Schur-Weyl duality and categorification

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Abstract. In some joint work with Kleshchev in 2008, we discovered a *higher level analog of Schur-Weyl duality*, relating parabolic category $\mathcal{O}$ for the general linear Lie algebra to certain cyclotomic Hecke algebras. Meanwhile Rouquier and others were developing a general axiomatic approach to the study of *categorical actions of Lie algebras*. In this survey, we recall aspects of these two theories, then explain some related recent developments due to Losev and Webster involving *tensor product categorifications*.

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1. Introduction

The double centralizer property. To set the scene in this introduction, we are going to briefly recall two classic topics in representation theory, namely:

- *Schur-Weyl duality* relating the representation theory of the general linear and symmetric groups;
- *Soergel's functor* relating the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for a semisimple Lie algebra to modules over coinvariant algebras.

First though we formulate some abstract *double centralizer property*. This applies to both of the above situations, as well as to the generalizations to be discussed in subsequent sections. (For the reader not familiar with the notion of a highest weight category, we will recall its meaning shortly.)

Theorem 1.1. *Let $\mathcal{M}$ be a highest weight category with a finite weight poset. Assume that the injective hulls of all its standard objects are projective. Let T be a* prinjective generator *for $\mathcal{M}$, that is, a prinjective (= both projective and injective) object such that every indecomposable prinjective object is isomorphic to a summand of T. Let $C := \mathrm{End}_{\mathcal{M}}(T)^{\mathrm{op}}$ and C-mod denote the category of finite dimensional left C-modules. Then the quotient functor*

$$\mathbb{V} := \mathrm{Hom}_{\mathcal{M}}(T, -) : \mathcal{M} \to C\text{-mod}$$

is fully faithful on projectives.

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Note to start with that if $0 \to P \to Q \to R \to 0$ is a short exact sequence in $\mathcal{M}$ with P projective and Q prinjective, then R has a Δ-flag; see [46, Proposition 2.3]. Now let P be any projective object in $\mathcal{M}$. Since P has a Δ-flag, the assumption on $\mathcal{M}$ implies that its injective hull Q is prinjective. Then we see that the cokernel of the inclusion $P \hookrightarrow Q$ has a Δ-flag, so its injective hull R is prinjective too. This proves the existence of an exact sequence $0 \to P \to Q \to R$ such that Q and R are direct sums of summands of T. Now argue as in [51, Corollary 1.7].

Remark 1.2. There is also a version of Theorem 1.1 for highest weight categories with infinite weight posets. For this, one needs to replace T by a family $\{T_d\}_{d \in D}$ of prinjective objects of $\mathcal{M}$ such that every indecomposable prinjective is a summand of at least one and at most finitely many of the T_d's. Then the algebra C becomes the vector space $\bigoplus_{d,d' \in D} \mathrm{Hom}_{\mathcal{M}}(T_d, T_{d'})$ viewed as a locally unital algebra via the opposite of composition, and C-mod means the category of locally unital finite dimensional left C-modules.

Schur-Weyl duality. The first situation in which the double centralizer property arises involves the representation theory of the general linear group $GL_n(\Bbbk)$ and the symmetric group S_d over an algebraically closed field $\Bbbk$. From our categorical perspective, this is only really interesting in the case that $\Bbbk$ is of positive characteristic. For simplicity we assume that $n \geq d$; the case $n < d$ is more subtle and is discussed in [39] from a similar point of view to this introduction.

Let $\mathcal{R}$ denote the category of *polynomial representations of degree d* for the algebraic group $G := GL_n(\Bbbk)$, that is, the Serre subcategory of its category of rational representations generated by the dth tensor power $V^{\otimes d}$ of the natural G-module V. It is a highest weight category with weight poset Λ being the set of all partitions of d, partially ordered by the usual dominance ordering. This poset indexes a set $\{L(\lambda)\}_{\lambda \in \Lambda}$ of representatives for the isomorphism classes of irreducible objects of $\mathcal{R}$; explicitly, one takes $L(\lambda)$ to be the irreducible highest weight module for G of highest weight λ. For $\lambda \in \Lambda$, we also have the projective cover $P(\lambda)$ of $L(\lambda)$ in $\mathcal{R}$, and the standard module

$$\Delta(\lambda) := P(\lambda) \Big/ \sum_{\substack{\mu \in \Lambda \text{ with } \mu \not\leq \lambda \\ f \in \mathrm{Hom}_G(P(\mu), \mathrm{rad}\, P(\lambda))}} \mathrm{im}\, f.$$

In this context $\Delta(\lambda)$ turns out to be isomorphic to the *Weyl module* whose character is given by the Schur polynomial s_λ. The statement that $\mathcal{R}$ is a highest weight category means that each $P(\lambda)$ has a finite filtration with top section $\Delta(\lambda)$ and other sections of the form $\Delta(\mu)$ for $\mu > \lambda$.

The symmetric group S_d acts on the right on the tensor space $T := V^{\otimes d}$ by permuting tensors, and this action induces an isomorphism between the group algebra $\Bbbk S_d$ and the endomorphism algebra $C := \mathrm{End}_G(T)^{\mathrm{op}}$. The module T is a projective module in $\mathcal{R}$, as more generally is the *divided power*

$$\Gamma^\mu(V) := \{v \in V^{\otimes d} \mid v \cdot g = v \text{ for all } g \in S_\mu\}$$

for any n-part composition $\mu \vDash_n d$, where S_μ denotes the parabolic subgroup $S_{\mu_1} \times \cdots \times S_{\mu_n}$ of S_d. As T is self-dual with respect to the natural duality on $\mathcal{R}$, it is also injective. Then it is a special feature of this situation that all of the standard modules $\Delta(\lambda)$ embed into T. Hence the injective hull of each standard module is a summand of T, so projective, and moreover T is a prinjective generator for $\mathcal{R}$. This verifies the hypotheses of Theorem 1.1. We deduce that the functor

$$\mathbb{V} := \mathrm{Hom}_G(T, -) : \mathcal{R} \to \Bbbk S_d\text{-mod}$$

is fully faithful on projectives.

In Green's monograph [31], this quotient functor $\mathbb{V}$ is called the *Schur functor*, and it is used in a systematic way to recover the representation theory of the symmetric group from that of the general linear group, thereby reversing the flow of information compared to Schur's classical work over $\mathbb{C}$. Green's approach also emphasizes the role of the *Schur algebra* $S(n, d)$, which is a certain finite dimensional algebra whose module category is equivalent to $\mathcal{R}$. From the perspective of the double centralizer property, the Schur algebra can be *defined* as the endomorphism algebra

$$S(n, d) := \mathrm{End}_{\Bbbk S_d}(T),$$

and then the equivalence $\mathcal{R} \xrightarrow{\sim} S(n, d)\text{-mod}$ can be seen as follows. Let

$$P := \bigoplus_{\mu \vDash_n d} \Gamma^\mu(V).$$

This is a projective generator for $\mathcal{R}$, hence the functor $\mathrm{Hom}_G(P, -)$ defines an equivalence of categories $\mathcal{R} \xrightarrow{\sim} A\text{-mod}$, where $A := \mathrm{End}_G(P)^{\mathrm{op}}$. Setting $Y := \mathbb{V}P$, it remains to observe that

$$A = \mathrm{End}_G(P)^{\mathrm{op}} \cong \mathrm{End}_{\Bbbk S_d}(Y)^{\mathrm{op}} \cong \mathrm{End}_{\Bbbk S_d}(Y^*) \cong \mathrm{End}_{\Bbbk S_d}(T) = S(n, d).$$

The non-trivial first isomorphism here is defined by applying the functor $\mathbb{V}$; the fact that it is an isomorphism follows from (indeed, is equivalent to) the double centralizer property. The second isomorphism is just taking linear duals to turn left modules into right modules. The final isomorphism follows on checking that $Y^* \cong \bigoplus_{\mu \vDash_n d} \Bbbk(S_\mu \backslash S_d) \cong T$ as right $\Bbbk S_d$-modules, i.e. both Y^* and T are direct sums of the same permutation modules.

