

TENSOR PRODUCT CATEGORIFICATIONS AND THE SUPER KAZHDAN-LUSZTIG CONJECTURE

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ABSTRACT. We give a new proof of the “super Kazhdan-Lusztig conjecture” for the Lie superalgebra $\mathfrak{gl}_{n|m}(\mathbb{C})$ as formulated originally by the first author. We also prove for the first time that any integral block of category \mathcal{O} for $\mathfrak{gl}_{n|m}(\mathbb{C})$ (and also all of its parabolic analogs) possesses a graded version which is Koszul. Our approach depends crucially on an application of the uniqueness of tensor product categorifications established recently by the second two authors.

1. INTRODUCTION

In this paper we explain how the uniqueness of tensor product categorifications established by the second two authors in [LW] yields a quick proof of the Kazhdan-Lusztig conjecture for the general linear Lie superalgebra $\mathfrak{gl}_{n|m}(\mathbb{C})$. This conjecture was formulated originally by the first author in [B1] and has been proved already by a different method by Cheng, Lam and Wang in [CLW]. We then prove the existence of a Koszul graded lift of the analog of the Bernstein-Gelfand-Gelfand category \mathcal{O} for $\mathfrak{gl}_{n|m}(\mathbb{C})$, in the spirit of the classic work [BGS].

Roughly, the super Kazhdan-Lusztig conjecture asserts that combinatorics in integral blocks of category \mathcal{O} for $\mathfrak{gl}_{n|m}(\mathbb{C})$ is controlled by various canonical bases in the \mathfrak{sl}_∞ -module $V^{\otimes n} \otimes W^{\otimes m}$, where V is the natural \mathfrak{sl}_∞ -module and W is its dual. In fact, like in [CLW], we prove the version of the conjecture suggested in the introduction of [Kuj] which is adapted to an arbitrary Borel subalgebra of $\mathfrak{gl}_{n|m}(\mathbb{C})$; for this the tensor factors in the mixed tensor space $V^{\otimes n} \otimes W^{\otimes m}$ are shuffled into more general orders. Note also the paper [CMW] which establishes an equivalence of categories from an arbitrary non-integral block of category \mathcal{O} for $\mathfrak{gl}_{n|m}(\mathbb{C})$ to an integral block of a direct sum of other general linear Lie superalgebras of the same total rank.

The basic idea of our proof is as follows. For a finite interval $I \subset \mathbb{Z}$ let \mathfrak{sl}_I be the special linear Lie algebra consisting of (complex) trace zero matrices with rows and columns indexed by integers from the set $I_+ := I \cup (I+1)$. Let V_I be the natural \mathfrak{sl}_I -module of column vectors and $W_I := V_I^*$. We construct a subquotient \mathcal{O}_I of the super category \mathcal{O} which is an \mathfrak{sl}_I -categorification of the tensor product $V_I^{\otimes n} \otimes W_I^{\otimes m}$ in the sense of Chuang and Rouquier [CR], [R]. Then, observing that

$$V_I^{\otimes n} \otimes W_I^{\otimes m} \cong V_I^{\otimes n} \otimes (\bigwedge^{|I|} V_I)^{\otimes m},$$

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one can apply the uniqueness of tensor product categorifications from [LW] to deduce that \mathcal{O}_I is equivalent to another well-known categorification \mathcal{O}'_I of this tensor product arising from the parabolic category \mathcal{O} associated to the Lie algebra $\mathfrak{gl}_{n+m|I}(\mathbb{C})$ and its Levi subalgebra $\mathfrak{gl}_1(\mathbb{C})^{\oplus n} \oplus \mathfrak{gl}_{|I|}(\mathbb{C})^{\oplus m}$. The combinatorics of the latter category is understood by the ordinary Kazhdan-Lusztig conjecture proved in [BB],[BrKa]. Since the finite interval I can be chosen freely, this gives enough information to deduce the super Kazhdan-Lusztig conjecture.

By well-known results from [BGS] and [B], the category \mathcal{O}'_I has a graded version which is Koszul. Hence so does the equivalent category \mathcal{O}_I . We show further that it is possible to choose the Koszul gradings on each \mathcal{O}_I in a compatible way so that they lift to a well-defined Koszul grading on the full category \mathcal{O} for $\mathfrak{gl}_{n|m}(\mathbb{C})$. This takes substantially more work. In fact we prove it simultaneously for the category \mathcal{O} and for all of its parabolic analogs, so that a very special case recovers the Koszul grading on the subcategory of \mathcal{O} consisting of all finite dimensional representations that was constructed explicitly in [BS]. Our main result here can be paraphrased as follows.

Theorem A. *Any block of parabolic category \mathcal{O} for $\mathfrak{gl}_{n|m}(\mathbb{C})$ with integral central character has a graded lift which is a standard Koszul highest weight category, and whose graded decomposition numbers are given by parabolic Kazhdan-Lusztig polynomials, as predicted in [B1].*

In the main body of the article we adopt a more axiomatic approach in the spirit of [LW]. In Section 2, we write down the formal definition of an \mathfrak{sl}_∞ -*tensor product categorification* of a tensor product of exterior powers of V and W . This is a category with

- an \mathfrak{sl}_∞ -action in the sense discussed in Definition 2.6,
- a highest weight category structure as in Definition 2.7, and
- some compatibility between these structures explained in Definition 2.9,

such that the complexified Grothendieck group of the underlying category of Δ -filtered objects is isomorphic to the given tensor product of exterior powers of V and W . Then the majority of the article is taken up with proving the following fundamental result about such categorifications.

Theorem B. *There exists a unique \mathfrak{sl}_∞ -tensor product categorification associated to any tensor product of exterior powers of V and W . Moreover such a category has a unique graded lift compatible with all the above structures.*

The existence part of this theorem is proved in Section 3, simply by verifying that parabolic category \mathcal{O} for the general linear Lie superalgebra satisfies the axioms; in fact this is the only time Lie superalgebras enter into the picture. The uniqueness is proved in Section 4. It is a non-trivial extension of the uniqueness theorem for finite \mathfrak{sl}_I -tensor product categorifications established in [LW]. The proof for \mathfrak{sl}_∞ depends on the construction of an interesting new category of *stable modules* for a certain tower of quiver Hecke algebras. Finally in Section 5, we incorporate gradings into the picture, defining the notion of a $U_q\mathfrak{sl}_\infty$ -*tensor product categorification* of a tensor product of q -deformed exterior powers of V and W ; see Definitions 5.3, 5.5 and 5.8. We prove the existence and uniqueness of these by exploiting *graded* stable modules over our tower of quiver Hecke algebras.

Then we prove that any such category is standard Koszul and deduce the graded version of the Kazhdan-Lusztig conjecture.

Conventions. We fix an algebraically closed field \mathbb{K} of characteristic 0 throughout the article. All categories and functors will be assumed to be \mathbb{K} -linear without further notice. Let \mathcal{Vec} be the category of finite dimensional vector spaces. For a finite dimensional graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$, we write $\dim_q V$ for its *graded dimension* $\sum_{n \in \mathbb{Z}} (\dim V_n) q^n \in \mathbb{Z}[q, q^{-1}]$.

2. TENSOR PRODUCT CATEGORIFICATIONS

In this section we review the definition of tensor product categorification from [LW, Definition 3.2] in the special case of tensor products of exterior powers of the natural and dual natural representations of \mathfrak{sl}_I . We include the possibility that the interval $I \subseteq \mathbb{Z}$ is infinite, when these are not highest weight modules. Then we state our first main result asserting the existence and uniqueness of such tensor product categorifications, extending the case of finite intervals from [LW]. After that we make some preparations for the proof, which actually takes place in Sections 3 and 4.

2.1. Schurian categories. By a *Schurian category* we mean an abelian category \mathcal{C} such that all objects are of finite length, there are enough projectives and injectives, and the endomorphism algebras of the irreducible objects are one dimensional. For example the category $\text{mod-}A$ of finite dimensional right modules over a finite dimensional \mathbb{K} -algebra A is Schurian. Note throughout this text we will work in terms of projectives, but obviously \mathcal{C} is Schurian if and only if \mathcal{C}^{op} is Schurian, so that everything could be expressed equivalently in terms of injectives. We use the made-up word *prinjective* for an object that is both projective and injective.

Given a Schurian category \mathcal{C} , we let $\mathcal{C}^{\text{proj}}$ be the full subcategory consisting of all projective objects. Let $\text{Fun}_f(\mathcal{C}^{\text{proj}}, \mathcal{Vec}^{\text{op}})$ denote the category of all contravariant functors from $\mathcal{C}^{\text{proj}}$ to \mathcal{Vec} which are zero on all but finitely many isomorphism classes of indecomposable projectives. The Yoneda functor

$$\mathcal{C} \rightarrow \text{Fun}_f(\mathcal{C}^{\text{proj}}, \mathcal{Vec}^{\text{op}}), \quad M \mapsto \text{Hom}_{\mathcal{C}}(-, M) \quad (2.1)$$

is an equivalence of categories. Hence \mathcal{C} can be recovered (up to equivalence) from $\mathcal{C}^{\text{proj}}$.

This assertion can be formulated in more algebraic terms as follows. Let $\{L(\lambda) \mid \lambda \in \Lambda\}$ be a complete set of pairwise non-isomorphic irreducible objects in \mathcal{C} , and fix a choice of a projective cover $P(\lambda)$ of each $L(\lambda)$. Let

$$A := \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_{\mathcal{C}}(P(\lambda), P(\mu)) \quad (2.2)$$

viewed as an associative algebra with multiplication coming from composition in \mathcal{C} . Let $1_\lambda \in A$ be the identity endomorphism of $P(\lambda)$. If Λ is finite then A is a unital algebra with $1 = \sum_{\lambda \in \Lambda} 1_\lambda$, indeed, A is the endomorphism algebra of the minimal projective generator $\bigoplus_{\lambda \in \Lambda} P(\lambda)$. However in general A is only *locally unital*, meaning that it is equipped with the system $\{1_\lambda \mid \lambda \in \Lambda\}$ of mutually orthogonal idempotents such that $A = \bigoplus_{\lambda, \mu \in \Lambda} 1_\mu A 1_\lambda$. Let $\text{mod-}A$ denote the

category of all finite dimensional locally unital right A -modules, that is, finite dimensional right A -modules M such that $M = \bigoplus_{\lambda \in \Lambda} M1_\lambda$. Then our earlier assertion about the Yoneda equivalence amounts to the statement that the functor

$$\mathbb{H} : \mathcal{C} \rightarrow \text{mod-}A, \quad M \mapsto \bigoplus_{\lambda \in \Lambda} \text{Hom}_{\mathcal{C}}(P(\lambda), M) \quad (2.3)$$

is an equivalence of categories. Of course this functor sends $P(\lambda)$ to the (necessarily finite dimensional) right ideal $1_\lambda A$; these are the indecomposable projective modules in $\text{mod-}A$. The linear duals of the indecomposable injective modules are isomorphic to the left ideals $A1_\lambda$, so that the latter are finite dimensional too. Conversely given any locally unital \mathbb{K} -algebra A with distinguished idempotents $\{1_\lambda \mid \lambda \in \Lambda\}$ such that all of the ideals $1_\lambda A$ and $A1_\lambda$ are finite dimensional, the category $\text{mod-}A$ is Schurian.

Let $K_0(\mathcal{C})$ (resp. $G_0(\mathcal{C})$) be the split Grothendieck group of the additive category $\mathcal{C}^{\text{proj}}$ (resp. the Grothendieck group of the abelian category \mathcal{C}). Set

$$[\mathcal{C}] := \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C}), \quad [\mathcal{C}]^* := \mathbb{C} \otimes_{\mathbb{Z}} G_0(\mathcal{C}).$$

So $[\mathcal{C}]$ is the complex vector space on basis $\{[P(\lambda)] \mid \lambda \in \Lambda\}$, while $[\mathcal{C}]^*$ has basis $\{[L(\lambda)] \mid \lambda \in \Lambda\}$. These bases are dual with respect to the bilinear *Cartan pairing* $(-, -) : [\mathcal{C}] \times [\mathcal{C}]^* \rightarrow \mathbb{C}$ defined from $([P], [L]) := \dim \text{Hom}_{\mathcal{C}}(P, L)$.

2.2. Combinatorics. Let $I \subseteq \mathbb{Z}$ be a non-empty interval and set

$$I_+ := I \cup (I + 1).$$

Let \mathfrak{sl}_I be the Lie algebra of (complex) trace zero matrices with rows and columns indexed by I_+ , all but finitely many of whose entries are zero. It is generated by the matrix units $f_i := e_{i+1, i}$ and $e_i := e_{i, i+1}$ for all $i \in I$. The *weight lattice* of \mathfrak{sl}_I is $P_I := \bigoplus_{i \in I} \mathbb{Z}\varpi_i$ where ϖ_i is the *i th fundamental weight*. The *root lattice* is $Q_I := \bigoplus_{i \in I} \mathbb{Z}\alpha_i < P_I$ where α_i is the *i th simple root* defined from

$$\alpha_i := 2\varpi_i - \varpi_{i-1} - \varpi_{i+1},$$

interpreting ϖ_i as 0 if $i \notin I$. Let $P_I \times Q_I \rightarrow \mathbb{Z}$, $(\varpi, \alpha) \mapsto \varpi \cdot \alpha$ be the bilinear pairing defined from $\varpi_i \cdot \alpha_j := \delta_{i, j}$, so that $(\alpha_i \cdot \alpha_j)_{i, j \in I}$ is the Cartan matrix. Let P_I^+ (resp. Q_I^+) be the positive cone in P_I (resp. Q_I) generated by the fundamental weights (resp. the simple roots). The *dominance order* \geq on P_I is defined by $\beta \geq \gamma$ if $\beta - \gamma \in Q_I^+$. For any $i \in I_+$ we set

$$\varepsilon_i := \varpi_i - \varpi_{i-1},$$

again interpreting ϖ_i as 0 for $i \notin I$. The following lemma is well known.

Lemma 2.1. *For $\beta = \sum_{i \in I_+} b_i \varepsilon_i$ and $\gamma = \sum_{i \in I_+} c_i \varepsilon_i$ in P_I with $\sum_i b_i = \sum_i c_i$, we have that $\beta \geq \gamma$ if and only if $\sum_{i \leq h} b_i \geq \sum_{i \leq h} c_i$ for all $h \in I$.*

An \mathfrak{sl}_I -module M is *integrable* if it decomposes into weight spaces as $M = \bigoplus_{\varpi \in P_I} M_{\varpi}$, and moreover each of the Chevalley generators f_i and e_i act locally nilpotently. Basic examples are the natural \mathfrak{sl}_I -module V_I of column vectors with

standard basis $\{v_i \mid i \in I_+\}$ and its dual W_I with basis $\{w_i \mid i \in I_+\}$; the Chevalley generators act on these basis vectors by

$$\begin{aligned} f_i v_j &= \delta_{i,j} v_{i+1}, & e_i v_j &= \delta_{i+1,j} v_i, \\ f_i w_j &= \delta_{i+1,j} w_i, & e_i w_j &= \delta_{i,j} w_{i+1}. \end{aligned}$$

The vector v_i is of weight ε_i while w_i is of weight $-\varepsilon_i$.

More generally we have the exterior powers $\bigwedge^n V_I$ and $\bigwedge^n W_I$ for $n \geq 0$; henceforth we denote these instead by $\bigwedge^{n,0} V_I$ and $\bigwedge^{n,1} V_I$, respectively. For $c \in \{0, 1\}$ let $\Lambda_{I;n,c}$ denote the set of 01-tuples $\lambda = (\lambda_i)_{i \in I_+}$ such that

$$|\{i \in I_+ \mid \lambda_i \neq c\}| = n.$$

This set parametrizes the natural monomial basis $\{v_\lambda \mid \lambda \in \Lambda_{I;n,c}\}$ of $\bigwedge^{n,c} V_I$ defined from

$$v_\lambda := \begin{cases} v_{i_1} \wedge \cdots \wedge v_{i_n} & \text{if } c = 0, \\ w_{i_1} \wedge \cdots \wedge w_{i_n} & \text{if } c = 1, \end{cases}$$

where $i_1 < \cdots < i_n$ are chosen so that $\lambda_{i_j} \neq c$ for each j . The actions of the Chevalley generators are given explicitly by

$$f_i v_\lambda := \begin{cases} v_{t_i(\lambda)} & \text{if } \lambda_i = 1 \text{ and } \lambda_{i+1} = 0, \\ 0 & \text{otherwise,} \end{cases} \quad (2.4)$$

$$e_i v_\lambda := \begin{cases} v_{t_i(\lambda)} & \text{if } \lambda_i = 0 \text{ and } \lambda_{i+1} = 1, \\ 0 & \text{otherwise,} \end{cases} \quad (2.5)$$

where $t_i(\lambda)$ denotes the tuple obtained from λ by switching λ_i and λ_{i+1} . Let $|\lambda| \in P_I$ denote the weight of the vector v_λ . We have that

$$|\lambda| = \sum_{i \in I_+} \lambda_i \varepsilon_i,$$

taking care to interpret the sum on the right hand side when I is infinite and $c = 1$ using the convention that $\cdots + \varepsilon_{i-1} + \varepsilon_i = \varpi_i$ and $\varepsilon_{i+1} + \varepsilon_{i+2} + \cdots = -\varpi_i$.

We are also going to be interested in tensor products of the modules $\bigwedge^{n,c} V_I$. Suppose that we are given $\underline{n} = (n_1, \dots, n_l) \in \mathbb{N}^l$ and $\underline{c} = (c_1, \dots, c_l) \in \{0, 1\}^l$; we refer to the pair $(\underline{n}, \underline{c})$ as a *type of level l* . Let

$$\bigwedge^{\underline{n}, \underline{c}} V_I := \bigwedge^{n_1, c_1} V_I \otimes \cdots \otimes \bigwedge^{n_l, c_l} V_I. \quad (2.6)$$

This module has the obvious basis of monomials $v_\lambda := v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_l}$ indexed by elements $\lambda = (\lambda_1, \dots, \lambda_l)$ of the set

$$\Lambda_{I; \underline{n}, \underline{c}} := \Lambda_{I; n_1, c_1} \times \cdots \times \Lambda_{I; n_l, c_l}. \quad (2.7)$$

The vector v_λ is of weight

$$|\lambda| := |\lambda_1| + \cdots + |\lambda_l|.$$

It is often convenient to regard $\lambda \in \Lambda_{I; \underline{n}, \underline{c}}$ as a 01-matrix $\lambda = (\lambda_{ij})_{1 \leq i \leq l, j \in I_+}$ with i th row $\lambda_i = (\lambda_{ij})_{j \in I_+}$. (There several other indexing conventions possible; for example earlier papers of the first and third authors have used the convention that λ is represented by a column-strict tableau with l columns such that the i th column is filled with all $j \in I_+$ such that $\lambda_{ij} = 1$.)

Assume for a moment that I is finite and that $\Lambda_{I; \underline{n}, \underline{c}}$ is non-empty. Let $\kappa = \kappa_{I; \underline{n}, \underline{c}}$ be the 01-matrix in $\Lambda_{I; \underline{n}, \underline{c}}$ in which all the entries 1 are as far to the left as

possible within each row. Thus $|\kappa| \in P_I$ is the unique highest weight of $\bigwedge^{n,c} V_I$ with respect to the dominance ordering. For any $\lambda \in \Lambda_{I;\underline{n},\underline{c}}$ define its *defect* by

$$\text{def}(\lambda) := \frac{1}{2}(|\kappa| \cdot |\kappa| - |\lambda| \cdot |\lambda|) = |\kappa| \cdot \alpha - \frac{1}{2}\alpha \cdot \alpha, \quad (2.8)$$

where $\alpha := |\kappa| - |\lambda|$. In combinatorial terms, this is $\frac{1}{2} \sum_{j \in I_+} (k_j^2 - l_j^2)$ where k_j (resp. l_j) counts the number of entries equal to 1 in the j th column of κ (resp. λ). The following lemma extends this definition to include infinite intervals I .

Lemma 2.2. *Suppose that I is an infinite interval and $\lambda \in \Lambda_{I;\underline{n},\underline{c}}$. Let $J \subset I$ be a finite subinterval such that $|J_+| \geq 2 \max(\underline{n})$ and $\lambda_{i,j} = c_i$ for all $1 \leq i \leq l$ and $j \in I_+ \setminus J_+$. Let $\lambda_J \in \Lambda_{J;\underline{n},\underline{c}}$ be the submatrix $(\lambda_{i,j})_{1 \leq i \leq l, j \in J_+}$ of λ . Let $\kappa_J := \kappa_{J;\underline{n},\underline{c}}$. Then the natural number*

$$\text{def}(\lambda) := \frac{1}{2}(|\kappa_J| \cdot |\kappa_J| - |\lambda_J| \cdot |\lambda_J|)$$

is independent of the particular choice of J .

Proof. Define the *trivial column* to be the column vector $(c_i)_{1 \leq i \leq l}$. Let J and J' be two intervals satisfying the hypotheses of the lemma with $J \subset J'$. The conditions imply that κ_J (resp. λ_J) can be obtained from $\kappa_{J'}$ (resp. $\lambda_{J'}$) by removing $|J'| - |J|$ trivial columns. The lemma follows easily from this using the combinatorial formulation of the definition of defect. \square

2.3. Hecke algebras. To prepare for the definition of an \mathfrak{sl}_I -categorification, we recall the definition of certain associative unital \mathbb{K} -algebras, namely, the (degenerate) affine Hecke algebra AH_d , and the quiver Hecke algebra $QH_{I,d}$ associated to the linear quiver with vertex set I and an edge $i \rightarrow j$ if $i = j + 1$. The latter is also known as a Khovanov-Lauda-Rouquier algebra after [KL1] and [R],

Definition 2.3. The *affine Hecke algebra* AH_d is the vector space $\mathbb{K}[x_1, \dots, x_d] \otimes \mathbb{K}S_d$ with multiplication defined so that the polynomial algebra $\mathbb{K}[x_1, \dots, x_d]$ and the group algebra $\mathbb{K}S_d$ of the symmetric group S_d are subalgebras and also

$$(AH) \quad t_j x_k - x_{t_j(k)} t_j = \begin{cases} 1 & \text{if } k = j + 1, \\ -1 & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

Here t_j denotes the transposition $(j \ j+1) \in S_d$.

Definition 2.4. The *quiver Hecke algebra* $QH_{I,d}$ is defined by generators

$$\{1_{\mathbf{i}} \mid \mathbf{i} \in I^d\} \cup \{\xi_1, \dots, \xi_d\} \cup \{\tau_1, \dots, \tau_{d-1}\}$$

subject to relations:

- (QH1) the elements ξ_1, \dots, ξ_d commute with each other and all $\{1_{\mathbf{i}} \mid \mathbf{i} \in I^d\}$;
- (QH2) the elements $\{1_{\mathbf{i}} \mid \mathbf{i} \in I^d\}$ are mutually orthogonal idempotents whose sum is the identity;
- (QH3) $\tau_j 1_{\mathbf{i}} = 1_{t_j(\mathbf{i})} \tau_j$ where $t_j(\mathbf{i})$ is the tuple obtained from $\mathbf{i} = (i_1, \dots, i_d)$ by flipping its j th and $(j+1)$ th entries;
- (QH4) $(\tau_j \xi_k - \xi_{t_j(k)} \tau_j) 1_{\mathbf{i}} = \begin{cases} 1_{\mathbf{i}} & \text{if } k = j + 1 \text{ and } i_j = i_{j+1}, \\ -1_{\mathbf{i}} & \text{if } k = j \text{ and } i_j = i_{j+1}, \\ 0 & \text{otherwise;} \end{cases}$
- (QH5) $\tau_j \tau_k = \tau_k \tau_j$ if $|j - k| > 1$;

$$\begin{aligned}
\text{(QH6)} \quad \tau_j^2 1_{\mathbf{i}} &= \begin{cases} 0 & \text{if } i_j = i_{j+1}, \\ (\xi_j - \xi_{j+1}) 1_{\mathbf{i}} & \text{if } i_j = i_{j+1} - 1, \\ (\xi_{j+1} - \xi_j) 1_{\mathbf{i}} & \text{if } i_j = i_{j+1} + 1, \\ 1_{\mathbf{i}} & \text{otherwise;} \end{cases} \\
\text{(QH7)} \quad (\tau_{j+1} \tau_j \tau_{j+1} - \tau_j \tau_{j+1} \tau_j) 1_{\mathbf{i}} &= \begin{cases} 1_{\mathbf{i}} & \text{if } i_j = i_{j+1} - 1 = i_{j+2}, \\ -1_{\mathbf{i}} & \text{if } i_j = i_{j+1} + 1 = i_{j+2}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

An important feature of $QH_{I,d}$ is that it possesses a non-trivial \mathbb{Z} -grading. This is defined by declaring that each idempotent $1_{\mathbf{i}}$ is in degree 0, each ξ_j in degree 2, and finally $\tau_k 1_{\mathbf{i}}$ is in degree $-\alpha_{i_k} \cdot \alpha_{i_{k+1}}$.

The algebras AH_d and $QH_{I,d}$ are closely related as explained in [R, Proposition 3.15]. This result can also be formulated as an isomorphism between certain *cyclotomic quotients* of AH_d and $QH_{I,d}$ as in [BK2]. Let $\varpi \in P_I^+$ be a dominant weight. Define AH_d^ϖ (resp. $QH_{I,d}^\varpi$) to be the quotient of AH_d (resp. $QH_{I,d}$) by the two-sided ideal generated by the polynomial $\prod_{i \in I} (x_i - i)^{\varpi \cdot \alpha_i}$ (resp. by the elements $\{\xi_1^{\varpi \cdot \alpha_{i_1}} 1_{\mathbf{i}} \mid \mathbf{i} \in I^d\}$). These are finite dimensional algebras. The image of the polynomial algebra $\mathbb{K}[x_1, \dots, x_d]$ in AH_d^ϖ is a finite dimensional commutative algebra, hence it contains mutually orthogonal idempotents $\{1_{\mathbf{i}} \mid \mathbf{i} \in \mathbb{K}^d\}$ such that $1_{\mathbf{i}}$ projects any module M onto its \mathbf{i} -th *word space*

$$M_{\mathbf{i}} := \{v \in M \mid (x_j - i_j)^N v = 0 \text{ for each } j = 1, \dots, d \text{ and } N \gg 0\}.$$

Then let

$$AH_{I,d}^\varpi := \bigoplus_{\mathbf{i}, \mathbf{j} \in I^d} 1_{\mathbf{i}} AH_d^\varpi 1_{\mathbf{j}} = AH_d^\varpi / \langle 1_{\mathbf{i}} \mid \mathbf{i} \notin I^d \rangle,$$

which is a sum of blocks of the algebra AH_d^ϖ . The following theorem gives an explicit choice of isomorphism between $QH_{I,d}^\varpi$ and $AH_{I,d}^\varpi$; any other reasonable choice of isomorphism such as the one from [R, Proposition 3.15] could be used instead throughout this article.

Theorem 2.5 ([BK2], [R]). *For $\varpi \in P_I^+$ there is an isomorphism $QH_{I,d}^\varpi \xrightarrow{\sim} AH_{I,d}^\varpi$ defined on generators by*

$$1_{\mathbf{i}} \mapsto 1_{\mathbf{i}}; \tag{2.9}$$

$$\xi_j 1_{\mathbf{i}} \mapsto (x_j - i_j) 1_{\mathbf{i}}; \tag{2.10}$$

$$\tau_j 1_{\mathbf{i}} \mapsto \begin{cases} (1 + t_j)(1 - x_j + x_{j+1})^{-1} 1_{\mathbf{i}} & \text{if } i_j = i_{j+1}, \\ (1 + t_j x_j - t_j x_{j+1}) 1_{\mathbf{i}} & \text{if } i_j = i_{j+1} + 1, \\ (1 + t_j x_j - t_j x_{j+1})(1 - x_j + x_{j+1})^{-1} 1_{\mathbf{i}} & \text{otherwise.} \end{cases} \tag{2.11}$$

(In fact, this isomorphism can be extended to the completions $\widehat{QH}_{I,d}$ and $\widehat{AH}_{I,d}$ with respect to these systems of quotients as discussed in [W4].)

Proof. This follows by [BK2, Main Theorem]. To get exactly this isomorphism one needs to choose the power series $q_j(\mathbf{i})$ of [BK2, (3.27)–(3.29)] so that $q_j(\mathbf{i}) = p_j(\mathbf{i})$ if $i_j = i_{j+1} + 1$ and $q_j(\mathbf{i}) = 1 - p_j(\mathbf{i})$ if $i_j \notin \{i_{j+1}, i_{j+1} + 1\}$. Note also that the opposite orientation of the quiver was used in [BK2] so that the elements $\psi_j e(\mathbf{i})$ in [BK2] are our $\tau_j 1_{\mathbf{i}}$ if $i_j \in \{i_{j+1}, i_{j+1} + 1\}$ and our $-\tau_j 1_{\mathbf{i}}$ otherwise; the elements $y_j e(\mathbf{i})$ in [BK2] are our elements $\xi_j 1_{\mathbf{i}}$. \square

Henceforth we will simply *identify* $QH_{I,d}^\varpi$ and $AH_{I,d}^\varpi$ via the isomorphism from the theorem.

2.4. Categorification. Following their work [CR], Chuang and Rouquier introduced the notion of an \mathfrak{sl}_I -categorification, also known as a categorical \mathfrak{sl}_I -action. The following is essentially [R, Definition 5.29] (taking $q = 1$ and switching the roles of E and F).

Definition 2.6. An \mathfrak{sl}_I -categorification is a Schurian category \mathcal{C} together with an endofunctor F , a right adjoint E to F (with a specified adjunction), and natural transformations $x \in \text{End}(F)$ and $t \in \text{End}(F^2)$ satisfying the axioms (SL1)–(SL4) formulated below. For the first axiom, we let F_i be the subfunctor of F defined by the generalized i -eigenspace of x , i.e. $F_i M = \sum_{k \geq 0} \ker(x_M - i)^k$ for each $M \in \mathcal{C}$.

(SL1) We have that $F = \bigoplus_{i \in I} F_i$, i.e. $FM = \bigoplus_{i \in I} F_i M$ for each $M \in \mathcal{C}$.

(SL2) For $d \geq 0$ the endomorphisms $x_j := F^{d-j} x F^{j-1}$ and $t_k := F^{d-k-1} t F^{k-1}$ of F^d satisfy the relations of the degenerate affine Hecke algebra AH_d .

(SL3) The functor F is isomorphic to a right adjoint of E .

For the final axiom, we let $c : \text{id} \rightarrow EF$ and $d : FE \rightarrow \text{id}$ be the unit and counit of the given adjunction, respectively. The endomorphisms x and t of F and F^2 induce endomorphisms x and t of E and E^2 too:

$$x : E \xrightarrow{cE} EFE \xrightarrow{ExE} EFE \xrightarrow{Ed} E, \quad (2.12)$$

$$t : E^2 \xrightarrow{cE^2} EFE^2 \xrightarrow{EcFE^2} E^2 F^2 E^2 \xrightarrow{E^2 t E^2} E^2 F^2 E^2 \xrightarrow{E^2 F d E} E^2 F E \xrightarrow{E^2 d} E^2. \quad (2.13)$$

Let E_i be the subfunctor of E defined by the generalized i -eigenspace of $x \in \text{End}(E)$. The axioms so far imply that $E = \bigoplus_{i \in I} E_i$ and moreover F_i and E_i are biadjoint, so they are both exact and send projectives to projectives.

(SL4) The endomorphisms f_i and e_i of $[\mathcal{C}] = \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C})$ induced by F_i and E_i , respectively, make $[\mathcal{C}]$ into an integrable representation of \mathfrak{sl}_I . Moreover the classes of the indecomposable projective objects are weight vectors.

The axiom (SL4) has the following equivalent dual formulation.

(SL4*) The endomorphisms f_i and e_i of $[\mathcal{C}]^* = \mathbb{C} \otimes_{\mathbb{Z}} G_0(\mathcal{C})$ induced by F_i and E_i , respectively, make $[\mathcal{C}]^*$ into an integrable representation of \mathfrak{sl}_I . Moreover the classes of the irreducible objects are weight vectors.

