Representations of the general linear Lie superalgebra in the BGG category $\mathscr{O}$

Jonathan Brundan

Abstract This is a survey of some recent developments in the highest weight repesentation theory of the general linear Lie superalgebra $\mathfrak{gl}_{n|m}(\mathbb{C})$. The main focus is on the analog of the Kazhdan-Lusztig conjecture as formulated by the author in 2002, which was finally proved in 2011 by Cheng, Lam and Wang. Recently another proof has been obtained by the author joint with Losev and Webster, by a method which leads moreover to the construction of a Koszul-graded lift of category $\mathscr{O}$ for this Lie superalgebra.

Keywords: General linear Lie superalgebra, category $\mathscr{O}$.

Mathematics Subject Classification (2010): 17B10, 17B37.

1 Introduction

The representation theory of the general linear Lie superalgebra (as well as the other classical families) was first investigated seriously by Victor Kac [30, 31] around 1976. Kac classified the finite dimensional irreducible representations and proved character formulae for the typical ones. Then in the 1980s work of Sergeev [44] and Berele-Regev [5] exploited the superalgebra analog of Schur-Weyl duality to work out character formulae for the irreducible polynomial representations. It took another decade before Serganova [43] explained how the characters of arbitrary finite dimensional irreducible representations could be approached. Subsequent work of the author and others [10, 47, 17, 16] means that by now the category of finite dimensional representations is well understood (although there remain interesting questions regarding the tensor structure).

One can also ask about the representation theory of the general linear Lie superalgebra in the analog of the Bernstein-Gelfand-Gelfand category $\mathscr{O}$ from [7]. This is the natural home for the irreducible highest weight representations. The classical theory of category $\mathscr{O}$ for a semisimple Lie algebra, as in for example Humphreys' book [27] which inspired this article, sits at the heart of modern geometric representation theory. Its combinatorics is controlled by the underlying Weyl group, and many beautiful results are deduced from the geometry of the

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Research supported in part by NSF grant no. DMS-1161094.

associated flag variety via the Beilinson-Bernstein localization theorem [3]. There still seems to be no satisfactory substitute for this geometric part of the story for $\mathfrak{gl}_{n|m}(\mathbb{C})$ but at least the combinatorics has now been worked out: in [10] it was proposed that the combinatorics of the Weyl group (specifically the Kazhdan-Lusztig polynomials arising from the associated Iwahori-Hecke algebra) should simply be replaced by the combinatorics of a canonical basis in a certain $U_q\mathfrak{sl}_\infty$-module $V^{\otimes n} \otimes W^{\otimes m}$. This idea led in particular to the formulation of a superalgebra analog of the Kazhdan-Lusztig conjecture.

The super Kazhdan-Lusztig conjecture is now a theorem. In fact there are two proofs, first by Cheng, Lam and Wang [18], then more recently in joint work of the author with Losev and Webster [15]. In some sense both proofs involve a reduction to the ordinary Kazhdan-Lusztig conjecture for the general linear Lie algebra. Cheng, Lam and Wang go via some infinite dimensional limiting versions of the underlying Lie (super)algebras using the technique of "super duality," which originated in [22, 17]. On the other hand the proof in [15] involves passing from category $\mathscr{O}$ for $\mathfrak{gl}_{n|m}(\mathbb{C})$ to some subquotients which, thanks to results of Losev and Webster from [36], are equivalent to sums of blocks of parabolic category $\mathscr{O}$ for some other general linear Lie algebra. The approach of [15] allows also for the construction of a graded lift of $\mathscr{O}$ which is Koszul, in the spirit of the famous results of Beilinson, Ginzburg and Soergel [4] in the classical setting. The theory of categorification developed by Rouquier [42] and others, and the idea of Schur-Weyl duality for higher levels from [14], both play a role in this work.

This article is an attempt to give a brief overview of these results. It might serve as a useful starting point for someone trying to learn about the combinatorics of category $\mathscr{O}$ for the general linear Lie superalgebra for the first time. We begin with the definition of $\mathscr{O}$ and the basic properties of Verma supermodules and their projective covers. Then we formulate the super Kazhdan-Lusztig conjecture precisely and give some examples, before fitting it into the general framework of tensor product categorifications. Finally we highlight one of the main ideas from [15] involving a double centralizer property (an analog of Soergel's Struktursatz from [45]), and suggest a related question which we believe should be investigated further. In an attempt to maximize the readability of the article, precise references to the literature have been deferred to notes at the end of each section.

We point out in conclusion that there is also an attractive Kazhdan-Lusztig conjecture for the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$ formulated in [11], which remains quite untouched. One can also ponder Kazhdan-Lusztig combinatorics for the other classical families of Lie superalgebra. Dramatic progress in the case of $\mathfrak{osp}_{n|2m}(\mathbb{C})$ has been made recently in [2]; see also [25].

Acknowledgements. Special thanks go to Catharina Stroppel for several discussions which influenced this exposition. I also thank Geoff Mason, Ivan Penkov and Joe Wolf for providing me the opportunity to write a survey article of this nature. In fact I gave a talk on exactly this topic at the West Coast Lie Theory Seminar at Riverside in November 2002, when the super Kazhdan-Lusztig conjecture was newborn.

2 Super category $\mathscr{O}$ and its blocks

Fix $n,m \geq 0$ and let $\mathfrak{g}$ denote the *general linear Lie superalgebra* $\mathfrak{gl}_{n|m}(\mathbb{C})$. As a vector superspace this consists of $(n+m)\times(n+m)$ complex matrices $\left(\begin{array}{c|c} * & * \\ \hline * & * \end{array}\right)$ with $\mathbb{Z}/2$-grading defined so that the ij-matrix unit $e_{i,j}$ is even for $1 \leq i,j \leq n$ or $n+1 \leq i,j \leq n+m$, and $e_{i,j}$ is odd otherwise. It is a Lie superalgebra via the *supercommutator*

$$[x,y] := xy - (-1)^{\bar{x}\bar{y}}yx$$

for homogeneous $x, y \in \mathfrak{g}$ of parities $\bar{x}, \bar{y} \in \mathbb{Z}/2$, respectively.

By a $\mathfrak{g}$-*supermodule* we mean a vector superspace $M = M_{\bar{0}} \oplus M_{\bar{1}}$ equipped with a graded linear left action of $\mathfrak{g}$, such that $[x,y]v = x(yv) - (-1)^{\bar{x}\bar{y}}y(xv)$ for all homogeneous $x, y \in \mathfrak{g}$ and $v \in M$. For example we have the *natural representation* U of $\mathfrak{g}$, which is just the superspace of column vectors on standard basis $u_1, \dots, u_{n+m}$, where $\bar{u}_i = \bar{0}$ for $1 \leq i \leq n$ and $\bar{u}_i = \bar{1}$ for $n+1 \leq i \leq n+m$. We write $\mathfrak{g}$-smod for the category of all $\mathfrak{g}$-supermodules. A morphism $f : M \to N$ in this category means a linear map such that $f(M_i) \subseteq N_i$ for $i \in \mathbb{Z}/2$ and $f(xv) = xf(v)$ for $x \in \mathfrak{g}, v \in M$. This is obviously a $\mathbb{C}$-linear abelian category. It is also a *supercategory*, that is, it is equipped with the additional data of an endofunctor $\Pi : \mathfrak{g}\text{-smod} \to \mathfrak{g}\text{-smod}$ with $\Pi^2 \cong \text{id}$. The functor Π here is the *parity switching functor*, which is defined on a supermodule M by declaring that ΠM is the same underlying vector space as M but with the opposite $\mathbb{Z}/2$-grading, viewed as a $\mathfrak{g}$-supermodule with the new action $x \cdot v := (-1)^{\bar{x}}xv$. On a morphism $f : M \to N$ we take $\Pi f : \Pi M \to \Pi N$ to be the same underlying linear map as f. Clearly $\Pi^2 = \text{id}$.

Remark 2.1 Given any $\mathbb{C}$-linear supercategory $\mathscr{C}$, one can form the *enriched category* $\widehat{\mathscr{C}}$. This is a category enriched in the monoidal category of vector superspaces. It has the same objects as in $\mathscr{C}$, and its morphisms are defined from $\text{Hom}_{\widehat{\mathscr{C}}}(M,N) := \text{Hom}_{\mathscr{C}}(M,N)_{\bar{0}} \oplus \text{Hom}_{\mathscr{C}}(M,N)_{\bar{1}}$ where

$$\text{Hom}_{\mathscr{C}}(M,N)_{\bar{0}} := \text{Hom}_{\mathscr{C}}(M,N), \qquad \text{Hom}_{\mathscr{C}}(M,N)_{\bar{1}} := \text{Hom}_{\mathscr{C}}(M,\Pi N).$$

The composition law is obvious (but involves the isomorphism $\Pi^2 \cong \text{id}$ which is given as part of the data of $\mathscr{C}$). This means one can talk about *even* and *odd* morphisms between objects of $\mathscr{C}$. In the case of $\mathfrak{g}$-smod, an odd homomorphism $f : M \to N$ is a linear map such that $f(M_i) \subseteq N_{i+\bar{1}}$ for $i \in \mathbb{Z}/2$ and $f(xv) = (-1)^{|x|}xf(v)$ for homogeneous $x \in \mathfrak{g}, v \in M$.

Let $\mathfrak{b}$ be the *standard Borel subalgebra* consisting of all upper triangular matrices in $\mathfrak{g}$. It is the stabilizer of the *standard flag* $\langle u_1 \rangle < \langle u_1, u_2 \rangle < \cdots < \langle u_1, \dots, u_{n+m} \rangle$ in the natural representation V. More generally a *Borel subalgebra* of $\mathfrak{g}$ is the stabilizer of an arbitrary homogeneous flag in V. Unlike in the purely even setting, it is not true that all Borel subalgebras are conjugate under the appropriate action of the general linear supergroup $G = \text{GL}_{n|m}$. This leads to some combinatorially interesting variants of the theory which are also well understood, but our focus in this article will just be on the standard choice of Borel.