Soergel's functor. Everything in the remainder of the article will be defined over the ground field $\mathbb{C}$. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Fix a Borel subalgebra $\mathfrak{b}$ containing a Cartan subalgebra $\mathfrak{t}$. Let $\mathcal{O}_0$ be the principal block of the BGG category $\mathcal{O}$ attached to this data. Thus $\mathcal{O}_0$ consists of all finitely generated $\mathfrak{g}$-modules which are locally finite over $\mathfrak{b}$, semisimple over $\mathfrak{t}$, and which have the same generalized central character as the trivial module. The irreducible modules in $\mathcal{O}_0$ are parametrized naturally by the Weyl group W of $\mathfrak{g}$. We denote them by $\{L(w)\}_{w \in W}$; explicitly, $L(w)$ is the irreducible highest weight module of highest weight $w\rho - \rho$ where ρ is the half-sum of the positive roots.

The category $\mathcal{O}_0$ is a highest weight category with weight poset W partially ordered by the opposite of the usual Bruhat order, i.e. the longest element $w_0 \in W$ is minimal. Its standard modules $\Delta(w)$ defined according to the general recipe explained above are better known as *Verma modules*, and may be denoted instead by $M(w)$. It is well known that the socle of each Verma module in $\mathcal{O}_0$ is isomorphic to $L(w_0)$. Moreover the only indecomposable projective module $P(w)$ that is also injective is the *antidominant projective* $P(w_0)$. This puts us in the situation of Theorem 1.1 with $T := P(w_0)$.

In [50], Soergel proved that the algebra $C := \mathrm{End}_{\mathfrak{g}}(T)^{\mathrm{op}}$ is canonically isomorphic to the *coinvariant algebra*, that is, the quotient of $S(\mathfrak{t})$ by the ideal generated by all homogeneous W-invariant polynomials of strictly positive degree. Equivalently, by a classical theorem of Borel, C is the cohomology algebra $H^*(G/B, \mathbb{C})$ of the flag variety associated to $\mathfrak{g}$. Soergel also showed that the functor

$$\mathbb{V} := \mathrm{Hom}_{\mathfrak{g}}(T, -) : \mathcal{O}_0 \to C\text{-mod}$$

is fully faithful on projectives, as asserted by Theorem 1.1. Moreover, all of the *Soergel modules* $Q(w) := \mathbb{V}P(w)$ admit unique (up to automorphism) gradings making them into self-dual graded modules over the naturally graded algebra C. Hence, letting $P := \bigoplus_{w\in W} P(w)$ and $Q := \bigoplus_{w\in W} Q(w)$ graded in this way, we get induced a grading on the endomorphism algebra

$$A := \mathrm{End}_{\mathfrak{g}}(P)^{\mathrm{op}} \cong \mathrm{End}_C(Q)^{\mathrm{op}},$$

where the isomorphism comes from the double centralizer property.

In fact, as shown by Beilinson, Ginzburg and Soergel in [4], the graded algebra A is a Koszul algebra. Since P is a projective generator for $\mathcal{O}_0$, the category $\mathcal{O}_0$ is equivalent to the category A-mod. This means that $\mathcal{O}_0$ has a natural graded lift, namely, the category A-grmod of finite dimensional *graded* left A-modules. This graded category is related intimately to the Iwahori-Hecke algebra associated to the Weyl group of $\mathfrak{g}$ and the Kazhdan-Lusztig conjecture.

Organization of the article. The rest of the article is an attempt to explain some generalizations of the above examples. The first of these, discussed in section 2, is the *Schur-Weyl duality for higher levels* introduced in [13]. This is built around a double centralizer property as above in which the category $\mathcal{M}$ is a sum of blocks of parabolic category $\mathcal{O}$ for the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, and the endomorphism algebra C of a suitably chosen prinjective generator T is some degenerate cyclotomic Hecke algebra. A key feature of this example is that the category $\mathcal{M}$ admits a categorical $\mathfrak{sl}_{\mathbb{Z}}$-action in the general sense of Chuang and Rouquier [24, 47].

In fact the category $\mathcal{M}$ fits into the axiomatic framework of *tensor product categorifications* introduced recently by Losev and Webster [44]. They show that all tensor product categorifications of integrable highest weight modules satisfy a double centralizer property in which the algebra C is some cyclotomic quiver Hecke algebra. From this they are able to deduce a striking uniqueness theorem. We sketch these results in section 3. When applied to our category $\mathcal{M}$, the Losev-Webster uniqueness theorem implies the equivalence of $\mathcal{M}$ with various other categories which have appeared elsewhere in the literature. Some examples are discussed in section 4, together with some further generalizations and possible future directions.

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2. Schur-Weyl duality for higher levels

Categorical actions. Throughout, we let $I \subseteq \mathbb{Z}$ be some non-empty interval, $I_+ := I \cup (I+1)$, and $\mathfrak{sl}_I$ be the (complex) special linear Lie algebra of $I_+ \times I_+$ matrices. It is generated by its matrix units $e_i := e_{i,i+1}$ and $f_i := e_{i+1,i}$ for each $i \in I$. Also let V_I be the natural $\mathfrak{sl}_I$-module of column vectors with standard basis $\{v_i\}_{i \in I_+}$. We denote the weight of v_i with respect to the Cartan subalgebra $\mathfrak{t}_I$ of $\mathfrak{sl}_I$ consisting of diagonal matrices by $\varepsilon_i \in \mathfrak{t}_I^*$.

The following is essentially [47, Definition 5.32].

Definition 2.1. Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category such that all its objects have finite length and there are enough projectives. A *categorical* $\mathfrak{sl}_I$*-action* on $\mathcal{C}$ is the data of an endofunctor F, a right adjoint E to F (with a specified adjunction), and natural transformations $x \in \mathrm{End}(F)$ and $s \in \mathrm{End}(F^2)$ satisfying the axioms (SL1)–(SL4) formulated below. For the first axiom, we let F_i be the subfunctor of F defined by the generalized i-eigenspace of x, i.e. $F_i M = \sum_{k \geq 0} \ker(x_M - i)^k$ for each $M \in \mathcal{C}$.

(SL1) We have that $F = \bigoplus_{i \in I} F_i$, i.e. $FM = \bigoplus_{i \in I} F_i M$ for each $M \in \mathcal{C}$.

(SL2) For $d \geq 0$ the endomorphisms $x_j := F^{d-j} x F^{j-1}$ and $s_k := F^{d-k-1} s F^{k-1}$ of F^d satisfy the relations of the degenerate affine Hecke algebra H_d, i.e. the x_j's commute like in the polynomial algebra $\mathbb{C}[x_1, \dots, x_d]$, the s_j's satisfy the Coxeter relations of the simple transpositions in the symmetric group S_d, $s_j x_{j+1} = x_j s_j + 1$, and $s_j x_k = x_k s_j$ for $k \neq j, j+1$.

(SL3) The functor F is isomorphic to a right adjoint of E.

For the final axiom, we let $c : \mathrm{id} \to EF$ and $d : FE \to \mathrm{id}$ be the unit and counit of the given adjunction, respectively. The endomorphisms x and s of F and F^2 induce endomorphisms x' and s' of E and E^2 too:

$$x' : E \stackrel{cE}{\to} EFE \stackrel{ExE}{\to} EFE \stackrel{Ed}{\to} E,$$

$$s' : E^2 \stackrel{cE^2}{\to} EFE^2 \stackrel{EcFE^2}{\to} E^2F^2E^2 \stackrel{E^2sE^2}{\to} E^2F^2E^2 \stackrel{E^2FdE}{\to} E^2FE \stackrel{E^2d}{\to} E^2.$$

Let E_i be the subfunctor of E defined by the generalized i-eigenspace of $x' \in \mathrm{End}(E)$. The axioms so far imply that $E = \bigoplus_{i \in I} E_i$ and moreover F_i and E_i are biadjoint, so they are both exact and send projectives to projectives.

(SL4) Let $K_0(\mathcal{C})$ be the split Grothendieck group of the category of projectives in $\mathcal{C}$. The endomorphisms f_i and e_i of $[\mathcal{C}] := \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C})$ induced by F_i and E_i, respectively, make $[\mathcal{C}]$ into an integrable representation of $\mathfrak{sl}_I$. Moreover the classes of the indecomposable projective objects are weight vectors.

There is also a much more general notion of a categorical action of an arbitrary Kac-Moody algebra $\mathfrak{g}$ on a category $\mathcal{C}$, which was introduced independently by Rouquier [47] and Khovanov and Lauda [38]. We will refer to this general notion in later discussion, but are not going to repeat its definition in full here. It involves a certain 2-category $\mathcal{U}(\mathfrak{g})$ defined in [47] by generators and relations, which is closely related to the diagrammatic category introduced in [38]. In particular, the degenerate affine Hecke algebra appearing in our axiom (SL2) gets replaced by the *quiver Hecke algebra* (or *Khovanov-Lauda-Rouquier algebra*) associated to $\mathfrak{g}$. The equivalence of the general definition with the special version stated above depends on the isomorphism theorem between affine Hecke algebras and quiver Hecke algebras in type A_n from [47, Proposition 3.15] (see also [14]), as well as on [47, Theorem 5.27].