The axiom (SL1) implies that F^d decomposes as $\bigoplus_{i \in I^d} F_i$ where

$$F_i := F_{i_d} \circ \cdots \circ F_{i_1}.$$

This further shows that the action of AH_d factors through the completion $\widehat{AH}_{I,d}$ of the inverse system of cyclotomic quotients $\{AH_{I,d}^\varpi \mid \varpi \in P_I^+\}$. Letting $1_i \in \text{End}(F^d)$ be the projection onto F_i we can then use the isomorphism of completions given by (2.9)–(2.11) to convert the homomorphism $AH_d \rightarrow \widehat{AH}_{I,d} \rightarrow \text{End}(F^d)$ into a homomorphism $QH_{I,d} \rightarrow \widehat{QH}_{I,d} \rightarrow \text{End}(F^d)$. In this way we see that the definition of an \mathfrak{sl}_I -categorification can be formulated equivalently using the quiver Hecke algebra $QH_{I,d}$ in place of the degenerate affine Hecke algebra AH_d . In this incarnation, \mathcal{C} should be equipped with adjoint pairs (F_i, E_i)

of endofunctors for all $i \in I$ (with specified adjunctions), together with natural transformations $\xi \in \text{End}(F)$ and $\tau \in \text{End}(F^2)$ where $F := \bigoplus_{i \in I} F_i$, satisfying the axioms (SL1')–(SL4').

- (SL1') The endomorphism ξ is locally nilpotent, i.e. $FM = \sum_{k \geq 0} \ker \xi_M^k$ for each $M \in \mathcal{C}$.
- (SL2') For $d \geq 0$ the endomorphisms $\xi_j := F^{d-j}\xi F^{j-1}$ and $\tau_k := F^{d-k-1}\tau F^{k-1}$ of F^d plus the projections 1_i of F^d onto its summands F_i for each $i \in I^d$ satisfy the relations of the quiver Hecke algebra $QH_{I,d}$.
- (SL3') Each functor F_i is isomorphic to a right adjoint of E_i .
- (SL4') Same as (SL4).

In fact this is just the first of several alternate definitions of \mathfrak{sl}_I -categorification in the literature. Notably in [R, Theorem 5.30] Rouquier proves that the data of an \mathfrak{sl}_I -categorification as above is equivalent to the data of an integrable 2-representation of the 2-Kac-Moody algebra associated to \mathfrak{sl}_I in the sense of [R, Definition 5.1]; see also [KL2] and [CL] for closely related notions.

Given two \mathfrak{sl}_I -categorifications \mathcal{C} and \mathcal{C}' , and denoting F, E, x, t for \mathcal{C}' instead by F', E', x', t' for clarity, a functor $\mathbb{G} : \mathcal{C} \rightarrow \mathcal{C}'$ is *strongly equivariant* if there exists an isomorphism of functors $\zeta^- : \mathbb{G} \circ F \xrightarrow{\sim} F' \circ \mathbb{G}$ with

$$x' \mathbb{G} \circ \zeta^- = \zeta^- \circ \mathbb{G}x \quad \text{in } \text{Hom}(\mathbb{G} \circ F, F' \circ \mathbb{G}), \quad (2.14)$$

$$t' \mathbb{G} \circ F' \zeta^- \circ \zeta^- F = F' \zeta^- \circ \zeta^- F \circ \mathbb{G}t \quad \text{in } \text{Hom}(\mathbb{G} \circ F^2, F'^2 \circ \mathbb{G}). \quad (2.15)$$

Equivalently, \mathbb{G} is strongly equivariant if there exists $\zeta^+ : \mathbb{G} \circ E \xrightarrow{\sim} E' \circ \mathbb{G}$ with

$$x' \mathbb{G} \circ \zeta^+ = \zeta^+ \circ \mathbb{G}x \quad \text{in } \text{Hom}(\mathbb{G} \circ E, E' \circ \mathbb{G}), \quad (2.16)$$

$$t' \mathbb{G} \circ E' \zeta^+ \circ \zeta^+ E = E' \zeta^+ \circ \zeta^+ E \circ \mathbb{G}t \quad \text{in } \text{Hom}(\mathbb{G} \circ E^2, E'^2 \circ \mathbb{G}). \quad (2.17)$$

See [CR, §5.1.2] where the recipe to pass between ζ^- and ζ^+ is explained; it uses the given adjunction but otherwise is purely formal and does not require the axioms (SL1)–(SL4) to hold. Of course this definition can also be formulated in the quiver Hecke algebra setup; one just has to replace x and t with ξ and τ in (2.14)–(2.15). Alternatively one can transport ξ and τ to $\text{End}(E)$ and $\text{End}(E^2)$ like in (2.12)–(2.13), and then work with the analogs of (2.16)–(2.17).

A *strongly equivariant equivalence* is a strongly equivariant functor $\mathbb{G} : \mathcal{C} \rightarrow \mathcal{C}'$ that is also an equivalence of categories. It is then automatic that $[\mathcal{C}] \cong [\mathcal{C}']$ as \mathfrak{sl}_I -modules.

2.5. Recollections about highest weight categories. We must also make a few reminders about (artinian) highest weight categories in the sense of [CPS]; see also [D, Appendix] which is a good source for proofs of all the results stated in this subsection (although it only treats finite weight posets).

Definition 2.7. A *highest weight category* is a Schurian category \mathcal{C} together with an interval-finite poset (Λ, \leq) indexing a complete set of pairwise non-isomorphic irreducible objects $\{L(\lambda) \mid \lambda \in \Lambda\}$ of \mathcal{C} , such that the following axiom holds.

- (HW) Let $P(\lambda)$ be a projective cover of $L(\lambda)$ in \mathcal{C} . Define the *standard object* $\Delta(\lambda)$ to be the largest quotient of $P(\lambda)$ such that $[\Delta(\lambda) : L(\mu)] = \delta_{\lambda,\mu}$ for $\mu \not\leq \lambda$. Then $P(\lambda)$ has a finite filtration with top section isomorphic to $\Delta(\lambda)$ and other sections of the form $\Delta(\mu)$ for $\mu > \lambda$.

It is well known that this is equivalent to the axiom (HW*) below; in other words \mathcal{C} is highest weight if and only if \mathcal{C}^{op} is highest weight.

(HW*) Let $I(\lambda)$ be an injective hull of $L(\lambda)$ in \mathcal{C} . Define the *costandard object* $\nabla(\lambda)$ to be the largest subobject of $I(\lambda)$ such that $[\nabla(\lambda) : L(\mu)] = \delta_{\lambda,\mu}$ for $\mu \not\leq \lambda$. Then $I(\lambda)$ has a finite filtration with bottom section isomorphic to $\nabla(\lambda)$ and other sections of the form $\nabla(\mu)$ for $\mu > \lambda$.

If \mathcal{C} is a highest weight category, we write \mathcal{C}^Δ and \mathcal{C}^∇ for the exact subcategories consisting of objects with a Δ -flag and objects with a ∇ -flag, respectively. Their complexified Grothendieck groups will be denoted $[\mathcal{C}^\Delta]$ and $[\mathcal{C}^\nabla]$; they have distinguished bases $\{[\Delta(\lambda)] \mid \lambda \in \Lambda\}$ and $\{[\nabla(\lambda)] \mid \lambda \in \Lambda\}$, respectively. The natural inclusion functors induce linear maps $[\mathcal{C}] \hookrightarrow [\mathcal{C}^\Delta] \hookrightarrow [\mathcal{C}]^* \hookleftarrow [\mathcal{C}^\nabla]$. When Λ is finite all these maps are actually isomorphisms so that all the Grothendieck groups are usually identified.

There are a couple of well-known constructions which will be essential later on. Suppose that we are given a decomposition $\Lambda = \Lambda_\vee \sqcup \Lambda_\wedge$ such that Λ_\vee is an ideal (lower set); equivalently Λ_\wedge is a coideal (upper set). Let \mathcal{C}_\vee be the Serre subcategory of \mathcal{C} generated by $\{L(\lambda) \mid \lambda \in \Lambda_\vee\}$. We write $\iota : \mathcal{C}_\vee \hookrightarrow \mathcal{C}$ for the natural inclusion, and $\iota^!$ (resp. ι^*) for the left (resp. right) adjoint to ι which sends an object M to its largest quotient (resp. subobject) belonging to \mathcal{C}_\vee . The category \mathcal{C}_\vee is itself a highest weight category with weight poset Λ_\vee . Its irreducible, standard and costandard objects are the same as the ones in \mathcal{C} indexed by the set Λ_\vee . For $\lambda \in \Lambda_\vee$ the projective cover (resp. injective hull) of $L(\lambda)$ in \mathcal{C}_\vee is $\iota^!P(\lambda)$ (resp. $\iota^*I(\lambda)$), which will in general be a proper quotient of $P(\lambda)$ (resp. a proper subobject of $I(\lambda)$). For any $M, N \in \mathcal{C}_\vee$ we have that

$$\text{Ext}_{\mathcal{C}}^n(M, N) \cong \text{Ext}_{\mathcal{C}_\vee}^n(M, N) \quad (2.18)$$

for all $n \geq 0$. This is proved by a Grothendieck spectral sequence argument exactly like in [D, A3.2–A3.3]. A key step is to check that the higher right derived functors $R^n\iota^*$ vanish on objects from \mathcal{C}^∇ ; dually the higher left derived functors $L_n\iota^!$ vanish on objects from \mathcal{C}^Δ .

As well as the subcategory \mathcal{C}_\vee , we can consider the Serre quotient category $\mathcal{C}_\wedge := \mathcal{C}/\mathcal{C}_\vee$; we stress that according to the definition of quotient category the objects of \mathcal{C}_\wedge are *the same* as the objects of \mathcal{C} ; morphisms $M \rightarrow N$ in \mathcal{C}_\wedge are obtained by taking a direct limit of the morphisms $M' \rightarrow N/N'$ in \mathcal{C} over all subobjects M' of M and N' of N such that M/M' and N' belong to \mathcal{C}_\vee . Let $\pi : \mathcal{C} \rightarrow \mathcal{C}_\wedge$ be the quotient functor, and fix a choice $\pi^!$ (resp. π^*) of a left (resp. right) adjoint to π . Note that the unit (resp. counit) of adjunction gives a canonical isomorphism $\text{id} \xrightarrow{\sim} \pi \circ \pi^!$ (resp. $\pi \circ \pi^* \xrightarrow{\sim} \text{id}$). The irreducible, standard, costandard, indecomposable projective and indecomposable injective objects in \mathcal{C}_\wedge are the same as the ones in \mathcal{C} indexed by weights from Λ_\wedge . Also for $\lambda \in \Lambda_\wedge$ we have that $\pi^!P(\lambda) \cong P(\lambda)$, $\pi^*I(\lambda) \cong I(\lambda)$, $\pi^!\Delta(\lambda) \cong \Delta(\lambda)$ and $\pi^*\nabla(\lambda) \cong \nabla(\lambda)$ in \mathcal{C} ; the first two isomorphisms here follow from properties of adjunctions; see Lemma 2.8 below for justification of the latter two. Finally for $M, N \in \mathcal{C}$ such that either M has a Δ -flag with sections of the form $\Delta(\lambda)$ indexed by weights $\lambda \in \mathcal{C}_\wedge$, or N has a ∇ -flag with sections $\nabla(\lambda)$ for $\lambda \in \mathcal{C}_\wedge$, we have that

$$\text{Ext}_{\mathcal{C}}^n(M, N) \cong \text{Ext}_{\mathcal{C}_\wedge}^n(M, N) \quad (2.19)$$

for all $n \geq 0$. This is [D, A3.13].

Lemma 2.8. *Let $\pi : \mathcal{C} \rightarrow \mathcal{C}_\wedge$ be the quotient associated to a coideal $\Lambda_\wedge \subseteq \Lambda$. For $\lambda \in \Lambda_\wedge$ there are canonical isomorphisms $\pi^! \Delta(\lambda) \cong \Delta(\lambda)$ and $\nabla(\lambda) \cong \pi^* \nabla(\lambda)$ in \mathcal{C} induced by the counit and unit of the fixed adjunctions.*

Proof. Let $\mathcal{C}_{\leq \lambda}$ (resp. $\mathcal{C}_{< \lambda}$) be the highest weight subcategory of \mathcal{C} associated to the ideal $\{\mu \in \Lambda \mid \mu \leq \lambda\}$ (resp. $\{\mu \in \Lambda \mid \mu < \lambda\}$). Let $\mathcal{C}_\lambda := \mathcal{C}_{\leq \lambda} / \mathcal{C}_{< \lambda}$. This category is a copy of $\mathcal{V}ec$ with unique (up to isomorphism) irreducible object $L(\lambda)$. Let $\pi_\lambda : \mathcal{C}_{\leq \lambda} \rightarrow \mathcal{C}_\lambda$ be the quotient functor with left adjoint $\pi_\lambda^!$. The projective cover of $L(\lambda)$ in $\mathcal{C}_{\leq \lambda}$ is $\Delta(\lambda)$, hence by properties of adjunctions we have that $\Delta(\lambda) \cong \pi_\lambda^! L(\lambda)$ in \mathcal{C} .

Similarly, working with \mathcal{C}_\wedge in place of \mathcal{C} , we define subcategories $\mathcal{C}_{\wedge, \leq \lambda}$ and $\mathcal{C}_{\wedge, < \lambda}$. The quotient $\mathcal{C}_{\wedge, \leq \lambda} / \mathcal{C}_{\wedge, < \lambda}$ is another copy of $\mathcal{V}ec$, hence is equivalent to \mathcal{C}_λ . This means that there is another quotient functor $\pi_{\wedge, \lambda} : \mathcal{C}_{\wedge, \leq \lambda} \rightarrow \mathcal{C}_\lambda$ such that $\pi_\lambda = \pi_{\wedge, \lambda} \circ \pi$, hence $\pi_\lambda^! \cong \pi^! \circ \pi_{\wedge, \lambda}^!$. Again we have that $\Delta(\lambda) \cong \pi_{\wedge, \lambda}^! L(\lambda)$ in \mathcal{C}_\wedge . Hence we get isomorphisms in \mathcal{C} :

$$\pi^! \Delta(\lambda) \cong \pi^! (\pi_{\wedge, \lambda}^! L(\lambda)) \cong \pi_{\wedge, \lambda}^! L(\lambda) \cong \Delta(\lambda).$$

It remains to observe that the counit $\pi^! \Delta(\lambda) = \pi^! (\pi \Delta(\lambda)) \rightarrow \Delta(\lambda)$ is an epimorphism as $\Delta(\lambda)$ has irreducible head $L(\lambda)$ and $\lambda \in \Lambda_\wedge$; hence this gives a canonical choice for the isomorphism.

The argument for ∇ is similar. \square

2.6. Existence and uniqueness of tensor product categorifications. Suppose we are given a type $(\underline{n}, \underline{c})$ of level l . Recall the \mathfrak{sl}_l -module $\bigwedge^{\underline{n}, \underline{c}} V_l$ from (2.6).

Definition 2.9. An \mathfrak{sl}_l -*tensor product categorification* of type $(\underline{n}, \underline{c})$ means a highest weight category \mathcal{C} together with an endofunctor F of \mathcal{C} , a right adjoint E to F (with specified adjunction), and natural transformations $x \in \text{End}(F)$ and $t \in \text{End}(F^2)$ satisfying axioms (SL1)–(SL3) and (TP1)–(TP3).

- (TP1) The weight poset Λ is the set $\Lambda_{I, \underline{n}, \underline{c}}$ from (2.7) partially ordered by $\lambda \leq \mu$ if and only if $|\lambda| = |\mu|$ and $|\lambda_1| + \cdots + |\lambda_k| \geq |\mu_1| + \cdots + |\mu_k|$ for all k .
- (TP2) The exact functors F_i and E_i send objects with Δ -flags to objects with Δ -flags.
- (TP3) The linear isomorphism $[\mathcal{C}^\Delta] \xrightarrow{\sim} \bigwedge^{\underline{n}, \underline{c}} V_l, [\Delta(\lambda)] \mapsto v_\lambda$ intertwines the endomorphisms f_i and e_i of $[\mathcal{C}^\Delta]$ induced by F_i and E_i with the endomorphisms of $\bigwedge^{\underline{n}, \underline{c}} V_l$ arising from the actions of the Chevalley generators f_i and e_i of \mathfrak{sl}_l .

Since $[\mathcal{C}]$ embeds into $[\mathcal{C}^\Delta] \cong \bigwedge^{\underline{n}, \underline{c}} V_l$, we deduce immediately from the axioms that $[\mathcal{C}]$ is itself an integrable \mathfrak{sl}_l -module, i.e. the axiom (SL4) holds automatically. Thus tensor product categorifications are categorifications in the sense of Definition 2.6 too.

Remark 2.10. This definition is a slightly modified version of [LW, Definition 3.2], where a general notion of tensor product categorification for arbitrary Kac-Moody algebras was introduced. The definition in [LW] is expressed in terms of quiver Hecke algebras rather than affine Hecke algebras; but of course the

above definition can be formulated equivalently with the axioms (SL1')–(SL3') replacing (SL1)–(SL3); so this is a superficial difference. More significantly in our formulation of the axioms (TP2)–(TP3) we have incorporated the explicit monomial basis $\{v_\lambda \mid \lambda \in \Lambda\}$ which is only available in our special minuscule situation. The analogous axioms (TPC2)–(TPC3) in [LW] are couched in terms of some commuting categorical \mathfrak{sl}_I -actions on the associated graded category $\text{gr } \mathcal{C} := \bigoplus_{\lambda \in \Lambda} \mathcal{C}_\lambda$ (where \mathcal{C}_λ is as in the proof of Lemma 2.8). The functors ${}_i F_j$ defining these actions can be recovered by taking a sum of equivalences $\mathcal{C}_\lambda \rightarrow \mathcal{C}_{t_{ij}(\lambda)}$ for all $\lambda \in \Lambda$ such that $\lambda_{ij} = 1$ and $\lambda_{i(j+1)} = 0$, where $t_{ij}(\lambda)$ is obtained from λ by interchanging λ_{ij} and $\lambda_{i(j+1)}$. Such functors exist since for a highest weight category each \mathcal{C}_λ is equivalent to Vec .

Any \mathfrak{sl}_I -tensor product categorification decomposes as

$$\mathcal{C} = \bigoplus_{\varpi \in P_I} \mathcal{C}_\varpi \quad (2.20)$$

where \mathcal{C}_ϖ is the Serre subcategory of \mathcal{C} generated by the irreducible objects $\{L(\lambda) \mid \lambda \in \Lambda, |\lambda| = \varpi\}$. In particular two irreducible objects $L(\lambda)$ and $L(\mu)$ belong to the same block of \mathcal{C} only if $|\lambda| = |\mu|$; see Theorem 2.20 for the converse.

Given another type $(\underline{n}', \underline{c}')$ of the same level, we say that $(\underline{n}, \underline{c})$ and $(\underline{n}', \underline{c}')$ are *equivalent* if one of the following holds for each i : either $c_i = c'_i$ and $n_i = n'_i$; or I is finite, $c_i \neq c'_i$ and $n_i = |I_+| - n'_i$. Observe in that case that the posets of 01-matrices $\Lambda_{I; \underline{n}, \underline{c}}$ and $\Lambda_{I; \underline{n}', \underline{c}'}$ are simply *equal*, and there is an \mathfrak{sl}_I -module isomorphism $\bigwedge^{\underline{n}, \underline{c}} V_I \xrightarrow{\sim} \bigwedge^{\underline{n}', \underline{c}'} V_I, v_\lambda \mapsto v_\lambda$.

Theorem 2.11. *For any interval $I \subseteq \mathbb{Z}$ and type $(\underline{n}, \underline{c})$, there exists an \mathfrak{sl}_I -tensor product categorification \mathcal{C} of type $(\underline{n}, \underline{c})$. Moreover \mathcal{C} is unique in the sense that if \mathcal{C}' is another tensor product categorification of an equivalent type $(\underline{n}', \underline{c}')$ then there is a strongly equivariant equivalence $\mathbb{G} : \mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ with $\mathbb{G}L(\lambda) \cong L'(\lambda)$ for each weight λ .*

In the case that I is finite, Theorem 2.11 is a special case of the main result of [LW]; see §2.7 for some further discussion of that. For infinite intervals, Theorem 2.11 is new and will be proved later in the article. Specifically we will establish existence for $I = \mathbb{Z}$ in §3.2 then existence for the other infinite but bounded above or below intervals follows by the truncation argument explained in §2.8. The uniqueness will be established in §4.4.

Corollary 2.12. *Any \mathfrak{sl}_I -tensor product categorification \mathcal{C} admits a duality \circledast such that $F_i \circ \circledast \cong \circledast \circ F_i, E_i \circ \circledast \cong \circledast \circ E_i$ and $L(\lambda) \cong L(\lambda)^{\circledast}$ for each weight λ . Similarly its category of projectives has a duality $\#$ such that $F_i \circ \# \cong \# \circ F_i, E_i \circ \# \cong \# \circ E_i$ and $P(\lambda) \cong P(\lambda)^\#$ for each λ .*

Proof. Using the homological criteria for Δ - and ∇ -flags, one checks that the axioms (TP2)–(TP3) are equivalent to the axioms (TP2*)–(TP3*) obtained from them by replacing all occurrences of Δ with ∇ . In other words \mathcal{C} is a tensor product categorification if and only if \mathcal{C}^{op} is one; when I is finite this assertion is [LW, Proposition 3.9]. Now apply the uniqueness from Theorem 2.11 with $\mathcal{C}' := \mathcal{C}^{\text{op}}$ to get \circledast .

To obtain the duality $\#$ on projectives, one can use (2.1) to reduce to the problem of defining a duality $\#$ on the subcategory of $\text{Fun}_f(\mathcal{C}^{\text{proj}}, \mathcal{V}ec^{\text{op}})$ consisting of all exact functors; there one sets $\text{Hom}_{\mathcal{C}}(P, -)^{\#} := * \circ \text{Hom}_{\mathcal{C}}(P, -) \circ \otimes$ (where the final $*$ is the duality on $\mathcal{V}ec$). Transporting through the Yoneda equivalence this yields a duality $\#$ on $\mathcal{C}^{\text{proj}}$ such that

$$\text{Hom}_{\mathcal{C}}(P^{\#}, M) \cong \text{Hom}_{\mathcal{C}}(P, M^{\otimes})^* \quad (2.21)$$

for all $M \in \mathcal{C}$. It is clear from (2.21) that $P(\lambda)^{\#} \cong P(\lambda)$, while the fact that $\#$ commutes with F_i and E_i follows by adjointness as \otimes commutes with E_i and F_i . (Alternatively this definition can be understood via (2.3) in terms of the algebra A : it corresponds to the composition $\otimes \circ \mathcal{N} : \text{proj-}A \rightarrow \text{proj-}A$ where \mathcal{N} is the Nakayama functor $\text{Hom}_A(-, A)^* : \text{proj-}A \rightarrow \text{inj-}A$.) \square

2.7. Review of the proof of Theorem 2.11 for finite intervals. In this subsection we assume that I is finite and recall for future reference some of the key ideas behind the proof of Theorem 2.11 from [LW]. Suppose we are given a type $(\underline{n}, \underline{c})$. To avoid trivialities we assume that $n_i \leq |I_+|$ for each i . There are two general approaches to the construction of the tensor product categorification \mathcal{C} in Theorem 2.11. First it can be realized in terms of certain blocks of the parabolic category \mathcal{O} associated to the general linear Lie algebra; see [LW, Definition 3.13]. Alternatively, \mathcal{C} can be constructed using the tensor product algebras of [W1]; see [LW, Theorem 3.12].

Turning our attention to uniqueness, let \mathcal{C} be some given \mathfrak{sl}_I -tensor product categorification with weight poset $\Lambda = \Lambda_{I, \underline{n}, \underline{c}}$. Let A be the algebra

$$A := \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_{\mathcal{C}}(P(\lambda), P(\mu)), \quad (2.22)$$

from (2.2), and $\mathbb{H} : \mathcal{C} \rightarrow \text{mod-}A$ be the canonical equivalence of categories from (2.3). There is a standard way to transport the categorical \mathfrak{sl}_I -action from \mathcal{C} to $\text{mod-}A$ in such a way that $\mathbb{H} : \mathcal{C} \rightarrow \text{mod-}A$ becomes a strongly equivariant equivalence. The appropriate functor $F : \text{mod-}A \rightarrow \text{mod-}A$ is the functor defined by tensoring over A with the (A, A) -bimodule

$$B := \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_{\mathcal{C}}(P(\lambda), FP(\mu)). \quad (2.23)$$

The natural transformations $x \in \text{End}(F)$ and $t \in \text{End}(F^2)$ come from bimodule endomorphisms $x : B \rightarrow B$ and $t : B \otimes_A B \rightarrow B \otimes_A B$ defined as follows: let $x : B \rightarrow B$ be defined on the summand $\text{Hom}_{\mathcal{C}}(P(\lambda), FP(\mu))$ of B by composing with $x_{P(\mu)} : FP(\mu) \rightarrow FP(\mu)$; let $t : B \otimes_A B \rightarrow B \otimes_A B$ be induced similarly by $t_{P(\mu)} : F^2P(\mu) \rightarrow F^2P(\mu)$ using also the following canonical isomorphism

$$B \otimes_A B \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_{\mathcal{C}}(P(\lambda), F^2P(\mu)).$$

Then we may take $E : \text{mod-}A \rightarrow \text{mod-}A$ to be the canonical right adjoint to F given by the functor $\text{Hom}_A(B, -)$. In this way we have made explicit the categorical \mathfrak{sl}_I -action on $\text{mod-}A$.

The strategy for the proof of uniqueness is as follows. Suppose that we are given another \mathfrak{sl}_I -categorification \mathcal{C}' of an equivalent type $(\underline{n}', \underline{c}')$. We repeat all of the above, defining its associated basic algebra

$$A' := \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_{\mathcal{C}'}(P'(\lambda), P'(\mu)), \quad (2.24)$$

and an (A', A') -bimodule

$$B' := \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_{\mathcal{C}'}(P'(\lambda), F'P'(\mu)) \quad (2.25)$$

together with endomorphisms $x' : B' \rightarrow B'$ and $t' : B' \otimes_{A'} B' \rightarrow B'$ leading to a categorical \mathfrak{sl}_I -action on $\text{mod-}A'$ too, such that the equivalence $\mathbb{H}' : \mathcal{C}' \rightarrow \text{mod-}A'$ is strongly equivariant. Then the point is to construct an algebra isomorphism $A \cong A'$, inducing an isomorphism of categories $\text{mod-}A \xrightarrow{\sim} \text{mod-}A'$. Moreover we prove that this isomorphism of categories is strongly equivariant, by defining an isomorphism $B \cong B'$ that intertwines the actions of A, x and t with A', x' and t' . Composing the isomorphism $\text{mod-}A \xrightarrow{\sim} \text{mod-}A'$ on one side with \mathbb{H} and with the canonical adjoint equivalence to \mathbb{H}' on the other, we obtain the desired strongly equivariant equivalence $\mathbb{G} : \mathcal{C} \rightarrow \mathcal{C}'$ from the statement of Theorem 2.11.

Let us begin. Recall that $\kappa = \kappa_{I; \underline{n}, \underline{c}}$ is the 01-matrix indexing the basis vector of maximal weight in $\bigwedge^{\underline{n}, \underline{c}} V_I$. The irreducible object $L(\kappa)$ is the only irreducible in its block, i.e. $L(\kappa) = \Delta(\kappa) = \nabla(\kappa) = P(\kappa) = I(\kappa)$. Since the functor F has both a left and right adjoint it sends projectives to projectives, hence the object

$$T = \bigoplus_{d \geq 0} T_d := \bigoplus_{d \geq 0} F^d L(\kappa) \in \mathcal{C} \quad (2.26)$$

is projective. Note also that $T_d = 0$ for $d \gg 0$ and $\text{Hom}_{\mathcal{C}}(T_d, T_{d'}) = 0$ for $d \neq d'$; both assertions follow from (2.20). Let

$$H = \bigoplus_{d \geq 0} H_d := \bigoplus_{d \geq 0} AH_{I, d}^{|\kappa|}. \quad (2.27)$$

The following theorem is at the heart of everything; see [LW, Proposition 3.2] for the first assertion, [LW, Theorem 5.1] for the second, and the proof of [LW, Theorem 6.1] for the final one; in the special case that \mathcal{C} is the tensor product categorification arising from parabolic category \mathcal{O} from [LW, Definition 3.13] the results here go back to [BK1].

Theorem 2.13 ([LW]). *The action of AH_d on T_d induces a canonical isomorphism between H_d and $\text{End}_{\mathcal{C}}(T_d)$; hence $H = \text{End}_{\mathcal{C}}(T)$. Moreover the exact functor*

$$\mathbb{V} := \text{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \rightarrow \text{mod-}H \quad (2.28)$$

is fully faithful on projectives. Finally for each weight $\lambda \in \Lambda$ the H -module

$$Y(\lambda) := \mathbb{V}P(\lambda) \quad (2.29)$$

is independent (up to isomorphism) of the particular choice of \mathcal{C} .

Remark 2.14. The proof of [LW, Theorem 5.1] establishes a slightly stronger result: the map $\mathbb{V} : \text{Hom}_{\mathcal{C}}(M, P) \rightarrow \text{Hom}_H(\mathbb{V}M, \mathbb{V}P)$ is an isomorphism for any $M, P \in \mathcal{C}$ with P projective.

Thus the functor \mathbb{V} has similar properties to Soergel's combinatorial functor from [So]. The modules $Y(\lambda)$ may be called *Young modules* by analogy with the modular representation theory of symmetric groups. The second assertion of Theorem 2.13 is known as the *double centralizer property*. It implies that the functor \mathbb{V} defines an algebra isomorphism

$$A \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y(\lambda), Y(\mu)). \quad (2.30)$$

Similarly for the primed category we get that

$$A' \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y'(\lambda), Y'(\mu)) \quad (2.31)$$

where $Y'(\lambda) := \mathbb{V}'P'(\lambda)$ for \mathbb{V}' defined analogously to \mathbb{V} . Then, applying the final assertion of Theorem 2.13, we choose H -module isomorphisms $Y(\lambda) \cong Y'(\lambda)$ for each λ . These choices induce the desired algebra isomorphism $A \cong A'$.

It remains to construct the bimodule isomorphism $B \cong B'$. This needs just a little more preparation. The category $\text{mod-}H$ is also equipped with a categorical \mathfrak{sl}_I -action. The endofunctors

$$F : \text{mod-}H \rightarrow \text{mod-}H, \quad E : \text{mod-}H \rightarrow \text{mod-}H \quad (2.32)$$

for this are the induction and restriction functors associated to the homomorphisms $H_d \rightarrow H_{d+1}$ induced by the natural inclusions $AH_d \hookrightarrow AH_{d+1}$; in particular, $FM := M \otimes_{H_d} H_{d+1}$ for a right H_d -module M . The canonical adjunction between tensor and hom makes (F, E) into an adjoint pair. Left multiplication by x_{d+1} defines an (H_d, H_{d+1}) -bimodule endomorphism of H_{d+1} , from which we obtain the natural transformation $x \in \text{End}(F)$. By transitivity of induction, F^2 is isomorphic to the functor sending a right H_d -module M to $M \otimes_{H_d} H_{d+2}$; then left multiplication by t_{d+1} defines an (H_d, H_{d+2}) -bimodule endomorphism of H_{d+2} inducing $t \in \text{End}(F^2)$. The next lemma was observed already in [LW].

Lemma 2.15. *The quotient functor $\mathbb{V} : \mathcal{C} \rightarrow \text{mod-}H$ is strongly equivariant.*

Proof. It suffices to show that there is an isomorphism $\zeta^+ : \mathbb{V} \circ E \xrightarrow{\sim} E \circ \mathbb{V}$ of functors from \mathcal{C} to $\text{mod-}H$ satisfying the properties (2.16)–(2.17).

Take $M \in \mathcal{C}$ such that $\mathbb{V}M = \text{Hom}_{\mathcal{C}}(T_d, M)$. As H_{d-1} -modules we have that $\text{Hom}_{\mathcal{C}}(T_{d-1}, EM) \cong \text{Hom}_{\mathcal{C}}(FT_{d-1}, M) = E \text{Hom}_{\mathcal{C}}(T_d, M)$, where the isomorphism comes from the adjunction between F and E . This defines ζ^+ .