Let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrices. Let $\delta_1, \dots, \delta_{n+m}$ be the basis for $\mathfrak{t}^*$ such that δ_i picks out the ith diagonal entry of a diagonal matrix. Define a non-degenerate symmetric bilinear form $(?,?)$ on $\mathfrak{t}^*$ by setting $(\delta_i, \delta_j) := (-1)^{\bar{u}_i}\delta_{i,j}$. The *root system* of $\mathfrak{g}$ is

$$R := \{\delta_i - \delta_j \mid 1 \leq i, j \leq n+m, i \neq j\},$$

which decomposes into even and odd roots $R = R_{\bar{0}} \sqcup R_{\bar{1}}$ so that $\delta_i - \delta_j$ is of parity $\bar{u}_i + \bar{u}_j$. Let $R^+ = R^+_{\bar{0}} \sqcup R^+_{\bar{1}}$ denote the positive roots associated to the Borel subalgebra $\mathfrak{b}$, i.e. $\delta_i - \delta_j$ is positive if and only if $i < j$. The *dominance order* $\trianglerighteq$ on $\mathfrak{t}^*$ is defined so that $\lambda \trianglerighteq \mu$ if $\lambda - \mu$ is a sum of positive roots. Let

$$\rho := -\delta_2 - 2\delta_3 - \cdots - (n-1)\delta_n + (n-1)\delta_{n+1} + (n-2)\delta_{n+2} + \cdots + (n-m)\delta_{n+m}.$$

One can check that 2ρ is congruent to the sum of the positive even roots minus the sum of the positive odd roots modulo $\delta := \delta_1 + \cdots + \delta_n - \delta_{n+1} - \cdots - \delta_{n+m}$.

Let $s\mathscr{O}$ be the full subcategory of $\mathfrak{g}$-smod consisting of all finitely generated $\mathfrak{g}$-supermodules which are locally finite dimensional over $\mathfrak{b}$ and satisfy

$$M = \bigoplus_{\lambda \in \mathfrak{t}^*} M_\lambda,$$

where for $\lambda \in \mathfrak{t}^*$ we write $M_\lambda = M_{\lambda,\bar{0}} \oplus M_{\lambda,\bar{1}}$ for the λ-weight space of M with respect to $\mathfrak{t}$ defined in the standard way. This is an abelian subcategory of $\mathfrak{g}$-smod closed under Π. It is the analog for $\mathfrak{gl}_{n|m}(\mathbb{C})$ of the Bernstein-Gelfand-Gelfand category $\mathscr{O}$ for a semisimple Lie algebra. All of the familiar basic properties from the purely even setting generalize rather easily to the super case. For example all supermodules in $s\mathscr{O}$ have finite length, there are enough projectives, and so on. An easy way to prove these statements is to compare $s\mathscr{O}$ to the classical BGG category $\mathscr{O}_{ev}$ for the even part $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_m(\mathbb{C})$ of $\mathfrak{g}$. One can restrict any supermodule in $s\mathscr{O}$ to $\mathfrak{g}_{\bar{0}}$ to get a module in $\mathscr{O}_{ev}$; conversely for any $M \in \mathscr{O}_{ev}$ we can view it as a supermodule concentrated in a single parity then induce to get $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} M \in \mathscr{O}$. This relies on the fact that $U(\mathfrak{g})$ is free of finite rank as a $U(\mathfrak{g}_{\bar{0}})$-module, thanks to the PBW theorem for Lie superalgebras. Then the fact that $s\mathscr{O}$ has enough projectives follows because $\mathscr{O}_{ev}$ does, and induction sends projectives to projectives as it is left adjoint to an exact functor.

In fact it is possible to eliminate the "super" in the supercategory $s\mathscr{O}$ entirely by passing to a certain subcategory $\mathscr{O}$. To explain this let $\widehat{\mathbb{C}}$ be some set of representatives for the cosets of $\mathbb{C}$ modulo $\mathbb{Z}$ such that $0 \in \widehat{\mathbb{C}}$. Then define $p_{z+n} := \bar{n} \in \mathbb{Z}/2$ for each $z \in \widehat{\mathbb{C}}$ and $n \in \mathbb{Z}$. Finally for $\lambda \in \mathfrak{t}^*$ let $p(\lambda) := p_{(\lambda,\delta_{n+1}+\cdots+\delta_{n+m})}$. This defines a *parity function* $p : \mathfrak{t}^* \to \mathbb{Z}/2$ with the key property that $p(\lambda+\delta_i) = p(\lambda) + \bar{u}_i$. If $M \in s\mathscr{O}$ then M decomposes as a direct sum of $\mathfrak{g}$-supermodules as

$$M = M_+ \oplus M_- \qquad \text{where} \qquad M_+ := \bigoplus_{\lambda \in \mathfrak{t}^*} M_{\lambda,p(\lambda)}, \quad M_- := \bigoplus_{\lambda \in \mathfrak{t}^*} M_{\lambda,p(\lambda)+\bar{1}}.$$

Let $\mathscr{O}$ (resp. $\Pi\mathscr{O}$) be the full subcategory of $s\mathscr{O}$ consisting of all supermodules M such that $M = M_+$ (resp. $M = M_-$). Both are Serre subcategories of $s\mathscr{O}$, hence they are abelian, and the functor Π defines an equivalence between $\mathscr{O}$ and $\Pi\mathscr{O}$. Moreover there are no non-zero odd homomorphisms between objects of $\mathscr{O}$; equivalently there are no non-zero even homomorphisms between an object of $\mathscr{O}$ and an object of $\Pi\mathscr{O}$. Hence:

Lemma 2.2 $s\mathscr{O} = \mathscr{O} \oplus \Pi\mathscr{O}$.

Remark 2.3 Let $\widehat{s\mathscr{O}}$ be the enriched category arising from the supercategory $s\mathscr{O}$ as in Remark 2.1. Lemma 2.2 implies that the natural inclusion functor $\mathscr{O} \to \widehat{s\mathscr{O}}$ is fully faithful and essentially surjective, hence it defines an equivalence between $\mathscr{O}$ and $\widehat{s\mathscr{O}}$. In particular $\widehat{s\mathscr{O}}$ is itself abelian, although the explicit construction of kernels and cokernels of inhomogeneous morphisms in $\widehat{s\mathscr{O}}$ is a bit awkward.

Henceforth we will work just with the category $\mathscr{O}$ rather than the supercategory $s\mathscr{O}$. Note in particular that $\mathscr{O}$ contains the natural supermodule U and its dual $U^\vee$, and it is closed under tensoring with these objects. For each $\lambda \in \mathfrak{t}^*$ we have the *Verma supermodule*

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda,p(\lambda)} \in \mathscr{O},$$

where $\mathbb{C}_{\lambda,p(\lambda)}$ is a one-dimensional $\mathfrak{b}$-supermodule of weight λ concentrated in parity $p(\lambda)$. The usual argument shows that $M(\lambda)$ has a unique irreducible quotient, which we denote by $L(\lambda)$. The supermodules $\{L(\lambda) \mid \lambda \in \mathfrak{t}^*\}$ give a complete set of pairwise non-isomorphic irreducibles in $\mathscr{O}$. We say that $\lambda \in \mathfrak{t}^*$ is *dominant* if

$$\begin{cases} (\lambda, \delta_i - \delta_{i+1}) \in \mathbb{Z}_{\geq 0} \text{ for } i = 1, \dots, n-1, \\ (\lambda, \delta_i - \delta_{i+1}) \in \mathbb{Z}_{\leq 0} \text{ for } i = n+1, \dots, n+m-1. \end{cases}$$

Then the supermodules $\{L(\lambda) \mid \text{for all dominant } \lambda \in \mathfrak{t}^*\}$ give a complete set of pairwise non-isomorphic finite dimensional irreducible $\mathfrak{g}$-supermodules (up to parity switch). This is an immediate consequence of the following elementary but important result.

Theorem 2.4 (Kac) *For $\lambda \in \mathfrak{t}^*$ the irreducible supermodule $L(\lambda)$ is finite dimensional if and only if λ is dominant.*

Proof. Let $L_{ev}(\lambda)$ be the irreducible highest weight module for $\mathfrak{g}_{\bar{0}}$ of highest weight λ. Classical theory tells us that $L_{ev}(\lambda)$ is finite dimensional if and only if λ is dominant. Since $L(\lambda)$ contains a highest weight vector of weight λ, its restriction to $\mathfrak{g}_{\bar{0}}$ has $L_{ev}(\lambda)$ as a composition factor, hence if $L(\lambda)$ is finite dimensional then λ is dominant. Conversely, let $\mathfrak{p}$ be the maximal parabolic subalgebra of $\mathfrak{g}$ consisting of block upper triangular matrices of the form $\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$. There is an obvious projection $\mathfrak{p} \twoheadrightarrow \mathfrak{g}_{\bar{0}}$, allowing us to view $L_{ev}(\lambda)$ as a $\mathfrak{p}$-supermodule concentrated in parity $p(\lambda)$. Then for any $\lambda \in \mathfrak{t}^*$ we can form the *Kac supermodule*

$$K(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_{ev}(\lambda) \in \mathscr{O}.$$

Since $K(\lambda)$ is a quotient of $M(\lambda)$, it has irreducible head $L(\lambda)$. Moreover the PBW theorem implies that $K(\lambda)$ is finite dimensional if and only if $L_{ev}(\lambda)$ is finite dimensional. Hence if λ is dominant we deduce that $L(\lambda)$ is finite dimensional.