Definition 2.2. Given two categorical $\mathfrak{sl}_I$-actions on categories $\mathcal{C}_1$ and $\mathcal{C}_2$, a functor $\mathbb{G} : \mathcal{C}_1 \to \mathcal{C}_2$ is *strongly equivariant* if there exists an isomorphism of functors $\zeta : \mathbb{G} \circ F \xrightarrow{\sim} F \circ \mathbb{G}$ with

$$x\mathbb{G} \circ \zeta = \zeta \circ \mathbb{G}x \qquad \text{in } \mathrm{Hom}(\mathbb{G} \circ F, F \circ \mathbb{G}),$$
$$s\mathbb{G} \circ F\zeta \circ \zeta F = F\zeta \circ \zeta F \circ \mathbb{G}s \qquad \text{in } \mathrm{Hom}(\mathbb{G} \circ F^2, F^2 \circ \mathbb{G}).$$

A *strongly equivariant equivalence* is a strongly equivariant functor $\mathbb{G} : \mathcal{C}_1 \to \mathcal{C}_2$ that is also an equivalence of categories. It is then automatic that $[\mathcal{C}_1] \cong [\mathcal{C}_2]$ as $\mathfrak{sl}_I$-modules.

First example of a categorical action. In this subsection, we explain our favorite example of a categorical action; for this the interval I will be $\mathbb{Z}$. Let $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{t}$ be the Cartan subalgebra consisting of diagonal matrices, and $\mathfrak{b}$ be the Borel subalgebra of upper triangular matrices. Inside $\mathfrak{t}^*$, we have the standard coordinate functions $\delta_1, \dots, \delta_n$, where δ_i picks out the ith diagonal entry of a diagonal matrix. Let $(-,-)$ be the symmetric bilinear form on $\mathfrak{t}^*$ defined from $(\delta_i, \delta_j) := \delta_{i,j}$. Also set

$$\rho := -\delta_2 - 2\delta_3 - \cdots - (n-1)\delta_n.$$

We identify the set $\mathfrak{t}^*_{\mathbb{Z}} := \mathbb{Z}\delta_1 \oplus \cdots \oplus \mathbb{Z}\delta_n$ of integral weights with $\mathbb{Z}^n$, so that $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$ is identified with the n-tuple $(\lambda_1, \dots, \lambda_n)$ defined from $\lambda_i := (\lambda + \rho, \delta_i)$. The *Bruhat order* $\leq$ on $\mathfrak{t}^*_{\mathbb{Z}}$ is the partial order generated by the basic relation that $\lambda < \mu$ if μ is obtained from the n-tuple λ by switching some pair of entries λ_i and λ_j for $i < j$ with $\lambda_i < \lambda_j$.

Let $\mathcal{O}$ be the BGG category of all finitely generated $\mathfrak{g}$-modules M that are locally finite over $\mathfrak{b}$ and satisfy

$$M = \bigoplus_{\lambda \in \mathfrak{t}^*_{\mathbb{Z}}} M_\lambda,$$

where M_λ denotes the λ-weight space with respect to $\mathfrak{t}$. The irreducible modules in $\mathcal{O}$ are the modules $\{L(\lambda)\}_{\lambda\in\mathfrak{t}^*_{\mathbb{Z}}}$, where $L(\lambda)$ is the irreducible highest weight module of highest weight λ. The category $\mathcal{O}$ is a highest weight category with weight poset $(\mathfrak{t}^*_{\mathbb{Z}}, \leq)$. Its standard modules are the Verma modules $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$.

In any highest weight category $\mathcal{C}$, projective objects have finite Δ-flags. Hence there is a map $K_0(\mathcal{C}) \hookrightarrow K_0(\mathcal{C}^\Delta)$, the right hand side denoting the Grothendieck group of the exact subcategory of $\mathcal{C}$ consisting of objects with a Δ-flag. In category $\mathcal{O}$, all the chains in the partial order $\leq$ are all finite, in which case this map is actually an isomorphism. This means that the classes $[M(\lambda)]$ of the Verma modules can be interpreted as elements of $[\mathcal{O}]$.

Now we define a categorical $\mathfrak{sl}_{\mathbb{Z}}$-action on $\mathcal{O}$. For the endofunctors F and E, we take the functors $F := -\otimes U$ and $E := -\otimes U^\vee$, where U is the natural $\mathfrak{g}$-module of column vectors and $U^\vee$ is its dual. These are both left and right adjoint to each other in a canonical way. For the natural transformation $x \in \mathrm{End}(F)$, we let $x_M : M \otimes U \to M \otimes U$ be the endomorphism defined by the action of the Casimir tensor $\Omega := \sum_{i,j=1}^n e_{i,j} \otimes e_{j,i}$. For $s \in \mathrm{End}(F^2)$, we let $s_M : M \otimes U \otimes U \to M \otimes U \otimes U$ be the map $m \otimes u \otimes v \mapsto m \otimes v \otimes u$. The axioms (SL1)–(SL4) are checked in [24, §7.4]. The hardest one is (SL4); for this one shows that the map

$$[\mathcal{O}] \to V_{\mathbb{Z}}^{\otimes n}, \qquad [M(\lambda)] \mapsto v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}$$

is an isomorphism of $\mathfrak{sl}_{\mathbb{Z}}$-modules. This follows from a slightly stronger statement, namely, that $F_i M(\lambda)$ has a Δ-flag with sections $M(\lambda + \delta_j)$ for all $j = 1, \dots, n$ such that $\lambda_j = i$; similarly $E_i M(\lambda)$ has a Δ-flag with sections $M(\lambda - \delta_j)$ for all $j = 1, \dots, n$ such that $\lambda_j = i+1$. On passing to the Grothendieck group, these two descriptions match the actions of f_i and e_i on the monomial $v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n} \in V_{\mathbb{Z}}^{\otimes n}$.

We have just identified $[\mathcal{O}]$ with $V_{\mathbb{Z}}^{\otimes n}$ so that the classes of the Verma modules correspond to the monomials. The classes of the indecomposable projective modules give another natural basis for $[\mathcal{O}]$. Under our identification it is known that this basis corresponds to *Lusztig's canonical basis* for $V_{\mathbb{Z}}^{\otimes n}$. We skip the precise definition of the canonical basis here; it arises by applying Lusztig's general construction of tensor product of based modules from [45, Ch. 27] to the n-fold tensor product of the q-analog of the module $V_{\mathbb{Z}}$ for quantized enveloping algebra $U_q\mathfrak{sl}_{\mathbb{Z}}$ (then specializing at $q = 1$). In fact the statement just made is an equivalent formulation of the Kazhdan-Lusztig conjecture for the Lie algebra $\mathfrak{g}$; see e.g. [12, Theorem 4.5] where the dual statement is explained.

Second example of a categorical action. Now take I to be an arbitrary nonempty interval and fix also a composition $\nu = (\nu_1, \dots, \nu_l)$ of integers with $1 \leq \nu_i \leq |I|$ for each i. Our next example of a categorical $\mathfrak{sl}_I$-action is going to have complexified Grothendieck group isomorphic to

$$\bigwedge^\nu V_I := \bigwedge^{\nu_1} V_I \otimes \cdots \otimes \bigwedge^{\nu_l} V_I.$$

Let us introduce some notation for the obvious monomial basis of this module: set

$$\Lambda := \left\{ \lambda = (\lambda_{i,j})_{1\leq i\leq l, 1\leq j\leq \nu_i} \,\middle|\, \lambda_{i,j} \in I_+, \lambda_{i,1} > \cdots > \lambda_{i,\nu_i} \text{ for each } i \right\},$$

$$v_\lambda := (v_{\lambda_{1,1}} \wedge \cdots \wedge v_{\lambda_{1,\nu_1}}) \otimes \cdots \otimes (v_{\lambda_{l,1}} \wedge \cdots \wedge v_{\lambda_{l,\nu_l}}) \in \bigwedge^\nu V_I.$$

Then $\{v_\lambda \mid \lambda \in \Lambda\}$ is a basis for $\bigwedge^\nu V_I$. Each of the modules $\bigwedge^{\nu_i} V_I$ is minuscule, so all of its weight spaces are one-dimensional. Hence the map

$$\Lambda \to (\mathfrak{t}_I^*)^{\oplus l}, \qquad \lambda \mapsto (|\lambda_1|, \dots, |\lambda_l|) \text{ where } |\lambda_i| := \sum_{j=1}^{\nu_i} \varepsilon_{\lambda_{i,j}}$$

is injective. Let $\leq$ be the usual dominance ordering on $\mathfrak{t}_I^*$ defined from $\lambda \leq \mu$ if $\mu - \lambda$ is a sum of simple roots $\varepsilon_i - \varepsilon_{i+1}$ $(i \in I)$. Then define a partial order $\leq$ on Λ by $\lambda \leq \mu$ if and only if $|\lambda_1| + \cdots + |\lambda_i| \geq |\mu_1| + \cdots + |\mu_i|$ for each $i = 1, \dots, l$, with equality in case $i = l$. We refer to this as the *reverse dominance ordering*.