We leave the easy check of the strong equivariance properties to the reader, just noting that the endomorphisms x and t of the restriction functors E and E^2 are given explicitly on a right H_d -module M simply by the actions of x_d and t_{d-1} , respectively. \square

Remark 2.16. We should also discuss the existence of a second adjunction making F into a right adjoint to E on the category $\text{mod-}H$. There are several explicit constructions in the literature, but actually the easiest way to see this in the present context is to note that any choice of adjunction making F into a right adjoint to E on \mathcal{C} can be pushed through the quotient functor \mathbb{V} to yield an adjunction at the level of $\text{mod-}H$ too. This just requires the isomorphism $\zeta^+ : \mathbb{V} \circ E \xrightarrow{\sim} E \circ \mathbb{V}$ just constructed and the isomorphism $\zeta^- : \mathbb{V} \circ F \xrightarrow{\sim} F \circ \mathbb{V}$ derived from ζ^+ via the first adjunction.

Applying Lemma 2.15, we deduce that the functor \mathbb{V} defines (A, A) -bimodule isomorphisms

$$B \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y(\lambda), FY(\mu)), \quad (2.33)$$

$$B \otimes_A B \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y(\lambda), F^2Y(\mu)). \quad (2.34)$$

Under these isomorphisms, $x : B \rightarrow B$ and $t : B \otimes_A B \rightarrow B \otimes_A B$ correspond to the endomorphisms of the bimodules on the right induced by all of the homomorphisms $x_{Y(\mu)} : FY(\mu) \rightarrow FY(\mu)$ and $t_{Y(\mu)} : F^2Y(\mu) \rightarrow F^2Y(\mu)$, respectively. Similarly

$$B' \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y'(\lambda), FY'(\mu)), \quad (2.35)$$

$$B' \otimes_A B' \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y'(\lambda), F^2Y'(\mu)). \quad (2.36)$$

Then the H -module isomorphisms $Y(\lambda) \cong Y'(\lambda)$ chosen earlier induce the desired isomorphism $B \cong B'$. It is immediate that it intertwines the actions of A, x and t with A', x' and t' . This completes our sketch of the proof of uniqueness in Theorem 2.11 for finite intervals.

2.8. Truncation. In this subsection we introduce our key tool for dealing with tensor product categorifications when I is infinite. Throughout we fix a type $(\underline{n}, \underline{c})$ and any interval I , and set $\Lambda := \Lambda_{I; \underline{n}, \underline{c}}$. Given a subinterval $J \subseteq I$, there is an obvious embedding $\mathfrak{sl}_J \hookrightarrow \mathfrak{sl}_I$. Let Λ_J be the subposet of Λ consisting of all 01-matrices λ such that $\lambda_{ij} = c_i$ whenever $j \notin J_+$. This is order-isomorphic to the poset $\Lambda_{J; \underline{n}, \underline{c}}$ via the map sending $\lambda = (\lambda_{ij})_{1 \leq i \leq l, j \in I_+} \in \Lambda_J$ to its submatrix $\lambda_J := (\lambda_{ij})_{1 \leq i \leq l, j \in J_+} \in \Lambda_{J; \underline{n}, \underline{c}}$. In turn, the \mathfrak{sl}_J -module $\bigwedge^{\underline{n}, \underline{c}} V_J$ can be identified with the \mathfrak{sl}_J -submodule of $\bigwedge^{\underline{n}, \underline{c}} V_I$ spanned by $\{v_\lambda \mid \lambda \in \Lambda_J\}$. We then have that

$$\bigwedge^{\underline{n}, \underline{c}} V_I = \bigcup_J \bigwedge^{\underline{n}, \underline{c}} V_J,$$

taking the union just over the finite subintervals $J \subseteq I$. We are going to develop a categorical analog of this decomposition.

Lemma 2.17. *For $\lambda, \mu \in \Lambda$, the following are equivalent:*

(i) $\lambda \leq \mu$;

(ii) *for all $h \in I$ and $1 \leq k \leq l$ we have that*
$$\sum_{i=1}^k \sum_{\substack{j \leq h \\ \lambda_{ij} \neq c_i}} (-1)^{c_i} \geq \sum_{i=1}^k \sum_{\substack{j \leq h \\ \mu_{ij} \neq c_i}} (-1)^{c_i},$$

with equality when $k = l$;

(iii) *for all $h \in I$ and $1 \leq k \leq l$ we have that*
$$\sum_{i=1}^k \sum_{\substack{j > h \\ \lambda_{ij} \neq c_i}} (-1)^{c_i} \leq \sum_{i=1}^k \sum_{\substack{j > h \\ \mu_{ij} \neq c_i}} (-1)^{c_i},$$

with equality when $k = l$.

Proof. The equivalence of (i) and (ii) follows from Lemma 2.1. The equivalence of (ii) and (iii) is obvious. \square

Again let $J \subseteq I$ be any subinterval. Let $\Lambda_{\leq J}$ denote the set of all $\lambda \in \Lambda$ which satisfy the conditions

$$\left\{ \begin{array}{l} \sum_{i=1}^k \sum_{\substack{j \leq h \\ \lambda_{ij} \neq c_i}} (-1)^{c_i} \geq 0 \quad \text{for all } h < \min(J) \text{ and } 1 \leq k \leq l, \\ \sum_{i=1}^k \sum_{\substack{j > h \\ \lambda_{ij} \neq c_i}} (-1)^{c_i} \leq 0 \quad \text{for all } h > \max(J) \text{ and } 1 \leq k \leq l. \end{array} \right. \quad (2.37)$$

Also let $\Lambda_{< J}$ denote the set of all $\lambda \in \Lambda_{\leq J}$ such that at least one of the above inequalities is strict. Lemma 2.17 implies that both $\Lambda_{\leq J}$ and $\Lambda_{< J}$ are ideals in the poset Λ . Moreover $\Lambda_J = \Lambda_{\leq J} \setminus \Lambda_{< J}$.

Suppose next that we are given an \mathfrak{sl}_I -tensor product categorification \mathcal{C} of type $(\underline{n}, \underline{c})$. Let $\mathcal{C}_{\leq J}$ (resp. $\mathcal{C}_{< J}$) be the highest weight subcategory of \mathcal{C} associated to the ideal $\Lambda_{\leq J}$ (resp. $\Lambda_{< J}$). Let $\mathcal{C}_J := \mathcal{C}_{\leq J} / \mathcal{C}_{< J}$. We denote the quotient functor by $\pi_J : \mathcal{C}_{\leq J} \rightarrow \mathcal{C}_J$.

Lemma 2.18. *For $j \in J$ the functors F_j and E_j preserve the subcategories $\mathcal{C}_{\leq J}$ and $\mathcal{C}_{< J}$ of \mathcal{C} .*

Proof. We just explain for $\mathcal{C}_{\leq J}$; the same argument works for $\mathcal{C}_{< J}$. Take any $\lambda \in \Lambda_{\leq J}$. We need to show that $F_j L(\lambda)$ and $E_j L(\lambda)$ both belong to $\mathcal{C}_{\leq J}$. Since $L(\lambda)$ is a quotient of $\Delta(\lambda)$, this follows if we can show that $F_j \Delta(\lambda)$ and $E_j \Delta(\lambda)$ belong to $\mathcal{C}_{\leq J}$. These objects have filtrations with sections of the form $\Delta(\mu)$ for weights μ obtained from λ by applying the transposition t_j to one of its rows. The integers on the left hand side of the inequalities (2.37) are the same for each of these μ as they are for λ , so that each μ arising is an element of $\Lambda_{\leq J}$ and $\Delta(\mu)$ does indeed belong to $\mathcal{C}_{\leq J}$. \square

Hence for $j \in J$ the functors F_j and E_j induce a well-defined biadjoint pair of endofunctors of \mathcal{C}_J . Let $F_J := \bigoplus_{j \in J} F_j$ and $E_J := \bigoplus_{j \in J} E_j$. The natural transformations x and s restrict to endomorphisms of F_J and F_J^2 , respectively, such that the associated endomorphisms x_j and s_k of F_J^d satisfy the degenerate affine Hecke algebra relations as in (SL2). The axioms (TP1)–(TP3) for \mathcal{C} imply the analogous statements for \mathcal{C}_J . Thus \mathcal{C}_J equipped with the endofunctors F_J and E_J is an \mathfrak{sl}_J -tensor product categorification of type $(\underline{n}, \underline{c})$.

Lemma 2.19. *Suppose that $J \subset I$ is a finite subinterval with $|J_+| \geq 2 \max(\underline{n})$. Let $\kappa \in \Lambda$ be the unique weight such that $\kappa_J = \kappa_{J; \underline{n}, \underline{c}} \in \Lambda_{J; \underline{n}, \underline{c}}$. Then $L(\kappa)$ is the unique indecomposable object in its block; in particular it is prinjective in \mathcal{C} .*

Proof. In view of (2.20) it suffices to show for $\lambda \in \Lambda$ that $|\lambda| = |\kappa| \Rightarrow \lambda = \kappa$. We proceed by induction on l , the case $l = 0$ being trivial.

For the induction step assume first that $c_i = 0$ for some i . Amongst all the i with $c_i = 0$ choose one for which n_i is minimal. Thus $n_i \leq n_j$ for all j with $c_j = 0$, and $n_i \leq 2 \max(\underline{n}) - n_j \leq |J_+| - n_j$ for all j with $c_j = 1$. Letting $s := \min(J_+) - 1$, it follows that the columns $s+1, \dots, s+n_i$ of the 01-matrix κ have all entries equal to 1. Since $|\lambda| = |\kappa|$ the number of entries 1 in each column of λ is the same as in κ . Hence all n_i of the entries 1 in the i th row of λ appear

in columns $s + 1, \dots, s + n_i$. Thus the i th row of λ is the same as the i th row of κ . Then we remove this row and proceed by induction.

This just leaves us with the case that $c_i = 1$ for all i . Choose i so that n_i is minimal and let $s := \max(J_+) - n_i$. Then columns $s + 1, \dots, s + n_i$ of κ , hence also of λ , have all entries equal to 0. So the i th row of λ is the same as in κ , and then we can induct as before. \square

2.9. Blocks, crystals and prinjectives. For any I and $(\underline{n}, \underline{c})$, let \mathcal{C} be an \mathfrak{sl}_I -tensor product categorification of type $(\underline{n}, \underline{c})$. To avoid trivialities assume that $\Lambda := \Lambda_{I; \underline{n}, \underline{c}}$ is non-empty. In the finite case the category \mathcal{C} is already well understood thanks to Theorem 2.11 and the extensive literature about parabolic category \mathcal{O} . In the infinite case many questions about the combinatorics of \mathcal{C} can be answered by picking a sufficiently large finite subinterval $J \subset I$ then passing to the subquotient \mathcal{C}_J introduced in the previous subsection.

For example, if $\lambda, \mu \in \Lambda$ are any two weights, we can choose a finite interval J so that they both belong to the subset Λ_J ; since the quotient functor π_J is exact the composition multiplicity $[\Delta(\lambda) : L(\mu)]$ in \mathcal{C} is the same as in \mathcal{C}_J ; hence it can be computed via the Kazhdan-Lusztig conjecture. See §5.9 for more about the explicit combinatorics here.

In this subsection we apply this technique to extend a couple of other basic combinatorial results from finite to infinite I . To start with we have the following result which determines the blocks of \mathcal{C} .

Theorem 2.20. *For $\lambda, \mu \in \Lambda$, the irreducible objects $L(\lambda)$ and $L(\mu)$ lie in the same block of \mathcal{C} if and only if $|\lambda| = |\mu|$ in the weight lattice P_I .*

Proof. When I is finite, the category \mathcal{C} is equivalent to a sum of integral blocks of parabolic category \mathcal{O} for the general linear Lie algebra, and the theorem has been proved already in [B4]. Now suppose that I is infinite. We observed already from (2.20) that λ and μ lie in the same block only if $|\lambda| = |\mu|$. Conversely suppose that $|\lambda| = |\mu|$. Pick a finite interval $J \subset I$ such that $\lambda, \mu \in \Lambda_J$. Then $|\lambda_J| = |\mu_J|$ in P_J , so by the finite result there exists a sequence of weights $\lambda = \lambda_0, \dots, \lambda_n = \mu$ in Λ_J such that one of $[\Delta(\lambda_i) : L(\lambda_{i-1})]$ or $[\Delta(\lambda_{i-1}) : L(\lambda_i)]$ is non-zero for each $i = 1, \dots, n$. Since these composition multiplicities are the same in \mathcal{C} or \mathcal{C}_J this does the job. \square

There is a crystal graph structure on Λ . This is an I -colored directed graph with vertex set Λ , such that there is at most one edge of each color entering and one edge of each color leaving any given vertex. To determine the edges of color i incident with vertex λ one proceeds as follows. First label rows of the matrix λ by the sign $-$ if the i th and $(i + 1)$ th entries of the row are 10, or by $+$ if these entries are 01; leave all the other rows unlabeled. Then reduce the labels by repeatedly erasing $+-$ pairs of labels whenever the $+$ -row is above the $-$ -row and all the rows in between are unlabeled. If at the end of this process a $-$ -row (resp. a $+$ -row) remains, then there is an edge $\lambda \xrightarrow{i} \mu$ (resp. $\lambda \xleftarrow{i} \mu$) in the crystal graph, where μ is obtained from λ by switching the i th and $(i + 1)$ th entries of the lowest $-$ -row (resp. the highest $+$ -row).

Theorem 2.21. *For $\lambda \in \Lambda$ and $i \in I$, we have that $F_i L(\lambda) = 0$ (resp. $E_i L(\lambda) = 0$) unless there is an edge $\lambda \xrightarrow{i} \mu$ (resp. $\lambda \xleftarrow{i} \mu$) in the crystal graph, in which*

case $F_i L(\lambda)$ (resp. $E_i L(\lambda)$) is indecomposable with irreducible head and socle isomorphic to $L(\mu)$.

Proof. This is already known for finite I ; see [LW, Theorem 7.2] for the most recent but also most conceptual proof (actually the arguments of [Lo] are sufficient here since \mathcal{C} is a highest weight category). In the infinite case we pick $J \subset I$ containing i such that λ and all the weights μ indexing the composition factors of $E_i L(\lambda)$ and $F_i L(\lambda)$ lie in Λ_J , and then pass to the subquotient \mathcal{C}_J . \square

Finally we classify the prinjective objects in \mathcal{C} , generalizing an old result of Irving [I] in the context of category \mathcal{O} and [BK1, Theorem 4.8] for parabolic category \mathcal{O} . The notation for this is slightly different according to whether I is finite or infinite.

In the finite case we let Λ° be the vertex set of the connected component of the crystal graph containing $\kappa := \kappa_{I; \underline{n}, \underline{c}}$. Let $T \in \mathcal{C}$ denote the object from (2.26).

In the infinite case we fix finite subintervals $I_1 \subset I_2 \subset \dots \subset I$ such that $I = \bigcup_{r \geq 1} I_r$ and $|I_r| = 2 \max(\underline{n}) + r$. Let $\Lambda_r := \Lambda_{I_r} \subset \Lambda$ and κ^r be the element of Λ_r corresponding to $\kappa_{I_r; \underline{n}, \underline{c}}$. Let Λ_r° be the vertex set of the connected subgraph of the crystal graph generated by κ^r and the edges of colors from I_r . Then set

$$\Lambda^\circ := \bigcup_{r \geq 1} \Lambda_r^\circ. \quad (2.38)$$

For each r we have that $\kappa^r \in \Lambda_{r+1}^\circ$; see (4.4) below for an explicit directed path $\kappa^{r+1} \rightarrow \kappa^r$. So we have that $\Lambda_1^\circ \subset \Lambda_2^\circ \subset \dots$, and Λ° can be described equivalently as the vertex set of the unique connected component of the entire crystal graph that contains every κ^r . Let T^r be the object

$$T^r = \bigoplus_{d \geq 0} T_d^r := \bigoplus_{d \geq 0} F_{I_r}^d L(\kappa^r) \in \mathcal{C}. \quad (2.39)$$

Theorem 2.22. *Let notation be as above. For $\lambda \in \Lambda$, the following are equivalent:*

- (i) $\lambda \in \Lambda^\circ$;
- (ii) $L(\lambda)$ is a constituent of both the socle and the head of T if I is finite, or the socle and head of T^r for some $r \geq 1$ if I is infinite;
- (iii) $L(\lambda)$ is a constituent of the socle of a standard object;
- (iv) $P(\lambda) \cong I(\lambda)$;
- (v) $P(\lambda)$ is injective;
- (vi) $I(\lambda)$ is projective.

Proof. (i) \Rightarrow (ii). In the finite case the connected component of the crystal graph with vertex set Λ° is a copy of Kashiwara's crystal graph associated to the irreducible \mathfrak{sl}_I -module of highest weight $|\kappa|$, and κ is its highest vertex. Hence there is a directed path $\kappa \xrightarrow{i_1} \dots \xrightarrow{i_d} \lambda$ in the crystal graph for some $d \geq 0$ and $i_1, \dots, i_d \in I$. Similarly in the infinite case there is a path $\kappa^r \xrightarrow{i_1} \dots \xrightarrow{i_d} \lambda$ in the crystal graph for some $r \geq 1$, $d \geq 0$ and $i_1, \dots, i_d \in I_r$. It remains to apply Theorem 2.21 to deduce that $L(\lambda)$ appears in both the head and the socle of $F_{i_d} \dots F_{i_1} L(\kappa)$ (resp. $F_{i_d} \dots F_{i_1} L(\kappa)$), which is a summand of T (resp. T^r).

(ii) \Rightarrow (iii). This follows because T (resp. T^r) has a Δ -flag.

(iii) \Rightarrow (i). We first prove this in the finite case. Suppose that $L(\lambda) \hookrightarrow \Delta(\nu)$ for some $\nu \in \Lambda$. We show that $\lambda \in \Lambda^\circ$ by induction on the height of $|\kappa| - |\nu| \in Q_I^+$. The base case is clear as then $\lambda = \nu = \kappa$. For the induction step we apply [LW, Proposition 5.2] to deduce that there exists $i \in I$ such that $E_i L(\lambda) \neq 0$ in \mathcal{C} . The socle of $E_i L(\lambda)$ is isomorphic to $L(\mu)$ where $\mu \xrightarrow{i} \lambda$ in the crystal graph. Now it suffices to show that $\mu \in \Lambda^\circ$. This follows by the induction hypothesis, noting that $L(\mu) \hookrightarrow E_i L(\lambda) \hookrightarrow E_i \Delta(\nu)$ and $E_i \Delta(\nu)$ has a Δ -flag.

For the infinite case suppose that $L(\lambda) \hookrightarrow \Delta(\mu)$. Then we pick $r \geq 1$ so that all composition factors of $\Delta(\mu)$ lie in $\Lambda_r := \Lambda_{I_r}$. Passing to the subquotient $\mathcal{C}_r := \mathcal{C}_{I_r}$ we still have that $L(\lambda)$ is a constituent of the socle of $\Delta(\mu)$ in \mathcal{C}_r . Hence $\lambda \in \Lambda_r^\circ \subset \Lambda^\circ$ thanks to the previous paragraph.

(iv) \Leftrightarrow (v) \Leftrightarrow (vi). This follows using the duality \otimes from Corollary 2.12.

(vi) \Rightarrow (iii). This is clear as $L(\lambda) \hookrightarrow I(\lambda)$ and projectives have Δ -flags.

(ii) \Rightarrow (vi). The injective $I(\lambda)$ is a summand of the injective object T (resp. T^r) and the latter is also projective; when I is infinite this uses Lemma 2.19. \square

3. THE GENERAL LINEAR LIE SUPERALGEBRA

Throughout the section we fix a type $(\underline{n}, \underline{c})$ of level l . The goal is to give an explicit construction of an $\mathfrak{sl}_{\mathbb{Z}}$ -tensor product categorification \mathcal{C} of type $(\underline{n}, \underline{c})$, thus establishing the existence in Theorem 2.11. We will obtain \mathcal{C} from the parabolic analog of the BGG category \mathcal{O} for the general linear Lie superalgebra. It is a (mostly known) generalization of [CR, §7.4] where category \mathcal{O} for the general linear Lie algebra was considered.

3.1. Super category \mathcal{O} . For $i = 1, \dots, l$ let U_i be a vector superspace of dimension n_i concentrated in degree $\bar{c}_i \in \mathbb{Z}/2$ (over the ground field \mathbb{K} as always). Then set

$$U := U_1 \oplus \dots \oplus U_l, \quad (3.1)$$

so that U is a vector superspace of even dimension $n := \sum_{c_i=0} n_i$ and odd dimension $m := \sum_{c_i=1} n_i$. Let \mathfrak{g} denote the Lie superalgebra $\mathfrak{gl}(U) \cong \mathfrak{gl}_{n|m}(\mathbb{K})$ consisting of all linear endomorphisms of U under the supercommutator $[-, -]$. We choose a homogeneous basis u_1, \dots, u_{m+n} for U by concatenating bases for U_1, \dots, U_l in order and let $\{e_{i,j} \mid 1 \leq i, j \leq m+n\}$ be the resulting basis of matrix units for \mathfrak{g} . We then have that

$$[e_{i,j}, e_{k,l}] = \delta_{k,j} e_{i,l} - (-1)^{(p_i+p_j)(p_k+p_l)} \delta_{i,l} e_{k,j},$$

where $p_i \in \mathbb{Z}/2$ is the parity of the i th basis vector u_i .

Let \mathfrak{t} (resp. \mathfrak{b}) be the Cartan (resp. Borel) subalgebra of \mathfrak{g} consisting of diagonal (resp. upper triangular) matrices relative to the ordered basis just chosen. Let $\delta_1, \dots, \delta_{m+n}$ be the basis for \mathfrak{t}^* dual to the basis $e_{1,1}, \dots, e_{m+n,m+n}$, and define a non-degenerate symmetric bilinear form $(-, -)$ on \mathfrak{t}^* by setting $(\delta_i, \delta_j) := (-1)^{p_i} \delta_{i,j}$. The *root system* of \mathfrak{g} is

$$R := \{\delta_i - \delta_j \mid 1 \leq i, j \leq m+n, i \neq j\},$$

which decomposes into even and odd roots $R = R_{\bar{0}} \sqcup R_{\bar{1}}$ so that $\delta_i - \delta_j$ is of parity $p_i + p_j$. Let $R^+ = R_{\bar{0}}^+ \sqcup R_{\bar{1}}^+$ denote the positive roots associated to the Borel

subalgebra \mathfrak{b} , i.e. $\delta_i - \delta_j$ is positive if and only if $i < j$. The *dominance order* \succeq on \mathfrak{t}^* is defined so that $\lambda \succeq \mu$ if $\lambda - \mu \in \mathbb{N}R^+$.

Let $\rho \in \mathfrak{t}^*$ be the unique weight such that

$$(\rho, \delta_1) = \begin{cases} 0 & \text{if } p_1 = \bar{0}, \\ 1 & \text{if } p_1 = \bar{1}; \end{cases} \quad (\rho, \delta_i - \delta_{i+1}) = \begin{cases} (-1)^{p_i} & \text{if } p_i = p_{i+1}, \\ 0 & \text{if } p_i \neq p_{i+1}. \end{cases}$$

Explicitly, we have that $\rho = -\sum_{i < j} (-1)^{p_i + p_j} \delta_j - \sum_{p_i = \bar{1}} \delta_i$. One can check that 2ρ is congruent to the sum of the positive even roots minus the sum of the positive odd roots modulo $\delta := \sum_i (-1)^{p_i} \delta_i$. We are ready to introduce the category \mathcal{O} associated to $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$, restricting from the outset to *integral weights* lying in the set $\mathfrak{t}_{\mathbb{Z}}^* := \mathbb{Z}\delta_1 \oplus \cdots \oplus \mathbb{Z}\delta_{m+n} \subset \mathfrak{t}^*$.

Definition 3.1. For $\lambda \in \mathfrak{t}_{\mathbb{Z}}^*$ let $p_\lambda := \sum_{p_i = \bar{1}} (\lambda, \delta_i) \in \mathbb{Z}/2$. Then define \mathcal{O} to be the category of all finitely generated \mathfrak{g} -supermodules $M = M_{\bar{0}} \oplus M_{\bar{1}}$ which are locally finite dimensional over \mathfrak{b} and satisfy

$$M = \bigoplus_{\lambda \in \mathfrak{t}_{\mathbb{Z}}^*} M_{\lambda, p_\lambda}, \quad (3.2)$$

where for $\lambda \in \mathfrak{t}^*$ and $p \in \mathbb{Z}/2$ we write $M_{\lambda, p}$ for the λ -weight space of M_p with respect to \mathfrak{t} defined in the standard way. Morphisms in \mathcal{O} mean arbitrary \mathfrak{g} -supermodule homomorphisms. The parity assumption in (3.2) ensures that all morphisms are automatically even, hence \mathcal{O} is an abelian category.

Note that both the natural \mathfrak{g} -supermodule U and its dual belong to \mathcal{O} , and \mathcal{O} is closed under tensoring with these objects. As usual, to classify the irreducible objects in \mathcal{O} , one starts from the *Verma supermodules* $\{M(\lambda) \mid \lambda \in \mathfrak{t}_{\mathbb{Z}}^*\}$ defined from

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{K}_{\lambda, p_\lambda}$$

where $\mathbb{K}_{\lambda, p_\lambda}$ is the one-dimensional \mathfrak{b} -supermodule of weight λ with $\mathbb{Z}/2$ -grading concentrated in degree p_λ . The weight λ is the highest weight of $M(\lambda)$ with respect to the dominance ordering, and the corresponding weight space is one dimensional. Therefore, by the usual arguments of highest weight theory, $M(\lambda)$ has a unique irreducible quotient $L(\lambda)$, and the supermodules $\{L(\lambda) \mid \lambda \in \mathfrak{t}_{\mathbb{Z}}^*\}$ give a complete set of pairwise non-isomorphic irreducible objects of \mathcal{O} .

We need one non-trivial result about linkage. For $\lambda \in \mathfrak{t}_{\mathbb{Z}}^*$ let

$$A(\lambda) := \{\alpha \in R_0^+ \mid (\lambda + \rho, \alpha) > 0\}, \quad B(\lambda) := \{\beta \in R_1^+ \mid (\lambda + \rho, \beta) = 0\}.$$

Then introduce a relation \uparrow on $\mathfrak{t}_{\mathbb{Z}}^*$ by declaring that $\mu \uparrow \lambda$ if we either have that $\mu = s_\alpha \cdot \lambda$ for some $\alpha \in A(\lambda)$ or we have that $\mu = \lambda - \beta$ for some $\beta \in B(\lambda)$; here, for $\alpha = \delta_i - \delta_j \in R_0^+$ and $\lambda \in \mathfrak{t}^*$, we write $s_\alpha \cdot \lambda$ for $s_{i,j}(\lambda + \rho) - \rho$, where $s_{i,j} : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ is the linear map transposing δ_i and δ_j and fixing all other δ_k .

Lemma 3.2. Suppose $\lambda, \mu \in \mathfrak{t}_{\mathbb{Z}}^*$ satisfy $[M(\lambda) : L(\mu)] \neq 0$. Then there exists $r \geq 0$ and weights $\nu_0, \dots, \nu_r \in \mathfrak{t}_{\mathbb{Z}}^*$ such that $\mu = \nu_0 \uparrow \nu_1 \uparrow \cdots \uparrow \nu_r = \lambda$.

Proof. This is a consequence of the superalgebra analog of the Jantzen sum formula from [M, Theorem 10.3.1]. In more detail, the Jantzen filtration on $M(\lambda)$ is a

certain exhaustive descending filtration $M(\lambda) = M(\lambda)_0 \supset M(\lambda)_1 \supseteq M(\lambda)_2 \supseteq \cdots$ such that $M(\lambda)_0/M(\lambda)_1 \cong L(\lambda)$, and the sum formula shows that

$$\sum_{k \geq 1} \text{ch } M(\lambda)_k = \sum_{\alpha \in A(\lambda)} \text{ch } M(s_\alpha \cdot \lambda) + \sum_{\beta \in B(\lambda)} \sum_{k \geq 1} (-1)^{k-1} \text{ch } M(\lambda - k\beta).$$

To deduce the lemma from this, suppose that $[M(\lambda) : L(\mu)] \neq 0$. Then $\mu \leq \lambda$, so that $\lambda - \mu$ is a sum of N simple roots $\delta_i - \delta_{i+1}$ for some $N \geq 0$. We proceed by induction on N , the case $N = 0$ being vacuous. If $N > 0$ then $L(\mu)$ is a composition factor of $M(\lambda)_1$ and the sum formula implies that $L(\mu)$ is a composition factor either of $M(s_\alpha \cdot \lambda)$ for some $\alpha \in A(\lambda)$ or that $L(\mu)$ is a composition factor of $M(\lambda - k\beta)$ for some odd $k \geq 1$ and $\beta \in B(\lambda)$. It remains to apply the induction hypothesis and the definition of \uparrow . \square

There is another partial order \leq on $\mathfrak{t}_{\mathbb{Z}}^*$, which we call the *Bruhat order*, defined from

$$\lambda \leq \mu \iff \sum_{\substack{1 \leq i \leq j \\ (\lambda + \rho, \delta_i) \leq h}} (-1)^{p_i} \geq \sum_{\substack{1 \leq i \leq j \\ (\mu + \rho, \delta_i) \leq h}} (-1)^{p_i} \quad (3.3)$$

for all $h \in \mathbb{Z}$ and $1 \leq j \leq m + n$, with equality whenever $j = m + n$.

Lemma 3.3. *For $\lambda, \mu \in \mathfrak{t}_{\mathbb{Z}}^*$ we have that $\lambda \uparrow \mu \Rightarrow \lambda \leq \mu \Rightarrow \lambda \leq \mu$.*

Proof. Exercise. \square

Remark 3.4. The Bruhat order \leq coincides with the transitive closure of the relation \uparrow if all the 0's appear either before or after all of the 1's in the sequence \underline{c} ; see [B1, Lemma 2.5]. However in general \leq is a proper refinement of the transitive closure of \uparrow ; see [CLW, Remark 2.3] for an example.

Theorem 3.5. *The category \mathcal{O} is a highest weight category with weight poset $(\mathfrak{t}_{\mathbb{Z}}^*, \leq)$. Its standard objects are the Verma supermodules $\{M(\lambda) \mid \lambda \in \mathfrak{t}_{\mathbb{Z}}^*\}$.*

Proof. By Lemmas 3.2 and 3.3, all composition factors of $M(\lambda)$ are of the form $L(\mu)$ for $\mu \leq \lambda$. The theorem follows from this and the BGG reciprocity established in a general graded Lie superalgebra setting in [B2, (6.6)]; one also needs to repeat the argument of [B2, Lemma 7.3] to check that projectives are of finite length. \square

3.2. Super parabolic category \mathcal{O} . Let $\mathfrak{h} \cong \mathfrak{gl}_{n_1}(\mathbb{K}) \oplus \cdots \oplus \mathfrak{gl}_{n_l}(\mathbb{K})$ be the subalgebra of \mathfrak{g} that is the stabilizer of the direct sum decomposition (3.1) and set $\mathfrak{p} := \mathfrak{h} + \mathfrak{b}$.

Definition 3.6. Let \mathcal{C} be the full subcategory of \mathcal{O} consisting of all objects that are locally finite dimensional over \mathfrak{p} . We refer to this as *super parabolic category \mathcal{O}* of type $(\underline{n}, \underline{c})$.

The isomorphism classes of irreducible objects in \mathcal{C} are represented by the supermodules $L(\lambda)$ for λ in the set

$$\Lambda := \left\{ \lambda \in \mathfrak{t}_{\mathbb{Z}}^* \mid \begin{array}{l} (-1)^{c_k}(\lambda + \rho, \delta_i - \delta_{i+1}) > 0 \text{ for } 1 \leq k \leq l \\ \text{and } n_1 + \cdots + n_{k-1} < i < n_1 + \cdots + n_k \end{array} \right\}. \quad (3.4)$$

To see this, note first that the condition in the definition of Λ is the usual dominance condition for finite dimensionality of irreducible highest weight modules

for \mathfrak{h} , so clearly for $L(\lambda)$ to belong to \mathcal{C} it is necessary that $\lambda \in \Lambda$. For the sufficiency, for each $\lambda \in \Lambda$, let $V(\lambda)$ be a finite dimensional irreducible \mathfrak{h} -supermodule of highest weight λ with $\mathbb{Z}/2$ -grading concentrated in degree p_λ . The corresponding *parabolic Verma supermodule*

$$\Delta(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V(\lambda)$$

belongs to \mathcal{C} and is a quotient of $M(\lambda)$. Hence $\Delta(\lambda)$ has irreducible head $L(\lambda)$, implying that $L(\lambda)$ belongs to \mathcal{C} . We see moreover from this argument (and our knowledge of composition factors of $M(\lambda)$) that all other composition factors of $\Delta(\lambda)$ are of the form $L(\mu)$ for $\mu < \lambda$ in the Bruhat order. The following theorem now follows on making another application of [B2, (6.6)].