The *degree of atypicality* of $\lambda \in \mathfrak{t}^*$ is defined to be the maximal number of mutually orthogonal odd roots $\beta \in R^+_{\bar{1}}$ such that $(\lambda + \rho, \beta) = 0$. In particular λ is *typical* if $(\lambda + \rho, \beta) \neq 0$ for all $\beta \in R^+_{\bar{1}}$. For typical $\lambda \in \mathfrak{t}^*$, Kac showed further that the Kac supermodules $K(\lambda)$ defined in the proof of Theorem 2.4 are actually irreducible. Thus most questions about typical irreducible supermodules in $\mathscr{O}$ reduce to the purely even case. For example using the Weyl character formula one can deduce in this way a simple formula for the character of an arbitrary typical finite dimensional irreducible $\mathfrak{g}$-supermodule. It is not so easy to compute the characters of *atypical* finite dimensional irreducible supermodules, but this has turned out still to be combinatorially quite tractable. We will say more about the much harder problem of finding characters of arbitrary (not necessarily typical or finite dimensional) irreducible supermodules in $\mathscr{O}$ in the next section; inevitably this involves some Kazhdan-Lusztig polynomials.

Let $P(\mu)$ be a projective cover of $L(\mu)$ in $\mathscr{O}$. We have the usual statement of *BGG reciprocity*: each $P(\mu)$ has a *Verma flag*, i.e. a finite filtration whose sections are Verma supermodules, and the multiplicity $(P(\mu) : M(\lambda))$ of $M(\lambda)$ as a section of a Verma flag of $P(\mu)$ is given by

$$(P(\mu) : M(\lambda)) = [M(\lambda) : L(\mu)],$$

where the right hand side denotes composition multiplicity. Of course $[M(\lambda) : L(\mu)]$ is zero unless $\mu \trianglelefteq \lambda$ in the dominance ordering, while $[M(\lambda) : L(\lambda)] = 1$. Thus $\mathscr{O}$ is a *highest weight category* in the formal sense of Cline, Parshall and Scott, with weight poset $(\mathfrak{t}^*, \trianglelefteq)$.

The partial order $\trianglelefteq$ on $\mathfrak{t}^*$ being used here is rather crude. It can be replaced with a more intelligent order $\leq$, called the *Bruhat order*. To define this, given $\lambda \in \mathfrak{t}^*$, let

$$A(\lambda) := \{\alpha \in R^+_{\bar{0}} \mid (\lambda + \rho, \alpha^\vee) \in \mathbb{Z}_{>0}\}, \qquad B(\lambda) := \{\beta \in R^+_{\bar{1}} \mid (\lambda + \rho, \beta) = 0\},$$

where $\alpha^\vee$ denotes $2\alpha/(\alpha, \alpha)$. Then introduce a relation $\uparrow$ on $\mathfrak{t}^*$ by declaring that $\mu \uparrow \lambda$ if we either have that $\mu = s_\alpha \cdot \lambda$ for some $\alpha \in A(\lambda)$ or we have that $\mu = \lambda - \beta$ for some

$\beta \in B(\lambda)$; here, for $\alpha = \delta_i - \delta_j \in R_{\bar{0}}^+$ and $\lambda \in \mathfrak{t}^*$, we write $s_\alpha \cdot \lambda$ for $s_\alpha(\lambda+\rho)-\rho$, where $s_\alpha : \mathfrak{t}^* \to \mathfrak{t}^*$ is the reflection transposing δ_i and δ_j and fixing all other δ_k. Finally define $\leq$ to be the transitive closure of the relation $\uparrow$, i.e. we have that $\mu \leq \lambda$ if there exists $r \geq 0$ and weights $\nu_0, \dots, \nu_r \in \mathfrak{t}^*$ with $\mu = \nu_0 \uparrow \nu_1 \uparrow \cdots \uparrow \mu_r = \lambda$.

Lemma 2.5 *If $[M(\lambda) : L(\mu)] \neq 0$ then $\mu \leq \lambda$ in the Bruhat order.*

Proof. This is a consequence of the super analog of the Jantzen sum formula from [39, §10.3]; see also [26]. In more detail, the Jantzen filtration on $M(\lambda)$ is a certain exhaustive descending filtration $M(\lambda) = M(\lambda)_0 \supset M(\lambda)_1 \supseteq M(\lambda)_2 \supseteq \cdots$ such that $M(\lambda)_0 / M(\lambda)_1 \cong L(\lambda)$, and the sum formula shows that

$$\sum_{k \geq 1} \mathrm{ch}\, M(\lambda)_k = \sum_{\alpha \in A(\lambda)} \mathrm{ch}\, M(s_\alpha \cdot \lambda) + \sum_{\beta \in B(\lambda)} \sum_{k \geq 1} (-1)^{k-1} \mathrm{ch}\, M(\lambda - k\beta).$$

To deduce the lemma from this, suppose that $[M(\lambda) : L(\mu)] \neq 0$. Then $\mu \trianglelefteq \lambda$, so that $\lambda - \mu$ is a sum of N simple roots $\delta_i - \delta_{i+1}$ for some $N \geq 0$. We proceed by induction on N, the case $N = 0$ being vacuous. If $N > 0$ then $L(\mu)$ is a composition factor of $M(\lambda)_1$ and the sum formula implies that $L(\mu)$ is a composition factor either of $M(s_\alpha \cdot \lambda)$ for some $\alpha \in A(\lambda)$ or that $L(\mu)$ is a composition factor of $M(\lambda - k\beta)$ for some odd $k \geq 1$ and $\beta \in B(\lambda)$. It remains to apply the induction hypothesis and the definition of $\uparrow$.

Let $\approx$ be the equivalence relation on $\mathfrak{t}^*$ generated by the Bruhat order $\leq$. We refer to the $\approx$-equivalence classes as *linkage classes*. For a linkage class $\xi \in \mathfrak{t}^*/\approx$, let $\mathscr{O}_\xi$ be the Serre subcategory of $\mathscr{O}$ generated by the irreducible supermodules $\{L(\lambda) \,|\, \lambda \in \xi\}$. Then, as a purely formal consequence of Lemma 2.5, we get that the category $\mathscr{O}$ decomposes as

$$\mathscr{O} = \bigoplus_{\xi \in \mathfrak{t}^*/\approx} \mathscr{O}_\xi.$$

In fact this is the finest possible such direct sum decomposition, i.e. each $\mathscr{O}_\xi$ is an indecomposable subcategory of $\mathscr{O}$. In other words, this is precisely the decomposition of $\mathscr{O}$ into *blocks*. An interesting open problem here is to classify the blocks $\mathscr{O}_\xi$ up to equivalence.

Let us describe the linkage class ξ of $\lambda \in \mathfrak{t}^*$ more explicitly. Let k be the degree of atypicality of λ and $\beta_1, \dots, \beta_k \in R_{\bar{1}}^+$ be distinct mutually orthogonal odd roots such that $(\lambda+\rho, \beta_i) = 0$ for each $i = 1, \dots, k$. Also let W_λ be the *integral Weyl group* corresponding to λ, that is, the subgroup of $GL(\mathfrak{t}^*)$ generated by the reflections s_α for $\alpha \in R_{\bar{0}}^+$ such that $(\lambda+\rho, \alpha) \in \mathbb{Z}$. Then

$$\xi = \left\{ w \cdot (\lambda + n_1\beta_1 + \cdots + n_k\beta_k) \,\middle|\, n_1, \dots, n_k \in \mathbb{Z}, w \in W_\lambda \right\},$$

where $w \cdot \nu = w(\nu+\rho) - \rho$ as before. Note in particular that all $\mu \approx \lambda$ have the same degree of atypicality k as λ.

The following useful result reduces many questions about $\mathscr{O}$ to the case of *integral blocks*, that is, blocks corresponding to linkage classes of *integral weights* belonging to the set

$$\mathfrak{t}^*_{\mathbb{Z}} := \mathbb{Z}\delta_1 \oplus \cdots \oplus \mathbb{Z}\delta_{n+m}.$$

Theorem 2.6 (Cheng, Mazorchuk, Wang) *Every block $\mathscr{O}_\xi$ of $\mathscr{O}$ is equivalent to a tensor product of integral blocks of general linear Lie superalgebras of the same total rank as $\mathfrak{g}$.*

If λ is atypical then the linkage class ξ containing λ is infinite. This is a key difference between the representation theory of Lie superalgebras and the classical representation theory of a semisimple Lie algebra, in which all blocks are finite (bounded by the order of the Weyl group). It means that the highest weight category $\mathscr{O}_\xi$ cannot be viewed as a category of modules over a finite dimensional quasi-hereditary algebra. Nevertheless one can still consider the underlying basic algebra

$$A_\xi := \bigoplus_{\lambda,\mu\in\xi} \mathrm{Hom}_{\mathfrak{g}}(P(\lambda),P(\mu))$$

with multiplication coming from composition. This is a *locally unital algebra*, meaning that it is equipped with the system of mutually orthogonal idempotents $\{1_\lambda \mid \lambda \in \xi\}$ such that

$$A_\xi = \bigoplus_{\lambda,\mu\in\xi} 1_\mu A_\xi 1_\lambda,$$

where 1_λ denotes the identity endomorphism of $P(\lambda)$. Writing mof-A_ξ for the category of finite dimensional locally unital right A_ξ-modules, i.e. modules M with $M = \bigoplus_{\lambda\in\xi} M1_\lambda$, the functor

$$\mathscr{O}_\xi \to \text{mof-}A_\xi, \qquad M \mapsto \bigoplus_{\lambda\in\xi} \mathrm{Hom}_{\mathfrak{g}}(P(\lambda),-)$$

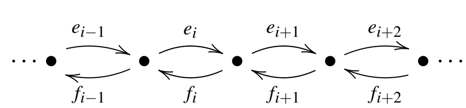
is an equivalence of categories. Note moreover that each right ideal $1_\lambda A_\xi$ and each left ideal $A_\xi 1_\lambda$ is finite dimensional; these are the indecomposable projectives and the linear duals of the indecomposable injectives in mof-A_ξ, respectively.