We identify the set Λ with a subset of the set $\mathfrak{t}_\mathbb{Z}^*$ of integral weights from the previous subsection so that $\lambda \in \Lambda$ corresponds to the weight $\lambda \in \mathfrak{t}_\mathbb{Z}^*$ with

$$(\lambda + \rho, \delta_{\nu_1+\cdots+\nu_{i-1}+j}) = \lambda_{i,j}$$

for each i, j, i.e. it is the tuple $(\lambda_{1,1}, \dots, \lambda_{1,\nu_1}, \dots, \lambda_{l,1}, \dots, \lambda_{l,\nu_l}) \in \mathbb{Z}^n$. Then let $\mathcal{M}$ be the Serre subcategory of the category $\mathcal{O}$ from the previous subsection generated by the modules $\{L(\lambda)\}_{\lambda\in\Lambda}$. In fact $\mathcal{M}$ is a sum of blocks of the parabolic category $\mathcal{O}$ associated to the standard parabolic subalgebra with Levi factor $\mathfrak{gl}_{\nu_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}_{\nu_l}(\mathbb{C})$. As is well known, $\mathcal{M}$ is again a highest weight category with weight poset Λ partially ordered by the reverse dominance ordering $\leq$ introduced in the previous subsection; this order is just the restriction of the Bruhat order $\leq$ on $\mathfrak{t}_\mathbb{Z}^*$ to Λ. The standard module $\Delta(\lambda) \in \mathcal{M}$ is the parabolic Verma module of highest weight λ. The functors F_i and E_i for $i \in I$ restrict to well-defined endofunctors of $\mathcal{M}$. Hence we can define a categorical $\mathfrak{sl}_I$-action on $\mathcal{M}$ with $F := \bigoplus_{i\in I} F_i$, $E := \bigoplus_{i\in I} E_i$, and x and s being the restrictions of the ones on $\mathcal{O}$. For (SL4), one checks that the map

$$[\mathcal{M}] \to \bigwedge^\nu V_\mathbb{Z}, \qquad [\Delta(\lambda)] \mapsto v_\lambda$$

is an isomorphism of $\mathfrak{sl}_I$-modules.

The natural inclusion $\mathcal{M} \hookrightarrow \mathcal{O}$ has a left adjoint $\pi : \mathcal{O} \to \mathcal{M}$, defined by taking the largest quotient that belongs to $\mathcal{M}$. As it is left adjoint to an exact functor, π sends projectives to projectives. In fact it is even the case that the restriction of π to $\mathcal{O}^\Delta$ is exact, with $\pi(M(\lambda)) \cong \Delta(\lambda)$ if $\lambda \in \Lambda$ and $\pi(M(\lambda)) = 0$ otherwise. This means that there is a commuting diagram of linear maps

$$\begin{array}{ccc} [\mathcal{O}] & \xrightarrow{[\pi]} & [\mathcal{M}] \\ \big\downarrow & & \big\downarrow \\ V_\mathbb{Z}^{\otimes n} & \longrightarrow & \bigwedge^\nu V_I, \end{array}$$

where the vertical maps are the isomorphisms introduced above, and the bottom map is an obvious surjection. The functor π sends the projective cover of $L(\lambda)$ in $\mathcal{O}$ to its projective cover $P(\lambda)$ in $\mathcal{M}$ if $\lambda \in \Lambda$, or to zero otherwise. This parallels the effect of the bottom map in the above commuting diagram on the canonical bases of $V_{\mathbb{Z}}^{\otimes n}$ and $\bigwedge^{\nu} V_I$. Hence the basis $\{[P(\lambda)]\}_{\lambda \in \Lambda}$ for $[\mathcal{M}]$ corresponds to the canonical basis $\{b_\lambda\}_{\lambda\in\Lambda}$ of the based module $\bigwedge^{\nu} V_I$.

Higher level Schur-Weyl duality. We continue with the notation of the previous subsection, assuming in addition that the interval I is *finite*. Set $o := \min(I)-1$ and $\varpi_i := \sum_{I \ni j \leq i} \varepsilon_j \in \mathfrak{t}_I^*$. The module $\bigwedge^{\nu} V_I$ has a unique highest weight in the dominance ordering, namely, the weight

$$\varpi := \sum_{i=1}^{l} \varpi_{o+\nu_i}.$$

We let $\kappa \in \Lambda$ be the unique element satisfying $|\kappa_1| + \cdots + |\kappa_l| = \varpi$, so that v_κ spans the highest weight space of $\bigwedge^{\nu} V_I$. The $\mathfrak{sl}_I$-submodule of $\bigwedge^{\nu} V_I$ generated by v_κ is a copy of the irreducible highest weight module $V(\varpi)$. Let

$$\iota : V(\varpi) \hookrightarrow \bigwedge\nolimits^{\nu} V_I$$

be the inclusion. Higher level Schur-Weyl duality categorifies this homomorphism.

Recall that $\bigwedge^{\nu} V_I$ is a based module with canonical basis $\{b_\lambda\}_{\lambda\in\Lambda}$ corresponding to the indecomposable projectives $\{P(\lambda)\}_{\lambda\in\Lambda}$ in $\mathcal{M}$. By the general theory of based modules, there is a subset $\Lambda^\circ \subset \Lambda$ such that $\{b_\lambda\}_{\lambda\in\Lambda^\circ}$ is the canonical basis of the irreducible submodule $V(\varpi)$. The best way to describe this set Λ° combinatorially is to note that the set Λ that labels the basis of our based module $\bigwedge^{\nu} V_I$ comes equipped with an explicit crystal structure defined via Kashiwara's tensor product rule; then Λ° is the connected component of this crystal generated by κ.

The representation theoretic significance of Λ° was first noticed in [13]: it is exactly the set of weights that index the indecomposable projective modules $P(\lambda) \in \mathcal{M}$ that are also injective; it is also the set of weights indexing the irreducible modules $L(\lambda)$ that are of maximal Gelfand-Kirillov dimension in $\mathcal{M}$. The hypotheses of Theorem 1.1 are all satisfied in the present situation. For the prinjective generator T, we take

$$T := \bigoplus_{d\geq 0} F^d L(\kappa) \in \mathcal{M}.$$

Setting $C := \operatorname{End}_{\mathfrak{g}}(T)^{\mathrm{op}}$, Theorem 1.1 implies that the the functor

$$\mathbb{V} : \mathcal{M} \to C\text{-mod}$$

is fully faithful on projectives. For this to be good for anything, we of course need to identify the algebra C explicitly.

To state the main result, let H_d^f be the quotient of the degenerate affine Hecke algebra H_d by the two-sided ideal generated by $f := \prod_{i=1}^{l}(x_1 - (o + \nu_i))$. This finite dimensional algebra is known as a degenerate *cyclotomic Hecke algebra*. It contains a system of mutually orthogonal idempotents $\{1_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathbb{C}^d}$ indexed by words $\boldsymbol{i} = i_1 \dots i_d \in \mathbb{C}^d$; these are defined so that $1_{\boldsymbol{i}}$ projects any H_d^f-module onto the generalized i_j-eigenspace of x_j for all j. Then let H_d^ω be the algebra $eH_d^f e$ where e is the central idempotent $\sum_{\boldsymbol{i}\in I^d} 1_{\boldsymbol{i}} \in H_d^f$.

