lambda, \mu \in X^+$. The numbers on the left hand side of this equation are computed in [B1].

Example 7.10. In the final example, we take $\mathfrak{g} = \mathfrak{q}(n)$. Thus, $\mathfrak{g}$ is the subalgebra of $\mathfrak{gl}(n|n)$ consisting of all matrices of the form $\left(\begin{array}{c|c} X & Y \\ \hline Y & X \end{array}\right)$. For $1 \le i, j \le n$, we will let $e_{i,j}$ resp. $e'_{i,j}$ denote the even resp. odd matrix unit, i.e. the matrix of the above form with the ij-entry of X resp. Y equal to 1 and all other entries equal to zero. The $\mathbb{Z}$-grading on $\mathfrak{g}$ is defined by $\deg(e_{i,j}) = \deg(e'_{i,j}) = (j - i)$. For this grading, $\mathfrak{h} := \mathfrak{g}_0$ is spanned by $\{e_{i,i}, e'_{i,i} \mid 1 \le i \le n\}$, and $\mathfrak{b} := \mathfrak{g}_{\ge 0}$ is spanned by $\{e_{i,j}, e'_{i,j} \mid 1 \le i \le j \le n\}$. We also let $\mathfrak{t} = \mathfrak{h}_{\bar{0}}$ and take the group X of admissible weights to be all of $\mathfrak{t}^*$.

As explained in [P, §3], the finite dimensional irreducible $\mathfrak{h}$-supermodules are parametrized by the set $\mathfrak{t}^*$. For $\lambda \in \mathfrak{t}^*$, we write $\mathfrak{u}(\lambda)$ for the corresponding irreducible $\mathfrak{h}$-supermodule. It is constructed in [P] as a certain Clifford module, of dimension a power of 2. The assumption (A5) can be checked from this construction and the fact that Clifford algebras are symmetric: one gets that

$$\widehat{\mathfrak{u}(\lambda)}^{*} \cong \widehat{\mathfrak{u}(-\lambda)}. \tag{7.11}$$

The remaining assumptions (A1)–(A4) and (A6) are easy, and one argues like in Lemma 7.3 to verify (A7) and (A8).

So now we can consider the category $\overline{\mathcal{O}}^{\text{fin}}$ as in section 6. Let

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathfrak{u}(\lambda), \qquad N(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \widehat{\mathfrak{u}(\lambda)},$$

for each $\lambda \in \mathfrak{t}^*$. Then $N(\lambda)$ is the standard module $\Delta(\lambda)$ in $\overline{\mathcal{O}}^{\text{fin}}$, while $M(\lambda)$ is dual to the costandard module $\nabla(\lambda)$ under the duality τ induced by the (unsigned) antiautomorphism

$$\left(\begin{array}{c|c} X & Y \\ \hline Y & X \end{array}\right) \mapsto \left(\begin{array}{c|c} X^T & Y^T \\ \hline Y^T & X^T \end{array}\right).$$

One checks that the trivial character $0 : \mathfrak{h} \to \mathbb{C}$ is an admissible semi-infinite character for $\mathfrak{g}$. So, writing $T(\lambda)$ resp. $L(\lambda)$ for the indecomposable tilting module resp. the irreducible module corresponding to $\lambda \in \mathfrak{t}^*$, (6.5) shows that

$$(T(\lambda) : N(\mu)) = [M(-\mu) : L(-\lambda)]. \tag{7.12}$$

A precise conjecture for these decomposition numbers in case λ, μ are integral weights is formulated in [B2].

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