Remark 2.7 It is also natural to view A_ξ as a superalgebra concentrated in parity $\bar{0}$. Then the block $s\mathscr{O}_\xi = \mathscr{O}_\xi \oplus \Pi\mathscr{O}_\xi$ of the supercategory $s\mathscr{O}$ associated to the linkage class ξ is equivalent to the category of finite dimensional locally unital right A_ξ-*supermodules*. This gives another point of view on Lemma 2.2.

Example 2.8 Let us work out in detail the example of $\mathfrak{gl}_{1|1}(\mathbb{C})$. This is easy but nevertheless very important: often $\mathfrak{gl}_{1|1}(\mathbb{C})$ plays a role parallel to that of $\mathfrak{sl}_2(\mathbb{C})$ in the classical theory. So now $\rho = 0$ and the only postive root is $\alpha = \delta_1 - \delta_2 \in R^+_{\bar{1}}$. The Verma supermodules $M(\lambda)$ are the same as the Kac supermodules $K(\lambda)$ from the proof of Theorem 2.4; they are two-dimensional with weights λ and $\lambda - \alpha$. Moreover $M(\lambda)$ is irreducible for typical λ. If λ is atypical then $\lambda = c\alpha$ for some $c \in \mathbb{C}$, and the irreducible supermodule $L(\lambda)$ comes from the one-dimensional representation $\mathfrak{g} \to \mathbb{C}, x \mapsto c\,\mathrm{str}\,x$ where str denotes *supertrace*. Finally let us restrict attention just to the *principal block* $\mathscr{O}_0$ containing the irreducible supermodules $L(i) := L(i\alpha)$ for each $i \in \mathbb{Z}$. We have shown that $M(i) := M(i\alpha)$ has length two with composition factors $L(i)$ and $L(i-1)$; hence by BGG reciprocity the projective indecomposable supermodule $P(i) := P(i\alpha)$ has a two-step Verma flag with sections $M(i)$ and $M(i+1)$. We deduce that the Loewy series of $P(i)$ looks like $P(i) = P^0(i) > P^1(i) > P^2(i) > 0$ with

$$P^0(i)/P^1(i) \cong L(i), \qquad P^1(i)/P^2(i) \cong L(i-1) \oplus L(i+1), \qquad P^2(i) \cong L(i).$$

From this one obtains the following presentation for the underlying basic algebra A_0: it is the path algebra of the quiver



with vertex set $\mathbb{Z}$, modulo the relations $e_i f_i + f_{i+1} e_{i+1} = 0, e_{i+1} e_i = f_i f_{i+1} = 0$ for all $i \in \mathbb{Z}$. We stress the similarity between these and the relations $ef + fe = c, e^2 = f^2 = 0$ in $U(\mathfrak{g})$ itself (where $c = e_{1,1} + e_{2,2} \in \mathfrak{z}(\mathfrak{g})$, $e = e_{1,2}$ and $f = e_{2,1}$). One should also observe at this point that these relations are homogeneous, so that A_0 can be viewed as a positively graded algebra, with grading coming from path length. In fact this grading makes A_0 into a (locally unital) Koszul algebra.

To conclude the section, we offer one piece of justification for focussing so much attention on category $\mathscr{O}$. The study of primitive ideals of universal enveloping algebras of Lie algebras, especially semisimple ones, has classically proved to be very rich and inspired many important discoveries. So it is natural to ask about the space of all primitive ideals $\operatorname{Prim} U(\mathfrak{g})$ in our setting too. It turns out for $\mathfrak{gl}_{n|m}(\mathbb{C})$ that all primitive ideals are automatically homogeneous. In fact one just needs to consider annihilators of irreducible supermodules in $\mathscr{O}$:

Theorem 2.9 (Musson) $\operatorname{Prim} U(\mathfrak{g}) = \{\operatorname{Ann}_{U(\mathfrak{g})} L(\lambda) \,|\, \lambda \in \mathfrak{t}^*\}$.

This is the analog of a famous theorem of Duflo in the context of semisimple Lie algebras. Letzter showed subsequently that there is a bijection

$$\operatorname{Prim} U(\mathfrak{g}_{\bar{0}}) \xrightarrow{\sim} \operatorname{Prim} U(\mathfrak{g}), \qquad \operatorname{Ann}_{U(\mathfrak{g}_{\bar{0}})} L_{ev}(\lambda) \mapsto \operatorname{Ann}_{U(\mathfrak{g})} L(\lambda).$$

Combined with classical results of Joseph, this means that the fibers of the map

$$\mathfrak{t}^* \to \operatorname{Prim} U(\mathfrak{g}), \qquad \lambda \mapsto \operatorname{Ann}_{U(\mathfrak{g})} L(\lambda)$$

can be described in terms of the Robinson-Schensted algorithm. Hence we get an explicit description of the set $\operatorname{Prim} U(\mathfrak{g})$.

Notes. For the basic facts about super category $\mathscr{O}$ for basic classical Lie superalgebras, see §8.2 of Musson's book [39]. Lemma 2.2 was pointed out originally in [10, §4-e]. The observation that $s\widehat{\mathscr{O}}$ is abelian from Remark 2.3 is due to Cheng and Lam [17]; in fact these authors work entirely with the equivalent category $s\widehat{\mathscr{O}}$ in place of our $\mathscr{O}$.

The classification of finite dimensional irreducible supermodules from Theorem 2.4 is due to Kac [30]. The irreducibilty of the typical Kac supermodules was established soon after in [31]. Kac only considered finite dimensional representations at the time but the same argument works in general. Composition multiplicities of atypical Kac supermodules were first computed as a certain alternating sum by Serganova in [43]. In fact, all Kac supermodules are multiplicity-free, so that Serganova's formula simplifies to 0 or 1. This was proved in [10] by a surprisingly direct representation theoretic argument, confirming a conjecture from [29]; see also [41] for a combinatorial proof of the equivalence of the formulae for composition multiplicies in [43] and [10]. Another approach to the finite dimensional representations via "super duality" was developed in [22, 17], showing in particular that the Kazhdan-Lusztig polynomials appearing in [43, 10] are the same as certain Kazhdan-Lusztig polynomials for Grassmannians as computed originally by Lascoux and Schützenberger [33]. Subsequently Su and Zhang [47] were able to use the explicit formula for these Kazhdan-Lusztig polynomials to extract some closed character and dimension formulae for the finite dimensional irreducibles. There is also an elegant diagrammatic description of the basic algebra that is Morita equivalent to the subcategory $\mathscr{F}$ of $\mathscr{O}$ consisting of all its finite dimensional supermodules in terms of Khovanov's arc algebra; see [16].

The analog of BGG reciprocity for $\mathfrak{gl}_{n|m}(\mathbb{C})$ as stated here was first established by Zou [48]; see also [12]. For the classification of blocks of $\mathscr{O}$ and proof of Theorem 2.6, see [19, Theorems 3.10–3.12]. A related problem is to determine when two irreducible highest weight supermodules have the same central character. This is solved via the explicit description of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ in terms the Harish-Chandra homomorphism and supersymmetric polynomials, which is due to Kac; see [39, §13.1] or [21, §2.2] for recent expositions. Lemma 2.5 is slightly more subtle and cannot be deduced just from central character considerations. Musson has recently proved a refinement of the sum formula recorded in the proof of Lemma 2.5, in which the right hand side is rewritten as a finite sum of characters of highest weight modules; details will appear in [40].

The results of Musson, Letzter and Joseph classifying primitive ideals of $U(\mathfrak{g})$ are in [38, 35, 28]; see also [39, Ch. 15]. The recent preprint [24] makes some further progress towards determining all inclusions between primitive ideals.

3 Kazhdan-Lusztig combinatorics and categorification

In this section we restrict attention just to the highest weight subcategory $\mathscr{O}_{\mathbb{Z}}$ of $\mathscr{O}$ consisting of supermodules M such that $M = \bigoplus_{\lambda \in \mathfrak{t}^*_{\mathbb{Z}}} M_{\lambda, p(\lambda)}$. In other words we only consider *integral blocks*. This is justified by Theorem 2.6. The goal is to understand the composition multiplicities

$$[M(\lambda) : L(\mu)]$$

of the Verma supermodules in $\mathscr{O}_{\mathbb{Z}}$. It will be convenient as we explain this to represent $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$ instead by the $n|m$-tuple $(\lambda_1, \dots, \lambda_n | \lambda_{n+1}, \dots, \lambda_{n+m})$ of integers defined from $\lambda_i := (\lambda + \rho, \delta_i)$.

Let P denote the free abelian group $\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}\varepsilon_i$ and $Q \subset P$ be the subgroup generated by the *simple roots* $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. Thus Q is the root lattice of the Lie algebra $\mathfrak{sl}_\infty$. Let $\leq$ be the usual dominance ordering on P defined by $\xi \leq \varpi$ if $\varpi - \xi$ is a sum of simple roots. For $\lambda = (\lambda_1, \dots, \lambda_n | \lambda_{n+1}, \dots, \lambda_{n+m}) \in \mathfrak{t}^*_{\mathbb{Z}}$ we let

$$|\lambda| := \varepsilon_{\lambda_1} + \cdots + \varepsilon_{\lambda_n} - \varepsilon_{\lambda_{n+1}} - \cdots - \varepsilon_{\lambda_{n+m}} \in P.$$

Then it is clear that two weights $\lambda, \mu \in \mathfrak{t}^*_{\mathbb{Z}}$ are linked if and only if $|\lambda| = |\mu|$, i.e. the fibers of the map $\mathfrak{t}^*_{\mathbb{Z}} \to P, \lambda \mapsto |\lambda|$ are exactly the linkage classes. The Bruhat order $\leq$ on $\mathfrak{t}^*_{\mathbb{Z}}$ can also be interpreted in these terms: let

$$|\lambda|_i := \begin{cases} \varepsilon_{\lambda_i} & \text{for } 1 \leq i \leq n, \\ -\varepsilon_{\lambda_i} & \text{for } n+1 \leq i \leq n+m, \end{cases}$$

so that $|\lambda| = |\lambda|_1 + \cdots + |\lambda|_{n+m}$. Then one can show that $\lambda \leq \mu$ in the Bruhat order if and only if $|\lambda|_1 + \cdots + |\lambda|_i \geq |\mu|_1 + \cdots + |\mu|_i$ in the dominance ordering on P for all $i = 1, \dots, n+m$, with equality when $i = n+m$.