Theorem 3.7. *The category \mathcal{C} is a highest weight category with weight poset (Λ, \leq) . Its standard objects are the parabolic Verma supermodules $\{\Delta(\lambda) \mid \lambda \in \Lambda\}$.*

It is time to switch to more combinatorial notation by *identifying* the set $\Lambda \subset \mathfrak{t}_{\mathbb{Z}}^*$ from (3.4) with the set $\Lambda_{\mathbb{Z}, n, \underline{c}}$ of 01-matrices from (2.7). We do this so that $\lambda \in \Lambda$ corresponds to the 01-matrix $(\lambda_{ij})_{1 \leq i \leq l, j \in \mathbb{Z}}$ defined from

$$\lambda_{ij} = \begin{cases} 1 - c_i & \text{if } j = (\lambda + \rho, \delta_{n_1 + \dots + n_{i-1} + k}) \text{ for some } k = 1, \dots, n_i, \\ c_i & \text{otherwise.} \end{cases} \quad (3.5)$$

When compared with Lemma 2.17(ii), the following lemma checks under this identification that the Bruhat order \leq on Λ agrees with the partial order \leq from the axiom (TP1) in the previous section.

Lemma 3.8. *For $\lambda, \mu \in \Lambda$ we have that $\lambda \leq \mu$ in the Bruhat order if and only if*

$$\sum_{\substack{1 \leq i \leq n_1 + \dots + n_k \\ (\lambda + \rho, \delta_i) \leq h}} (-1)^{p_i} \geq \sum_{\substack{1 \leq i \leq n_1 + \dots + n_k \\ (\mu + \rho, \delta_i) \leq h}} (-1)^{p_i}$$

for all $h \in \mathbb{Z}$ and $1 \leq k \leq l$, with equality whenever $k = l$.

Proof. The forward implication is clear from the definition (3.3). For the converse suppose that $\lambda \not\leq \mu$ in the Bruhat order. Let $1 \leq j \leq m + n$ be minimal so that

$$\sum_{\substack{1 \leq i \leq j \\ (\lambda + \rho, \delta_i) \leq h}} (-1)^{p_i} < \sum_{\substack{1 \leq i \leq j \\ (\mu + \rho, \delta_i) \leq h}} (-1)^{p_i}$$

for some $h \in \mathbb{Z}$. To complete the proof, we show for k defined from $n_1 + \dots + n_{k-1} < j \leq n_1 + \dots + n_k$ that

$$\sum_{\substack{1 \leq i \leq n_1 + \dots + n_k \\ (\lambda + \rho, \delta_i) \leq h}} (-1)^{p_i} < \sum_{\substack{1 \leq i \leq n_1 + \dots + n_k \\ (\mu + \rho, \delta_i) \leq h}} (-1)^{p_i}.$$

Suppose first that $p_j = \bar{0}$. Then by the minimality of j we must have that $(\lambda + \rho, \delta_j) > h$ and $(\mu + \rho, \delta_j) = h$, hence $(\mu + \rho, \delta_i) < h$ for all $j < i \leq n_1 + \dots + n_k$.

We deduce that

$$\begin{aligned} \sum_{\substack{1 \leq i \leq n_1 + \dots + n_k \\ (\lambda + \rho, \delta_i) \leq h}} (-1)^{p_i} &\leq n_1 + \dots + n_k - j + \sum_{\substack{1 \leq i \leq j \\ (\lambda + \rho, \delta_i) \leq h}} (-1)^{p_i} \\ &< n_1 + \dots + n_k - j + \sum_{\substack{1 \leq i \leq j \\ (\mu + \rho, \delta_i) \leq h}} (-1)^{p_i} = \sum_{\substack{1 \leq i \leq n_1 + \dots + n_k \\ (\mu + \rho, \delta_i) \leq h}} (-1)^{p_i}. \end{aligned}$$

Instead suppose that $p_j = \bar{1}$. Then $(\lambda + \rho, \delta_j) = h$ and $(\mu + \rho, \delta_j) > h$, hence $(\lambda + \rho, \delta_i) > h$ and $(\mu + \rho, \delta_i) > h$ for all $j < i \leq n_1 + \dots + n_k$. We deduce that

$$\sum_{\substack{1 \leq i \leq n_1 + \dots + n_k \\ (\lambda + \rho, \delta_i) \leq h}} (-1)^{p_i} = \sum_{\substack{1 \leq i \leq j \\ (\lambda + \rho, \delta_i) \leq h}} (-1)^{p_i} < \sum_{\substack{1 \leq i \leq j \\ (\mu + \rho, \delta_i) \leq h}} (-1)^{p_i} = \sum_{\substack{1 \leq i \leq n_1 + \dots + n_k \\ (\mu + \rho, \delta_i) \leq h}} (-1)^{p_i}.$$

We are done. \square

Finally we introduce a categorical $\mathfrak{sl}_{\mathbb{Z}}$ -action on \mathcal{C} . Let F (resp. E) be the endofunctor of \mathcal{C} defined by tensoring with U (resp. its dual U^*). Let $x \in \text{End}(F)$ be the endomorphism defined on a supermodule M by letting $x : M \otimes U \rightarrow M \otimes U$ be the endomorphism defined by multiplication by

$$\Omega := \sum_{1 \leq r, s \leq m+n} (-1)^{p_s} e_{r,s} \otimes e_{s,r}.$$

Let $s \in \text{End}(F^2)$ be the endomorphism arising from the flip $U \otimes U \rightarrow U \otimes U$, $u_r \otimes u_s \mapsto (-1)^{p_r p_s} u_s \otimes u_r$.

Theorem 3.9. *The preceding definitions make \mathcal{C} into an $\mathfrak{sl}_{\mathbb{Z}}$ -tensor product categorification of type $(\underline{n}, \underline{c})$.*

Proof. The verification of the axioms (SL1)–(SL3) is standard; cf. [CR, §7.4] and also [CW, Proposition 5.1] where the degenerate affine Hecke algebra relations are checked in the super case. Also we have seen already that \mathcal{C} is a highest weight category with weight poset (Λ, \leq) as required by axiom (TP1).

Now we take $\lambda \in \Lambda$ and consider the supermodule $F\Delta(\lambda)$. By the tensor identity we have that

$$F\Delta(\lambda) = (U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V(\lambda)) \otimes U \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (V(\lambda) \otimes U).$$

Since U has an obvious \mathfrak{p} -filtration with sections U_1, \dots, U_l , we deduce that $F\Delta(\lambda)$ has a filtration with sections $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (V(\lambda) \otimes U_i)$ for $i = 1, \dots, l$. Then the classical Pieri rule gives a decomposition of each $V(\lambda) \otimes U_i$ as a direct sum of irreducible \mathfrak{h} -supermodules. We conclude that $F\Delta(\lambda)$ has a multiplicity-free Δ -flag with sections $\Delta(\lambda + \delta_{n_1 + \dots + n_{i-1} + k})$ for $1 \leq i \leq l$ and $1 \leq k \leq n_i$ such that $\lambda + \delta_{n_1 + \dots + n_{i-1} + k} \in \Lambda$. The final condition here means either that $k = 1$; or that $k > 1$, $c_i = 0$ and $(\lambda + \rho, \delta_{n_1 + \dots + n_{i-1} + k}) < (\lambda + \rho, \delta_{n_1 + \dots + n_{i-1} + k-1}) - 1$; or that $k > 1$, $c_i = 1$ and $(\lambda + \rho, \delta_{n_1 + \dots + n_{i-1} + k}) > (\lambda + \rho, \delta_{n_1 + \dots + n_{i-1} + k-1}) + 1$. Converting to the 01-matrix notation using (3.5) this shows that $F\Delta(\lambda)$ has a Δ -flag with sections

$$\{\Delta(t_{ij}(\lambda)) \mid \text{for all } 1 \leq i \leq l \text{ and } j \in \mathbb{Z} \text{ such that } \lambda_{ij} = 1 \text{ and } \lambda_{i(j+1)} = 0\},$$

where $t_{ij}(\lambda)$ denotes the 01-matrix obtained by applying the transposition t_j to the i th row of λ ; the index j in this description is related to the k in the previous

sentence by $j = (\lambda + \rho, \delta_{n_1} + \cdots + \delta_{n_{i-1}} + k)$ if $c_i = 0$ or $j = (\lambda + \rho, \delta_{n_1} + \cdots + \delta_{n_{i-1}} + k) - 1$ if $c_i = 1$.

We claim further that the endomorphism Ω preserves the filtration just constructed and induces the endomorphism of the section $\Delta(t_{ij}(\lambda))$ that is multiplication by the scalar $j \in \mathbb{Z}$. To see this note that $\Omega = (\Delta(c) - c \otimes 1 - 1 \otimes c)/2$ where Δ is the comultiplication on $U(\mathfrak{g})$ and $c := \sum_{1 \leq r, s \leq m+n} (-1)^{p_s} e_{r,s} e_{s,r} \in U(\mathfrak{g})$ is Casimir. This shows that Ω preserves the filtration. Also c acts on $\Delta(\lambda)$ by a scalar c_λ . Hence the endomorphism of $\Delta(t_{ij}(\lambda))$ induced by Ω is the scalar $(c_{t_{ij}(\lambda)} - c_\lambda - n + m)/2$, which magically simplifies to j .

Now we have shown that $F_j \Delta(\lambda)$ has a Δ -flag and that

$$[F_j \Delta(\lambda)] = \sum_i [\Delta(t_{ij}(\lambda))]$$

summing over all $i = 1, \dots, l$ such that $\lambda_{ij} = 1$ and $\lambda_{i(j+1)} = 0$. Similarly one checks that $E_j \Delta(\lambda)$ has a Δ -flag and that

$$E_j [\Delta(\lambda)] = \sum_i [\Delta(t_{ij}(\lambda))]$$

summing over all $i = 1, \dots, l$ such that $\lambda_{ij} = 0$ and $\lambda_{i(j+1)} = 1$; it helps to know for this that $E_j M$ is the generalized $(m - n - j)$ -eigenspace of Ω on $M \otimes U^*$. These formulae are consistent with the formulae for the actions of f_j and e_j on $v_\lambda \in \bigwedge^{\underline{n}, \underline{c}} V_{\mathbb{Z}}$. So we have proved (TP2)–(TP3). \square

With Theorem 3.9 in hand, one can transport all of the results from §2.9 to super parabolic category \mathcal{O} . In particular Theorem 2.20 gives a classification of blocks generalizing [CMW, Theorem 3.12], while Theorem 2.21 gives another proof of the main result of [Kuj]. The classification of projectives resulting from Theorem 2.22 is new.

In fact we now have in place all of the theory needed to prove the super Kazhdan-Lusztig conjecture: one can compute any multiplicity $[\Delta(\lambda) : L(\mu)]$ in super parabolic category \mathcal{O} by passing to a sufficiently large subquotient \mathcal{O}_I and then using the uniqueness theorem to relate that to a purely even parabolic category \mathcal{O} . We postpone the detailed discussion of this until §5.9.

4. STABLE MODULES

In this section the goal is to complete the proof of Theorem 2.11 by establishing the uniqueness for infinite intervals. The strategy is almost exactly the same as the strategy for finite intervals recalled in §2.7. The main issue is to find a suitable substitute for the quotient functor $\mathbb{V} : \mathcal{C} \rightarrow \text{mod-}H$ from Theorem 2.13. This arises from a new category $\text{mod-}H$ of “stable modules,” which seems quite interesting in its own right. Throughout the section $(\underline{n}, \underline{c})$ will be a fixed type and I will be an infinite interval. Like in Theorem 2.22, we pick finite subintervals $I_1 \subset I_2 \subset \cdots$ of I such that $I = \bigcup_{r \geq 1} I_r$ and $|I_r| = 2 \max(\underline{n}) + r$.

4.1. Tower of Hecke algebras. Let \mathcal{C} be some given \mathfrak{sl}_I -tensor product categorification of type $(\underline{n}, \underline{c})$. For each r we denote the subcategories $\mathcal{C}_{\leq I_r}$ and $\mathcal{C}_{< I_r}$ from §2.8 by $\mathcal{C}_{\leq r}$ and $\mathcal{C}_{< r}$, respectively. Then each subquotient $\mathcal{C}_r := \mathcal{C}_{\leq r} / \mathcal{C}_{< r}$ gets the induced structure of an \mathfrak{sl}_{I_r} -tensor product categorification with weight

poset $\Lambda_r := \Lambda_{I_r} \subset \Lambda$. Denote the element of Λ_r corresponding to $\kappa_{I_r; \underline{n}, \underline{c}} \in \Lambda_{I_r; \underline{n}, \underline{c}}$ by κ^r . Thus v_{κ^r} is the highest weight vector of $\bigwedge^{\underline{n}, \underline{c}} V_{I_r} < \bigwedge^{\underline{n}, \underline{c}} V_I$.

Let T^r be the r th “tensor space” from (2.39). It has a Δ -flag with sections of the form $\Delta(\mu)$ for $\mu \in \Lambda_r$. Hence by (2.19) the quotient functor $\pi_r : \mathcal{C}_{\leq r} \rightarrow \mathcal{C}_r$ defines a isomorphism between $\text{End}_{\mathcal{C}}(T^r)$ and $\text{End}_{\mathcal{C}_r}(T^r)$. So by the first part of Theorem 2.13 we can identify $\text{End}_{\mathcal{C}}(T^r)$ with the algebra

$$H^r = \bigoplus_{d \geq 0} H_d^r := \bigoplus_{d \geq 0} QH_{I_r, d}^{|\kappa^r|}. \quad (4.1)$$

Note here and subsequently we prefer to view this as a cyclotomic quiver Hecke algebra rather than a cyclotomic affine Hecke algebra.

For each $r \geq 1$ and $i \in I_r$, we introduce the idempotent

$$1_{d; i}^r := \sum_{\substack{\mathbf{i} \in I_r^{d+1} \\ i_{d+1} = i}} 1_{\mathbf{i}} \in H_{d+1}^r. \quad (4.2)$$

There is a natural algebra homomorphism

$$\iota_{d; i}^r : H_d^r \rightarrow 1_{d; i}^r H_{d+1}^r 1_{d; i}^r, \quad 1_j \mapsto 1_{\mathbf{j}i}, \quad \xi_j \mapsto \xi_j 1_{d; i}^r, \quad \tau_j \mapsto \tau_j 1_{d; i}^r. \quad (4.3)$$

In diagrammatic terms this map is given by tensoring with a single string $i \rightarrow i$ on the right hand side. With the following lemma we construct some much less familiar maps between Hecke algebras for different r . These are given by tensoring with some explicitly known diagram $\mathbf{i} \rightarrow \mathbf{i}$ on the *left*.

Lemma 4.1. *Fix $r \geq 1$ and $d \geq 0$. If $\max(I_r) = \max(I_{r+1})$ we define*

$$\begin{aligned} p_j &:= |\{i \mid c_i = 0, n_i \geq j\}|, & a &:= \max\{n_i \mid c_i = 0\}, \\ s &:= \min(I_{r+1}) - 1, & d_r &:= \sum_{c_i=0} n_i. \end{aligned}$$

Instead if $\min(I_r) = \min(I_{r+1})$, we let

$$\begin{aligned} p_j &:= |\{i \mid c_i = 1, n_i \geq j\}|, & a &:= \max\{n_i \mid c_i = 1\}, \\ s &:= \max(I_{r+1}) + 1 - a, & d_r &:= \sum_{c_i=1} n_i. \end{aligned}$$

In either case we let \mathbf{i} denote the word $(s+a)^{p_a} \cdots (s+2)^{p_2} (s+1)^{p_1} \in I_{r+1}^{d_r}$.

(i) *There exists an explicit idempotent $e_d^r \in H_{d_r+d}^{r+1}$ such that the map*

$$\phi_d^r : H_d^r \xrightarrow{\sim} e_d^r H_{d_r+d}^{r+1} e_d^r, \quad 1_j \mapsto 1_{\mathbf{j}} e_d^r, \quad \xi_j \mapsto \xi_{d_r+j} e_d^r, \quad \tau_j \mapsto \tau_{d_r+j} e_d^r$$

is a well-defined algebra isomorphism. Moreover $e_d^r 1_{\mathbf{k}} = 1_{\mathbf{k}} e_d^r = 0$ unless $\mathbf{k} = \mathbf{i} \mathbf{j}$ for some $\mathbf{j} \in I_r^d$.

(ii) *There exists an isomorphism $\theta_d^r : T_d^r \xrightarrow{\sim} e_d^r T_{d_r+d}^{r+1}$ in \mathcal{C} , such that the following diagram commutes for all $h \in H_d^r$:*

$$\begin{array}{ccc} T_d^r & \xrightarrow{h} & T_d^r \\ \theta_d^r \downarrow & & \downarrow \theta_d^r \\ e_d^r T_{d_r+d}^{r+1} & \xrightarrow{\phi_d^r(h)} & e_d^r T_{d_r+d}^{r+1}. \end{array}$$

- (iii) For $i \in I_r$ the map $\iota_{d_r+d;i}^{r+1}$ from (4.3) sends e_d^r to $1_{d_r+d;i}^{r+1}e_{d+1}^r$.
- (iv) For $i \in I_r$ we have that $\phi_{d+1}^r(1_{d;i}^r) = 1_{d_r+d;i}^{r+1}e_{d+1}^r$.

Proof. A straightforward check reveals in $\bigwedge^{\underline{n}, \underline{c}} V_I$ that

$$f_{s+1}^{(p_1)} \cdots f_{s+a}^{(p_a)} v_{\kappa^{r+1}} = v_{\kappa^r}, \quad (4.4)$$

where we write $f_i^{(m)}$ for the divided power $f_i^m/m!$. Now recall for each $i \in I_{r+1}$ and $m \geq 1$ that there is an explicit idempotent $b_m \in 1_{i^m} QH_{I_{r+1}, m} 1_{i^m}$ such that the summand $F_i^{(m)} := b_m F_i^m$ of the functor F_i^m induces $f_i^{(m)}$ at the level of the Grothendieck group; see [R, Lemma 4.1]. Hence for each $d \geq 0$ there is an explicit idempotent $e_d^r \in QH_{I_{r+1}, d_r+d}^{|\kappa^{r+1}|} = H_{d_r+d}^{r+1}$ such that

$$e_d^r F_{I_{r+1}}^{d_r+d} L(\kappa^{r+1}) = F_{I_r}^d F_{s+1}^{(p_1)} \cdots F_{s+a}^{(p_a)} L(\kappa^{r+1}). \quad (4.5)$$

Since $L(\kappa^{r+1})$ and $L(\kappa^r)$ are standard objects, the identity (4.4) implies that there exists a (unique up to scalars) isomorphism $L(\kappa^r) \xrightarrow{\sim} F_{s+1}^{(p_1)} \cdots F_{s+a}^{(p_a)} L(\kappa^{r+1})$. Applying the functor $F_{I_r}^d$ to this and using the above equality we obtain the isomorphism $\theta_d^r : T_d^r \xrightarrow{\sim} T_{d_r+d}^{r+1} e_d^r$. This definition ensures that θ_d^r intertwines the actions of $1_j, \xi_j, \tau_j \in H_d^r$ and $1_{ij} e_d^r, \xi_{d_r+j} e_d^r, \tau_{d_r+j} e_d^r \in e_d^r H_{d_r+d}^{r+1} e_d^r$. Also $e_d^r H_{d_r+d}^{r+1} e_d^r = \text{End}_{\mathcal{C}}(e_d^r T_{d_r+d}^{r+1}) \cong \text{End}_{\mathcal{C}}(T_d^r) = H_d^r$. Parts (i) and (ii) follow.

For the final parts of the lemma, the homomorphism $\iota_{d_r+d;i}^{r+1}$ can be viewed simply as an application of the functor F_i , so it maps e_d^r to $F_i e_d^r$, while $e_{d+1}^r = F_{I_r} e_d^r$ by its analogous definition. This implies (iii). Part (iv) is clear. \square

In the notation of the lemma, we set $e^r := \sum_{d \geq 0} e_d^r$, $\phi^r := \sum_{d \geq 0} \phi_d^r$ and $\theta^r := \sum_{d \geq 0} \theta_d^r$. Thus $e^r \in H^{r+1}$ is an idempotent, $\phi^r : H^r \xrightarrow{\sim} e^r H^{r+1} e^r$ is an algebra isomorphism, and $\theta^r : T^r \xrightarrow{\sim} e^r T^{r+1}$ is an isomorphism in \mathcal{C} . In particular this gives us a tower

$$H^1 \hookrightarrow H^2 \hookrightarrow H^3 \hookrightarrow \cdots$$

of cyclotomic quiver Hecke algebras which will play a key role. If M is a right (resp. left) H^{r+1} -module we will implicitly view $M e^r$ (resp. $e^r M$) as a right (resp. left) H^r -module via ϕ^r .

More generally for $r \leq s$ we set

$$\phi^{r,s} := \phi^{s-1} \circ \cdots \circ \phi^r, \quad e^{r,s} := \phi^{r,s}(1_{H_r}), \quad \theta^{r,s} := \theta^{s-1} \circ \cdots \circ \theta^r \quad (4.6)$$

Thus $e^{r,s}$ is an idempotent in H^s , $\phi^{r,s} : H^r \xrightarrow{\sim} e^{r,s} H^s e^{r,s}$ is an algebra isomorphism, and $\theta^{r,s} : T^r \rightarrow e^{r,s} T^s$ is an isomorphism in \mathcal{C} . If M is a right (resp. left) H^s -module we will implicitly view $M e^{r,s}$ (resp. $e^{r,s} M$) as an H^r -module via $\phi^{r,s}$.

4.2. Stable modules and the double centralizer property. With the following definition we introduce an auxiliary category which is actually a little too big; we will cut it down to size in Definition 4.4 below.

Definition 4.2. Let $\text{mod-}H^\infty$ be the category whose objects are diagrams

$$M = (M^1 \xrightarrow{\iota^1} M^2 \xrightarrow{\iota^2} M^3 \xrightarrow{\iota^3} \cdots)$$

such that $M^r \in \text{mod-}H^r$ for each r and ι^r gives an H^r -module isomorphism $M^r \xrightarrow{\sim} M^{r+1}e^r$ for all $r \geq 1$. A morphism $f : M \rightarrow N$ in $\text{mod-}H^\infty$ means a sequence $(f^r)_{r \geq 1}$ of H^r -module homomorphisms $f^r : M^r \rightarrow N^r$ such that the following diagram commutes:

$$\begin{array}{ccccccc} M^1 & \longrightarrow & M^2 & \longrightarrow & M^3 & \longrightarrow & \dots \\ f^1 \downarrow & & f^2 \downarrow & & f^3 \downarrow & & \\ N^1 & \longrightarrow & N^2 & \longrightarrow & N^3 & \longrightarrow & \dots \end{array} \quad (4.7)$$

We write simply $\text{Hom}_H(M, N)$ for the morphisms in $\text{mod-}H^\infty$. It is easy to see that $\text{mod-}H^\infty$ is an abelian category. We stress however that there is no algebra H in sight here.

For any $s \geq 1$, an object $M \in \text{mod-}H^\infty$ is determined uniquely up to isomorphism just by knowledge of its tail $(M^s \rightarrow M^{s+1} \rightarrow \dots)$. We make this statement precise by introducing the category $\text{Mod-}H^{\geq s}$ consisting of diagrams $M = (M^s \xrightarrow{\iota^s} M^{s+1} \xrightarrow{\iota^{s+1}} \dots)$ with $M^r \in \text{mod-}H^r$ and $\iota^r : M^r \xrightarrow{\sim} M^{r+1}e^r$ for each $r \geq s$ (just like in Definition 4.2 but starting at r not 1). Then there is a forgetful functor

$$\text{tail}_s : \text{mod-}H^\infty \rightarrow \text{mod-}H^{\geq s} \quad (4.8)$$

sending $(M^1 \rightarrow M^2 \rightarrow \dots)$ to its tail $(M^s \rightarrow M^{s+1} \rightarrow \dots)$. Let

$$\text{head}_s : \text{mod-}H^{\geq s} \rightarrow \text{mod-}H^\infty \quad (4.9)$$

be the functor sending $(M^s \xrightarrow{\iota^s} M^{s+1} \xrightarrow{\iota^{s+1}} \dots)$ to $(M^1 \xrightarrow{\iota^1} \dots \xrightarrow{\iota^{s-1}} M^s \xrightarrow{\iota^s} \dots)$, where for $r < s$ we let $M^r := M^s e^{r,s} \in \text{mod-}H^r$; the maps $\iota^r : M^r \rightarrow M^{r+1}$ for $r < s$ are simply the inclusions. Obviously $\text{tail}_s \circ \text{head}_s = \text{id}$. It is an easy exercise to show moreover that $\text{head}_s \circ \text{tail}_s \cong \text{id}$. Thus the functors tail_s and head_s are quasi-inverse equivalences of categories.

For each $r \geq 1$, we define two more functors

$$\text{top}_r : \text{mod-}H^{\geq r} \rightarrow \text{mod-}H^r, \quad \text{top}_r^! : \text{mod-}H^r \rightarrow \text{mod-}H^{\geq r}. \quad (4.10)$$

The first of these is defined simply by projecting M onto its top term M^r . The second is defined on $M \in \text{mod-}H^r$ by $\text{top}_r^! M := (M^r \rightarrow M^{r+1} \rightarrow \dots) \in \text{mod-}H^{\geq r}$, where $M^s := M \otimes_{H^r} e^{r,s} H^s \in \text{mod-}H^s$. The linear maps $\iota^s : M^s \rightarrow M^{s+1}$ are the maps $M^s \rightarrow M^{s+1}, v \otimes h \mapsto v \otimes \phi^s(h)$. Finally we set

$$\text{pr}_r := \text{top}_r \circ \text{tail}_r : \text{mod-}H^\infty \rightarrow \text{mod-}H^r, \quad (4.11)$$

$$\text{pr}_r^! := \text{head}_r \circ \text{top}_r^! : \text{mod-}H^r \rightarrow \text{mod-}H^\infty. \quad (4.12)$$

The first of these pr_r is of course just the obvious projection onto the r th component. It is also clear that $\text{pr}_r \circ \text{pr}_r^! \cong \text{id}$.

Lemma 4.3. *The functor $\text{pr}_r^!$ is left adjoint to pr_r .*

Proof. It suffices to check that $\text{top}_r^!$ is left adjoint to top_r . The counit of the adjunction on object $M = (M^r \xrightarrow{\iota^r} M^{r+1} \xrightarrow{\iota^{r+1}} \dots)$ is $(\eta^{r,s})_{s \geq r} : \text{top}_r^!(\text{top}_r M) \rightarrow M$ defined from

$$\eta^{r,s} : M^r \otimes_{H^r} e^{r,s} H^s \mapsto M^s, \quad v \otimes h \mapsto \iota^{r,s}(v)h, \quad (4.13)$$

setting $\iota^{r,s} := \iota^{s-1} \circ \dots \circ \iota^r$. We leave the routine checks to the reader. \square

Definition 4.4. We say that $M \in \text{mod-}H^\infty$ is r -stable if it is in the essential image of the functor $\text{pr}_r^!$; equivalently the maps (4.13) are isomorphisms for all $s > r$. Then M is *stable* if it is r -stable for some $r \geq 1$. Finally let $\text{mod-}H$ be the full subcategory of $\text{mod-}H^\infty$ consisting of all stable objects. (We will see soon that $\text{mod-}H$ is itself an abelian category but this is not obvious as it is not a Serre subcategory of $\text{mod-}H^\infty$.)

Lemma 4.5. *If $M \in \text{mod-}H$ is stable at level r and $N \in \text{mod-}H^\infty$ is any object then $\text{pr}_r : \text{Hom}_H(M, N) \rightarrow \text{Hom}_{H^r}(M^r, N^r)$ is an isomorphism.*

Proof. We have that $M \cong \text{pr}_r^!(M_r)$. Now use Lemma 4.3. \square

Now we bring the category \mathcal{C} back into the picture. Let

$$\mathbb{V}^r := \text{Hom}_{\mathcal{C}}(T^r, -) : \mathcal{C} \rightarrow \text{mod-}H^r. \quad (4.14)$$

Then let $\mathbb{V} : \mathcal{C} \rightarrow \text{mod-}H^\infty$ be the functor sending $M \in \mathcal{C}$ to

$$\mathbb{V}M := (\mathbb{V}^1 M \xrightarrow{\iota^1} \mathbb{V}^2 M \xrightarrow{\iota^2} \mathbb{V}^3 M \xrightarrow{\iota^3} \dots), \quad (4.15)$$

where the isomorphism $\iota^r : \text{Hom}_{\mathcal{C}}(T^r, M) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(e^r T^{r+1}, M)$ here is the map $\phi \mapsto \phi \circ (\theta^r)^{-1}$. On a morphism $f : M \rightarrow N$ we let $\mathbb{V}f := (\mathbb{V}^r f)_{r \geq 1}$. By Lemma 2.19, each T^r is a prinjective object of \mathcal{C} , hence the functors \mathbb{V}^r and \mathbb{V} are both exact. Also $\mathbb{V}^r = \text{pr}_r \circ \mathbb{V}$. Recalling the definition (2.38), we let

$$Y(\lambda) = (Y^1(\lambda) \rightarrow Y^2(\lambda) \rightarrow \dots) := \mathbb{V}P(\lambda), \quad (4.16)$$

$$D(\mu) = (D^1(\mu) \rightarrow D^2(\mu) \rightarrow \dots) := \mathbb{V}L(\mu), \quad (4.17)$$

for $\lambda \in \Lambda$ and $\mu \in \Lambda^\circ$. By Theorem 2.22 for the category \mathcal{C}_r (and the usual theory of quotient functors) the objects $\{D^r(\mu) \mid \mu \in \Lambda_r^\circ\}$ give a complete set of pairwise non-isomorphic irreducible H^r -modules. Moreover $Y^r(\mu)$ is the projective cover of $D^r(\mu)$ in $\text{mod-}H^r$ for each $\mu \in \Lambda_r^\circ$.

Theorem 4.6. *The essential image of the functor \mathbb{V} is the subcategory $\text{mod-}H$ of $\text{mod-}H^\infty$.*

Proof. We first show that $\mathbb{V}M$ is stable for any $M \in \mathcal{C}$. Given M pick $r \geq 1$ so that

$$\bigcup_{\substack{\mu \in \Lambda \\ [M:L(\mu)] \neq 0}} \{\nu \in \Lambda \mid [P(\mu) : L(\nu)] \neq 0\} \subseteq \Lambda_r. \quad (4.18)$$

This is possible as the set on the left hand side here is finite. We claim for this r that $\mathbb{V}M$ is r -stable. This amounts to showing for each $s > r$ that the H^s -module homomorphism

$$\text{Hom}_{\mathcal{C}}(T^r, M) \otimes_{H^r} e^{r,s} H^s \rightarrow \text{Hom}_{\mathcal{C}}(T^s, M), \quad f \otimes h \mapsto f \circ (\theta^{r,s})^{-1} \circ h$$

is an isomorphism. For surjectivity, we split T^s and T^r into indecomposables $T^s = P_1 \oplus \dots \oplus P_n$ and $T^r = Q_1 \oplus \dots \oplus Q_m$, so that

$$\text{Hom}_{\mathcal{C}}(T^s, M) = \bigoplus_{i=1}^n \text{Hom}_{\mathcal{C}}(P_i, M), \quad \text{Hom}_{\mathcal{C}}(T^r, M) = \bigoplus_{j=1}^m \text{Hom}_{\mathcal{C}}(Q_j, M).$$

By the assumption on r all composition factors of M are of the form $L(\mu)$ for $\mu \in \Lambda_r$. Hence $\text{Hom}_{\mathcal{C}}(P_i, M) = 0$ unless $P_i \cong P(\mu)$ for some $\mu \in \Lambda_r^\circ$. In that

case there is a summand Q_j with $Q_j \cong P_i$. Thus any $f \in \text{Hom}_{\mathcal{C}}(P_i, M) \subseteq \text{Hom}_{\mathcal{C}}(T^s, M)$ factors as $g \circ k$ for some $g \in \text{Hom}_{\mathcal{C}}(Q_j, M) \subseteq \text{Hom}_{\mathcal{C}}(T^r, M)$ and $k \in \text{Hom}_{\mathcal{C}}(P_i, Q_j) \subseteq \text{Hom}_{\mathcal{C}}(T^s, T^r)$. Since $\text{Hom}_{\mathcal{C}}(T^s, T^r) \cong e^{r,s}H^s$, we deduce that $f = g \circ (\theta^{r,s})^{-1} \circ h$ for some $h \in e^{r,s}H^s$, and surjectivity follows.