Theorem 2.3 (Brundan-Kleshchev). *There is a well-defined right action of H_d^ω on $F^d L(\kappa)$ defined so that each of its generators x_j and s_k act via the natural transformations from (SL2). This action induces an isomorphism $H_d^\omega \xrightarrow{\sim} \mathrm{End}_{\mathfrak{g}}(F^d L(\kappa))^{\mathrm{op}}$. Hence $C \cong \bigoplus_{d\geq 0} H_d^\omega$.*

Remark 2.4. Theorem 2.3 was first proved in [13] under the assumption that $\nu_1 \geq \cdots \geq \nu_l$; this restriction was removed in [8]. The original proof goes via finite W-algebras and a result of Vust establishing some generalization of classical Schur-Weyl duality for centralizers in the general linear Lie algebra. Vust's result itself is quite non-trivial; its proof was completed in [40] by an invariant theory argument depending on the normality of closures of conjugacy classes of matrices. As discussed further in Remark 3.5, Losev and Webster have subsequently found a completely different proof of Theorem 2.3 based on the uniqueness of minimal categorifications of integrable highest weight modules established in [47].

By works of Ariki [1] and Grojnowski [32], the category

$$C\text{-mod} = \bigoplus_{d\geq 0} H_d^\omega\text{-mod}$$

admits a categorical $\mathfrak{sl}_I$-action making it into a minimal categorification of the irreducible $\mathfrak{sl}_I$-module $V(\omega)$. The appropriate functors F and E are the induction and restriction functors going between H_d^ω-mod and H_{d+1}^ω-mod. The functor $\mathbb{V} : \mathcal{M} \to C$-mod is then strongly equivariant in the sense of Definition 2.2.

The left adjoint to the quotient functor $\mathbb{V}$ sends the indecomposable projectives in C-mod to the ones in $\mathcal{M}$ indexed by the set Λ°. It induces a linear map $[C\text{-mod}] \hookrightarrow [\mathcal{M}]$ which corresponds exactly to the inclusion $\iota : V(\omega) \hookrightarrow \bigwedge^\nu V_I$ mentioned already above. Thus the classes of the indecomposable projectives in $[C\text{-mod}]$ coincide with the canonical basis $\{b_\lambda\}_{\lambda\in\Lambda^\circ}$ for $V(\omega)$. There are also certain *Specht modules* $\{S(\lambda)\}_{\lambda\in\Lambda}$ which have an intrinsic definition in terms of H_d^ω. In fact, as show in [13], the Specht module $S(\lambda)$ is the image of the parabolic Verma module $\Delta(\lambda)$ under the quotient functor $\mathbb{V}$. Then one can deduce almost everything known about the representation theory of the degenerate cyclotomic Hecke algebras H_d^ω from that of $\mathcal{M}$. This is done systematically in [15], leading to another proof of *Ariki's categorification theorem* from [1] for a generic parameter. This argument is similar to the way that Green recovers the representation theory of the symmetric group from the general linear group in [31].

The point of the double centralizer property is that it gives a way to recover the category $\mathcal{M}$ (up to equivalence) from the algebra C and knowledge of the *Young modules* $Y(\lambda) := \mathbb{V}P(\lambda)$ for each $\lambda \in \Lambda$. Indeed if Y is any *Young generator* for C-mod, that is, a direct sum of Young modules with each occuring at least once, then the double centralizer property shows that $\mathcal{M}$ is equivalent to the category A-mod where $A := \mathrm{End}_C(Y)^{\mathrm{op}}$. One application of this is given in [13]: it is shown there that a particular Young generator Y may be obtained by taking a direct sum of all of the so-called *permutation modules* introduced by Dipper, James and Mathas in [25] (or rather, their degenerate analogs). For this choice, the algebra A is the *cyclotomic Schur algebra*, i.e. the degenerate version of the algebra introduced in [25]. Hence by the double centralizer property the category $\mathcal{M}$ is equivalent to the category of finite dimensional modules over the cyclotomic Schur algebra. This argument is similar to the proof of the equivalence of the categories $\mathcal{R}$ and $S(n,d)$-mod from the introduction.

3. Tensor product categorifications

Discussion of the definition. In this section we are going to focus on some results of Losev and Webster from [44]. These put the Schur-Weyl duality for higher levels discussed above into a general axiomatic framework. We begin by formulating their definition in a very special case, namely, for tensor products of minuscule representations of $\mathfrak{sl}_I$ for a finite interval $I \subset \mathbb{Z}$. The following is exactly like in [16, Definition 2.9]. Note also that the category $\mathcal{M}$ defined in the previous section is an example, thus establishing the existence of such structures.

Definition 3.1. Let $\nu = (\nu_1, \dots, \nu_l)$ be a composition of n and $I \subset \mathbb{Z}$ be a finite interval. A *tensor product categorification* of $\bigwedge^\nu V_I$ means a highest weight category $\mathcal{M}$ together with an endofunctor F of $\mathcal{M}$, a right adjoint E to F (with specified adjunction), and natural transformations $x \in \mathrm{End}(F)$ and $s \in \mathrm{End}(F^2)$ satisfying the axioms (SL1)–(SL3) from Definition 2.1 and the axioms (TP1)–(TP2) below.

(TP1) The weight poset Λ is the set of tuples $(\lambda_1, \dots, \lambda_l) \in (\mathfrak{t}_I^*)^{\oplus l}$ such that each λ_i is a weight of $\bigwedge^{\nu_i} V_I$, ordered by the *reverse dominance ordering* $\lambda \leq \mu$ if and only if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for each i with equality when $i = l$.

(TP2) The exact functors F_i and E_i send objects with Δ-flags to objects with Δ-flags. Moreover the linear isomorphism $[\mathcal{M}] \xrightarrow{\sim} \bigwedge^\nu V_I, [\Delta(\lambda)] \mapsto v_\lambda$ intertwines the endomorphisms f_i and e_i of $[\mathcal{M}]$ induced by F_i and E_i with the endomorphisms of $\bigwedge^\nu V_I$ arising from the actions of the Chevalley generators f_i and e_i of $\mathfrak{sl}_I$.

In [44], Losev and Webster have introduced a substantially more general notion of tensor product categorication of $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$ for arbitrary integrable highest weight modules $V(\nu_1), \dots, V(\nu_l)$ for an arbitrary Kac-Moody algebra $\mathfrak{g}$. We are going to explain their definition somewhat informally. To start with, since they work with an arbitrary $\mathfrak{g}$ rather than $\mathfrak{sl}_I$, the axioms (SL1)–(SL3) are replaced by the corresponding axioms for a categorical $\mathfrak{g}$-action, i.e. the natural transformations defining the degenerate affine Hecke algebra action on F^d are replaced by natural transformations defining a quiver Hecke algebra action.

A more significant issue is that the modules $V(\nu_i)$ are no longer assumed to be minuscule, so their weight spaces are not all one-dimensional. We still have a natural poset Ξ, namely, the set of l-tuples $\xi = (\xi_1, \dots, \xi_l)$ such that ξ_i is a weight of the $\mathfrak{g}$-module $V(\nu_i)$, ordered by the reverse dominance ordering as above. The theory of based modules also produces a couple of natural bases for $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$, both indexed by the set Λ that is the Cartesian product of the underlying highest weight crystals: the *monomial basis* $\{v_\lambda\}_{\lambda \in \Lambda}$ arising from the naive tensor product of the canonical bases in each $V(\nu_i)$, and the *canonical basis* $\{b_\lambda\}_{\lambda \in \Lambda}$ for $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$ itself defined via Lusztig's general construction from [45, Ch. 27]. However now there is only a surjection

$$\rho : \Lambda \twoheadrightarrow \Xi,$$

rather than the isomorphism that we exploited in the previous section. This is a shadow of the problem at a categorical level: the category $\mathcal{M}$ is no longer going to be a highest weight category. Rather, it is the following weakening of the notion of highest weight category introduced by Losev and Webster (building on an earlier notion of standardly stratified algebra studied by a number of authors in the literature).

Definition 3.2. Let Ξ be an interval-finite poset and $\rho : \Lambda \twoheadrightarrow \Xi$ be a surjective function with finite fibers. A *standardly stratified category* of this type is a $\mathbb{C}$-linear abelian category $\mathcal{M}$ together with a given set of representatives $\{L(\lambda)\}_{\lambda \in \Lambda}$ for its irreducible objects, satisfying the axioms (SS1)–(SS3) below.

(SS1) All objects of $\mathcal{M}$ are of finite length, there are enough projectives and injectives, and $\operatorname{End}_{\mathcal{M}}(L(\lambda)) \cong \mathbb{C}$ for each λ.