Let V be the natural $\mathfrak{sl}_\infty$-module on basis $\{v_i \,|\, i \in \mathbb{Z}\}$ and W be its dual on basis $\{w_i \,|\, i \in \mathbb{Z}\}$. The Chevalley generators $\{f_i, e_i \,|\, i \in \mathbb{Z}\}$ of $\mathfrak{sl}_\infty$ act by

$$f_i v_j = \delta_{i,j} v_{i+1}, \quad e_i v_j = \delta_{i+1,j} v_i, \quad f_i w_j = \delta_{i+1,j} w_i, \quad e_i w_j = \delta_{i,j} w_{i+1}.$$

The tensor space $V^{\otimes n} \otimes W^{\otimes m}$ has the obvious basis of monomials

$$v_\lambda := v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n} \otimes w_{\lambda_{n+1}} \otimes \cdots \otimes w_{\lambda_{n+m}}$$

indexed by $n|m$-tuples $\lambda = (\lambda_1, \dots, \lambda_n | \lambda_{n+1}, \dots, \lambda_{n+m})$ of integers. In other words the monomial basis of $V^{\otimes n} \otimes W^{\otimes m}$ is parametrized by the set $\mathfrak{t}^*_{\mathbb{Z}}$ of integral weights for $\mathfrak{g} = \mathfrak{gl}_{n|m}(\mathbb{C})$.

This prompts us to bring category $\mathscr{O}$ back into the picture. Let $\mathscr{O}_{\mathbb{Z}}^{\Delta}$ be the exact subcategory of $\mathscr{O}_{\mathbb{Z}}$ consisting of all supermodules with a Verma flag, and denote its complexified Grothendieck group by $K(\mathscr{O}_{\mathbb{Z}}^{\Delta})$. Thus $K(\mathscr{O}_{\mathbb{Z}}^{\Delta})$ is the complex vector space on basis $\{[M(\lambda)]\,|\,\lambda \in \mathfrak{t}_{\mathbb{Z}}^*\}$. Henceforth we *identify*

$$K(\mathscr{O}_{\mathbb{Z}}^{\Delta}) \leftrightarrow V^{\otimes n} \otimes W^{\otimes m}, \qquad [M(\lambda)] \leftrightarrow v_\lambda.$$

Since projectives have Verma flags we have that $P(\mu) \in \mathscr{O}_{\mathbb{Z}}^{\Delta}$; let $b_\mu \in V^{\otimes n} \otimes W^{\otimes m}$ be the corresponding tensor under the above identification, i.e.

$$[P(\mu)] \leftrightarrow b_\mu.$$

By BGG reciprocity we have that

$$b_\mu = \sum_{\lambda \in \mathfrak{t}_{\mathbb{Z}}^*} [M(\lambda) : L(\mu)] v_\lambda.$$

Now the punchline is that the vectors $\{b_\mu\,|\,\mu \in \mathfrak{t}_{\mathbb{Z}}^*\}$ turn out to coincide with Lusztig's *canonical basis* for the tensor space $V^{\otimes n} \otimes W^{\otimes m}$. The definition of the latter goes via some quantum algebra introduced in the next few paragraphs.

Let $U_q\mathfrak{sl}_\infty$ be the quantized enveloping algebra associated to $\mathfrak{sl}_\infty$. This is the $\mathbb{Q}(q)$-algebra on generators $\{\dot{f}_i, \dot{e}_i, \dot{k}_i, \dot{k}_i^{-1}\,|\,i \in \mathbb{Z}\}$[1] subject to well-known relations. We view $U_q\mathfrak{sl}_\infty$ as a Hopf algebra with comultiplication

$$\Delta(\dot{f}_i) = 1 \otimes \dot{f}_i + \dot{f}_i \otimes \dot{k}_i, \quad \Delta(\dot{e}_i) = \dot{k}_i^{-1} \otimes \dot{e}_i + \dot{e}_i \otimes 1, \quad \Delta(\dot{k}_i) = \dot{k}_i \otimes \dot{k}_i.$$

We have the natural $U_q\mathfrak{sl}_\infty$-module $\dot{V}$ on basis $\{\dot{v}_i\,|\,i \in \mathbb{Z}\}$ and its dual $\dot{W}$ on basis $\{\dot{w}_i\,|\,i \in \mathbb{Z}\}$. The Chevalley generators $\dot{f}_i$ and $\dot{e}_i$ of $U_q\mathfrak{sl}_\infty$ act on these basis vectors by exactly the same formulae as at $q = 1$, and also $\dot{k}_i \dot{v}_j = q^{\delta_{i,j} - \delta_{i+1,j}} \dot{v}_j$ and $\dot{k}_i \dot{w}_j = q^{\delta_{i+1,j} - \delta_{i,j}} \dot{w}_j$. There is also an R-matrix giving some distinguished intertwiners $\dot{V} \otimes \dot{V} \xrightarrow{\sim} \dot{V} \otimes \dot{V}$ and $\dot{W} \otimes \dot{W} \xrightarrow{\sim} \dot{W} \otimes \dot{W}$, from which we produce the following $U_q\mathfrak{sl}_\infty$-module homomorphisms:

$$\dot{c} : \dot{V} \otimes \dot{V} \to \dot{V} \otimes \dot{V}, \qquad \dot{v}_i \otimes \dot{v}_j \mapsto \begin{cases} \dot{v}_j \otimes \dot{v}_i + q^{-1} \dot{v}_i \otimes \dot{v}_j & \text{if } i < j, \\ (q + q^{-1}) \dot{v}_j \otimes \dot{v}_i & \text{if } i = j, \\ \dot{v}_j \otimes \dot{v}_i + q \dot{v}_i \otimes \dot{v}_j & \text{if } i > j; \end{cases}$$

$$\dot{c} : \dot{W} \otimes \dot{W} \to \dot{W} \otimes \dot{W}, \qquad \dot{w}_i \otimes \dot{w}_j \mapsto \begin{cases} \dot{w}_j \otimes \dot{w}_i + q \dot{w}_i \otimes \dot{w}_j & \text{if } i < j, \\ (q + q^{-1}) \dot{w}_j \otimes \dot{w}_i & \text{if } i = j, \\ \dot{w}_j \otimes \dot{w}_i + q^{-1} \dot{w}_i \otimes \dot{w}_j & \text{if } i > j. \end{cases}$$

Then we form the tensor space $\dot{V}^{\otimes n} \otimes \dot{W}^{\otimes m}$, which is a $U_q\mathfrak{sl}_\infty$-module with its monomial basis $\{\dot{v}_\lambda\,|\,\lambda \in \mathfrak{t}_{\mathbb{Z}}^*\}$ defined just like above. Let $\dot{c}_k := 1^{\otimes(k-1)} \otimes \dot{c} \otimes 1^{n+m-1-k}$ for $k \neq n$, which is a $U_q\mathfrak{sl}_\infty$-module endomorphism of $\dot{V}^{\otimes n} \otimes \dot{W}^{\otimes m}$.

Next we must pass to a formal completion $\dot{V}^{\otimes n} \widehat{\otimes} \dot{W}^{\otimes m}$ of our q-tensor space. Let $I \subset \mathbb{Z}$ be a finite subinterval and $I_+ := I \cup (I+1)$. Let $\dot{V}_I$ and $\dot{W}_I$ be the subspaces of $\dot{V}$ and $\dot{W}$ spanned by the basis vectors $\{\dot{v}_i\,|\,i \in I_+\}$ and $\{\dot{w}_i\,|\,i \in I_+\}$, respectively. Then $\dot{V}_I^{\otimes n} \otimes \dot{W}_I^{\otimes m}$ is a subspace of $\dot{V}^{\otimes n} \otimes \dot{W}^{\otimes m}$. For $J \subseteq I$ there is an obvious projection $\pi_J : \dot{V}_I^{\otimes n} \otimes \dot{W}_I^{\otimes m} \twoheadrightarrow \dot{V}_J^{\otimes n} \otimes \dot{W}_J^{\otimes m}$ mapping $\dot{v}_\lambda$ to $\dot{v}_\lambda$ if all the entries of the tuple λ lie in J_+, or to zero otherwise. Then we set

[1] We follow the convention of adding a dot to all q-analogs to distinguish them from their classical counterparts.

$$\dot V^{\otimes n}\widehat{\otimes}\dot W^{\otimes m} := \varprojlim \dot V_I^{\otimes n}\otimes \dot W_I^{\otimes m},$$

taking the inverse limit over all finite subintervals $I \subset \mathbb{Z}$ with respect to the projections π_J just defined. The action of $U_q\mathfrak{sl}_\infty$ and of each $\dot c_k$ extend naturally to the completion.