For injectivity, let K be the kernel of the map, so that there is a short exact sequence

$$0 \rightarrow K \rightarrow \text{Hom}_{\mathcal{C}}(T^r, M) \otimes_{H^r} e^{r,s}H^s \rightarrow \text{Hom}_{\mathcal{C}}(T^s, M) \rightarrow 0.$$

On truncating with the idempotent $e^{r,s}$, the second map becomes an isomorphism, hence $Ke^{r,s} = 0$. Thus all composition factors of K are of the form $\{D^s(\nu) \mid \nu \in \Lambda_s^\circ \setminus \Lambda_r^\circ\}$. On the other hand all composition factors of $\text{Hom}_{\mathcal{C}}(T^s, M)$ are of the form $D^s(\mu)$ for $\mu \in \Lambda_s^\circ$ such that $[M : L(\mu)] \neq 0$. Now the choice of r ensures for such ν and μ that $L(\nu)$ is not a composition factor of $P(\mu)$. Hence $D^s(\nu)$ is not a composition factor of $Y^s(\mu)$, which is the projective cover of $L^s(\mu)$. It follows that $\text{Ext}_{H^s}^1(D^s(\mu), D^s(\nu)) = 0$, and we have proved that the above short exact sequence splits. But then we get that

$$\begin{aligned} \text{Hom}_{H^s}(K, D^s(\nu)) &\leftarrow \text{Hom}_{H^s}(\text{Hom}_{\mathcal{C}}(T^r, M) \otimes_{H^r} e^{r,s}H^s, D^s(\nu)) \\ &\cong \text{Hom}_{H^r}(\text{Hom}_{\mathcal{C}}(T^r, M), D^s(\nu)e^{r,s}) = 0 \end{aligned}$$

for any $\nu \in \Lambda_s^\circ \setminus \Lambda_r^\circ$. This implies that $K = 0$.

So \mathbb{V} restricts to a well-defined functor $\mathbb{V} : \mathcal{C} \rightarrow \text{mod-}H$. We next construct a left adjoint $\mathbb{V}^! : \text{mod-}H \rightarrow \mathcal{C}$ such that $\mathbb{V} \circ \mathbb{V}^! \cong \text{id}$, implying in particular that this new \mathbb{V} is essentially surjective. On an object $M \in \text{mod-}H$ we let $r \geq 1$ be minimal such that M is r -stable then set $\mathbb{V}^!M := M^r \otimes_{H^r} T^r \in \mathcal{C}_{\leq r}$, where $M^r \otimes_{H^r} -$ here is the tensor product functor from the category of H^r -module objects in \mathcal{C} to \mathcal{C} . Note for $s \geq r$ that there is an isomorphism

$$f^{r,s} : M^r \otimes_{H^r} T^r \xrightarrow{\sim} M^s \otimes_{H^s} T^s$$

defined by the composition of the following canonical isomorphisms

$$M^r \otimes_{H^r} T^r \xrightarrow{\text{id} \otimes \theta^{r,s}} M^r \otimes_{H^r} e^{r,s}T^s \cong M^r \otimes_{H^r} e^{r,s}H^s \otimes_{H^s} T^s \xrightarrow{\eta^{r,s} \otimes \text{id}} M^s \otimes_{H^s} T^s,$$

for $\eta^{r,s}$ coming from (4.13). Then on a morphism $f = (f^r)_{r \geq 1} : M \rightarrow N$ we define $\mathbb{V}^!f$ by picking t such that both M and N are t -stable then setting $\mathbb{V}^!f := (f^{s,t})^{-1} \circ f^t \circ f^{r,t} : M^r \otimes_{H^r} T^r \rightarrow M^s \otimes_{H^s} T^s$. One needs to observe that this is well defined independent of the choice of t as $\theta^{s,t} \circ \theta^{r,s} = \theta^{r,t}$ and $\eta^{s,t} \circ \eta^{r,s} = \eta^{r,t}$. Then it follows easily that this is a functor. To see that $\text{id} \cong \mathbb{V} \circ \mathbb{V}^!$, take the natural isomorphism defined on M by the canonical isomorphisms

$$\begin{aligned} M &\xrightarrow{\sim} \text{pr}_r^!(\text{pr}_r M) = \text{pr}_r^!(M^r) \\ &\cong \text{pr}_r^!(M^r \otimes_{H^r} \text{Hom}_{\mathcal{C}}(T^r, T^r)) \xrightarrow{\sim} \text{pr}_r^!(\text{Hom}_{\mathcal{C}}(T^r, M^r \otimes_{H^r} T^r)) \\ &= \text{pr}_r^!(\mathbb{V}_r(\mathbb{V}^!M)) = \text{pr}_r^!(\text{pr}_r(\mathbb{V}(\mathbb{V}^!M))) \xrightarrow{\sim} \mathbb{V}(\mathbb{V}^!M) \end{aligned}$$

where r is minimal so that M is r -stable. Here the morphisms on the first and last lines come from the counit of adjunction $\text{pr}_r^! \circ \text{pr}_r \cong \text{id}$ and the middle one is the obvious isomorphism. In particular this isomorphism gives the unit $\epsilon : \text{id} \rightarrow \mathbb{V} \circ \mathbb{V}^!$ of the claimed adjunction. We proceed write down the counit $\eta : \mathbb{V}^! \circ \mathbb{V} \rightarrow \text{id}$. To

define this on $M \in \mathcal{C}$, let r be minimal such that $\mathbb{V}M$ is r -stable. Then we take η_M to be the obvious evaluation

$$\mathbb{V}^!(\mathbb{V}M) = \mathrm{Hom}_{H^r}(T^r, M) \otimes_{H^r} T^r \xrightarrow{\mathrm{ev}_r} M. \quad (4.19)$$

This can also be obtained as the composition

$$\mathbb{V}^!(\mathbb{V}M) = \mathrm{Hom}_{H^r}(T^r, M) \otimes_{H^r} T^r \xrightarrow{f^{r,s}} \mathrm{Hom}_{H^s}(T^s, M) \otimes_{H^s} T^s \xrightarrow{\mathrm{ev}_s} M$$

for any $s \geq r$; in particular this makes it clear that it is surjective. We leave the remaining checks to the reader. \square

Remark 4.7. For $M \in \mathcal{C}$, one can consider the integer

$$r_M := \min\{r \geq 1 \mid \mathbb{V}M \text{ is } r\text{-stable}\}.$$

Of course this depends implicitly on the choices of the intervals $I_1 \subset I_2 \subset \dots$. We observed during the proof of Theorem 4.6 that the map (4.19) with $r = r_M$ is surjective. Hence all constituents of the head of M are of the form $L(\lambda)$ for $\lambda \in \Lambda_r$ and $M \in \mathcal{C}_{\leq r}$; these properties give a lower bound for r_M . For a (surely much too big) upper bound one can take the smallest r satisfying (4.18).

Theorem 4.8. *The category $\mathrm{mod}\text{-}H$ is Schurian with a complete set of pairwise non-isomorphic irreducibles given by the objects $\{D(\mu) \mid \mu \in \Lambda^\circ\}$. Moreover $Y(\mu)$ is the projective cover of $D(\mu)$ in $\mathrm{mod}\text{-}H$ for each $\mu \in \Lambda^\circ$. Finally*

$$\mathbb{V} : \mathcal{C} \rightarrow \mathrm{mod}\text{-}H \quad (4.20)$$

satisfies the universal property of the quotient of \mathcal{C} by the Serre subcategory generated by the irreducible objects $\{L(\lambda) \mid \lambda \in \Lambda \setminus \Lambda^\circ\}$.

Proof. Now we show that the functor $\mathbb{V} : \mathcal{C} \rightarrow \mathrm{mod}\text{-}H$ has the universal property of quotients. We need to show for any abelian category \mathcal{C}' and any exact functor $\mathbb{U} : \mathcal{C} \rightarrow \mathcal{C}'$ such that $\mathbb{U}(L(\lambda)) = 0$ for all $\lambda \in \Lambda \setminus \Lambda^\circ$ that there exists a unique (up to isomorphism) functor $\bar{\mathbb{U}} : \mathrm{mod}\text{-}H \rightarrow \mathcal{C}'$ such that $\bar{\mathbb{U}} \circ \mathbb{V} \cong \mathbb{U}$. Composing on the right with $\mathbb{V}^!$ we see at once that the only choice (up to isomorphism) is to take $\bar{\mathbb{U}} := \mathbb{U} \circ \mathbb{V}^!$. The counit of adjunction η gives a natural transformation $\mathbb{U}\eta : \mathbb{U} \circ \mathbb{V}^! \circ \mathbb{V} = \bar{\mathbb{U}} \circ \mathbb{V} \rightarrow \mathbb{U}$. To see this is an isomorphism, take $M \in \mathcal{C}$ and let r be minimal such that $\mathbb{V}M$ is r -stable. Then $(\mathbb{U}\eta)_M : \bar{\mathbb{U}}(\mathbb{V}M) \rightarrow \mathbb{U}M$ is the morphism obtained by applying \mathbb{U} to the second arrow in the following short exact sequence:

$$0 \rightarrow K \rightarrow \mathrm{Hom}_{\mathcal{C}}(T^r, M) \otimes_{H^r} T^r \xrightarrow{\eta_M} M \rightarrow 0.$$

To see that this map is an isomorphism it suffices by exactness of \mathbb{U} to show that $\mathbb{U}K = 0$. This follows because $\mathbb{V}K = 0$.

Finally we let \mathcal{C}° be the quotient of \mathcal{C} by the Serre subcategory generated by the objects $\{L(\lambda) \mid \lambda \in \Lambda \setminus \Lambda^\circ\}$ and $\pi : \mathcal{C} \rightarrow \mathcal{C}^\circ$ be the quotient functor. By the general theory of quotients this is a Schurian category with irreducibles $\{\pi L(\mu) \mid \mu \in \Lambda^\circ\}$; the projective cover of $\pi L(\lambda)$ is $\pi P(\lambda)$. The universal property established in the previous paragraph gives us a functor $\bar{\pi} : \mathrm{mod}\text{-}H \rightarrow \mathcal{C}^\circ$ such that $\pi \cong \bar{\pi} \circ \mathbb{V}$. On the other hand by the universal property of \mathcal{C}° there is a functor $\bar{\mathbb{V}} : \mathcal{C}^\circ \rightarrow \mathrm{mod}\text{-}H^\infty$ such that $\mathbb{V} \cong \bar{\mathbb{V}} \circ \pi$; we are being careful here since we do not yet know that $\mathrm{mod}\text{-}H$ is itself abelian. The essential image of $\bar{\mathbb{V}}$ is $\mathrm{mod}\text{-}H$, i.e. it is actually a functor $\mathcal{C}^\circ \rightarrow \mathrm{mod}\text{-}H$. Then the usual argument with

uniqueness shows that $\bar{\mathbb{V}}$ and $\bar{\pi}$ are quasi-inverse equivalences. The theorem now follows directly. \square

At last we can prove the appropriate analog of the double centralizer property from Theorem 2.13.

Theorem 4.9. *The functor $\mathbb{V} : \mathcal{C} \rightarrow \text{mod-}H$ is fully faithful on projectives. Moreover for each $\lambda \in \Lambda$ the object $Y(\lambda) = \mathbb{V}P(\lambda) \in \text{mod-}H$ is independent (up to isomorphism) of the particular choice of \mathcal{C} .*

Proof. Take projectives $P, Q \in \mathcal{C}$ and choose r so that both $\mathbb{V}P$ and $\mathbb{V}Q$ are r -stable. By Remark 4.7 this means that P and Q are projective objects also in \mathcal{C}_r . Hence by the double centralizer property from Theorem 2.13 the composition \mathbb{V}^r of the following two maps is an isomorphism:

$$\text{Hom}_{\mathcal{C}}(P, Q) \xrightarrow{\mathbb{V}} \text{Hom}_H(\mathbb{V}P, \mathbb{V}Q) \xrightarrow{\text{pr}_r^!} \text{Hom}_{H^r}(\mathbb{V}^r P, \mathbb{V}^r Q).$$

Also the second map here is an isomorphism because of Lemma 4.5. Hence the first map is an isomorphism, proving that \mathbb{V} is fully faithful on projectives.

Now suppose we are given another tensor product categorification \mathcal{C}' . Define $\mathbb{V}' : \mathcal{C}' \rightarrow \text{mod-}H$ exactly as above then set

$$Y'(\lambda) = (Y'^1(\lambda) \rightarrow Y'^2(\lambda) \rightarrow \cdots) := \mathbb{V}'P'(\lambda).$$

Pick r so that both $Y(\lambda)$ and $Y'(\lambda)$ are r -stable. Then $P(\lambda)$ is projective both in \mathcal{C}_r and in the analogously defined subquotient \mathcal{C}'_r . By the final part of Theorem 2.13 we get that $Y^r(\lambda) \cong Y'^r(\lambda)$ in $\text{mod-}H^r$, hence $Y(\lambda) \cong \text{pr}_r^! Y^r(\lambda) \cong \text{pr}_r^! Y'^r(\lambda) \cong Y'(\lambda)$. \square

Remark 4.10. The slightly stronger result from Remark 2.14 also holds in the present situation; the proof is the same.

4.3. Categorical action on stable modules. Next we are going to introduce a categorical \mathfrak{sl}_I -action onto the category $\text{mod-}H$ in such a way that the quotient functor $\mathbb{V} : \mathcal{C} \rightarrow \text{mod-}H$ is strongly equivariant. For each $r \geq 1$ we have the usual induction and restriction functors $F^r, E^r : \text{mod-}H^r \rightarrow \text{mod-}H^r$ from (2.32). In terms of cyclotomic quiver Hecke algebras, these are the direct sums over all $i \in I_r$ of the i -induction and i -restriction functors

$$F_i^r : \text{mod-}H^r \rightarrow \text{mod-}H^r, \quad E_i^r : \text{mod-}H^r \rightarrow \text{mod-}H^r \quad (4.21)$$

defined as follows. Recalling the idempotent (4.2), F_i^r is given by tensoring over H_d^r with the bimodule $1_{d;i}^r H_{d+1}^r$, viewing $1_{d;i}^r H_{d+1}^r$ as an (H_d^r, H_{d+1}^r) -bimodule via the homomorphism (4.3). Its canonical right adjoint E_i^r is given on a right H_{d+1}^r -module simply by right multiplication by this idempotent, viewing the result as a right H_d^r -module via $\iota_{d;i}^r$ again. The endomorphism $\xi \in \text{End}(F_i^r)$ is induced by the endomorphism of the bimodule $1_{d;i}^r H_{d+1}^r$ defined by left multiplication by ξ_{d+1} . To define $\tau \in \text{Hom}(F_j^r \circ F_i^r, F_i^r \circ F_j^r)$, note obviously that

$$1_{d;i}^r H_{d+1}^r \otimes_{H_{d+1}^r} 1_{d+1;j}^r H_{d+2}^r \cong 1_{d;j}^r H_{d+2}^r \quad \text{where} \quad 1_{d;j} := \sum_{\substack{\mathbf{i} \in I_r^{d+2} \\ i_{d+1}=i \\ i_{d+2}=j}} 1_{\mathbf{i}} \in H_{d+2}^r.$$

Then τ comes from the bimodule homomorphism $1_{d;i}^r H_{d+2}^r \rightarrow 1_{d;i}^r H_{d+2}^r$ defined by left multiplication by τ_{d+1} . The canonical adjunction making each (F_i^r, E_i^r) into an adjoint pair comes simply from adjointness of tensor and hom. When we use this as in (2.12)–(2.13) to transfer $\xi \in \text{End}(F_i^r)$ and $\tau \in \text{Hom}(F_j^r \circ F_i^r, F_i^r \circ F_j^r)$ to $\xi \in \text{End}(E_i^r)$ and $\tau \in \text{Hom}(E_i^r \circ E_j^r, E_j^r \circ E_i^r)$ we get the natural transformations which are defined on a right H_d^r -module simply by right multiplication by ξ_d and τ_{d-1} , respectively.

We are ready to define $F_i : \text{mod-}H \rightarrow \text{mod-}H$. Take $M = (M^1 \xrightarrow{\iota^1} M^2 \xrightarrow{\iota^2} \dots) \in \text{mod-}H$. Suppose to start with that r is chosen so that $i \in I_r$. By Lemma 4.1(iv), the restriction of ϕ_{d+1}^r gives a right H_{d+1}^r -module homomorphism

$$\psi_{d;i}^r : 1_{d;i}^r H_{d+1}^r \hookrightarrow 1_{d_r+d;i}^{r+1} H_{d_r+d+1}^{r+1} e_{d+1}^r.$$

Let $\psi_i^r := \bigoplus_{d \geq 0} \psi_{d;i}^r$ so that the map $\iota^r \otimes \psi^r : F_i^r M^r \rightarrow (F_i^{r+1} M^{r+1}) e^r$ is an H^r -module homomorphism. Now we assume that r is minimal such that M is r -stable and $i \in I_r$. Then define

$$F_i M := \text{head}_r(F_i^r M^r \xrightarrow{\iota^r \otimes \psi^r} F_i^{r+1} M^{r+1} \xrightarrow{\iota^{r+1} \otimes \psi^{r+1}} \dots). \quad (4.22)$$

For this to even make sense we need to justify that the maps $\iota^s \otimes \psi^s : F_i^s M^s \rightarrow (F_i^{s+1} M^{s+1}) e^s$ are isomorphisms for all $s \geq r$, so that the object in parentheses really is an object of $\text{mod-}H^{\geq r}$. This follows at once from the next lemma, which shows moreover that $F_i M$ is r -stable.

Lemma 4.11. *If $M \in \text{mod-}H$ is r -stable and $i \in I_r$ then the maps*

$$\eta^s : F_i^s M^s \otimes_{H^s} e^s H^{s+1} \rightarrow F_i^{s+1} M^{s+1}, \quad v \otimes h' \otimes h \mapsto \iota^s(v) \otimes \psi^s(h')h$$

are isomorphisms for all $s \geq r$. Hence F_i sends r -stable objects to r -stable objects.

Proof. Suppose that $M^s \in \text{mod-}H_d^s$. To prove the lemma we need to show that the map

$$\begin{aligned} M^s \otimes_{H_d^s} 1_{d;i}^s H_{d+1}^s \otimes_{H_{d+1}^s} e_{d+1}^s H_{d_r+d+1}^{s+1} &\rightarrow M^{s+1} \otimes_{H_{d_r+d}^{s+1}} 1_{d_r+d;i}^{s+1} H_{d_r+d+1}^{s+1} \\ v \otimes h' \otimes h &\mapsto \iota^s(v) \otimes \psi^s(h')h \end{aligned}$$

is an isomorphism. Taking $h' = 1_{d;i}^s$ and contracting the second tensor product using Lemma 4.1(iv), this is equivalent to showing that the map

$$\iota^s \otimes \text{id} : M^s \otimes_{H_d^s} 1_{d_r+d;i}^{s+1} e_{d+1}^s H_{d_r+d+1}^{s+1} \rightarrow M^{s+1} \otimes_{H_{d_r+d}^{s+1}} 1_{d_r+d;i}^{s+1} H_{d_r+d+1}^{s+1}$$

is an isomorphism. As M is r -stable the following map is an isomorphism:

$$\begin{aligned} M^s \otimes_{H_d^s} e_d^s H_{d_r+d}^{s+1} \otimes_{H_{d_r+d}^{s+1}} 1_{d_r+d;i}^{s+1} H_{d_r+d+1}^{s+1} &\rightarrow M^{s+1} \otimes_{H_{d_r+d}^{s+1}} 1_{d_r+d;i}^{s+1} H_{d_r+d+1}^{s+1} \\ v \otimes h' \otimes h &\mapsto \iota^s(v) h' \otimes h. \end{aligned}$$

Taking $h' = e_d^s$ and contracting the second tensor product using Lemma 4.1(iii), gives exactly the desired isomorphism. \square

So now by (4.22) we have defined the functor F_i on an r -stable object. On a morphism $f : M \rightarrow N$ we just pick r so that $i \in I_r$ and M is r -stable. Then $F_i M$ is r -stable too, so by Lemma 4.5 there is a unique morphism $F_i f : F_i M \rightarrow F_i N$

such that $(F_i f)^r = F_i^r f^r$. This is independent of the choice of r , which is all that is needed to check that F_i is a well-defined functor.

We also must define natural transformations

$$\xi \in \text{End}(F_i), \quad \tau \in \text{Hom}(F_j \circ F_i, F_i \circ F_j).$$

Take some $M \in \text{mod-}H$ and pick r so that $i \in I_r$ and M is r -stable. Then by Lemma 4.5 again there is a unique morphism $\xi_M : F_i M \rightarrow F_i M$ such that $(\xi_M)^r = \xi_{M^r}$; similarly, there is a unique $\tau_M : F_j F_i M \rightarrow F_i F_j M$ such that $(\tau_M)^r = \tau_{M^r}$. The naturality of ξ and τ follows because these definitions of ξ_M and τ_M are independent of the particular choice of r , as may be checked using Lemma 4.1.

We turn our attention to $E_i : \text{mod-}H \rightarrow \text{mod-}H$. For this we start by defining a functor $E_i : \text{mod-}H^\infty \rightarrow \text{mod-}H^\infty$. Let r be minimal such that $i \in I_r$. Then for $M = (M^1 \rightarrow M^2 \rightarrow \dots) \in \text{mod-}H^\infty$ we let

$$E_i M := (0 \rightarrow \dots \rightarrow 0 \rightarrow E_i^r M^r \rightarrow E_i^{r+1} M^{r+1} \rightarrow \dots) \quad (4.23)$$

where the maps are simply the restrictions. Lemma 4.12 below verifies that this is indeed an object of $\text{mod-}H^\infty$. On a morphism $f : M \rightarrow N$ we define $(E_i f)^s := E_i^s f^s$ if $i \in I_s$ and $(E_i f)^s := 0$ otherwise. Similarly we can introduce natural transformations $\xi : E_i \rightarrow E_i$ and $\tau : E_i \circ E_j \rightarrow E_j \circ E_i$; the homomorphisms ξ_M and τ_M are defined so that $(\xi_M)^s = \xi_{M^s}$ and $(\tau_M)^s = \tau_{M^s}$ whenever $i \in I_s$.

Lemma 4.12. *Suppose we are given $M \in \text{mod-}H^\infty$, $i \in I$ and $r \geq 1$. If $i \in I_r$ then the restriction of $\iota^r : M^r \xrightarrow{\sim} M^{r+1} e^r$ is an isomorphism $E_i^r M^r \xrightarrow{\sim} (E_i^{r+1} M^{r+1}) e^r$. If $i \in I_{r+1} \setminus I_r$ then $(E_i^{r+1} M^{r+1}) e^r = 0$.*

Proof. We may assume that $M^r \in \text{mod-}H_{d+1}^r$. Then we just have to recall that $E_i^r M^r = M^r 1_{d,i}^r$. So using also Lemma 4.1(iv) we deduce that ι^r maps it isomorphically to $M^{r+1} 1_{d_r+d,i}^{r+1} e_{d+1}^r$, which is all of $(E_i^{r+1} M^{r+1}) e_{d+1}^r$ as required. For the second statement just note that $(E_i^{r+1} M^{r+1}) e_{d+1}^r = M^{r+1} 1_{d_r+d,i}^{r+1} e_{d+1}^r$ which is zero by Lemma 4.1(i) since $i \notin I_r$. \square

Lemma 4.13. *For each $i \in I$ there is an isomorphism $\zeta_i^+ : \mathbb{V} \circ E_i \xrightarrow{\sim} E_i \circ \mathbb{V}$ of functors from \mathcal{C} to $\text{mod-}H^\infty$, such that $\xi \mathbb{V} \circ \zeta_i^+ = \zeta_i^+ \circ \mathbb{V} \xi$ in $\text{Hom}(\mathbb{V} \circ E_i, E_i \circ \mathbb{V})$ and $\tau \mathbb{V} \circ E_i \zeta_j^+ \circ \zeta_i^+ E_j = E_j \zeta_i^+ \circ \zeta_j^+ E_i \circ \mathbb{V} \tau$ in $\text{Hom}(\mathbb{V} \circ E_i \circ E_j, E_j \circ E_i \circ \mathbb{V})$.*

Proof. To define ζ_i^+ on an object $M \in \mathcal{C}$, take any r such that $i \in I_r$ and both $\mathbb{V}(E_i M)$ and $E_i(\mathbb{V} M)$ are r -stable. By Lemma 2.15 there exists a canonical isomorphism $(\zeta_i^+)^r : \mathbb{V}^r \circ E_i^r \xrightarrow{\sim} E_i^r \circ \mathbb{V}^r$ with the analogous strong equivariance properties. Then we define $(\zeta_i^+)_M : \mathbb{V}(E_i M) \rightarrow E_i(\mathbb{V} M)$ to be the unique morphism with r th component equal to $(\zeta_i^+)^r_M$. For the naturality one just needs to observe that this is independent of the choice of r . The strong equivariance properties are immediate from the ones for $(\zeta_i^+)^r$. \square

Lemma 4.14. *The functor $E_i : \text{mod-}H^\infty \rightarrow \text{mod-}H^\infty$ sends stable modules to stable modules, hence it restricts to $E_i : \text{mod-}H \rightarrow \text{mod-}H$.*

Proof. This is immediate from Lemma 4.13 since $\mathbb{V} : \mathcal{C} \rightarrow \text{mod-}H$ is dense by Theorem 4.6. \square

So now we have defined the endofunctors F_i and E_i on $\text{mod-}H$. It remains to define an adjunction making (F_i, E_i) into an adjoint pair. Given an object M we pick r large enough so that $i \in I_r$, and $M, E_i M$ and $E_i F_i M$ are r -stable. Then we take the unit and counit of the adjunction on object M to be induced by the ones for E_i^r and F_i^r on $M^r \in \text{mod-}H^r$. As usual to prove naturality one needs to observe that the resulting morphisms are independent of the choice of r . The natural transformations ξ and τ on E_i and $E_i \circ E_j$ are compatible under this choice of adjunction with the corresponding natural transformations on F_i and $F_j \circ F_i$.

Theorem 4.15. *The category $\text{mod-}H$, together with the adjoint pairs (F_i, E_i) for each $i \in I$ and the natural transformations ξ and τ , is an \mathfrak{sl}_I -categorification in the sense of Definition 2.6. Moreover the quotient functor $\mathbb{V} : \mathcal{C} \rightarrow \text{mod-}H$ is strongly equivariant.*

Proof. We have all of the required data in hand, so just need to check the axioms (SL1')–(SL4'). The first two follow via Lemma 4.5 and the truth of the corresponding axioms on each $\text{mod-}H^r$. Also Lemma 4.13 checks exactly the properties for \mathbb{V} to be strongly equivariant. Hence we obtain a second adjunction making (E_i, F_i) into an adjoint pair by pushing some choice of such an adjunction in \mathcal{C} through \mathbb{V} exactly like in Remark 5.13. Finally (SL4') holds because the left adjoint functor $\mathbb{V}^!$ embeds $[\text{mod-}H]$ into $[\mathcal{C}]$, and the latter is integrable. \square

4.4. Proof of uniqueness for infinite intervals. Now we can complete the proof of the uniqueness part of Theorem 2.11 for infinite intervals by mimicking the arguments explained for finite intervals in §2.7. Suppose we are given another \mathfrak{sl}_I -tensor product categorification \mathcal{C}' of the same type as \mathcal{C} . Introduce the primed analog \mathbb{V}' of the quotient functor \mathbb{V} , and set $Y'(\lambda) := \mathbb{V}' P'(\lambda)$ for each $\lambda \in \Lambda$. Letting A and A' be the basic algebras underlying \mathcal{C} and \mathcal{C}' as in (2.22) and (2.24), the double centralizer property from Theorem 4.9 implies the existence of isomorphisms (2.30)–(2.31). Applying the last assertion of Theorem 4.9, we pick isomorphisms $Y(\lambda) \cong Y'(\lambda)$ in $\text{mod-}H$ for each $\lambda \in \Lambda$. These induce algebra isomorphisms $A \cong A'$ hence an isomorphism of categories $\text{mod-}A \xrightarrow{\sim} \text{mod-}A'$. Since \mathcal{C} is equivalent to $\text{mod-}A$ and \mathcal{C}' is equivalent to $\text{mod-}A'$, we get the desired equivalence $\mathbb{G} : \mathcal{C} \rightarrow \mathcal{C}'$. To see that this equivalence is strongly equivariant we just need to introduce the bimodules B and B' from (2.23) and (2.25). These satisfy (2.33)–(2.36) for the same reasons as in §2.7, using Theorem 4.15 in place of Lemma 2.15.

5. GRADED TENSOR PRODUCT CATEGORIFICATIONS

We are in the business now of constructing graded lifts of the structures introduced so far. We are going to use the same notation in this section for graded versions as was used in the earlier sections in the ungraded setting. To avoid confusion we add bars to all our earlier notation. For example we denote the natural \mathfrak{sl}_I -module V_I now by \bar{V}_I , so that the notation V_I can be reused for its quantum analog.

5.1. Quantum \mathfrak{sl}_I . Consider the field $\mathbb{Q}(q)$ equipped with the *bar involution* defined by $\overline{f(q)} := f(q^{-1})$. For an interval I , the quantized enveloping algebra $U_q \mathfrak{sl}_I$ is the $\mathbb{Q}(q)$ -algebra with generators $\{f_i, e_i, k_i \mid i \in I\}$ subject to well-known relations. We will often appeal to facts from [Lu]; note for this that our q is Lusztig's v^{-1} while our f_i, e_i, k_i are Lusztig's F_i, E_i, K_i^{-1} . We make $U_q \mathfrak{sl}_I$ into a Hopf algebra as in [Lu] with comultiplication Δ defined from

$$\Delta(f_i) := 1 \otimes f_i + f_i \otimes k_i, \quad \Delta(e_i) := k_i^{-1} \otimes e_i + e_i \otimes 1, \quad \Delta(k_i) := k_i \otimes k_i.$$

There is a linear algebra antiautomorphism $*$: $U_q \mathfrak{sl}_I \rightarrow U_q \mathfrak{sl}_I$ such that

$$f_i^* := q e_i k_i, \quad e_i^* := q f_i k_i^{-1}, \quad k_i^* := k_i.$$

This is a coalgebra automorphism whose square is the identity. Also $U_q \mathfrak{sl}_I$ possesses an antilinear algebra automorphism ψ , also usually called the bar involution, which is defined from

$$\psi(f_i) := f_i, \quad \psi(e_i) := e_i, \quad \psi(k_i) := k_i^{-1}. \quad (5.1)$$

Let $\psi^* := * \circ \psi \circ *$, so that ψ^* is another antilinear algebra automorphism with

$$\psi^*(f_i^*) = f_i^*, \quad \psi^*(e_i^*) = e_i^*, \quad \psi^*(k_i^*) = k_i^{*-1}. \quad (5.2)$$

Equivalently

$$\psi^*(f_i) = q^2 f_i k_i^{-2}, \quad \psi^*(e_i) = q^2 e_i k_i^2, \quad \psi^*(k_i) = k_i^{-1}.$$

For $\varpi \in P_I$ the ϖ -weight space of a $U_q \mathfrak{sl}_I$ -module M is the subspace

$$M_\varpi := \{v \in M \mid k_i v = q^{\varpi \cdot \alpha_i} v \text{ for each } i \in I\}.$$

Then the notion of integrable module is defined in the same way as for \mathfrak{sl}_I . For $n \geq 0$ and $c \in \{0, 1\}$, let $\bigwedge^{n,c} V_I$ be the integrable $U_q \mathfrak{sl}_I$ -module on basis $\{v_\lambda \mid \lambda \in \Lambda_{I;n,c}\}$, with f_i and e_i acting by the same formulae (2.4)–(2.5) as before and $k_i v_\lambda := q^{\lambda_i - \lambda_{i+1}} v_\lambda$. Note that f_i^* and e_i^* act on $\bigwedge^{n,c} V_I$ in exactly the same way as e_i and f_i , respectively.