For $\xi \in \Xi$, let $\mathcal{M}_{\leq \xi}$ be the Serre subcategory of $\mathcal{M}$ generated by $\{L(\lambda)\}_{\lambda \in \Lambda, \rho(\lambda) \leq \xi}$. Define $\mathcal{M}_{< \xi}$ similarly, and let $\pi_\xi : \mathcal{M}_{\leq \xi} \to \mathcal{M}_\xi$ be the quotient of $\mathcal{M}_{\leq \xi}$ by $\mathcal{M}_{< \xi}$. The *associated graded category* is $\operatorname{gr} \mathcal{M} := \bigoplus_{\xi \in \Xi} \mathcal{M}_\xi$. The *standardization functor* is $\Delta := \bigoplus_{\xi \in \Xi} \Delta_\xi : \operatorname{gr} \mathcal{M} \to \mathcal{M}$ where $\Delta_\xi : \mathcal{M}_\xi \to \mathcal{M}_{\leq \xi}$ is some choice of a left adjoint to π_ξ.

(SS2) The standardization functor is exact.

Let $P(\lambda)$ be the projective cover of $L(\lambda)$ in $\mathcal{M}$ and $\Delta(\lambda)$ be the projective cover of $L(\lambda)$ in $\mathcal{M}_{\leq \rho(\lambda)}$. In other words, $\Delta(\lambda)$ is the largest quotient of $P(\lambda)$ that belongs to $\mathcal{M}_{\leq \rho(\lambda)}$:

$$\Delta(\lambda) = P(\lambda) \Big/ \sum_{\substack{\mu \in \Lambda \text{ with } \rho(\mu) \not\leq \rho(\lambda) \\ f \in \operatorname{Hom}_G(P(\mu), P(\lambda))}} \operatorname{im} f.$$

(SS3) Each $P(\lambda)$ admits a finite Δ-flag with $\Delta(\lambda)$ at the top and lower sections of the form $\Delta(\mu)$ for $\mu \in \Lambda$ with $\rho(\mu) > \rho(\lambda)$.

Now we can complete our sketch of what it means for $\mathcal{M}$ to be a tensor product categorification of $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$. Of course it should be standardly stratified with $\rho : \Lambda \twoheadrightarrow \Xi$ as defined just before Definition 3.2. Moreover $\operatorname{gr} \mathcal{M}$ should admit a categorical action of $\mathfrak{g}^{\oplus l}$ making it into a minimal categorification of the irreducible $\mathfrak{g}^{\oplus l}$-module $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$; we denote the functors F_i and E_i for the action of the jth copy of $\mathfrak{g}$ here by ${}_jF_i$ and ${}_jE_i$, respectively. By the general theory of categorifications of integrable highest weight modules, the isomorphism classes of indecomposable projectives in $\operatorname{gr} \mathcal{M}$ are canonically labelled by the $\mathfrak{g}^{\oplus l}$-crystal Λ; this is explained in [41] or follows from the theory of perfect bases from [5]. Then there are two axioms which give some compatibility between $\operatorname{gr} \mathcal{M}$ and $\mathcal{M}$:

(TP1′) The indecomposable projective object of $\operatorname{gr} \mathcal{M}$ labelled by $\lambda \in \Lambda$ is isomorphic to the one arising by taking the image of $\Delta(\lambda)$ under the functor $\pi_{\rho(\lambda)}$.

(TP2′) For each $M \in \operatorname{gr} \mathcal{M}$, the object $F_i\Delta(M)$ (resp. $E_i\Delta(M)$) admits a filtration with sections $\Delta({}_jF_iM)$ (resp. $\Delta({}_jE_iM)$) for $j = 1, \dots, l$.

(This formulation of the definition looks slightly different but is equivalent to the one in [44].)

The problem of *existence* of such general tensor product categorifications was addressed already in earlier work of Webster [60]. In this, he introduced certain *tensor product algebras*, which can naturally be viewed as generalizations of cyclotomic quotients of quiver Hecke algebras. Then he uses the category of finite dimensional modules over these algebras to construct arbitrary tensor product categorifications.

The Losev-Webster uniqueness theorem. Having sketched the definition of tensor product categorification, we can now paraphrase the main result established in [44] as follows. Recall Definition 2.2 (which has an analog for arbitrary $\mathfrak{g}$).

Theorem 3.3 (Losev-Webster). *Let $V(\nu_1), \dots, V(\nu_l)$ be integrable highest weight for some Kac-Moody algebra $\mathfrak{g}$. Any tensor product categorification of $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$ is unique up to strongly equivariant equivalence.*

Here we restate this in a special case:

Corollary 3.4. *Let $I \subset \mathbb{Z}$ be a finite interval. Any tensor product categorification of the $\mathfrak{sl}_I$-module $\bigwedge^\nu V_I$ in the sense of Definition 3.1 is strongly equivariantly equivalent to the category $\mathcal{M}$ constructed from parabolic category $\mathcal{O}$ in the previous section.*

In order to emphasize the similarity between the present situation and the Schur-Weyl duality for higher levels from the previous section, let us say a few words about the strategy behind the proof of Theorem 3.3. As we mentioned earlier, the tensor product $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$ has a canonical basis $\{b_\lambda\}_{\lambda \in \Lambda}$ parametrized by the Cartesian product Λ of the underlying highest weight crystals. Kashiwara's tensor product rule gives the set Λ a canonical structure of $\mathfrak{g}$-crystal. (Building on earlier arguments from [42], Losev and Webster even give an interpretation of this crystal structure in terms of tensor product categorifications, which is the key to the proof of property (P1) stated in the next paragraph.) Let $\kappa \in \Lambda$ be the label of the highest weight vector of $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$ of weight

$\omega := \nu_1 + \cdots + \nu_l$, and let Λ° be the connected component of the $\mathfrak{g}$-crystal Λ generated by κ. Then the vectors $\{b_\lambda\}_{\lambda\in\Lambda^\circ}$ span a $\mathfrak{g}$-submodule of $V(\nu_1)\otimes\cdots\otimes V(\nu_l)$ isomorphic to $V(\omega)$.

Now let $\mathcal{M}$ be a tensor product categorification of $V(\nu_1)\otimes\cdots\otimes V(\nu_l)$. Losev and Webster show:

(P1) For $\lambda \in \Lambda$, the projective object $P(\lambda)$ is injective if and only if $\lambda \in \Lambda^\circ$. Moreover $T := \bigoplus_{d\geq 0} F^d L(\kappa)$ is a prinjective generator for $\mathcal{M}$.

(P2) The algebra $C := \mathrm{End}_{\mathcal{M}}(T)^{\mathrm{op}}$ is the direct sum $\bigoplus_{d\geq 0} H_d^\omega$ of the cyclotomic quiver Hecke algebras attached to $\mathfrak{g}$ and the dominant weight ω.

(P3) The double centralizer property holds, i.e. $\mathbb{V} := \mathrm{Hom}_{\mathcal{M}}(T, -) : \mathcal{M} \to C\text{-mod}$ is fully faithful on projectives.

(P4) For $\lambda \in \Lambda$ the isomorphism type of the C-module $Y(\lambda) := \mathbb{V}P(\lambda)$ is independent of the particular choice of the tensor product categorification $\mathcal{M}$.

(These statements make sense as written only in the case that $\mathfrak{g}$ is of finite type; for the general case one needs to modify them in an analogous way to Remark 1.2, taking $T_d := F^d L(\kappa)$ for $d \in \mathbb{N}$.) By the double centralizer property (P3), the category $\mathcal{M}$ can be recovered from the algebra C and its modules $\{Y(\lambda)\}_{\lambda\in\Lambda}$. Hence (P2) and (P4) establish the uniqueness of $\mathcal{M}$ up to equivalence. The strong equivariance follows by some further considerations in a similar vein.