Lemma 3.1 *There is a unique continuous antilinear involution*

$$\psi : \dot V^{\otimes n}\widehat{\otimes}\dot W^{\otimes m} \to \dot V^{\otimes n}\widehat{\otimes}\dot W^{\otimes m}$$

such that

- *ψ commutes with the actions of $\dot f_i$ and $\dot e_i$ for all $i \in \mathbb{Z}$ and with the endomorphisms $\dot c_k$ for all $k \neq n$;*
- *each $\psi(\dot v_\lambda)$ is equal to $\dot v_\lambda$ plus a (possibly infinite) $\mathbb{Z}[q,q^{-1}]$-linear combination of $\dot v_\mu$ for $\mu > \lambda$ in the Bruhat order.*

Proof. For each finite subinterval $I \subset \mathbb{Z}$, let $U_q\mathfrak{sl}_I$ be the subalgebra of $U_q\mathfrak{sl}_\infty$ generated by $\{\dot f_i, \dot e_i, \dot k_i^{\pm 1} \mid i \in I\}$. A construction of Lusztig [37, §27.3] involving the quasi-R-matrix Θ_I for $U_q\mathfrak{sl}_I$ gives an antilinear involution $\psi_I : \dot V_I^{\otimes n}\otimes \dot W_I^{\otimes m} \to \dot V_I^{\otimes n}\otimes \dot W_I^{\otimes m}$ commuting with the actions of $\dot f_i$ and $\dot e_i$ for $i \in I$. Moreover for $J \subset I$ the involutions ψ_I and ψ_J are intertwined by the projection $\pi_J : \dot V_I^{\otimes n}\otimes \dot W_I^{\otimes m} \twoheadrightarrow \dot V_J^{\otimes n}\otimes \dot W_J^{\otimes m}$, as follows easily from the explicit form of the quasi-R-matrix. Hence the involutions ψ_I for all I induce a well-defined involution ψ on the inverse limit. The fact that the resulting involution commutes with each $\dot c_k$ can be deduced from the formal definition of the latter in terms of the R-matrix. Finally the uniqueness is a consequence of the existence of an algorithm to uniquely compute the canonical basis using just the given two properties (as sketched below).

This puts us in position to apply Lusztig's lemma to deduce for each $\mu \in \mathfrak{t}_\mathbb{Z}^*$ that there is a unique vector $\dot b_\mu \in \dot V^{\otimes n}\widehat{\otimes}\dot W^{\otimes m}$ such that

- $\psi(\dot b_\mu) = \dot b_\mu$;
- $\dot b_\mu$ is equal to $\dot v_\mu$ plus a (possibly infinite) $q\mathbb{Z}[q]$-linear combination of $\dot v_\lambda$ for $\lambda > \mu$.

We refer to the resulting topological basis $\{\dot b_\mu \mid \mu \in \mathfrak{t}_\mathbb{Z}^*\}$ for $\dot V^{\otimes n}\widehat{\otimes}\dot W^{\otimes m}$ as the *canonical basis*. In fact, but this is in no way obvious from the above definition, each $\dot b_\mu$ is always a *finite* sum of $\dot v_\lambda$'s, i.e. $\dot b_\mu \in \dot V^{\otimes n}\otimes \dot W^{\otimes m}$ before completion. Moreover the polynomials $d_{\lambda,\mu}(q)$ arising from the expansion

$$\dot b_\mu = \sum_{\lambda \in \mathfrak{t}_\mathbb{Z}^*} d_{\lambda,\mu}(q)\dot v_\lambda$$

are known always to be some finite type A parabolic Kazhdan-Lusztig polynomials (suitably normalized). In particular $d_{\lambda,\mu}(q) \in \mathbb{N}[q]$.

Now we can state the following fundamental theorem, formerly known as the "super Kazhdan-Lusztig conjecture."

Theorem 3.2 (Cheng, Lam, Wang) *For any $\lambda,\mu \in \mathfrak{t}_\mathbb{Z}^*$ we have that $[M(\lambda):L(\mu)] = d_{\lambda,\mu}(1)$. In other words, the vectors $\{b_\mu \mid \mu \in \mathfrak{t}_\mathbb{Z}^*\}$ arising from the projective indecomposable supermodules in $\mathscr{O}_\mathbb{Z}$ via the identification $K(\mathscr{O}_\mathbb{Z}^\Delta) \leftrightarrow V^{\otimes n}\otimes W^{\otimes m}$ coincide with the specialization of Lusztig's canonical basis $\{\dot b_\mu \mid \mu \in \mathfrak{t}_\mathbb{Z}^*\}$ at $q = 1$.*

We are going to do two more things in this section. First we sketch briefly how one can compute the canonical basis algorithmically. Then we will explain how Theorem 3.2 should really be understood in terms of a certain graded lift $\dot{\mathscr{O}}_{\mathbb{Z}}$ of $\mathscr{O}_{\mathbb{Z}}$, using the language of categorification.

The algorithm to compute the canonical basis goes by induction on the degree of atypicality. Recall that a weight $\mu \in \mathfrak{t}_{\mathbb{Z}}^*$ is *typical* if $\{\mu_1,\dots,\mu_n\} \cap \{\mu_{n+1},\dots,\mu_{n+m}\} = \varnothing$. We also say it is *weakly dominant* if $\mu_1 \geq \cdots \geq \mu_n$ and $\mu_{n+1} \leq \cdots \leq \mu_{n+m}$ (equivalently $\mu + \rho$ is dominant in the earlier sense). The weights that are both typical and weakly dominant are maximal in the Bruhat ordering, so that $\dot{b}_\mu = \dot{v}_\mu$. Then to compute $\dot{b}_\mu$ for an arbitrary typical but not weakly dominant μ we just have to follow the usual algorithm to compute Kazhdan-Lusztig polynomials. Pick $k \neq n$ such that *either* $k < n$ and $\mu_k < \mu_{k+1}$ *or* $k > n$ and $\mu_k > \mu_{k+1}$. Let λ be the weight obtained from μ by interchanging μ_k and μ_{k+1}. By induction on the Bruhat ordering we may assume that $\dot{b}_\lambda$ is already computed. Then $\dot{c}_k \dot{b}_\lambda$ is ψ-invariant and has $\dot{v}_\mu$ as its leading term with coefficient 1, i.e. it equals $\dot{v}_\mu$ plus a $\mathbb{Z}[q,q^{-1}]$-linear combination of $\dot{v}_\nu$ for $\nu > \mu$. It just remains to adjust this vector by subtracting bar-invariant multiples of inductively computed canonical basis vectors $\dot{b}_\nu$ for $\nu > \mu$ to obtain a vector that is both ψ-invariant and lies in $\dot{v}_\mu + \sum_{\lambda > \mu} q\mathbb{Z}[q]\dot{v}_\lambda$. This must equal $\dot{b}_\mu$ by the uniqueness.

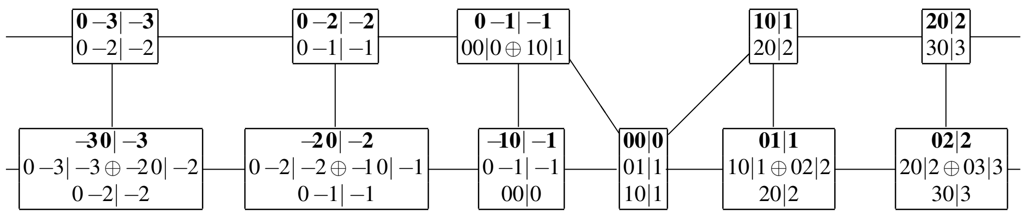
Now suppose that $\mu \in \mathfrak{t}_{\mathbb{Z}}^*$ is not typical. The idea to compute $\dot{b}_\mu$ then is to apply a certain *bumping procedure* to produce from μ another weight λ of strictly smaller atypicality, together with a monomial $\dot{x}$ of quantum divided powers of Chevalley generators of $U_q\mathfrak{sl}_\infty$, such that $\dot{x}\dot{b}_\lambda$ has $\dot{v}_\mu$ as its leading term with coefficient 1. Then we can adjust this ψ-invariant vector by subtracting bar-invariant multiples of recursively computed canonical basis vectors $\dot{b}_\nu$ for $\nu > \mu$, to obtain $\dot{b}_\mu$ as before. The catch is that (unlike the situation in the previous paragraph) there are infinitely many weights $\nu > \mu$ so that it is not clear that the recursion always terminates in finitely many steps. Examples computed using a GAP implementation of the algorithm suggest that it always does; our source code is available at [13]. (In any case one can always find a finite interval I such that $\dot{x}\dot{b}_\lambda \in \dot{V}_I^{\otimes n} \otimes \dot{W}_I^{\otimes m}$; then by some non-trivial but known positivity of structure constants we get that $\dot{b}_\mu \in \dot{V}_I^{\otimes n} \otimes \dot{W}_I^{\otimes m}$ too; hence one can apply π_I prior to making any subsequent adjustments to guarantee that the algorithm terminates in finitely many steps.)

Example 3.3 With this example we outline the bumping procedure. Given an atypical μ we let i be the largest integer that appears both to the left and to the right of the separator $|$ in the tuple μ. Pick one of the two sides of the separator and let $j \geq i$ be maximal such that all of $i, i+1, \dots, j$ appear on this side of μ. Add 1 to all occurrences of $i, i+1, \dots, j$ on the chosen side. Then if $j+1$ also appears on the other side of μ we repeat the bumping procedure on that side with i replaced by $j+1$. We continue in this way until $j+1$ is not repeated on the other side. This produces the desired output weight λ of strictly smaller atypicality. For example if $\mu = (0,5,2,2|0,1,3,4)$ of atypicality one we bump as follows:

$$(1,6,3,3|0,2,4,5) \xleftarrow{\dot{e}_5} (1,5,3,3|0,2,4,5) \xleftarrow{\dot{f}_4\dot{f}_3} (1,5,3,3|0,2,3,4)$$
$$\xleftarrow{\dot{e}_2^{(2)}} (1,5,2,2|0,2,3,4) \xleftarrow{\dot{f}_1} (1,5,2,2|0,1,3,4) \xleftarrow{\dot{e}_0} (0,5,2,2|0,1,3,4).$$

The labels on the edges here are the appropriate monomials that reverse the bumping procedure; then the final monomial $\dot{x}$ output by the bumping procedure is the product $\dot{e}_5\dot{f}_4\dot{f}_3\dot{e}_2^{(2)}\dot{f}_1\dot{e}_0$ of all of these labels. Thus we should compute $\dot{e}_5\dot{f}_4\dot{f}_3\dot{e}_2^{(2)}\dot{f}_1\dot{e}_0\dot{b}_{(1,6,3,3|0,2,4,5)}$, where $\dot{b}_{(1,6,3,3|0,2,4,5)}$ can be worked out using the typical algorithm. The result is a ψ-invariant vector equal to $\dot{b}_{(0,5,2,2|0,1,3,4)}$ plus some higher terms which can be computed recursively (specifically one finds that $(q+q^{-1})\dot{b}_{(2,5,2,2|1,2,3,4)}$ needs to be subtracted).