More generally given a type $(\underline{n}, \underline{c})$ of level l we have the tensor product $\bigwedge^{\underline{n}, \underline{c}} V_I := \bigwedge^{n_1, c_1} V_I \otimes \cdots \otimes \bigwedge^{n_l, c_l} V_I$. It has a monomial basis $\{v_\lambda \mid \lambda \in \Lambda_{I; \underline{n}, \underline{c}}\}$ just like before. The actions of f_j and e_j on these basis vectors are given explicitly by the formulae

$$f_j v_\lambda = \sum_{\substack{1 \leq i \leq l \\ \lambda_{ij}=1 \\ \lambda_{i(j+1)}=0}} q^{(|\lambda_{i+1}| + \cdots + |\lambda_l|) \cdot \alpha_j} v_{t_{ij}(\lambda)}, \quad (5.3)$$

$$e_j v_\lambda = \sum_{\substack{1 \leq i \leq l \\ \lambda_{ij}=0 \\ \lambda_{i(j+1)}=1}} q^{-(|\lambda_1| + \cdots + |\lambda_{i-1}|) \cdot \alpha_j} v_{t_{ij}(\lambda)}, \quad (5.4)$$

writing $t_{ij}(\lambda)$ for the 01-matrix obtained from λ by flipping its entries λ_{ij} and $\lambda_{i(j+1)}$. In general it is no longer the case that f_i^* and e_i^* act in the same way as e_i and f_i . Instead we have that

$$(uv, w) = (v, u^* w) \quad (5.5)$$

for $u \in U_q \mathfrak{sl}_I$ and $v, w \in \bigwedge^{\mathbf{n}, \mathbf{c}} V_I$, where $(-, -)$ is the symmetric bilinear form on $\bigwedge^{\mathbf{n}, \mathbf{c}} V_I$ with respect to which the monomial basis is orthonormal.

5.2. Graded lifts. By a *graded category* we mean a category \mathcal{C} equipped with an adjoint pair (q, q^{-1}) of self-equivalences; the adjunction induces canonical isomorphisms $q^m \circ q^n \xrightarrow{\sim} q^{m+n}$ for all $m, n \in \mathbb{Z}$ making the obvious square of isomorphisms from $q^m \circ q^n \circ q^l$ to q^{m+n+l} commute. Let $\widehat{\mathcal{C}}$ denote the category with the same objects as \mathcal{C} and

$$\mathrm{Hom}_{\widehat{\mathcal{C}}}(M, N) := \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(M, N)_m$$

where $\mathrm{Hom}_{\mathcal{C}}(M, N)_m$ denotes $\mathrm{Hom}_{\mathcal{C}}(q^m M, N) \cong \mathrm{Hom}_{\mathcal{C}}(M, q^{-m} N)$; composition is induced by that of \mathcal{C} making $\widehat{\mathcal{C}}$ into a category enriched in graded vector spaces. We refer to elements of $\mathrm{Hom}_{\mathcal{C}}(M, N)_m$ as *homogeneous morphisms of degree m* . Thus morphisms in \mathcal{C} itself are the homogeneous morphisms of degree zero in $\widehat{\mathcal{C}}$. Assuming \mathcal{C} is abelian, we define

$$\mathrm{Ext}_{\widehat{\mathcal{C}}}^n(M, N) := \bigoplus_{m \in \mathbb{Z}} \mathrm{Ext}_{\mathcal{C}}^n(M, N)_m$$

similarly.

A *graded functor* $F : \mathcal{C} \rightarrow \mathcal{C}'$ between two graded categories means a functor between the underlying categories plus the additional data of an isomorphism of functors $q' \circ F \xrightarrow{\sim} F \circ q$; using adjunctions we get from this canonical isomorphisms $q'^m \circ F \xrightarrow{\sim} F \circ q^n$ for all $n \in \mathbb{Z}$ making the obvious pentagon of isomorphisms from $q'^{m+n} \circ F$ to $F \circ q^m \circ q^n$ commute for all $m, n \in \mathbb{Z}$. There is an induced functor $\widehat{F} : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}'}$ which is equal to F on objects; on a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)_m$ we define $\widehat{F}f \in \mathrm{Hom}_{\mathcal{C}'}(FM, FN)_m$ from $(q')^m FM \xrightarrow{\sim} F(q^m M) \xrightarrow{Ff} FN$. A *graded equivalence* is a graded functor that is also an equivalence of categories. A *graded duality* is a graded equivalence $D : \mathcal{C} \rightarrow \mathcal{C}^{\mathrm{op}}$, viewing $\mathcal{C}^{\mathrm{op}}$ as a graded category via the inverse adjoint pair (q^{-1}, q) of self-equivalences to (q, q^{-1}) .

Suppose \mathcal{C} is a graded abelian category such that all objects have finite length, there are enough projectives, and the endomorphism algebras of the irreducible objects are one dimensional. Assume further that \mathcal{C} is *acyclic* in the sense that $L \not\cong q^n L$ for each irreducible L and $n \neq 0$. Fix a choice of representatives $\{L(\lambda) \mid \lambda \in \Lambda\}$ for the homogeneous isomorphism classes of irreducible objects in \mathcal{C} , and let $P(\lambda)$ be a projective cover of $L(\lambda)$. Then the functor q makes $K_0(\mathcal{C})$ and $G_0(\mathcal{C})$ into free $\mathbb{Z}[q, q^{-1}]$ -modules with bases $\{[P(\lambda)] \mid \lambda \in \Lambda\}$ and $\{[L(\lambda)] \mid \lambda \in \Lambda\}$, respectively. Set $[\mathcal{C}]_q := \mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} K_0(\mathcal{C})$ and $[\mathcal{C}]_q^* := \mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} G_0(\mathcal{C})$. Given $M, L \in \mathcal{C}$ with L irreducible, we write $[M : L]_q$ for $\sum_{n \in \mathbb{Z}} q^n [M : q^n L]$; in particular $[M : L(\lambda)]_q = \dim_q \mathrm{Hom}_{\widehat{\mathcal{C}}}(P(\lambda), M)$.

Continuing with \mathcal{C} as in the previous paragraph, let $\mathrm{grmod}\text{-}A$ be the category of finite dimensional locally unital graded right modules over the locally unital graded algebra

$$A := \bigoplus_{\lambda, \mu \in \Lambda} \mathrm{Hom}_{\widehat{\mathcal{C}}}(P(\lambda), P(\mu)). \quad (5.6)$$

This can be viewed as a graded category with q being the grading shift functor, i.e. qM is the same underlying module with new grading defined from $(qM)_n :=$

$\widehat{M_{n-1}}$. We write simply $\text{Hom}_A(M, N)$ for morphisms in the enriched category $\text{grmod-}A$. Then the functor

$$\mathbb{H} : \mathcal{C} \rightarrow \text{grmod-}A, \quad M \mapsto \bigoplus_{\lambda \in \Lambda} \text{Hom}_{\widehat{\mathcal{C}}}(P(\lambda), M) \quad (5.7)$$

is a graded equivalence. The right ideals $1_\lambda A$ are all finite dimensional. The left ideals $A1_\lambda$ are all finite dimensional if and only if \mathcal{C} has enough injectives; in that case \mathcal{C} is an *acyclic graded Schurian category*.

Definition 5.1. By a *graded lift* of a Schurian category $\overline{\mathcal{C}}$, we mean a graded abelian category \mathcal{C} together with a fully faithful functor $\nu : \widehat{\mathcal{C}} \rightarrow \overline{\mathcal{C}}$ such that

(GL1) ν is dense on projectives, i.e. every projective object $\overline{P} \in \overline{\mathcal{C}}$ is isomorphic to νP for some projective object $P \in \mathcal{C}$;

(GL2) $\nu \circ \widehat{q} \cong \nu$.

The hypotheses that ν is fully faithful and dense on projectives imply that the restriction of ν to \mathcal{C} is exact.

Given a graded lift \mathcal{C} of $\overline{\mathcal{C}}$, the *gradable objects* of $\overline{\mathcal{C}}$ mean the objects in the essential image of ν . All projective objects of $\overline{\mathcal{C}}$ are gradable by the definition (GL1). We will see shortly that all irreducible and all injective objects of $\overline{\mathcal{C}}$ are gradable too. An important point is that graded lifts of an indecomposable object are unique up to homogeneous isomorphism; see [BGS, Lemma 2.5.3]. Note also that the functor ν induces a canonical isomorphism

$$\text{Ext}_{\widehat{\mathcal{C}}}^n(M, N) \xrightarrow{\sim} \text{Ext}_{\overline{\mathcal{C}}}^n(\nu M, \nu N)$$

for any $M, N \in \mathcal{C}$ and $n \geq 0$.

The problem of finding a graded lift \mathcal{C} of a Schurian category $\overline{\mathcal{C}}$ is easy to understand in terms of algebras as follows. Let $\{\overline{L}(\lambda) \mid \lambda \in \Lambda\}$ be a complete set of pairwise non-isomorphic irreducible objects of $\overline{\mathcal{C}}$. Let $\overline{P}(\lambda)$ be a projective cover of $\overline{L}(\lambda)$ in $\overline{\mathcal{C}}$. Recall from (2.3) that $\overline{\mathcal{C}}$ is equivalent to the category $\text{mod-}A$ where

$$A := \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_{\overline{\mathcal{C}}}(\overline{P}(\lambda), \overline{P}(\mu)). \quad (5.8)$$

Now suppose that \mathcal{C} is a graded lift of $\overline{\mathcal{C}}$, and pick graded lifts $P(\lambda)$ of each $\overline{P}(\lambda)$. As $\text{End}_{\widehat{\mathcal{C}}}(P(\lambda)) \cong \text{End}_{\overline{\mathcal{C}}}(\overline{P}(\lambda))$ which is local, $P(\lambda)$ is an indecomposable projective object of \mathcal{C} with irreducible head denoted $L(\lambda)$. It is easy to see that $L(\lambda)$ is a graded lift of $\overline{L}(\lambda)$, i.e. all irreducible objects of $\overline{\mathcal{C}}$ are gradable. Moreover if L is any irreducible object of \mathcal{C} then νL is irreducible in $\overline{\mathcal{C}}$, from which we get that $L \cong q^n L(\lambda)$ for some $n \in \mathbb{Z}$ and $\lambda \in \Lambda$. Also each $\text{End}_{\widehat{\mathcal{C}}}(L(\lambda))$ is one-dimensional, hence using also (GL2) we see that \mathcal{C} is acyclic. Thus

$$\{q^n L(\lambda) \mid \lambda \in \Lambda, n \in \mathbb{Z}\}$$

is a complete set of pairwise non-isomorphic irreducible objects in \mathcal{C} . Finally note that every object of \mathcal{C} has finite length by the exactness of ν . This puts us in the setup of (5.7). Using ν to identify the algebras A from (5.6) and (5.8), the

following diagram of functors commutes up to isomorphism:

$$\begin{array}{ccc}
 \widehat{\mathcal{C}} & \xrightarrow{\widehat{\mathbb{H}}} & \widehat{\text{grmod-}A} \\
 \nu \downarrow & & \downarrow \nu \\
 \overline{\mathcal{C}} & \xrightarrow[\mathbb{H}]{} & \text{mod-}A.
 \end{array} \tag{5.9}$$

Here, the functor ν on the right is the obvious functor that forgets the grading. In this way we see that a choice of a graded lift of $\overline{\mathcal{C}}$ amounts to choosing a \mathbb{Z} -grading on the underlying basic algebra A with respect to which the idempotents $1_\lambda \in A$ are homogeneous.

Lemma 5.2. *Let \mathcal{C} be a graded lift of Schurian category $\overline{\mathcal{C}}$ with notation as above. For each $\lambda \in \Lambda$ the irreducible object $L(\lambda)$ has an injective hull $I(\lambda)$ in \mathcal{C} , which is a graded lift of the injective hull of $\overline{L}(\lambda)$ in $\overline{\mathcal{C}}$. Hence \mathcal{C} is an acyclic graded Schurian category.*

Proof. Working in terms of the equivalent categories $\text{mod-}A$ and $\text{grmod-}A$ as in (5.9), the indecomposable injectives in $\text{mod-}A$ are the linear duals of the (necessarily finite dimensional) left ideals $A1_\lambda$. They are naturally graded, hence give indecomposable injective objects in $\text{grmod-}A$ too. This proves the first statement. Hence \mathcal{C} has enough injectives. All the other properties of an acyclic graded Schurian category have already been verified above. \square

Finally let $\overline{F} : \overline{\mathcal{C}} \rightarrow \overline{\mathcal{C}}'$ be a functor between two Schurian categories, and \mathcal{C} and \mathcal{C}' be graded lifts of $\overline{\mathcal{C}}$ and $\overline{\mathcal{C}}'$, respectively. A *graded lift* of \overline{F} means a graded functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ such that $\nu' \circ \widehat{F} \cong \overline{F} \circ \nu$. Assuming \overline{F} has a right adjoint, it corresponds to a functor between the underlying module categories $\text{mod-}A$ and $\text{mod-}A'$ that is defined by tensoring with an (A, A') -bimodule. Then a choice of graded lift amounts to choosing a \mathbb{Z} -grading on this bimodule making it into a graded (A, A') -bimodule.

5.3. Graded highest weight categories. The next definition is the obvious graded analog of Definition 2.7.

Definition 5.3. A *graded highest weight category* is an acyclic graded Schurian category \mathcal{C} plus the data of a distinguished set of irreducible objects $\{L(\lambda) | \lambda \in \Lambda\}$ indexed by some interval-finite poset Λ such that the following two axioms hold.

- (GHW1) Every irreducible object of \mathcal{C} is isomorphic to $q^n L(\lambda)$ for unique $\lambda \in \Lambda$ and $n \in \mathbb{Z}$. Hence letting $P(\lambda)$ be a projective cover of $L(\lambda)$ in \mathcal{C} , the objects $\{q^n P(\lambda) | \lambda \in \Lambda, n \in \mathbb{Z}\}$ give a complete set of pairwise non-isomorphic indecomposable projective objects in \mathcal{C} .
- (GHW2) Define the *standard object* $\Delta(\lambda)$ to be the largest quotient of $P(\lambda)$ such that $[\Delta(\lambda) : L(\mu)]_q = \delta_{\lambda, \mu}$ for $\mu \not\prec \lambda$. Then $P(\lambda)$ has a filtration with top section isomorphic to $\Delta(\lambda)$ and other sections of the form $q^n \Delta(\mu)$ for $n \in \mathbb{Z}$ and $\mu \succ \lambda$.

There is also an equivalent dual formulation of this definition in terms of indecomposable injective objects $I(\lambda)$ and costandard objects $\nabla(\lambda)$.

An object M of a graded highest weight category \mathcal{C} has a *graded Δ -flag* if it has a filtration with sections of the form $q^n \Delta(\lambda)$ for $n \in \mathbb{Z}$ and $\lambda \in \Lambda$. The notion of a *graded ∇ -flag* is defined similarly. We let \mathcal{C}^Δ be the exact subcategory of \mathcal{C} consisting of all objects with a graded Δ -flag. The $\mathbb{Q}(q)$ -form $[\mathcal{C}^\Delta]_q$ of its Grothendieck group has basis $\{[\Delta(\lambda)] \mid \lambda \in \Lambda\}$.

The other basic facts about highest weight categories from §2.5 also extend to the graded setting.

If \mathcal{C} is a graded highest weight category and $\bar{\mathcal{C}}$ is the underlying Schurian category, i.e. \mathcal{C} is a graded lift of $\bar{\mathcal{C}}$, then it is easy to check that $\bar{\mathcal{C}}$ is a highest weight category in the sense of Definition 2.7 with the same weight poset and standard objects $\bar{\Delta}(\lambda) := \nu \Delta(\lambda)$. The following lemma establishes the converse of this statement.

Lemma 5.4. *Suppose that \mathcal{C} is a graded lift of a highest weight category $\bar{\mathcal{C}}$. For each $\lambda \in \Lambda$ let $L(\lambda)$ be some choice of graded lift of the irreducible object $\bar{L}(\lambda)$ of $\bar{\mathcal{C}}$. Then \mathcal{C} is a graded highest weight category with $\{L(\lambda) \mid \lambda \in \Lambda\}$ as its distinguished irreducible objects. Moreover its standard objects are graded lifts of the ones in $\bar{\mathcal{C}}$.*

Proof. We know that \mathcal{C} is an acyclic graded Schurian category by Lemma 5.2. Let A be as in (5.6). We may as well assume that $\bar{\mathcal{C}}$ is $\text{mod-}A$ and \mathcal{C} is $\text{grmod-}A$. So we can take

$$\bar{P}(\lambda) = 1_\lambda A = P(\lambda), \quad \text{Hom}_{\bar{\mathcal{C}}}(\bar{P}(\mu), \bar{P}(\lambda)) = 1_\lambda A 1_\mu = \text{Hom}_{\bar{\mathcal{C}}}(P(\mu), P(\lambda)).$$

By (HW) the standard object $\bar{\Delta}(\lambda)$ in $\bar{\mathcal{C}}$ is the quotient of $P(\lambda)$ by the submodule generated by the images of all (not necessarily homogeneous) homomorphisms $P(\mu) \rightarrow P(\lambda)$ for $\mu > \lambda$. The image of an arbitrary homomorphism is contained in the sum of the images of its homogeneous pieces, so $\bar{\Delta}(\lambda)$ can also be described as the quotient of $P(\lambda)$ by the submodule generated by the images of all homogeneous homomorphisms $q^n P(\mu) \rightarrow P(\lambda)$ of degree zero for $\mu > \lambda$ and $n \in \mathbb{Z}$. Thus it is gradable and a graded lift is given by $\Delta(\lambda)$ as defined by (GHW2). Similarly each $\bar{\nabla}(\lambda)$ is a graded lift of the corresponding costandard object $\bar{\nabla}(\lambda)$ of $\bar{\mathcal{C}}$. Then an argument mimicking the usual proof of the criterion for a module to have Δ -flag shows that $P(\lambda)$ has a graded Δ -flag. \square

5.4. Graded categorifications. Next we formulate the graded version of the definition of \mathfrak{sl}_I -categorification from Definition 2.6.

Definition 5.5. A $U_q \mathfrak{sl}_I$ -categorification is an acyclic graded Schurian category \mathcal{C} together with the data of graded endofunctors F_i, E_i, K_i and K_i^{-1} for each $i \in I$, an adjunction making $F_i^* := q E_i K_i$ into a right adjoint to F_i , and homogeneous endomorphisms $\xi \in \text{Hom}(F_i, F_i)_2$ and $\tau \in \text{Hom}(F_j \circ F_i, F_i \circ F_j)_{-\alpha_i \cdot \alpha_j}$ for each $i, j \in I$, such that the following axioms hold.

- (GSL1) There is a decomposition $\mathcal{C} = \bigoplus_{\varpi \in P_I} \mathcal{C}_\varpi$ such that $K_i|_{\mathcal{C}_\varpi} \cong q^{\varpi \cdot \alpha_i}$ and $K_i^{-1}|_{\mathcal{C}_\varpi} \cong q^{-\varpi \cdot \alpha_i}$.
- (GSL2) Letting $F := \bigoplus_{i \in I} F_i$, the endomorphisms $\xi_j := \widehat{F}^{d-j} \xi \widehat{F}^{j-1}$ and $\tau_k := \widehat{F}^{d-k-1} \tau \widehat{F}^{k-1}$ of \widehat{F}^d plus the projections 1_i of \widehat{F}^d onto its summands $\widehat{F}_i := \widehat{F}_{i_d} \circ \cdots \circ \widehat{F}_{i_1}$ for each $i \in I^d$ satisfy the relations of the quiver Hecke algebra $QH_{I,d}$.

- (GSL3) Each functor $E_i^* := qF_iK_i^{-1}$ is isomorphic to a right adjoint of E_i .
 (GSL4) The endomorphisms f_i, e_i and k_i of $[\mathcal{C}]_q$ induced by the functors F_i, E_i and K_i make $[\mathcal{C}]_q$ into an integrable $U_q\mathfrak{sl}_I$ -module.

The last axiom here has the following equivalent dual formulation.

- (GSL4*) The endomorphisms f_i, e_i and k_i of $[\mathcal{C}]_q^*$ induced by the functors F_i, E_i and K_i make $[\mathcal{C}]_q^*$ into an integrable $U_q\mathfrak{sl}_I$ -module.

The above axioms are equivalent to saying that the given data defines a homogeneous action of the 2-Kac-Moody algebra associated to \mathfrak{sl}_I on \mathcal{C} making it into a graded integrable 2-representation in the sense of [R]. This is a variation on [R, Theorem 5.30]; see the proofs of the next two lemmas.

Lemma 5.6. *If \mathcal{C} is a $U_q\mathfrak{sl}_I$ -categorification that is a graded lift of a Schurian category $\bar{\mathcal{C}}$, then there is an induced structure of \mathfrak{sl}_I -categorification on $\bar{\mathcal{C}}$.*

Proof. Let us work in terms of graded modules over the algebra A from (5.6). Then F and E are defined by tensoring with certain graded (A, A) -bimodules, and ξ, τ and the various adjunctions become graded bimodule homomorphisms. Forgetting the grading we get endofunctors \bar{F} and \bar{E} of $\text{mod-}A$, natural transformations $\bar{\xi}$ and $\bar{\tau}$ satisfying the relations of (SL2'), and adjunctions both ways round between \bar{F} and \bar{E} . It remains to verify the axioms (SL1') and (SL4'). The first follows as ξ acts nilpotently on any graded projective by degree considerations, hence $\bar{\xi}$ is locally nilpotent too. Finally (SL4') follows from (GSL4) specialized at $q = 1$. \square

Lemma 5.7. *Let $\bar{\mathcal{C}}$ be an \mathfrak{sl}_I -categorification, denoting its various functors and natural transformations by $\bar{F}_i, \bar{E}_i, \bar{\xi}$ and $\bar{\tau}$. For each $\varpi \in P_I$ let $\bar{\mathcal{C}}_\varpi$ be the full subcategory of $\bar{\mathcal{C}}$ consisting of all objects \bar{M} such that $[\bar{M}]$ lies in the ϖ -weight space of the integrable \mathfrak{sl}_I -module $[\bar{\mathcal{C}}]$. Suppose that we are given the following additional data:*

- (i) *graded lifts \mathcal{C}_ϖ of each $\bar{\mathcal{C}}_\varpi$, hence a graded lift $\mathcal{C} = \bigoplus_{\varpi \in P_I} \mathcal{C}_\varpi$ of $\bar{\mathcal{C}}$;*
- (ii) *graded functors K_i and K_i^{-1} satisfying (GSL1);*
- (iii) *graded lifts F_i and E_i of the functors \bar{F}_i and \bar{E}_i together with an adjunction making $F_i^* = qE_iK_i$ into a right adjoint to F_i ;*
- (iv) *graded lifts $\xi \in \text{Hom}(F_i, F_i)_2$ and $\tau \in \text{Hom}(F_j \circ F_i, F_i \circ F_j)_{-\alpha_i \cdot \alpha_j}$ of $\bar{\xi}$ and $\bar{\tau}$, meaning that $\chi \circ \nu \xi = \bar{\xi} \nu \circ \chi$ in $\text{Hom}(\nu \circ \hat{F}, \bar{F} \circ \nu)$ and $\bar{F} \chi \circ \chi \hat{F} \circ \nu \tau = \bar{\tau} \nu \circ \bar{F} \chi \circ \chi \hat{F}$ in $\text{Hom}(\nu \circ \hat{F}^2, \bar{F}^2 \circ \nu)$ for some choice of the isomorphism $\chi : \nu \circ \hat{F} \xrightarrow{\sim} \bar{F} \circ \nu$.*

Then \mathcal{C} is a $U_q\mathfrak{sl}_I$ -categorification. (Note we do not insist that the adjunction in (iii) is a lift of the given adjunction between \bar{F}_i and \bar{E}_i .)

Proof. We know that \mathcal{C} is an acyclic graded Schurian category by Lemma 5.2. In view of [R, Theorem 5.30], we have the data required to define a homogeneous action of the 2-Kac-Moody algebra associated to \mathfrak{sl}_I on \mathcal{C} , making it into a graded integrable 2-representation. The axioms (GSL1)–(GSL2) follow immediately from the definition of the latter, while (GSL3) follows from [R, Theorem 5.16]. To check (GSL4) we need to verify the relations of $U_q\mathfrak{sl}_I$. Almost all of them follow from the 2-relations in the 2-Kac-Moody algebra. We are just left with the quantum Serre relations which follow from [R, Lemma 3.13]. \square

5.5. Existence and uniqueness of graded tensor product categorifications. Given a type $(\underline{n}, \underline{c})$ of level l , we can at last formulate the graded analog of Definition 2.9.

Definition 5.8. A $U_q \mathfrak{sl}_I$ -tensor product categorification of type $(\underline{n}, \underline{c})$ is a graded highest weight category \mathcal{C} together with graded endofunctors F_i, E_i, K_i and K_i^{-1} , an adjunction making $F_i^* := qE_i K_i$ into a right adjoint to F_i , and homogeneous natural transformations ξ and τ as above such that (GSL1)–(GSL3) and (GTP1)–(GTP3) hold.

(GTP1) Same as (TP1).

(GTP2) The exact functors F_i and E_i send objects with graded Δ -flags to objects with graded Δ -flags.

(GTP3) The linear isomorphism $[\mathcal{C}^\Delta] \xrightarrow{\sim} \bigwedge^{\underline{n}, \underline{c}} V_I, [\Delta(\lambda)] \mapsto v_\lambda$ intertwines the endomorphisms f_i, e_i and k_i of $[\mathcal{C}^\Delta]$ induced by F_i, E_i and K_i with the endomorphisms of $\bigwedge^{\underline{n}, \underline{c}} V_I$ arising from the actions of $f_i, e_i, k_i \in U_q \mathfrak{sl}_I$.

Since $[\mathcal{C}]_q$ embeds into $[\mathcal{C}^\Delta]_q = \bigwedge^{\underline{n}, \underline{c}} V_I$, these axioms imply that (GSL4) holds, so that \mathcal{C} is a $U_q \mathfrak{sl}_I$ -categorification in the sense of Definition 5.5 too.

Lemma 5.9. Suppose that \mathcal{C} is a $U_q \mathfrak{sl}_I$ -tensor product categorification as above. Then \mathcal{C}^{op} is a $U_q \mathfrak{sl}_I$ -tensor product categorification with categorification functors $F_i^{\text{op}} := q^2 F_i K_i^{-2}$, $E_i^{\text{op}} := q^2 E_i K_i^2$, $K_i^{\text{op}} := K_i^{-1}$ and $(K_i^{\text{op}})^{-1} := K_i$, taking the distinguished irreducible objects and other required adjunctions and natural transformations to be the same as in \mathcal{C} .

Proof. All of the axioms follow immediately except for (GTP2)–(GTP3). To see these note using (GTP2)–(GTP3) for \mathcal{C} and (5.4) that $F_j \Delta(\lambda)$ has a graded Δ -flag with sections $q^{(|\lambda_{i+1}| + \dots + |\lambda_l|) \cdot \alpha_j} \Delta(t_{ij}(\lambda))$ for $i = 1, \dots, l$ such that $\lambda_{ij} = 1$ and $\lambda_{i(j+1)} = 0$. Hence, by an argument involving the adjoint pair (F_j, F_j^*) and the homological criteria for graded Δ - and ∇ -flags, $F_j^* \nabla(\lambda)$ has a graded ∇ -flag with sections $q^{-(|\lambda_{i+1}| + \dots + |\lambda_l|) \cdot \alpha_j} \nabla(t_{ij}(\lambda))$ for $i = 1, \dots, l$ such that $\lambda_{ij} = 0$ and $\lambda_{i(j+1)} = 1$. Now rescale to deduce that $E_j^{\text{op}} \nabla(\lambda)$ has a filtration with sections $q^{(|\lambda_1| + \dots + |\lambda_{i-1}|) \cdot \alpha_j} \nabla(t_{ij}(\lambda))$ for $i = 1, \dots, l$ such that $\lambda_{ij} = 0$ and $\lambda_{i(j+1)} = 1$. Comparing with (5.3) and bearing in mind that it is the shift functor q^{-1} on \mathcal{C}^{op} that induces q on its Grothendieck group, this checks (GTP2) and the equivariance from (GTP3) for e_j . A similar argument works for f_j , while the equivariance with respect to each k_j is obvious. \square

The truncation construction of §2.8 can obviously be applied also in the graded setting. Thus, given some $U_q \mathfrak{sl}_I$ -tensor product categorification \mathcal{C} and a subinterval $J \subset I$, the subquotient $\mathcal{C}_J = \mathcal{C}_{\leq J} / \mathcal{C}_{< J}$ has a naturally induced structure of $U_q \mathfrak{sl}_J$ -tensor product categorification.

If \mathcal{C} is a $U_q \mathfrak{sl}_I$ -tensor product categorification then the underlying Schurian category $\overline{\mathcal{C}}$ is an \mathfrak{sl}_I -tensor product categorification in sense of Definition 2.9; this is just like in Lemma 5.6. The main goal now is to prove the following theorem going in the other direction; cf. Lemma 5.7.

Theorem 5.10. Let I be an interval and $\overline{\mathcal{C}}$ be an \mathfrak{sl}_I -tensor product categorification of any type.

- (i) *There exists a graded lift \mathcal{C} of $\bar{\mathcal{C}}$ together with graded functors F_i , E_i , K_i and K_i^{-1} , an adjunction making $F_i^* = qE_iK_i$ into a right adjoint to F_i , and homogeneous natural transformations ξ and τ , satisfying all the hypotheses of Lemma 5.7(i)–(iv).*
- (ii) *Given any choice for the data in (i), there exist unique (up to isomorphism and a global shift) graded lifts $L(\lambda) \in \mathcal{C}$ of the irreducible objects of $\bar{\mathcal{C}}$ such that \mathcal{C} satisfies (GTP2)–(GTP3), viewing \mathcal{C} as a graded highest weight category with these lifts as its distinguished irreducible objects. Thus \mathcal{C} becomes a $U_q\mathfrak{sl}_I$ -tensor product categorification.*
- (iii) *If \mathcal{C}' is another $U_q\mathfrak{sl}_I$ -tensor product categorification lifting $\bar{\mathcal{C}}$ as in (i)–(ii), there is a strongly equivariant graded equivalence $\mathbb{G} : \mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ with $\nu' \circ \mathbb{G} \cong \nu$ and $\mathbb{G}L(\lambda) \cong L'(\lambda)$ for each weight λ .*

We will prove Theorem 5.10 in the next two subsections; see also [LW, Corollary 6.3] for a closely related result when I is finite. Before we do that we record the following variation which combines Theorems 2.11 and 5.10.