Remark 3.5. Here we sketch the Losev-Webster proof of (P2); note in view of [14] that this generalizes Theorem 2.3 above. By (P1), the left adjoint to the quotient functor $\mathbb{V}$ induces an embedding $[C\text{-mod}] \hookrightarrow [\mathcal{M}]$ which corresponds to the inclusion $V(\omega) \hookrightarrow V(\nu_1)\otimes\cdots\otimes V(\nu_l)$ at the level of Grothendieck groups. Thus the category of projectives in C-mod is equivalent to the bottom section of Rouquier's canonical filtration of the category of projectives in $\mathcal{M}$ from [47, Theorem 5.8]. (Indeed, Rouquier's filtration parallels Lusztig's canonical filtration of the based module $V(\nu_1)\otimes\cdots\otimes V(\nu_l)$.) Since $\mathrm{End}_{\mathcal{M}}(L(\kappa)) \cong \mathbb{C}$, it follows that this category is a realization of the minimal categorication of $V(\omega)$. But it is also known that the cyclotomic quiver Hecke algebras H_d^ω give such a realization with $F := H_{d+1}^\omega \otimes_{H_d^\omega} -$ and $L(\kappa) := H_0^\omega$ (e.g. see [34]); in this setting it is obvious that $\mathrm{End}(F^d L(\kappa))^{\mathrm{op}} \cong H_d^\omega$. The property (P2) is now clear from the uniqueness of minimal categorifications established in [47].

Graded lifts. The following theorem should by now come as no surprise. Note this builds essentially on the base case $l = 1$, where the identification of the canonical basis with the basis arising from indecomposable projectives in the minimal categorification of an integrable highest weight module was established already by Rouquier [48] and Varagnolo and Vasserot [55] (for symmetric Cartan matrices over $\mathbb{C}$ only).

Theorem 3.6 (Webster). *Let $V(\nu_1), \ldots, V(\nu_l)$ be integrable highest weight modules for some Kac-Moody algebra $\mathfrak{g}$ with a* symmetric *Cartan matrix. Let $\mathcal{M}$ be* *a tensor product categorification of $V(\nu_1) \otimes \cdots \otimes V(\nu_l)$ (over the ground field $\mathbb{C}$). Identify $[\mathcal{M}]$ with $V(\nu_1)\otimes\cdots\otimes V(\nu_l)$ so that each $[\Delta(\lambda)]$ is identified with the monomial v_λ (= the tensor product of canonical basis vectors in each $V(\nu_i)$). Then each $[P(\lambda)]$ coincides with the canonical basis vector b_λ.*

This is proved in [58] using the realization of $\mathcal{M}$ given by the tensor product algebras. In the special case that $\mathfrak{g}$ is of finite type A and each ν_i is minuscule, it follows already from Corollary 3.4 and the Kazhdan-Lusztig conjecture for the general linear Lie algebra, as we discussed already in the previous section; see also [16, Corollary 5.29] for a more direct argument in this case along the lines of [58].

The basic idea of Webster's proof of Theorem 3.6 involves the construction of certain *graded lifts* of tensor product categorifications. For Webster, these are no problem since his tensor product algebras are naturally graded. One can also understand them along similar lines to Soergel's definition of the graded lift of category $\mathcal{O}_0$ sketched in the introduction. This depends on the existence of a grading on C arising from the natural grading on quiver Hecke algebras. The images $Y(\lambda) := \mathbb{V}P(\lambda)$ all turn out to admit a unique grading (up to automorphism) with respect to which they are graded-self-dual modules over the graded algebra C. Then, setting $Y := \bigoplus_{\lambda\in\Lambda} Y(\lambda)$, the graded lift of $\mathcal{M}$ arises from the category A-grmod where $A := \mathrm{End}_C(Y)^{\mathrm{op}}$.

The point then is that the graded category A-grmod admits a categorical action of the quantized enveloping algebra $U_q(\mathfrak{g})$; see e.g. [16, Definition 5.5] where this definition is spelled out in the special case that $\mathfrak{g} = \mathfrak{sl}_I$. The grading shift functor makes its Grothendieck group into a $\mathbb{Z}[q, q^{-1}]$-module, hence tensoring over $\mathbb{Z}[q, q^{-1}]$ with $\mathbb{Q}(q)$, we obtain a $\mathbb{Q}(q)$-vector space. The categorical action of $U_q(\mathfrak{g})$ on A-grmod makes this into a $U_q(\mathfrak{g})$-module isomorphic to the q-analog of $V(\nu_1)\otimes\cdots\otimes V(\nu_l)$. The next step is to show that the category of graded projectives in A-grmod admits a duality which corresponds to Lusztig's bar involution on the q-analog of $V(\nu_1)\otimes\cdots\otimes V(\nu_l)$. This machinery reduces the proof of Theorem 3.6 to verifying that the grading on the algebra A is positive (with A_0 being semisimple); for Webster this is the property that the graded lift of $\mathcal{M}$ is *mixed*. Finally that is established by appealing to some geometric construction involving quiver varieties; see [58, Theorem 6.8].

In the special case that $\mathfrak{g} = \mathfrak{sl}_I$ for a finite interval I and all the weights ν_i are minuscule, the grading on the algebra A defined in the previous paragraph makes it into a *Koszul algebra* (hence it is mixed). The proof of this can actually be deduced from the Koszulity of blocks of parabolic category $\mathcal{O}$ established already by Beilinson, Ginzburg and Soergel [4] (also Backelin [2] for the singular-singular case). This is explained by Webster in [60], and independently by Hu and Mathas in [33]; see also [59] which suggests an entirely different approach to see the Koszulity via quiver varieties.

4. Some applications and future directions

Level two examples arising from Khovanov's arc algebra. There is a completely explicit description of the basic algebra underlying all tensor product categorifications in the sense of Definition 3.1 in which the level l is two. This arises from Khovanov's arc algebra, which was introduced originally in [36, 37] in the course of his work on categorification of the Jones polynomial. We give a very brief sketch here.

Fix an interval $I \subseteq \mathbb{Z}$ and let $I_+ := I \cup (I+1)$ as before. A *weight diagram* means a horizontal axis with vertices at each of the integers in the set I_+, each of which is labelled by one of the symbols $\vee, \wedge, \circ$ or $\times$; on excluding finitely many of the vertices, we require that no $\vee$ should appear to the left of an $\wedge$. Assume that we are given some set Λ of weight diagrams that is closed under the following swaps of labels of any pair of adjacent vertices:

$$\circ\times \leftrightarrow \vee\wedge \leftrightarrow \wedge\vee \leftrightarrow \times\circ, \quad \circ\vee \leftrightarrow \vee\circ, \quad \circ\wedge \leftrightarrow \wedge\circ, \quad \times\vee \leftrightarrow \vee\times, \quad \times\wedge \leftrightarrow \wedge\times\,.$$

To the set Λ, we associated in [18] a certain positively graded basic algebra K_Λ, which is some generalization of Khovanov's arc algebra. The algebra K_Λ has a (graded cellular) basis corresponding to certain diagrams. Its multiplication is defined by some explicit combinatorial procedure which arises ultimately from the two-dimensional TQFT associated to the Frobenius algebra $H^*(\mathbb{P}^1;\mathbb{C})$.

Then in [19] we defined some endofunctors of K_Λ-mod defined by some explicit bimodules (generalizing Khovanov's geometric bimodules from [37]), and used these to prove directly that K_Λ is Koszul. Moreover in [20, (3.11)] we used certain of these functors to define biadjoint endofunctors F_i and E_i of K_Λ-mod for each $i \in I$. Finally in [20, (5.3)–(5.4)] we defined natural transformations $F_i \to F_i$ and $F_i \circ F_j \to F_j \circ F_i$ which, when suitably signed, satisfy the quiver Hecke algebra relations. Thus we obtain all of the data needed for a categorical $\mathfrak{sl}_I$-action on K_Λ-mod. (In [20] we considered only the case that I is finite but the constructions there apply in any case.)

Now we specialize to the case that the set I is finite and that Λ consists of all weight diagrams such that exactly m of the labels are either $\vee$ or $\times$, and exactly n of the labels are either $\wedge$ or $\times$. Then it is straightforward to check that K_Λ-mod is actually a tensor product categorification of $\bigwedge^m V_I \otimes \bigwedge^n V_I$ in the sense of Definition 3.1. Applying Corollary 3.4, we deduce that K_Λ-mod is equivalent to the parabolic category $\mathcal{O}$ denoted $\mathcal{M}$ in section 2 for $\nu = (m, n)$. In this way, one quite easily reproves an equivalence of categories established originally in [52, 20].

The super Kazhdan-Lusztig conjecture. In Definition 3.1, we assumed that the interval I was finite. The case $I = \mathbb{Z}$ is also interesting. Note for this that the Lie algebra $\mathfrak{sl}_\mathbb{Z}$ has four natural families of minuscule representations: the integrable highest weight modules $V(\omega_m)$ indexed by the fundamental dominant weights ω_m, the integrable lowest weight modules $V(-\omega_m)$, the exterior powers $\bigwedge^n V_\mathbb{Z}$ where $V_\mathbb{Z}$ is the natural $\mathfrak{sl}_\mathbb{Z}$-module, and the exterior powers $\bigwedge^n W_\mathbb{Z}$ where $W_\mathbb{Z}$ is dual to $V_\mathbb{Z}$.