Example 3.4 Here we work out the combinatorics in the principal block for $\mathfrak{gl}_{2|1}(\mathbb{C})$. The weights are $\{(0,i|i),(i,0|i) \mid i \in \mathbb{Z}\}$. The corresponding canonical basis vectors $\dot{b}_\mu$ are represented in the following diagram which is arranged according to the Bruhat graph; we show just enough vertices for the generic pattern to be apparent.



For example the center node of this diagram encodes $\dot{b}_{(0,0|0)} = \dot{v}_{(0,0|0)} + q\dot{v}_{(0,1|1)} + q^2\dot{v}_{(1,0|1)}$; the node to the right of that encodes $\dot{b}_{(0,1|1)} = \dot{v}_{(0,1|1)} + q\dot{v}_{(1,0|1)} + q\dot{v}_{(0,2|2)} + q^2\dot{v}_{(2,0|2)}$. Let us explain in more detail how we computed $\dot{b}_{(-1,0|-1)}$ here. The bumping procedure tells us to look at $\dot{e}_0\dot{e}_{-1}\dot{b}_{(0,1|-1)}$. As $(0,1|-1)$ is typical we get easily from the typical algorithm that $\dot{b}_{(0,1|-1)} = \dot{c}_1\dot{v}_{(1,0|-1)} = \dot{v}_{(0,1|-1)} + q\dot{v}_{(1,0|-1)}$. Hence

$$\dot{e}_0\dot{e}_{-1}\dot{b}_{(0,1|-1)} = \dot{v}_{(-1,0|-1)} + (1+q^2)\dot{v}_{(0,0|0)} + q\dot{v}_{(0,-1|-1)} + q\dot{v}_{(0,1|1)} + q^2\dot{v}_{(1,0|1)}.$$

This vector is ψ-invariant with the right leading term $\dot{v}_{(-1,0|-1)}$, but we must make one correction to remove a term $\dot{v}_{(0,0|0)}$, i.e. we must subtract $\dot{b}_{(0,0|0)}$ as already computed, to obtain that $\dot{b}_{(-1,0|-1)} = \dot{v}_{(-1,0|-1)} + q\dot{v}_{(0,-1|-1)} + q^2\dot{v}_{(0,0|0)}$.

Returning to more theoretical considerations, the key point is that the category $\mathscr{O}_\mathbb{Z}$ is an example of an $\mathfrak{sl}_\infty$-tensor product categorification of $V^{\otimes n} \otimes W^{\otimes m}$. This means in particular that there exist some exact endofunctors F_i and E_i of $\mathscr{O}_\mathbb{Z}^\Delta$ whose induced actions on $K(\mathscr{O}_\mathbb{Z}^\Delta)$ match the actions of the Chevalley generators f_i and e_i on $V^{\otimes n} \otimes W^{\otimes m}$ under our identification. To define these functors, recall that U denotes the natural $\mathfrak{g}$-module of column vectors. Let $U^\vee$ be its dual. Introduce the biadjoint projective functors

$$F := - \otimes U : \mathscr{O}_\mathbb{Z} \to \mathscr{O}_\mathbb{Z}, \qquad E := - \otimes U^\vee : \mathscr{O}_\mathbb{Z} \to \mathscr{O}_\mathbb{Z}.$$

The action of the Casimir tensor

$$\Omega := \sum_{i,j=1}^{n+m} (-1)^{\bar{u}_j} e_{i,j} \otimes e_{j,i} \in \mathfrak{g} \otimes \mathfrak{g}$$

defines an endomorphism of $FM = M \otimes U$ for each $M \in \mathscr{O}_\mathbb{Z}$. Let F_i be the summand of the functor F defined so that F_iM is the generalized i-eigenspace of Ω for each $i \in \mathbb{Z}$. We then have that $F = \bigoplus_{i\in\mathbb{Z}} F_i$. Similarly the functor E decomposes as $E = \bigoplus_{i\in\mathbb{Z}} E_i$ where each E_i is biadjoint to F_i; explicitly one can check that E_iM is the generalized $(m-n-i)$-eigenspace of Ω on $EM = M \otimes U^\vee$. Now it is an instructive exercise to prove:

Lemma 3.5 *The exact functors F_i and E_i send supermodules with Verma flags to supermodules with Verma flags. Moreover the induced endomorphisms $[F_i]$ and $[E_i]$ of $K(\mathscr{O}_\mathbb{Z}^\Delta)$ agree under the above identification with the endomorphisms f_i and e_i of $V^{\otimes n} \otimes W^{\otimes m}$ defined by the action of the Chevalley generators of $\mathfrak{sl}_\infty$.*

In fact much more is true here. The action of Ω on each FM defines a natural transformation $x \in \mathrm{End}(F)$. Also let $t \in \mathrm{End}(F^2)$ be such that $t_M : F^2M \to F^2M$ is the endomorphism

$$t_M : M \otimes U \otimes U \to M \otimes U \otimes U, \qquad v \otimes u_i \otimes u_j \mapsto (-1)^{\bar{u}_i \bar{u}_j} v \otimes u_j \otimes u_i.$$

From x and t one obtains $x_i := F^{d-i}xF^{i-1} \in \mathrm{End}(F^d)$ and $t_j := F^{d-j-1}tF^{j-1} \in \mathrm{End}(F^d)$ for each $d \geq 0, 1 \leq i \leq d$ and $1 \leq j \leq d-1$. It is straightforward to check that these natural transformations satisfy the defining relations of the degenerate affine Hecke algebra H_d. This shows that the category $\mathcal{O}_{\mathbb{Z}}$ equipped with the biadjoint pair of endofunctors F and E, plus the endomorphisms $x \in \mathrm{End}(F)$ and $t \in \mathrm{End}(F^2)$, is an $\mathfrak{sl}_\infty$*-categorification* in the sense of Chuang and Rouquier. In addition $\mathcal{O}_{\mathbb{Z}}$ is a highest weight category, and Lemma 3.5 checks some appropriate compatibility of the categorical action with this highest weight structure. The conclusion is that $\mathcal{O}_{\mathbb{Z}}$ is actually an $\mathfrak{sl}_\infty$*-tensor product categorification* of $V^{\otimes n} \otimes W^{\otimes m}$ in a formal sense introduced by Losev and Webster.

We are ready to state the following extension of the super Kazhdan-Lusztig conjecture, which incorporates a $\mathbb{Z}$-grading in the spirit of the classic work of Beilinson, Ginzburg and Soergel on Koszulity of category $\mathcal{O}$ in the purely even setting.

Theorem 3.6 (Brundan, Losev, Webster) *There exists a unique (up to equivalence) graded lift $\dot{\mathcal{O}}_{\mathbb{Z}}$ of $\mathcal{O}_{\mathbb{Z}}$ that is a $U_q\mathfrak{sl}_\infty$-tensor product categorification of $\dot{V}^{\otimes n} \otimes \dot{W}^{\otimes m}$. Moreover the category $\dot{\mathcal{O}}_{\mathbb{Z}}$ is a standard Koszul highest weight category, and its graded decomposition numbers $[\dot{M}(\lambda) : \dot{L}(\mu)]_q$ are given by the parabolic Kazhdan-Lusztig polynomials $d_{\lambda,\mu}(q)$ as defined above.*

A few more explanations are in order. To start with we should clarify what it means to say that $\dot{\mathcal{O}}_{\mathbb{Z}}$ is a graded lift of $\mathcal{O}_{\mathbb{Z}}$. The easiest way to understand this is to remember as discussed in the previous section that $\mathcal{O}_{\mathbb{Z}}$ is equivalent to the category mof-A of finite dimensional locally unital right A-modules, where A is the locally unital algebra

$$A := \bigoplus_{\lambda,\mu \in \mathfrak{t}_{\mathbb{Z}}^*} \mathrm{Hom}_{\mathfrak{g}}(P(\lambda), P(\mu)).$$

To give a graded lift $\dot{\mathcal{O}}_{\mathbb{Z}}$ of $\mathcal{O}_{\mathbb{Z}}$ amounts to exhibiting some $\mathbb{Z}$-grading on the algebra A with respect to which each of its distinguished idempotents 1_λ are homogeneous; then the category grmof-A of *graded* finite dimensional locally unital right A-modules gives a graded lift of $\mathcal{O}_{\mathbb{Z}}$. Of course there can be many ways to do this, including the trivial way that puts all of A in degree zero! Theorem 3.6 asserts in particular that the algebra A admits a positive grading making it into a (locally unital) Koszul algebra; as is well known such a grading (if it exists) is unique up to automorphism.