Theorem 5.11. *Let I be any interval and $(\underline{n}, \underline{c})$ be any type. Then there exists a $U_q\mathfrak{sl}_I$ -tensor product categorification \mathcal{C} of type $(\underline{n}, \underline{c})$. It is unique in the sense that if \mathcal{C}' is another $U_q\mathfrak{sl}_I$ -tensor product categorification of an equivalent type then there exists a strongly equivariant graded equivalence $\mathbb{G} : \mathcal{C} \rightarrow \mathcal{C}'$ such that $\mathbb{G}L(\lambda) \cong L'(\lambda)$ for each weight λ .*

Proof. The existence is clear from Theorems 2.11 and 5.10(i)–(ii). For the uniqueness, given \mathcal{C} and \mathcal{C}' , the underlying Schurian categories $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}}'$ are tensor product categorifications of equivalent types. Hence by Theorem 2.11 there exists a strongly equivariant equivalence $\bar{\mathbb{G}} : \bar{\mathcal{C}} \xrightarrow{\sim} \bar{\mathcal{C}}'$. Thus \mathcal{C} together with the functor $\bar{\mathbb{G}} \circ \nu : \hat{\mathcal{C}} \rightarrow \bar{\mathcal{C}}'$ is a graded lift of $\bar{\mathcal{C}}'$ in the sense of Theorem 5.10(i)–(ii), as of course is \mathcal{C}' with its given forgetful functor $\nu' : \hat{\mathcal{C}}' \rightarrow \bar{\mathcal{C}}'$. Hence Theorem 5.10(iii) gives us the desired strongly equivariant graded equivalence $\mathbb{G} : \mathcal{C} \rightarrow \mathcal{C}'$ with $\nu' \circ \mathbb{G} \cong \bar{\mathbb{G}} \circ \nu$ and $\mathbb{G}L(\lambda) \cong L'(\lambda)$. \square

Corollary 5.12. *Any $U_q\mathfrak{sl}_I$ -tensor product categorification \mathcal{C} admits a graded duality \circledast with $\circledast \circ F_i^* \cong F_i^* \circ \circledast$, $\circledast \circ E_i^* \cong E_i^* \circ \circledast$ and $L(\lambda) \cong L(\lambda)^{\circledast}$ for each weight λ . Similarly its category of projectives has a graded duality $\#$ with $\# \circ F_i \cong F_i \circ \#$, $\# \circ E_i \cong E_i \circ \#$ and $P(\lambda) \cong P(\lambda)^{\#}$ for each λ .*

Proof. View \mathcal{C}^{op} as a $U_q\mathfrak{sl}_I$ -tensor product categorification as in Lemma 5.9. Then Theorem 5.11 implies that there exists a strongly equivariant graded equivalence $\circledast : \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$. In particular this means that $\circledast \circ F_i \cong q^2 F_i K_i^{-2} \circ \circledast$, which is equivalent to the assertion that $\circledast \circ E_i^* \cong E_i^* \circ \circledast$. Similarly $\circledast \circ F_i^* \cong F_i^* \circ \circledast$. The graded duality $\#$ on projectives is defined by the formula (2.21) as before. \square

5.6. Proof of Theorem 5.10 for finite intervals. Let I be finite and $\bar{\mathcal{C}}$ be an \mathfrak{sl}_I -tensor product categorification of some fixed type. The hardest part of the proof of Theorem 5.10 in this situation is to show that there exists a $U_q\mathfrak{sl}_I$ -tensor product categorification \mathcal{C} lifting $\bar{\mathcal{C}}$ as in Theorem 5.10(i)–(ii). Fortunately this is already established in the literature. Briefly, in view of Theorem 2.11 and [LW, Theorem 3.12], we may assume that $\bar{\mathcal{C}}$ is the category of modules over a tensor product algebra in the sense of [W1]. This algebra is naturally graded and

its category of graded modules gives us the desired graded lift \mathcal{C} . The graded functors F_i and E_i and the other data of a $U_q\mathfrak{sl}_I$ -categorification are constructed in [W1]. The axioms (GTP2)–(GTP3) follow from [W1, Proposition 4.4].

Remark 5.13. There are at least two other approaches to the construction of \mathcal{C} in the literature. It can be realized following [BGS],[B] in terms of Soergel’s graded lift of parabolic category \mathcal{O} ; see [W1, Corollary 8.12] for the equivalence of graded parabolic category \mathcal{O} with the realization of \mathcal{C} arising from the tensor product algebras. The appropriate definition of the graded functors F_i and E_i on graded parabolic category \mathcal{O} is recorded in [FKS], but a direct proof of the axiom (GTP3) via this approach is still missing in the literature. Also Hu and Mathas [HM] have given another construction of \mathcal{C} in terms of their version of quiver Schur algebras; these are graded Morita equivalent to the tensor product algebras by [W1, Theorem 4.30]. However again this is not sufficient by itself for our purposes as Hu and Mathas do not discuss the graded categorical actions.

This proves Theorem 5.10(i), and for this particular choice of \mathcal{C} it also establishes the existence of the graded lifts $L(\lambda)$ in (ii) making \mathcal{C} into a $U_q\mathfrak{sl}_I$ -tensor product categorification. The uniqueness of the graded lifts in (ii) follows by passing to the graded Grothendieck group $[\mathcal{C}^\Delta]_q$ then applying the following elementary combinatorial lemma.

Lemma 5.14. *Suppose that we are given integers $\{n_\lambda \mid \lambda \in \Lambda\}$ such that the map $\bigwedge^{\underline{n}, \underline{c}} V_I \rightarrow \bigwedge^{\underline{n}, \underline{c}} V_I, v_\lambda \mapsto q^{n_\lambda} v_\lambda$ is a $U_q\mathfrak{sl}_I$ -module homomorphism. Then all the integers n_λ are equal.*

Proof. Exercise. □

To finish the proof, let \mathcal{C} be some fixed $U_q\mathfrak{sl}_I$ -tensor product categorification lifting $\bar{\mathcal{C}}$ in the sense of Theorem 5.10(i)–(ii). Let $\kappa := \kappa_{I; \underline{n}, \underline{c}}$ and

$$T = \bigoplus_{d \geq 0} T_d := \bigoplus_{d \geq 0} F^d L(\kappa) \in \mathcal{C}, \quad (5.10)$$

which is a graded lift of $\bar{T} = \bigoplus_{d \geq 0} \bar{T}_d = \bigoplus_{d \geq 0} \bar{F}^d \bar{L}(\kappa) \in \bar{\mathcal{C}}$. We can identify $\text{End}_{\hat{\mathcal{C}}}(T_d)$ with $\text{End}_{\bar{\mathcal{C}}}(\bar{T}_d)$. Hence, applying the first part of Theorem 2.13, we can identify $\text{End}_{\hat{\mathcal{C}}}(T) = \text{End}_{\bar{\mathcal{C}}}(T)$ with the graded algebra

$$H = \bigoplus_{d \geq 0} H_d := \bigoplus_{d \geq 0} QH_{I, d}^{|\kappa|}. \quad (5.11)$$

Note by the definition from Lemma 5.7(iv) that the actions of $1_i, \xi_j, \tau_k \in H_d$ on T_d obtained in this way agree with the endomorphisms induced by the action of $QH_{I, d}$ on \hat{F}^d . Now we are going to exploit the graded functor

$$\mathbb{V} := \text{Hom}_{\hat{\mathcal{C}}}(T, -) : \mathcal{C} \rightarrow \text{grmod-}H. \quad (5.12)$$

This is a graded lift of $\bar{\mathbb{V}} := \text{Hom}_{\bar{\mathcal{C}}}(\bar{T}, -) : \bar{\mathcal{C}} \rightarrow \text{mod-}H$, i.e. the following diagram of functors commutes up to equivalence:

$$\begin{array}{ccc} \hat{\mathcal{C}} & \xrightarrow{\hat{\mathbb{V}}} & \widehat{\text{grmod-}H} \\ \nu \downarrow & & \downarrow \nu \\ \bar{\mathcal{C}} & \xrightarrow{\bar{\mathbb{V}}} & \text{mod-}H. \end{array} \quad (5.13)$$

The bottom functor is fully faithful on projectives by the second part of Theorem 2.13, as are the vertical functors, hence so is the top functor.

The category $\text{grmod-}H$ has the structure of a $U_q\mathfrak{sl}_I$ -categorification, which lifts the categorical \mathfrak{sl}_I -action on $\text{mod-}H$ in the sense of Lemma 5.7. On $\text{grmod-}H_d$ the functor F_i is the graded induction functor $- \otimes_{H_d} 1_{d;i} H_{d+1}$, where $1_{d;i} := \sum_{i \in I^{d+1}, i_{d+1}=i} 1_i$. Its natural transformations ξ and τ defined exactly as explained in the paragraph after (4.21) are automatically homogeneous of the right degree. The right adjoint F_i^* is the restriction functor defined on $M \in \text{grmod-}H_{d+1}$ by right multiplication by $1_{d;i}$. Then E_i is defined so that $F_i^* = qE_i K_i$, where

$$K_i M = \bigoplus_{j \in I^d} q^{(|\kappa| - \alpha_{j_1} - \dots - \alpha_{j_d}) \cdot \alpha_i} M_j$$

for $M \in \text{grmod-}H_d$. The graded functor $\mathbb{V} : \mathcal{C} \rightarrow \text{grmod-}H$ is strongly equivariant in the usual sense; this follows by the proof of Lemma 2.15.

For $\lambda \in \Lambda$ we let

$$Y(\lambda) := \mathbb{V}P(\lambda) \in \text{grmod-}H, \quad (5.14)$$

$$\bar{Y}(\lambda) := \bar{\mathbb{V}}\bar{P}(\lambda) \in \text{mod-}H. \quad (5.15)$$

Thus $Y(\lambda)$ is a graded lift of the indecomposable module $\bar{Y}(\lambda)$. Letting A be the graded algebra from (5.6), the functor $\hat{\mathbb{V}}$ defines a canonical isomorphism of graded algebras

$$A \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y(\lambda), Y(\mu)).$$

From (5.7) we get a canonical graded equivalence $\mathbb{H} : \mathcal{C} \xrightarrow{\sim} \text{grmod-}A$ fitting into a commuting square analogous to (5.9). Just like we did in the ungraded case, we lift the $U_q\mathfrak{sl}_I$ -categorification structure on \mathcal{C} to $\text{grmod-}A$ so that \mathbb{H} becomes a strongly equivariant graded equivalence. The endofunctor $F_i : \text{grmod-}A \rightarrow \text{grmod-}A$ is given by tensoring over A with the graded (A, A) -bimodule

$$B_i := \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_{\mathcal{C}}(P(\lambda), F_i P(\mu)).$$

As in (2.33)–(2.34) we then apply $\hat{\mathbb{V}}$ and the strong equivariance of \mathbb{V} to obtain isomorphisms

$$B_i \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y(\lambda), F_i Y(\mu)), \quad (5.16)$$

$$B_i \otimes_A B_j \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y(\lambda), F_j F_i Y(\mu)). \quad (5.17)$$

Now the natural transformations ξ and τ come from the homogeneous bimodule homomorphisms $\xi : B_i \rightarrow B_i$ and $\tau : B_i \otimes_A B_j \rightarrow B_j \otimes B_i$ induced by ξ and τ at the level of $\text{grmod-}H$.

Suppose that \mathcal{C}' is another graded lift of $\bar{\mathcal{C}}$ equipped with graded endofunctors $F'_i, E'_i, K'_i, K'^{-1}_i$, and an adjunction and natural transformations ξ', τ' just like in Theorem 5.10(i). Fix a (unique up to homogeneous isomorphism) graded lift $L'(\kappa) \in \mathcal{C}'$ of $\bar{L}(\kappa)$. Then we repeat the above definitions to get a graded lift $T' \in \mathcal{C}'$ of \bar{T} such that $\text{End}_{\hat{\mathcal{C}}'}(T') = H$, and a graded functor $\mathbb{V}' : \mathcal{C}' \rightarrow \text{grmod-}H$ that is fully faithful on projectives and fits into another commuting square like (5.13). Next we must make a coherent choice of graded lifts $L'(\lambda) \in \mathcal{C}'$ of the other irreducibles $\bar{L}(\lambda) \in \bar{\mathcal{C}}$, thereby making \mathcal{C}' into a graded highest weight category; cf. Lemma 5.4. Equivalently we choose graded lifts $P'(\lambda)$ of the indecomposable projectives according to the following lemma.

Lemma 5.15. *For each $\lambda \in \Lambda$ there exists a (unique up to isomorphism) graded lift $P'(\lambda) \in \mathcal{C}'$ of $\bar{P}(\lambda)$ such that $Y'(\lambda) := \mathbb{V}'P'(\lambda)$ is isomorphic to $Y(\lambda)$ as a graded H -module.*

Proof. Let $P' \in \mathcal{C}'$ be some arbitrary choice of graded lift of $\bar{P}(\lambda)$. Then $Y' := \mathbb{V}'P' \in \text{grmod-}H$ is a graded lift of $\bar{Y}(\lambda)$, as is $Y(\lambda)$. As $\bar{Y}(\lambda)$ is indecomposable there exists a unique $n \in \mathbb{Z}$ such that $q^n Y' \cong Y(\lambda)$. We then define $P'(\lambda)$ to be $q^n P'$. \square

For $Y'(\lambda)$ as in Lemma 5.15, we then introduce the graded algebra A' and graded (A', A') -bimodules B'_i as above so that

$$\begin{aligned} A' &\cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y'(\lambda), Y'(\mu)), \\ B'_i &\cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y'(\lambda), F_i Y'(\mu)), \\ B'_i \otimes_{A'} B'_j &\cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y'(\lambda), F_j F_i Y'(\mu)). \end{aligned}$$

We make $\text{grmod-}A'$ into a $U_q \mathfrak{sl}_I$ -categorification with $F_i := - \otimes_{A'} B'_i$ just like before, so that there is a strongly equivariant graded equivalence $\mathbb{H}' : \mathcal{C}' \xrightarrow{\sim} \text{grmod-}A'$ fitting into another analog of (5.9). Next choose graded H -module isomorphisms $Y(\lambda) \cong Y'(\lambda)$ for each $\lambda \in \Lambda$. These induce graded algebra isomorphisms between A and A' , and isomorphisms between the graded bimodules B_i and B'_i with appropriate equivariance properties. Hence we get a strongly equivariant graded isomorphism of categories $\text{grmod-}A \xrightarrow{\sim} \text{grmod-}A'$ sending each $1_\lambda A$ to $1_\lambda A'$. Since \mathcal{C} is strongly equivariantly equivalent to $\text{grmod-}A$ and \mathcal{C}' is strongly equivariantly equivalent to $\text{grmod-}A'$, we deduce that there is a strongly equivariant graded equivalence $\mathbb{G} : \mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ with $\nu' \circ \mathbb{G} \cong \nu$ and $\mathbb{G}L(\lambda) \cong L'(\lambda)$ for each $\lambda \in \Lambda$.

The existence of \mathbb{G} implies that \mathcal{C}' also satisfies (GTP2)–(GTP3), so that it is indeed a tensor product categorification, completing the proof of Theorem 5.10(ii) for an arbitrary choice of \mathcal{C} . We have also essentially proved (iii). More precisely, under the assumptions of (iii), we have proved that there exists a strongly equivariant graded equivalence $\mathbb{G} : \mathcal{C} \rightarrow \mathcal{C}'$ with $\nu' \circ \mathbb{G} \cong \nu$. By the uniqueness in

(ii) there exists $n \in \mathbb{Z}$ such that $L'(\lambda) \cong q^n \mathbb{G}L(\lambda)$ for all λ . Now replace \mathbb{G} by $q'^n \circ \mathbb{G}$. This completes the proof of Theorem 5.10 for finite I .

The uniqueness just established implies that the *graded Young module* $Y(\lambda)$ from (5.14) does not depend on the particular choice of \mathcal{C} (up to isomorphism in $\text{grmod-}H$). The following lemma gives a slightly different characterization of this important module. It also gives a first glimpse of the significance of the combinatorial statistic *defect* introduced way back in (2.8). Note in the proof of the lemma we use some definitions from §5.9 below, but none of the intermediate theorems. (In fact more is true here: $Y(\lambda)$ is self-dual with respect to a natural graded duality $\#$ on $\text{grmod-}H$.)

Lemma 5.16. *For $\lambda \in \Lambda$, the graded H -module $Y(\lambda)$ from (5.14) is the unique (up to isomorphism) graded lift of $\bar{Y}(\lambda)$ such that $q^{-\text{def}(\lambda)}Y(\lambda)$ is self-dual as a graded vector space. In fact each word space $q^{-\text{def}(\lambda)}Y(\lambda)1_{\mathbf{i}}$ is self-dual as a graded vector space.*

Proof. We just check that each $q^{-\text{def}(\lambda)}Y(\lambda)1_{\mathbf{i}}$ is graded-self-dual. Let $E_{\mathbf{i}}^! := q^{-1}F_{\mathbf{i}}K_{\mathbf{i}}^{-1}$, which is left adjoint to $E_{\mathbf{i}}$. We may assume that $\mathbf{i} = (i_1, \dots, i_d) \in I^d$ satisfies $\alpha_{i_1} + \dots + \alpha_{i_d} = |\kappa| - |\lambda|$, since otherwise $Y(\lambda)1_{\mathbf{i}}$ is zero. Then a little calculation shows that

$$q^{-\text{def}(\lambda)}Y(\lambda)1_{\mathbf{i}} \cong \text{Hom}_{\hat{\mathcal{C}}}(q^{\text{def}(\lambda)}F_{\mathbf{i}}L(\kappa), P(\lambda)) \cong \text{Hom}_{\hat{\mathcal{C}}}(E_{i_d}^! \cdots E_{i_1}^! L(\kappa), P(\lambda)).$$

Hence we are reduced to showing that

$$\dim_q \text{Hom}_{\hat{\mathcal{C}}}(E_{i_d}^! \cdots E_{i_1}^! L(\kappa), P(\lambda)) = \dim_q \text{Hom}_{\hat{\mathcal{C}}}(L(\kappa), E_{i_1} \cdots E_{i_d} P(\lambda))$$

is bar-invariant. Using the notation from (5.30)–(5.35) below, setting $e := e_{i_1} \cdots e_{i_d}$ for short, we have that

$$\dim_q \text{Hom}_{\hat{\mathcal{C}}}(L(\kappa), E_{i_1} \cdots E_{i_d} P(\lambda)) = \overline{(v_{\kappa}, \psi^*(eb_{\lambda}))} = (\psi(v_{\kappa}), eb_{\lambda}) = (v_{\kappa}, eb_{\lambda}).$$

By the symmetry of this form, this equals

$$(eb_{\lambda}, v_{\kappa}) = (\psi(eb_{\lambda}), v_{\kappa}) = \overline{(eb_{\lambda}, \psi^*(v_{\kappa}))} = \overline{(eb_{\lambda}, v_{\kappa})}.$$

So it is bar-invariant. \square

Corollary 5.17. *If $Y'(\lambda)$ is any graded lift of $\bar{Y}(\lambda)$ such that $q^{-\text{def}(\lambda)}Y'(\lambda)1_{\mathbf{i}}$ is a non-zero self-dual graded vector space for some word \mathbf{i} then $Y'(\lambda) \cong Y(\lambda)$.*

5.7. Proof of Theorem 5.10 for infinite intervals. Now the interval I is infinite. Fixing a type $(\underline{n}, \underline{c})$ we set $\Lambda := \Lambda_{I; \underline{n}, \underline{c}}$ and let $\bar{\mathcal{C}}$ be some given \mathfrak{sl}_I -tensor product categorification of type $(\underline{n}, \underline{c})$. Apart from adding bars to almost everything in sight, we are going to adopt all of the notation from Section 4. Thus we have chosen subintervals $I_1 \subset I_2 \subset \dots \subset I$, leading to subquotients $\bar{\mathcal{C}}_r$ of $\bar{\mathcal{C}}$ with weight posets $\Lambda_1 \subset \Lambda_2 \subset \dots \subset \Lambda$. Also $\kappa^r := \kappa_{I_r; \underline{n}, \underline{c}} \in \Lambda_r$ and we have the tensor spaces $\bar{T}^r = \bigoplus_{d \geq 0} \bar{T}_d^r \in \bar{\mathcal{C}}$ defined like in (2.39). Each $\text{End}_{\bar{\mathcal{C}}}(\bar{T}^r)$ is identified with the algebra H^r from (4.1). We have the abelian category $\text{mod-}H$ of stable modules from Definition 4.4, and the exact functor $\bar{\mathbb{V}} : \bar{\mathcal{C}} \rightarrow \text{mod-}H$ from Theorem 4.6. This functor is fully faithful on projectives by Theorem 4.9. We have the objects $\bar{Y}(\lambda) = (\bar{Y}^1(\lambda) \rightarrow \bar{Y}^2(\lambda) \rightarrow \dots) := \bar{\mathbb{V}}\bar{P}(\lambda) \in \text{mod-}H$ as in (4.16). We have the functor $\bar{\text{pr}}_r : \text{mod-}H \rightarrow \text{mod-}H^r$ and its left adjoint $\bar{\text{pr}}_r^!$ from (4.11)–(4.12). Thus $\bar{\text{pr}}_r \circ \bar{\mathbb{V}}$ is equal to $\bar{\mathbb{V}}^r := \text{Hom}_{\bar{\mathcal{C}}}(\bar{T}^r, -) : \bar{\mathcal{C}} \rightarrow \text{mod-}H^r$.

The idempotents e_d^r and the isomorphisms ϕ_d^r from Lemma 4.1 are evidently homogeneous. Thus $\phi^r : H^r \xrightarrow{\sim} e^r H^{r+1} e^r$ is an isomorphism of graded algebras. So it makes sense to introduce a graded version $\text{grmod-}H$ of the category $\text{mod-}H$ of stable modules from Definition 4.4. Its objects are diagrams

$$M = (M^1 \xrightarrow{\iota^1} M^2 \xrightarrow{\iota^2} M^3 \xrightarrow{\iota^3} \dots)$$

such that $M^r \in \text{grmod-}H^r$, $\iota^r : M^r \xrightarrow{\sim} M^{r+1} e^r$ is an isomorphism of graded H^r -modules for each $r \geq 1$, and the maps (4.13) are isomorphisms for $r \gg 1$. Morphisms are tuples $(f^r)_{r \geq 1}$ such that each f^r is a morphism in $\text{grmod-}H^r$ making the analogous diagram to (4.7) commute. The category $\widehat{\text{grmod-}H}$ is graded with q being the obvious grading shift functor. Let $\nu : \widehat{\text{grmod-}H} \rightarrow \text{mod-}H$ be the obvious functor that forgets the grading on each M^r . Let $\text{pr}_r : \widehat{\text{grmod-}H} \rightarrow \text{grmod-}H^r$ be the r th projection. It has an obvious left adjoint $\text{pr}_r^!$ which is a graded lift of $\overline{\text{pr}}_r^!$; it is defined in exactly the same way as before just working in the graded module categories.

Lemma 5.18. *The category $\widehat{\text{grmod-}H}$ is a graded lift of $\text{mod-}H$ in the sense of Definition 5.1.*

Proof. We need to check that $\widehat{\text{grmod-}H}$ is an abelian category and that ν is dense on projectives.

Let $f : M \rightarrow N$ be a morphism in $\widehat{\text{grmod-}H}$. Let $\overline{f} : \overline{M} \rightarrow \overline{N}$ be the corresponding morphism in $\text{mod-}H$ obtained by applying the forgetful functor ν . We know already that $\text{mod-}H$ is an abelian category. To show that $\widehat{\text{grmod-}H}$ is abelian too it suffices to check that $\ker \overline{f} \rightarrow \overline{M}$ and $\overline{N} \rightarrow \text{coker } \overline{f}$ are gradable. Pick r so that all of $\overline{M}, \overline{N}, \ker \overline{f}$ and $\text{coker } \overline{f}$ are r -stable. Then we have simply that $(\ker \overline{f} \rightarrow \overline{M}) \cong \overline{\text{pr}}_r^!(\ker \overline{f}^r \rightarrow \overline{M}^r)$. Hence it is graded by $\text{pr}_r^!(\ker f^r \rightarrow M^r)$. Similarly the cokernel is gradable.

To see that ν is dense on projectives, note that every projective in $\text{mod-}H$ is a direct sum of summands of $\overline{\nabla} \overline{T}^r \cong \overline{\text{pr}}_r^! H^r$, where H^r is the regular right H^r -module. Each $\overline{\text{pr}}_r^! H^r$ admits the graded lift $\text{pr}_r^! H^r$, which is projective in $\widehat{\text{grmod-}H}$ by properties of adjoints. \square

Hence by Lemma 5.2 we see that $\widehat{\text{grmod-}H}$ is a graded Schurian category. It also has a structure of $U_q \mathfrak{sl}_I$ -categorification lifting the categorical action on $\text{mod-}H$ as in Lemma 5.7. The graded lifts F_i and E_i on $\widehat{\text{grmod-}H}$ are defined in exactly the same way as the original functors \overline{F}_i and \overline{E}_i on $\text{mod-}H$ were defined in §4.3, just replacing each \overline{F}_i^r and \overline{E}_i^r with its graded version as defined in the previous subsection. It is worth noting that $K_i : \widehat{\text{grmod-}H} \rightarrow \widehat{\text{grmod-}H}$ is defined on $M = (M^1 \xrightarrow{\iota^1} M^2 \xrightarrow{\iota^2} \dots)$ so that it is the degree shift $q^{(|\kappa^r| - \alpha_{j_1} - \dots - \alpha_{j_d}) \cdot \alpha_i}$ on $1_j M^r$ for each $r \geq 1$ and $j \in I_r^d$. One needs to check here that the maps ι^r remain homogeneous of degree zero after these shifts are performed; this follows because $|\kappa^r| = |\kappa^{r+1}| - \alpha_{i_1} - \dots - \alpha_{i_{d_r}}$ where $\mathbf{i} = i_1 \dots i_{d_r}$ is the word appearing in the definition of ϕ^r from Lemma 4.1.

Lemma 5.19. *Suppose we are given $\overline{M} \in \overline{\mathcal{C}}$ and $r \geq 1$ such that $\overline{\nabla} \overline{M}$ is r -stable. Then $\overline{\nabla} \overline{M}$ is a gradable object of $\text{mod-}H$ if and only if $\overline{\nabla}^r \overline{M}$ is a gradable object of $\text{mod-}H^r$.*

Proof. The forward direction is clear. For the converse, let M^r be a graded lift of $\overline{M}^r := \text{Hom}_{\overline{\mathcal{C}}}(\overline{T}^r, \overline{M})$. Since $\overline{\mathcal{V}}\overline{M}$ is r -stable it is isomorphic to $\text{pr}_r^! \overline{M}^r \in \text{mod-}H$. Hence $\text{pr}_r^! M^r \in \text{grmod-}H$ gives the desired graded lift. \square

Lemma 5.19 applies in particular to the objects $\overline{Y}(\lambda)$, since we know already that $\overline{Y}^r(\lambda)$ is gradable for sufficiently large r by (5.14)–(5.15). We need one more piece of book-keeping in order to make a canonical choice of such a graded lift.

Lemma 5.20. *Fix $r \geq 1$. Let i and p_1, \dots, p_a be as in Lemma 4.1. Set*

$$\sigma_r := \frac{1}{2}p_1 + \dots + \frac{1}{2}p_a. \quad (5.18)$$

Then $\dim_q M^r 1_j = q^{\sigma_r} \dim_q M^{r+1} 1_{ij} / [p_1]! \dots [p_a]!$ for each $M \in \text{grmod-}H$.

Proof. This follows from the explicit form of the idempotent e_d^r constructed in the proof of Lemma 4.1. The essential point is that $b_m \in 1_{i^m} QH_{I_{r+1}, m} 1_{i^m}$ has the property for any finite dimensional graded $QH_{I_{r+1}, m}$ -module M that $\dim_q Mb_m = q^{\frac{1}{2}m(m-1)} \dim_q M 1_{i^m} / [m]!$. To see this note that $1_{i^m} QH_{I_{r+1}, m} 1_{i^m}$ is a nil-Hecke algebra. So it has a unique irreducible module \overline{L} , a graded lift L of which has graded dimension $[m]!$. Since $b_m = \tau_{w_0} \xi_1^{m-1} \xi_2^{m-2} \dots \xi_{m-1}$ we deduce by degree considerations that $\dim_q Lb_m = q^{\frac{1}{2}m(m-1)}$. \square

Then for σ_r as in (5.18) we define

$$\Sigma_r := \sigma_1 + \dots + \sigma_{r-1}. \quad (5.19)$$

Recall also the definition of $\text{def}(\lambda)$ for infinite intervals from Lemma 2.2.

Theorem 5.21. *For $\lambda \in \Lambda$, there exists a unique (up to isomorphism) graded lift $Y(\lambda) = (Y^1(\lambda) \rightarrow Y^2(\lambda) \rightarrow \dots) \in \text{grmod-}H$ of $\overline{Y}(\lambda)$ such that $q^{\Sigma_r - \text{def}(\lambda)} Y^r(\lambda)$ is self-dual as a graded vector space for each $r \geq 1$. In fact if $Y'(\lambda)$ is any graded lift of $\overline{Y}(\lambda)$ such that $q^{\Sigma_s - \text{def}(\lambda)} Y'^s(\lambda)$ is non-zero and self-dual as a graded vector space for some $s \geq 1$, then we have that $Y'(\lambda) \cong Y(\lambda)$ in $\text{grmod-}H$.*

Proof. Choose r so that $\overline{Y}(\lambda)$ is r -stable. We already observed by Lemma 5.19 that $\overline{Y}(\lambda)$ is gradable. In view of Lemma 5.16 we can pick a graded lift $Y(\lambda)$ so that each word space of $q^{\Sigma_r - \text{def}(\lambda)} Y^r(\lambda)$ is graded-self-dual. Lemma 5.20 then implies immediately that $q^{\Sigma_s - \text{def}(\lambda)} Y^s(\lambda)$ is graded-self-dual for each $s < r$; this depends also on the independence from Lemma 2.2. The same argument together also with Corollary 5.17 proves the same thing for $s > r$ too.

To establish the claim about uniqueness, let $Y'(\lambda)$ be another graded lift of $\overline{Y}(\lambda)$ such that $q^{\Sigma_s - \text{def}(\lambda)} Y'^s(\lambda)$ is a non-zero self-dual graded vector space for some $s \geq 1$. Then $Y''(\lambda)$ is a graded lift of $\overline{Y}^r(\lambda)$ and the argument just given using Lemmas 5.20, 5.16 and Corollary 5.17 imply that $Y''(\lambda)$ is also graded-self-dual. Hence actually $Y''(\lambda) \cong Y^r(\lambda)$. But then the r -stability implies that $Y'(\lambda) \cong \text{pr}_r^!(Y''(\lambda)) \cong \text{pr}_r^!(Y^r(\lambda)) \cong Y(\lambda)$. \square

With Theorem 5.21 in hand it is clear how to construct a graded lift \mathcal{C} of $\overline{\mathcal{C}}$. We fix a choice of object $Y(\lambda) = (Y^1(\lambda) \rightarrow Y^2(\lambda) \rightarrow \dots) \in \text{grmod-}H$ as in Theorem 5.21 for each $\lambda \in \Lambda$, then define

$$A := \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y(\lambda), Y(\mu)). \quad (5.20)$$

This is a graded lift of the basic algebra underlying the original category $\bar{\mathcal{C}}$. Thus $\mathcal{C} := \text{grmod-}A$ is a graded lift of $\bar{\mathcal{C}}$, with forgetful functor $\nu : \widehat{\mathcal{C}} \rightarrow \bar{\mathcal{C}}$ that is the composite of the forgetful functor $\widehat{\text{grmod-}A} \rightarrow \text{mod-}A$ and the adjoint equivalence $\text{mod-}A \rightarrow \bar{\mathcal{C}}$ to the usual equivalence $\mathbb{H} : \bar{\mathcal{C}} \rightarrow \text{mod-}A$.

Next we introduce a categorical $U_q\mathfrak{sl}_I$ -action on \mathcal{C} . Let

$$B_i := \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y(\lambda), F_i Y(\mu)), \quad (5.21)$$

$$\text{so that} \quad B_i \otimes_A B_j \cong \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_H(Y(\lambda), F_j F_i Y(\mu)). \quad (5.22)$$

The natural transformations ξ and τ from the categorical action on $\text{grmod-}H$ induce homogeneous bimodule homomorphisms $\xi : B_i \rightarrow B_i$ and $\tau : B_i \otimes_A B_j \rightarrow B_j \otimes_A B_i$ as usual. Let $F_i : \text{grmod-}A \rightarrow \text{grmod-}A$ be the functor defined by tensoring with B_i , with its homogeneous natural transformations ξ and τ induced by the preceding bimodule endomorphisms. This gives us a choice of graded lifts of $\bar{F}_i : \bar{\mathcal{C}} \rightarrow \bar{\mathcal{C}}$, $\bar{\xi} \in \text{End}(\bar{F}_i)$ and $\bar{\tau} \in \text{Hom}(\bar{F}_j \circ \bar{F}_i, \bar{F}_i \circ \bar{F}_j)$. For F_i^* we take the canonical graded right adjoint $\text{Hom}_A(B_i, -)$ to F_i . All the other required lifts are obvious, and then Lemma 5.7 tells us that all this data makes \mathcal{C} into a $U_q\mathfrak{sl}_I$ -categorification.