In [16] we proved a version of the Losev-Webster uniqueness theorem for $\mathfrak{sl}_{\mathbb{Z}}$-tensor product categorifications involving tensor products of the exterior powers $\bigwedge^n V_{\mathbb{Z}}$ and $\bigwedge^n W_{\mathbb{Z}}$ (which are neither highest nor lowest weight). To set this up formally, one needs to modify Definition 3.1 slightly since the poset Λ defined exactly as in (TP1) need no longer have finite chains; the fix is to replace the Grothendieck group $[\mathcal{M}]$ in (TP2) with $[\mathcal{M}^\Delta]$. In all cases, projectives have finite Δ-flags, so that $[\mathcal{M}]$ still embeds naturally into $[\mathcal{M}^\Delta]$. We also proved that any such tensor product categorification admits a unique (up to equivalence) graded lift, and this graded lift is Koszul. Both of these results were deduced ultimately as applications of Corollary 3.4 and the known Koszulity of the graded lifts for finite intervals.

The main example of such $\mathfrak{sl}_{\mathbb{Z}}$-tensor product categorifications comes by considering parabolic category $\mathcal{O}$ for the general linear Lie superalgebra. Using this one can define a category $\mathcal{M}$ admitting a categorical $\mathfrak{sl}_{\mathbb{Z}}$-action in a very similar way to the second example from section 2. This leads to a construction of tensor product categorifications of any number of the modules of the form $\bigwedge^n V_{\mathbb{Z}}$ or $\bigwedge^n W_{\mathbb{Z}}$, with tensor factors appearing in any order. In particular the (integral part of) full category $\mathcal{O}$ for $\mathfrak{gl}_{m|n}(\mathbb{Z})$ relative to the standard Borel gives a tensor product categorification of $V_{\mathbb{Z}}^{\otimes m} \otimes W_{\mathbb{Z}}^{\otimes n}$, while its category of (finite dimensional) integrable representations gives a tensor product categorification of $\bigwedge^m V_{\mathbb{Z}} \otimes \bigwedge^n W_{\mathbb{Z}}$. The results from the previous paragraph imply at once that these categories all admit Koszul graded lifts. Moreover the super Kazhdan-Lusztig conjecture formulated originally in [6] and first proved by Cheng, Lam and Wang in [23] falls out easily from Corollary 3.4. See also [9] for a recent survey.

The existence of a Koszul graded lift of the category of integrable representations of $\mathfrak{gl}_{m|n}(\mathbb{C})$ had been proved earlier in [21]. In fact there is an explicit construction of this category in terms of Khovanov's arc algebra from the previous subsection: one just applies the results sketched there to $I = \mathbb{Z}$ with Λ consisting of all weights in which exactly m vertices are labelled $\times$ or $\vee$ and exactly n vertices are labelled $\circ$ or $\vee$. This produces another tensor product categorification of $\bigwedge^m V_{\mathbb{Z}} \otimes \bigwedge^n W_{\mathbb{Z}}$. Then the uniqueness of such tensor product categorifications implies that this is strongly equivariantly equivalent to the category of representations of $\mathfrak{gl}_{m|n}(\mathbb{C})$. In this way, one can recover the main theorem of [21].

Lowest tensored highest weight modules. There is one more interesting family of examples coming from Khovanov's arc algebra. Take the interval I to be $\mathbb{Z}$. Fix also integers $m, n \in \mathbb{Z}$. Let Λ be the set of all bipartitions $\lambda = (\lambda^\vee, \lambda^\wedge)$. We identify bipartition $\lambda \in \Lambda$ with the weight diagram having label $\vee$ at vertices $n+1-\lambda_1^\vee, n+2-\lambda_2^\vee, n+3-\lambda_3^\vee, \dots$ and label $\wedge$ at vertices $m+\lambda_1^\wedge, m-1+\lambda_2^\wedge, m-2+\lambda_3^\wedge, \dots$ (both $\vee$ and $\wedge$ means $\times$, neither means $\circ$). Then as above we get associated an arc algebra K_Λ and the data of a categorical $\mathfrak{sl}_{\mathbb{Z}}$-action on a suitable category of K_Λ-modules. This turns out to be a tensor product categorification of $V(-\omega_n) \otimes V(\omega_m)$, i.e. lowest weight tensored highest weight minuscule representations of $\mathfrak{sl}_{\mathbb{Z}}$. Actually, there is some further loss of finiteness here: although finitely generated projective modules still have finite Δ-

flags, the standard modules in this category have infinite length in general. This means that one needs to modify Definition 3.1 again, allowing certain direct limits of highest weight categories.

There is another naturally occurring example of such a tensor product categorification of $V(-\omega_n) \otimes V(\omega_m)$. This is given by Deligne's category $\underline{\mathrm{Re}}\mathrm{p}(GL_\delta)$ where $\delta := m - n$. By definition, Deligne's catgory is the Karoubification of the *oriented Brauer category* $\mathcal{OB}(\delta)$ as defined in [10]. As conjectured in [22], this categorification is expected to be strongly equivariantly equivalent to one arising from the arc algebra K_Λ from the previous paragraph.

In [58], Webster has also introduced categorifications of integrable lowest tensored highest weight representations associated to arbitrary Kac-Moody algebras. These arise as certain generalized cyclotomic quotients of the 2-Kac-Moody algebra $\mathcal{U}(\mathfrak{g})$. Yet more examples, which should be isomorphic to special cases of Webster's categories, arise from the cyclotomic oriented Brauer categories $\mathcal{OB}^{f,f'}$ defined in [10] and studied further in [17]. These are attached to a pair f, f' of monic polynomials of degree ℓ and produce $\mathfrak{sl}_\mathbb{Z}$-tensor product categorifications of the form $V(-\omega') \otimes V(\omega)$ where ω and ω' are level ℓ dominant weights defined from f and f', respectively.

Other sorts of categorical actions. We end by listing several recent works which hint at the existence of various undeveloped (or at least underdeveloped) parallel theories of categorical actions.

In [35], Kang, Kashiwara and Tsuchioka have introduced *quiver Hecke superalgebras*, and proved some isomorphism theorems relating them to the affine Sergeev superalgebras and affine Hecke-Clifford superalgebras which arose in [11, 53]. The quiver Hecke superalgebra for the trivial quiver with one (odd) vertex is closely related to the spin Hecke algebra of [57] and the odd nil-Hecke algebra of [28]. There is slowly emerging a parallel theory of super categorical actions based around these algebras. It seems reasonable to expect that there should be a version of Rouquier's canonical filtration in this setting, and results like the uniqueness of minimal categorifications and more generally of tensor product categorifications. An interesting example comes from the category $\mathcal{O}$ for the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$; we hope this new point of view will one day shed light on the Kazhdan-Lusztig conjecture for $\mathfrak{q}_n(\mathbb{C})$ formulated in [7].

In [3], some new canonical bases have been defined which are related to category $\mathcal{O}$ for the symplectic and orthogonal Lie algebras and the orthosymplectic Lie superalgebras. This points towards another twisted theory of categorification, in which the role of degenerate affine Hecke algebras is played by the generalized Wenzl (VW) algebra as suggested in [27]. In [26], Ehrig and Stroppel have also introduced some twisted version of the Khovanov arc algebra which should fit into this picture.

There is also a completely different sort of twisted quiver Hecke algebra related to affine Hecke algebras of types B and C. These were introduced by Varagnolo and Vasserot in [56], who used them to prove the Lascoux-Leclerc-Thibon-type conjecture formulated in [29].

Finally we mention very briefly another very rich example of a categorical action. This arises from the category $\mathcal{O}$ in the sense of [30] attached to the rational Cherednik algebras associated to the complex reflection groups $S_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n$. In [54], Varagnolo and Vasserot conjectured this category to be equivalent to a truncation of parabolic category $\mathcal{O}$ for the affine general linear Lie algebra. Independent proofs of this conjecture have recently been given by Losev [43] and Rouquier, Shan, Varagnolo and Vasserot [49]. Losev's proof makes essential use of the theory of categorical actions, which he extends to something he calls a *Schur categorification*.

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