Theorem 5.22. *There exists a (unique up to isomorphism and global shift) choice of distinguished irreducible objects $\{L(\lambda) \mid \lambda \in \Lambda\}$ making the graded lift \mathcal{C} of $\bar{\mathcal{C}}$ just constructed into a $U_q\mathfrak{sl}_I$ -tensor product categorification.*

Proof. We apply the truncation construction to define subquotients $\mathcal{C}_r := \mathcal{C}_{I_r}$ for each $r \geq 1$. The structures defined on \mathcal{C} make each \mathcal{C}_r into a graded lift of $\bar{\mathcal{C}}_r$ in the sense of Theorem 5.10(i). Thus by Theorem 5.10(ii) for the finite interval I_r there exists a unique (up to isomorphism and global shift) set $\{L_r(\lambda) \mid \lambda \in \Lambda_r\}$ of distinguished irreducible objects in \mathcal{C}_r making it into a $U_q\mathfrak{sl}_{I_r}$ -tensor product categorification.

Now \mathcal{C}_r is a subquotient of \mathcal{C}_{r+1} , and the objects $\{L_{r+1}(\lambda) \mid \lambda \in \Lambda_r\}$ give another choice of distinguished irreducible objects of \mathcal{C}_r making it into a $U_q\mathfrak{sl}_{I_r}$ -tensor product categorification. By the uniqueness in Theorem 5.10(ii) this means that $L_r(\lambda) \cong q^n L_{r+1}(\lambda)$ for some $n \in \mathbb{Z}$ and all $\lambda \in \Lambda_r$. Replacing $L_{r+1}(\lambda)$ by $q^n L_{r+1}(\lambda)$ this shows that we may assume that $L_r(\lambda) \cong L_{r+1}(\lambda)$ for each $\lambda \in \Lambda_r$. Starting at $r = 1$ and proceeding recursively in this way we can ensure this is the case for all $r \geq 1$.

Then we define irreducible objects $L(\lambda) \in \mathcal{C}$ for $\lambda \in \Lambda$ by taking $L(\lambda)$ to be any irreducible object of \mathcal{C} such that $L(\lambda) \cong L_r(\lambda)$ in \mathcal{C}_r whenever $\lambda \in \Lambda_r$. This gives the desired set of distinguished irreducible objects $\{L(\lambda) \mid \lambda \in \Lambda\}$ for \mathcal{C} . This makes \mathcal{C} into a graded highest weight category by Lemma 5.4. Moreover it satisfies (GTP2)–(GTP3) because they hold in each subquotient \mathcal{C}_r . \square

Remark 5.23. We will see momentarily that the irreducible objects $L(\lambda)$ in Theorem 5.22 can be taken to be the irreducible heads of the indecomposable projectives $P(\lambda) := 1_\lambda A \in \text{grmod-}A$.

We have now done the hard work, proving Theorem 5.10(i) and establishing the existence of graded lifts $L(\lambda)$ as in (ii) for one particular choice of \mathcal{C} . The proof of the remainder of Theorem 5.10 follows the same argument as explained for finite intervals in the previous subsection. So let \mathcal{C} be some fixed $U_q \mathfrak{sl}_I$ -tensor product categorification lifting $\bar{\mathcal{C}}$ in the sense of Theorem 5.10(i)–(ii). Let

$$T^r = \bigoplus_{d \geq 0} T_d^r := q^{\Sigma_r} \bigoplus_{d \geq 0} F_{I_r}^d L(\kappa^r) \in \mathcal{C}, \quad (5.23)$$

which is a graded lift of \bar{T}^r , hence $\text{End}_{\hat{\mathcal{C}}}(T^r) = H^r$. The shifts Σ_r here are as in (5.19), and their appearance is explained this time by the following lemma.

Lemma 5.24. *For $r \geq 1$ there is an isomorphism $\theta^r : T^r \xrightarrow{\sim} e^r T^{r+1}$ in \mathcal{C} such that $\theta^r \circ h = \phi^r(h) \circ \theta^r$ for each $h \in H^r$.*

Proof. This follows from the proof of Lemma 4.1. The only new observation in the graded setting is that the quantum divided power functor $F_i^{(m)}$ is defined from $F_i^{(m)} := q^{-\frac{1}{2}m(m-1)} b_m F_i^m$, where $b_m \in 1_{i^m} QH_{I_{r+1}, m} 1_{i^m}$ is the distinguished idempotent from [R, Lemma 4.1]. This implies that $F_{I_r}^d L(\kappa^r) \cong q^{-\sigma_r} e^r F_{I_{r+1}}^{d_r+d} L(\kappa^{r+1})$. \square

Then we can introduce the graded functors

$$\mathbb{V}^r : \mathcal{C} \rightarrow \text{grmod-}H^r, \quad \mathbb{V} : \mathcal{C} \rightarrow \text{grmod-}H, \quad (5.24)$$

exactly like (4.14)–(4.15) but working in the graded categories. Thus $\mathbb{V}^r = \text{pr}_r \circ \mathbb{V} = \text{Hom}_{\hat{\mathcal{C}}}(T^r, -)$ is a graded lift of $\bar{\mathbb{V}}^r$, and \mathbb{V} is a graded lift of $\bar{\mathbb{V}}$, i.e. the analog of the diagram (5.13) commutes up to equivalence. Applying Theorem 4.9 we deduce that \mathbb{V} is fully faithful on projectives. Moreover Lemma 4.13 holds also in the graded setup, so \mathbb{V} is a strongly equivariant graded functor.

For $\lambda \in \Lambda$ we let

$$Y(\lambda) = (Y^1(\lambda) \rightarrow Y^2(\lambda) \rightarrow \cdots) := \mathbb{V}P(\lambda) \in \text{grmod-}H. \quad (5.25)$$

Thus $Y(\lambda)$ is a graded lift of $\bar{Y}(\lambda)$. For any sufficiently large r , Lemmas 2.2 and 5.16 imply that $q^{\Sigma_r - \text{def}(\lambda)} Y^r(\lambda)$ is self-dual as a graded vector space. Hence the uniqueness assertion of Theorem 5.21 implies that the object $Y(\lambda)$ just defined is isomorphic to the object $Y(\lambda)$ from that theorem. Hence the graded algebra from (5.6) is isomorphic to the algebra (5.20). As usual \mathcal{C} and $\text{grmod-}A$ are strongly equivariantly equivalent with respect to the graded categorical action on $\text{grmod-}A$ arising from the bimodules (5.16)–(5.17); these bimodules are equivariantly isomorphic to the ones appearing in (5.21)–(5.22). In the present situation it is clear that the projective indecomposable object $P(\lambda) \in \mathcal{C}$ corresponds under this equivalence to the graded A -module $1_\lambda A$, so that this justifies the claim made in Remark 5.23.

To complete the proof of Theorem 5.10 we take another graded lift \mathcal{C}' of $\bar{\mathcal{C}}$ as in Theorem 5.10(i). We fix choices of graded lifts $L'(\kappa^r) \in \mathcal{C}'$ of each $\bar{L}(\kappa^r)$ in $\bar{\mathcal{C}}$ such that $F'_{s+1}{}^{(p_1)} \cdots F'_{s+a}{}^{(p_a)} L'(\kappa^{r+1}) \cong L'(\kappa^r)$ for each $r \geq 1$, notation as in Lemma 4.1. Then we define T'^r just like in (5.23). The careful choices just made ensure that the analog of Lemma 5.24 holds for \mathcal{C}' . So we can continue to define $\mathbb{V}' : \mathcal{C}' \rightarrow \text{grmod-}H$ like in (5.24), which is fully faithful on projectives. Using

this repeat the rest of the above constructions to get an algebra A' as in (5.25), bimodules B'_i , etc Then run the logic from the previous subsection again to finish the proof.

5.8. Koszulity. We say that a graded Schurian category \mathcal{C} is *mixed* if the graded algebra A defined by (5.6) is strictly positively graded, i.e. $A = \bigoplus_{n \geq 0} A_n$ with $A_0 = \bigoplus_{\lambda \in \Lambda} \mathbb{K}1_\lambda$. (The terminology “mixed” here goes back to [BGS]; it is equivalent to property that $\text{Ext}_{\mathcal{C}}^1(L, L') = 0$ for irreducibles L, L' with $\text{wt}(L) \leq \text{wt}(L')$, where the *weight* of $L \cong q^n L(\lambda)$ is defined to be $-n$.) A graded highest weight category is mixed if and only if both of the following holding for all weights λ and μ , respectively:

$$[P(\lambda)] = [\Delta(\lambda)] + (\text{a } q\mathbb{N}[q]\text{-linear combination of } [\Delta(\mu)] \text{ for } \mu > \lambda), \quad (5.26)$$

$$[\Delta(\mu)] = [L(\mu)] + (\text{a } q\mathbb{N}[q]\text{-linear combination of } [L(\lambda)] \text{ for } \lambda < \mu). \quad (5.27)$$

If \mathcal{C} possesses a graded duality fixing the distinguished irreducible objects, then we can use graded BGG reciprocity to deduce further that (5.26) and (5.27) are equivalent.

Now assume instead just that \mathcal{C} is a graded highest weight category with a graded duality. Then \mathcal{C} is *standard Koszul* if the minimal projective resolution

$$\cdots \rightarrow P^1(\lambda) \rightarrow P^0(\lambda) \rightarrow \Delta(\lambda) \rightarrow 0$$

of each standard object $\Delta(\lambda)$ is linear, i.e. $P^0(\lambda) \cong P(\lambda)$ and for each $n \geq 1$ the object $P^n(\lambda)$ is a direct sum of objects $q^n P(\mu)$ for $\mu > \lambda$. This implies that (5.26) holds for all λ , hence that \mathcal{C} is mixed.

In fact standard Koszul implies Koszul in the usual sense, i.e. \mathcal{C} is mixed and $\text{Ext}_{\mathcal{C}}^n(L(\lambda), L(\mu))_m = 0$ unless $m + n = 0$ for any $\lambda, \mu \in \Lambda$. This is proved for finite weight posets in [ADL, Theorem 1]; the general case follows on passing to sufficiently large subquotients.

Theorem 5.25. *Let \mathcal{C} be a $U_q \mathfrak{sl}_I$ -tensor product categorification of any type and for any interval I . Then \mathcal{C} is standard Koszul.*

Proof. When I is finite this is already known. In fact there are a couple of proofs available in the literature depending on which realization of \mathcal{C} is adopted; see [W2] and [HM]. Both proofs depend ultimately on the known standard Koszulity of graded parabolic category \mathcal{O} which follows from [BGS], [B] and [ADL, Corollary 3.8]. (If the gap mentioned in Remark 5.13 related to checking directly that graded parabolic category \mathcal{O} satisfies (GTP3) can be filled, then Theorem 5.10 would allow [W2] or [HM] to be bypassed entirely here.)

For the general case, take $\lambda \in \Lambda$ and consider a minimal projective resolution

$$\cdots \rightarrow P^2(\lambda) \rightarrow P^1(\lambda) \rightarrow P^0(\lambda) \rightarrow \Delta(\lambda) \rightarrow 0$$

of $\Delta(\lambda)$. Suppose for some n that $P^n(\lambda)$ is not a direct sum of $q^n P(\mu)$. Pick a finite subinterval $J \subset I$ such that all composition factors of $P^m(\lambda)$ belong to Λ_J for all $m \leq n$. Then $P^n(\lambda) \rightarrow \cdots \rightarrow P^0(\lambda) \rightarrow \Delta(\lambda) \rightarrow 0$ is the start of a non-linear minimal projective resolution for $\Delta(\lambda)$ in the subquotient \mathcal{C}_J too. This contradicts the standard Koszulity of \mathcal{C}_J . \square

Corollary 5.26. *All $U_q \mathfrak{sl}_I$ -tensor product categorifications in the sense of Definition 5.8 are mixed.*

Remark 5.27. One very interesting question is whether any reasonable description of the Koszul dual of \mathcal{C} is possible. In the finite case, the usual singular-parabolic duality of [B] shows that the Koszul dual of a weight space in an \mathfrak{sl}_n -tensor product categorification of level l is a weight space in an \mathfrak{sl}_l -tensor product categorification of level n ; this is a categorical version of skew Howe duality. It is not clear how much of this structure survives in the infinite case.

5.9. Kazhdan-Lusztig polynomials. Let \mathcal{C} be a $U_q\mathfrak{sl}_I$ -tensor product categorification of type $(\underline{n}, \underline{c})$ and set $\Lambda := \Lambda_{I; \underline{n}, \underline{c}}$. We conclude by proving the graded analog of the Kazhdan-Lusztig conjecture. This allows the polynomials

$$d_{\lambda, \mu}(q) := [\Delta(\mu) : L(\lambda)]_q, \quad (5.28)$$

$$p_{\lambda, \mu}(q) := \sum_{n \geq 0} \dim \operatorname{Ext}_{\hat{\mathcal{C}}}^n(\Delta(\lambda), L(\mu)) q^n \quad (5.29)$$

to be computed in principle in terms of parabolic Kazhdan-Lusztig polynomials in finite type A. Note by the general theory of standard Koszul categories that the two families of polynomials here are closely related: the matrices $(d_{\lambda, \mu}(q))_{\lambda, \mu \in \Lambda}$ and $(p_{\lambda, \mu}(-q))_{\lambda, \mu \in \Lambda}$ are inverse to each other.

To start with we have the graded Grothendieck groups $[\mathcal{C}]_q \hookrightarrow [\mathcal{C}^\Delta]_q \hookrightarrow [\mathcal{C}]_q^*$. The functors F_i, E_i and K_i induce endomorphisms f_i, e_i and k_i making these Grothendieck groups into $U_q\mathfrak{sl}_I$ -modules in such a way that the maps between them are module homomorphisms. Recall also that $[\mathcal{C}^\Delta]_q$ is identified with the $U_q\mathfrak{sl}_I$ -module $\bigwedge^{\underline{n}, \underline{c}} V_I$ so that $v_\lambda = [\Delta(\lambda)]$. We let $b_\lambda := [P(\lambda)]$ and $b_\lambda^* := [L(\lambda)]$. Thus we have constructed $U_q\mathfrak{sl}_I$ -modules $[\mathcal{C}]_q \leq \bigwedge^{\underline{n}, \underline{c}} V_I \leq [\mathcal{C}]_q^*$ with the distinguished bases $\{b_\lambda \mid \lambda \in \Lambda\}, \{v_\lambda \mid \lambda \in \Lambda\}$ and $\{b_\lambda^* \mid \lambda \in \Lambda\}$, respectively. When I is finite we have simply that $[\mathcal{C}]_q = \bigwedge^{\underline{n}, \underline{c}} V_I = [\mathcal{C}]_q^*$, but in general this is not the case. We also point out that the vectors $b_\lambda \in \bigwedge^{\underline{n}, \underline{c}} V_I$ do not depend on the particular choice of \mathcal{C} ; this follows from the uniqueness in Theorem 5.11.

Let $\otimes : \mathcal{C} \rightarrow \mathcal{C}$ and $\# : \mathcal{C}^{\text{proj}} \rightarrow \mathcal{C}^{\text{proj}}$ be graded dualities as in Corollary 5.12. Then define a bilinear pairing $(-, -) : [\mathcal{C}]_q \times [\mathcal{C}]_q^* \rightarrow \mathbb{Q}(q)$ by setting

$$([P], [L]) := \dim_q \operatorname{Hom}_{\hat{\mathcal{C}}}(P^\#, L) = \overline{\dim_q \operatorname{Hom}_{\hat{\mathcal{C}}}(P, L^*)}. \quad (5.30)$$

Since F_i^* is right adjoint to F_i and E_i^* is right adjoint to E_i this pairing has the property that $(uv, w) = (v, u^*w)$ for any $u \in U_q\mathfrak{sl}_I$. Also it is immediate that the bases $\{b_\lambda \mid \lambda \in \Lambda\}$ and $\{b_\lambda^* \mid \lambda \in \Lambda\}$ are dual to each other. Recall from (5.5) that we have also already introduced a symmetric bilinear form $(-, -)$ on $\bigwedge^{\underline{n}, \underline{c}} V_I$ such that $(v_\lambda, v_\mu) = \delta_{\lambda, \mu}$. Our choice of notation here is not ambiguous as the two forms restrict to the same thing on the intersection of their domains. This follows because

$$(b_\lambda, v_\mu) = \overline{\dim_q \operatorname{Hom}_{\hat{\mathcal{C}}}(P(\lambda), \nabla(\mu))} = \dim_q \operatorname{Hom}_{\hat{\mathcal{C}}}(P(\lambda), \Delta(\mu)) = [\Delta(\mu) : L(\lambda)]_q$$

is the graded multiplicity of $\Delta(\mu)$ in a graded Δ -flag of P . This argument recovers graded BGG reciprocity: we have that

$$b_\lambda = \sum_{\mu \in \Lambda} d_{\lambda, \mu}(q) v_\mu, \quad v_\mu = \sum_{\lambda \in \Lambda} d_{\lambda, \mu}(q) b_\lambda^*, \quad (5.31)$$

recalling (5.28). When I is finite we can invert these formulae to obtain also

$$v_\lambda = \sum_{\mu \in \Lambda} p_{\lambda, \mu}(-q) b_\mu, \quad b_\mu^* = \sum_{\lambda \in \Lambda} p_{\lambda, \mu}(-q) v_\lambda. \quad (5.32)$$

However the last two formulae do not make sense in general when I is infinite as the sums become infinite too; this issue can be bypassed by introducing some suitable completion of $\bigwedge^{\underline{n}, \underline{c}} V_I$ but there is no need for us to do that here.

The dualities $\#$ and \otimes induce antilinear involutions

$$\psi : [\mathcal{C}]_q \rightarrow [\mathcal{C}]_q, \quad \psi^* : [\mathcal{C}]_q^* \rightarrow [\mathcal{C}]_q^*, \quad (5.33)$$

i.e. $\psi([P]) := [P^\#]$ and $\psi^*([L]) := [L^\otimes]$. Equivalently, these are the unique antilinear involutions satisfying

$$\psi(b_\lambda) = b_\lambda, \quad \psi^*(b_\lambda^*) = b_\lambda^* \quad (5.34)$$

for each $\lambda \in \Lambda$. In view of (2.21), ψ and ψ^* are adjoint antilinear maps, i.e. we have that $(\psi(v), w) = \overline{(v, \psi^*(w))}$. Also we have for any vector v and any $u \in U_q \mathfrak{sl}_I$ that

$$\psi(uv) = \psi(u)\psi(v), \quad \psi^*(uv) = \psi^*(u)\psi^*(v), \quad (5.35)$$

thanks to the equivariance properties from Corollary 5.12.

The following theorem is a special case of [W3, Proposition 3.14], where it is proved using the tensor product algebra realization of \mathcal{C} . We give a slightly different proof here in terms of the axiomatic framework of [LW].

Theorem 5.28. *Assume that I is finite so that $[\mathcal{C}]_q = \bigwedge^{\underline{n}, \underline{c}} V_I = [\mathcal{C}]_q^*$. Then $\psi : \bigwedge^{\underline{n}, \underline{c}} V_I \rightarrow \bigwedge^{\underline{n}, \underline{c}} V_I$ coincides with Lusztig's bar involution from [Lu, §27.3].*

Proof. We prove this by induction on the level l of $(\underline{n}, \underline{c})$. When $l = 1$ the result is clear as both ψ and Lusztig's bar involution fix the highest weight vector v_κ and commute with each f_i . Now suppose that $l > 1$. Let $\underline{n}^+ := (n_1, \dots, n_{l-1})$ and $\underline{c}^+ := (c_1, \dots, c_{l-1})$, so that $\bigwedge^{\underline{n}, \underline{c}} V_I = \bigwedge^{\underline{n}^+, \underline{c}^+} V_I \otimes \bigwedge^{n_l, c_l} V_I$. Let $\Lambda^+ := \Lambda_{I, \underline{n}^+, \underline{c}^+}$. To avoid potential confusion later on we denote the monomial basis vectors for $\bigwedge^{\underline{n}^+, \underline{c}^+} V_I$ by v_λ^+ instead of v_λ . Let $\tilde{\psi} : \bigwedge^{\underline{n}, \underline{c}} V_I \rightarrow \bigwedge^{\underline{n}, \underline{c}} V_I$ and $\tilde{\psi}^+ : \bigwedge^{\underline{n}^+, \underline{c}^+} V_I \rightarrow \bigwedge^{\underline{n}^+, \underline{c}^+} V_I$ be Lusztig's bar involutions. It is easy to see from Lusztig's definition that the following two properties are satisfied:

- (i) $\tilde{\psi}(f_i v) = f_i \tilde{\psi}(v)$ for all $i \in I$ and $v \in \bigwedge^{\underline{n}, \underline{c}} V_I$;
- (ii) $\tilde{\psi}(v \otimes v_{\kappa_l}) = \tilde{\psi}^+(v) \otimes v_{\kappa_l}$ for each $v \in \bigwedge^{\underline{n}^+, \underline{c}^+} V_I$.

Moreover these two properties characterize $\tilde{\psi}$ uniquely. Thus to prove the theorem we must show that $\psi(v \otimes v_{\kappa_l}) = \tilde{\psi}^+(v) \otimes v_{\kappa_l}$ too. Let \mathcal{C}^+ be a $U_q \mathfrak{sl}_I$ -tensor product categorification of type $(\underline{n}^+, \underline{c}^+)$. Its Grothendieck group $[\mathcal{C}^+] = \bigwedge^{\underline{n}^+, \underline{c}^+} V_I$ is equipped with the antilinear involution ψ^+ defined as above, so that $\psi^+(b_\lambda^+) = b_\lambda^+$ for each $\lambda \in \Lambda^+$ where $b_\lambda^+ \in \bigwedge^{\underline{n}^+, \underline{c}^+} V_I$ denotes the class of the indecomposable projective $P(\lambda) \in \mathcal{C}^+$. By the induction hypothesis $\psi^+ = \tilde{\psi}^+$. So we are reduced to showing that $\psi(v \otimes v_{\kappa_l}) = \psi^+(v) \otimes v_{\kappa_l}$; equivalently we will show that $\psi(b_\lambda^+ \otimes v_{\kappa_l}) = b_\lambda^+ \otimes v_{\kappa_l}$ for each $\lambda \in \Lambda^+$.

Now we make a particular choice for the category \mathcal{C}^+ . Let us identify Λ^+ with the coideal $\{\lambda \in \Lambda \mid \lambda_l = \kappa_l\}$ of Λ so that $(\lambda_1, \dots, \lambda_{l-1}) \in \Lambda^+$ is identified

with $(\lambda_1, \dots, \lambda_{l-1}, \kappa_l) \in \Lambda$. Let \mathcal{C}^+ be the quotient category of \mathcal{C} associated to this coideal. It is a graded highest weight category with distinguished irreducible objects $\{L(\lambda) \mid \lambda \in \Lambda^+\}$. The functor F_i obviously leaves invariant the Serre subcategory of \mathcal{C} generated by the irreducible objects $\{q^m L(\lambda) \mid m \in \mathbb{Z}, \lambda \in \Lambda, |\lambda_l| < |\kappa_l|\}$. Hence it induces a well-defined graded endofunctor $F_i^+ : \mathcal{C}^+ \rightarrow \mathcal{C}^+$. The natural transformations ξ and τ restrict to give $\xi \in \text{End}(F_i^+)_2$ and $\tau \in \text{Hom}(F_j^+ \circ F_i^+, F_i^+ \circ F_j^+)_{-\alpha_i \cdot \alpha_j}$. The functors K_i and K_i^{-1} obviously descend to \mathcal{C}^+ too. We claim further that there exists a graded endofunctor $E_i^+ : \mathcal{C}^+ \rightarrow \mathcal{C}^+$ such that $qE_i^+ K_i$ is right adjoint to F_i^+ , and that this data endows the graded highest weight category \mathcal{C}^+ with the structure of a $U_q \mathfrak{sl}_I$ -tensor product categorification of type $(\underline{n}^+, \underline{c}^+)$.

The proof of the claim depends on the categorical splitting construction from [LW]. To construct E_i^+ for a fixed $i \in I$, we let \mathcal{C}_i be the quotient of \mathcal{C} associated to the coideal $\Lambda_i := \{\lambda \in \Lambda \mid |\kappa_l| - |\lambda_l| \in \mathbb{N}\alpha_i\}$. Both of the functors F_i and E_i descend to endofunctors of \mathcal{C}_i . Let \mathcal{C}_i^- be the subcategory of \mathcal{C}_i associated to the ideal $\{\lambda \in \Lambda_i \mid |\lambda_l| = |\kappa_l| - r\alpha_i\}$ where $r := \kappa_l \cdot \alpha_i$ (which happens in our special minuscule situation to be either 0 or 1). Thus we have constructed categories and functors $\mathcal{C}_i^- \xrightarrow{\iota} \mathcal{C}_i \xrightarrow{\pi} \mathcal{C}^+$. The functor F_i on \mathcal{C}_i restricts to an endofunctor F_i^- of \mathcal{C}_i^- . Moreover [LW, Proposition 4.3] implies that $\pi \circ E_i^{(r)} \circ \iota : \mathcal{C}_i^- \rightarrow \mathcal{C}^+$ is a graded equivalence of categories, which intertwines F_i^- and F_i^+ by [LW, Lemma 4.7]. Let $E_i^- := \iota^* \circ E_i \circ \iota : \mathcal{C}_i^- \rightarrow \mathcal{C}_i^-$. Since $qE_i K_i$ is right adjoint to F_i it is immediate that $qE_i^- K_i$ is right adjoint to F_i^- . Then we transfer E_i^- through the equivalence to obtain the desired functor $E_i^+ : \mathcal{C}^+ \rightarrow \mathcal{C}^+$ such that $qE_i^+ K_i$ is right adjoint to F_i^+ . Finally let $\bar{\mathcal{C}}^+$ be the underlying ungraded category. In [LW, Theorem 4.10] it is shown that \bar{F}_i^+ and \bar{E}_i^+ (together with the various other natural transformations induced by the ones constructed above) make $\bar{\mathcal{C}}^+$ into an \mathfrak{sl}_I -tensor product categorification of type $(\underline{n}^+, \underline{c}^+)$. We have in front of us graded lifts as in Theorem 5.10(i). Then we apply Theorem 5.10(ii) to deduce that \mathcal{C}^+ is a $U_q \mathfrak{sl}_I$ -tensor product categorification of type $(\underline{n}^+, \underline{c}^+)$. So we have proved the claim.

We can now complete the proof of the theorem. Let $\pi : \mathcal{C} \rightarrow \mathcal{C}^+$ be the quotient functor and $\pi^! : \mathcal{C}^+ \rightarrow \mathcal{C}$ be a left adjoint. In \mathcal{C} we have that $\pi^! P(\lambda) \cong P(\lambda)$ and $\pi^! \Delta(\lambda) \cong \Delta(\lambda)$ for each $\lambda \in \Lambda^+$; the latter isomorphism follows by the graded analog of Lemma 2.8. Thus $\pi^!$ induces a linear map $\bigwedge^{\underline{n}^+, \underline{c}^+} V_I \hookrightarrow \bigwedge^{\underline{n}, \underline{c}} V_I$ such that $b_\lambda^+ \mapsto b_\lambda$ and $v_\lambda^+ \mapsto v_\lambda$ for each $\lambda \in \Lambda^+$. It follows immediately that $b_\lambda = b_\lambda^+ \otimes v_{\kappa_l}$ for each $\lambda \in \Lambda^+$. Hence $\psi(b_\lambda^+ \otimes v_{\kappa_l}) = b_\lambda^+ \otimes v_{\kappa_l}$ as required. \square

Now we apply the positivity from Corollary 5.26 (which we recall depended itself on the results from [BGS], [B] which exploit relations to geometry of flag varieties) to get the following; see also [W3, Theorem 6.8] for the generalization of this to more general tensor products (with a different proof via geometry of certain quiver varieties).

Corollary 5.29. *For finite I , $\{b_\lambda \mid \lambda \in \Lambda\}$ is Lusztig's canonical basis for $\bigwedge^{\underline{n}, \underline{c}} V_I$ from [Lu, §27.3], while $\{b_\lambda^* \mid \lambda \in \Lambda\}$ is the dual canonical basis.*

Proof. Corollary 5.26, (5.27) and (5.34) show that b_λ is a ψ -invariant vector in $v_\lambda + \sum_{\mu > \lambda} q\mathbb{Z}[q]v_\mu$. \square

This shows for finite I that the polynomials $d_{\lambda,\mu}(q)$ from (5.28) are the entries of the transition matrix from the canonical to the monomial basis of $\bigwedge^{n,\varepsilon} V_I$, while the polynomials $p_{\lambda,\mu}(-q)$ are the entries of the inverse transition matrix. Hence both are certain finite type A parabolic Kazhdan-Lusztig polynomials; see e.g. [FKK] or [B3] where this elementary combinatorial identification is made explicitly. (Alternatively one could use [BGS, Theorem 3.11.4] here to see all of this directly in a way that bypasses canonical bases completely.)

To determine the polynomials $d_{\lambda,\mu}(q)$ and $p_{\lambda,\mu}(q)$ when I is infinite it just remains to pick a finite subinterval $J \subset I$ such that $\lambda, \mu \in \Lambda_J$. Then it is immediate from the definition (5.28) and exactness of the quotient functor that $d_{\lambda,\mu}(q)$ computed in \mathcal{C} is the same as in \mathcal{C}_J . The same thing holds for $p_{\lambda,\mu}(q)$ in view of (2.18)–(2.19). Thus again all of these polynomials for infinite I are identified with some finite type A parabolic Kazhdan-Lusztig polynomials. If we specialize to $q = 1$ this proves the super Kazhdan-Lusztig conjecture as formulated in [B1, Conjecture 4.32], and all of its subsequent generalizations to other Borels and parabolics.

Remark 5.30. The basis called “canonical basis” in [B1] is a twisted version of the canonical basis here. It corresponds to the indecomposable tilting objects rather than the indecomposable projectives in \mathcal{C} . In more detail, let $T(\lambda) \in \mathcal{C}$ be the unique (up to isomorphism) \otimes -self-dual object possessing a graded Δ -flag with $\Delta(\lambda)$ at the bottom and other sections of the form $q^n \Delta(\mu)$ for $\mu < \lambda$ and $n \in \mathbb{Z}$. The existence of such an object follows by a construction due to Ringel involving taking iterated extensions of standard objects; cf. [B2] which justifies in the context of super parabolic category \mathcal{O} that Ringel’s construction terminates after finitely many steps. Since \mathcal{C} is mixed the higher sections of a graded Δ -flag of $T(\lambda)$ are actually all of the form $q^n \Delta(\mu)$ for $\mu < \lambda$ and $n < 0$. Let

$$\tilde{b}_\lambda := [T(\lambda)] \in \bigwedge^{n,\varepsilon} V_I.$$

When I is finite this gives us the *twisted canonical basis* $\{\tilde{b}_\lambda \mid \lambda \in \Lambda\}$ for $\bigwedge^{n,\varepsilon} V_I$; each \tilde{b}_λ here is the unique ψ^* -invariant vector in $v_\lambda + \sum_{\mu < \lambda} q^{-1}\mathbb{Z}[q^{-1}]v_\mu$. In any case we have that

$$\tilde{b}_\lambda = \sum_{\mu \in \Lambda} d_{\tilde{\lambda},\mu}(q^{-1})v_\mu,$$

where we write $\tilde{\lambda}$ for the 01-matrix obtained from λ by reversing the order of its rows. This follows from [LW, Remark 3.10], which implies that the Ringel dual of \mathcal{C} has the induced structure of a $U_q \mathfrak{sl}_I$ -tensor product categorification of type (\tilde{n}, \tilde{c}) where $\tilde{n} = (n_l, \dots, n_1)$ and $\tilde{c} = (c_l, \dots, c_1)$.

Remark 5.31. The structure constants describing the actions of the quantum divided powers $f_i^{(r)}/[r]!$ and $e_i^{(r)}/[r]!$ on the bases $\{b_\lambda \mid \lambda \in \Lambda\}$ and $\{b_\lambda^* \mid \lambda \in \Lambda\}$ all belong to $\mathbb{N}[q, q^{-1}]$. This is immediate because $f_i^{(r)}$ and $e_i^{(r)}$ have been categorified by $F_i^{(r)}$ and $E_i^{(r)}$. For finite I this was observed originally by Sussan in [Su] (although the argument there is incomplete). In particular this implies the truth of [B1, Conjecture 2.28]. In fact this argument generalizes to show that any

element of Lusztig's canonical basis of the modified quantum algebra \dot{U} acts with coefficients in $\mathbb{N}[q, q^{-1}]$, since by [W3, Theorem A] each of these basis vectors can be lifted to a functor acting on any tensor product categorification.

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