For this choice of grading, the category $\dot{\mathcal{O}}_{\mathbb{Z}} :=$ grmof-A is a graded highest weight category with distinguished irreducible objects $\{\dot{L}(\lambda) \mid \lambda \in \mathfrak{t}_{\mathbb{Z}}^*\}$, standard objects $\{\dot{M}(\lambda) \mid \lambda \in \mathfrak{t}_{\mathbb{Z}}^*\}$ and indecomposable projective objects $\{\dot{P}(\lambda) := 1_\lambda A \mid \lambda \in \mathfrak{t}_{\mathbb{Z}}^*\}$; these are graded lifts of the modules $L(\lambda), M(\lambda)$ and $P(\lambda)$, respectively, such that the canonical maps $\dot{P}(\lambda) \twoheadrightarrow \dot{M}(\lambda) \twoheadrightarrow \dot{L}(\lambda)$ are all homogeneous of degree zero. Then the assertion from Theorem 3.6 that $\dot{\mathcal{O}}_{\mathbb{Z}}$ is standard Koszul means that each $\dot{M}(\lambda)$ possesses a *linear projective resolution*, that is, there is an exact sequence

$$\cdots \to \dot{P}^2(\lambda) \to \dot{P}^1(\lambda) \to \dot{P}(\lambda) \to \dot{M}(\lambda) \to 0$$

such that for each $i \geq 1$ the module $\dot{P}^i(\lambda)$ is a direct sum of graded modules $q^i \dot{P}(\mu)$ for $\mu > \lambda$. Here q denotes the degree shift functor defined on a graded module M by letting qM be the same underlying module with new grading defined from $(qM)_i := M_{i-1}$.

Let $\dot{\mathcal{O}}_{\mathbb{Z}}^{\Delta}$ be the exact subcategory of $\dot{\mathcal{O}}_{\mathbb{Z}}$ consisting of modules with a graded Δ-flag. Its Grothendieck group is a $\mathbb{Z}[q,q^{-1}]$-module with q acting by degree shift. Let $K_q(\dot{\mathcal{O}}_{\mathbb{Z}}^{\Delta})$ be the $\mathbb{Q}(q)$-vector space obtained by extending scalars, i.e. it is the $\mathbb{Q}(q)$-vector space on basis $\{[\dot{M}(\lambda)] \mid \lambda \in \mathfrak{t}_{\mathbb{Z}}^*\}$. Then again we *identify*

$$K_q(\dot{\mathcal{O}}_{\mathbb{Z}}^{\Delta}) \leftrightarrow \dot{V}^{\otimes n} \otimes \dot{W}^{\otimes m}, \qquad [\dot{M}(\lambda)] \leftrightarrow \dot{v}_{\lambda}.$$

The assertion about graded decomposition numbers in Theorem 3.6 means under this identification that

$$[\dot{P}(\mu)] \leftrightarrow \dot{b}_{\mu}.$$

The assertion that $\dot{\mathcal{O}}_{\mathbb{Z}}$ is a $U_q\mathfrak{sl}_\infty$-tensor product categorification means in particular that the biadjoint endofunctors F_i and E_i of $\mathcal{O}_{\mathbb{Z}}$ admit graded lifts $\dot{F}_i$ and $\dot{E}_i$, which are also biadjoint up to approriate degree shifts. Moreover these graded functors preserve modules with a graded Verma flag, and their induced actions on $K_q(\dot{\mathcal{O}}_{\mathbb{Z}}^{\Delta})$ agree with the actions of $\dot{f}_i, \dot{e}_i \in U_q\mathfrak{sl}_\infty$ under our identification.

We will say more about the proof of Theorem 3.6 in the next section.

Notes. The identification of the Bruhat order on $\mathfrak{t}_{\mathbb{Z}}^*$ with the "reverse dominance ordering" is justified in [10, Lemma 2.5]. Our Lemma 3.1 is a variation on [10, Theorem 2.14]; the latter theorem was used in [10] to define a twisted version of the canonical basis which corresponds to the indecomposable tilting supermodules rather than the indecomposable projectives in $\mathcal{O}_{\mathbb{Z}}$. The super Kazhdan-Lusztig conjecture as formulated here is equivalent to [10, Conjecture 4.32]; again the latter was expressed in terms of tilting supermodules. The equivalence of the two versions of the conjecture can be deduced from the Ringel duality established in [12, (7.4)]; see also [15, Remark 5.30]. The algorithm for computing the canonical basis sketched here is a variation on an algorithm described in detail in [10, §2-h]; the latter algorithm computes the twisted canonical basis rather than the canonical basis. Example 3.4 was worked out already in [20, §9.5].

Theorems 3.2 and 3.6 are proved in [18] and [15], respectively. In fact both of these articles also prove a more general form of the super Kazhdan-Lusztig conjecture which is adapted to arbitrary Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$; at the level of combinatorics this amounts to shuffling the tensor factors in the tensor product $\dot{V}^{\otimes n} \otimes \dot{W}^{\otimes m}$ into more general orders. The article [15] also considers parabolic analogs. The idea that blocks of category $\mathcal{O}$ should possess Koszul graded lifts goes back to the seminal work of Beilinson, Ginzburg and Soergel [4] in the context of semisimple Lie algebras. The notion of $\mathfrak{sl}$-categorification was introduced by Chuang and Rouquier following their joint work [23]. The definition was recorded for the first time in the literature in [42, Definition 5.29]. For the definition of tensor product categorification, see [36, Definition 3.2] and also [15, Definition 2.9]. A full proof of Lemma 3.5 (and its generalization to the parabolic setting) can be found in [15, Theorem 3.9].

4 Principal W-algebras and the double centralizer property

By a *prinjective object* we mean an object that is both projective and injective. To set the scene for this section we recall a couple of classical results. Let $\mathcal{O}_0$ be the principal block of category $\mathcal{O}$ for a semisimple Lie algebra $\mathfrak{g}$, and recall that the irreducible modules in $\mathcal{O}_0$ are the modules $\{L(w \cdot 0) \mid w \in W\}$ parametrized by the Weyl group W. There is a unique indecomposable prinjective module in $\mathcal{O}_0$ up to isomorphism, namely, the projective cover

$P(w_0 \cdot 0)$ of the "antidominant" Verma module $L(w_0 \cdot 0)$; here w_0 is the longest element of the Weyl group.

Theorem 4.1 (Soergel's Endomorphismensatz) *The endomorphism algebra*

$$C_0 := \mathrm{End}_{\mathfrak{g}}(P(w_0 \cdot 0))$$

is generated by the center $Z(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$. Moreover C_0 is canonically isomorphic to the coinvariant algebra, i.e. the cohomology algebra $H^*(G/B, \mathbb{C})$ of the flag variety associated to $\mathfrak{g}$.

Theorem 4.2 (Soergel's Struktursatz) *The functor*

$$\mathbb{V}_0 := \mathrm{Hom}_{\mathfrak{g}}(P(w_0 \cdot 0), -) : \mathscr{O}_0 \to \text{mof-}C_0$$

is fully faithful on projectives.

With these two theorems in hand, we can explain Soergel's approach to the construction of the Koszul graded lift of the category $\mathscr{O}_0$. Introduce the *Soergel modules*

$$Q(w) := \mathbb{V}_0 P(w \cdot 0) \in \text{mof-}C_0$$

for each $w \in W$. The Struktursatz implies that the finite dimensional algebra

$$A_0 := \bigoplus_{x,y \in W} \mathrm{Hom}_{C_0}(Q(x), Q(y))$$

is isomorphic to the endomorphism algebra of a minimal projective generator for $\mathscr{O}_0$. The algebra C_0 is naturally graded as it is a cohomology algebra. It turns out that each Soergel module $Q(w)$ also admits a unique graded lift $\dot{Q}(w)$ that is a self-dual graded C_0-module. Hence we get induced a grading on the algebra A_0. This is the grading making A_0 into a Koszul algebra. The resulting category grmof-A_0 is the appropriate graded lift $\dot{\mathscr{O}}_0$ of $\mathscr{O}_0$.

Now we return to the situation of the previous section, so $\mathscr{O}_{\mathbb{Z}}$ is the integral part of category $\mathscr{O}$ for $\mathfrak{g} = \mathfrak{gl}_{n|m}(\mathbb{C})$ and we represent integral weights $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$ as $n|m$-tuples of integers. The proof of Theorem 3.6 stated above follows a similar strategy to Soergel's construction in the classical case but there are several complications. To start with, in any atypical block, there turn out to be infinitely many isomorphism classes of indecomposable prinjective supermodules:

Lemma 4.3 *For $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$, the projective supermodule $P(\lambda) \in \mathscr{O}_{\mathbb{Z}}$ is injective if and only if λ is* antidominant, *i.e. $\lambda_1 \leq \cdots \leq \lambda_n$ and $\lambda_{n+1} \geq \cdots \geq \lambda_{n+m}$. (Recall λ_i denotes $(\lambda + \rho, \delta_i) \in \mathbb{Z}$.)*

Proof. This follows by a special case of [15, Theorem 2.22]. More precisely, there is an $\mathfrak{sl}_\infty$-crystal with vertex set $\mathfrak{t}^*_{\mathbb{Z}}$, namely, Kashiwara's crystal associated to the $\mathfrak{sl}_\infty$-module $V^{\otimes n} \otimes W^{\otimes m}$. Then [15, Theorem 2.22] shows that the set of $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$ such that $P(\lambda)$ is injective is the vertex set of the connected component of this crystal containing any weight $(i, \ldots, i | j, \ldots, j)$ for $i < j$. Now it is a simple combinatorial exercise to see that the vertices in this connected component are exactly the antidominant $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$.

Remark 4.4 More generally, for $\lambda \in \mathfrak{t}^*$, the projective $P(\lambda) \in \mathscr{O}$ is injective if and only if λ is antidominant in the sense that $(\lambda, \delta_i - \delta_j) \notin \mathbb{Z}_{\geq 0}$ for $1 \leq i < j \leq n$ and $(\lambda, \delta_i - \delta_j) \notin \mathbb{Z}_{\leq 0}$ for $n+1 \leq i < j \leq n+m$. This follows from Lemma 4.3 and Theorem 2.6. In other words, the projective $P(\lambda)$ is injective if and only if the irreducible supermodule $L(\lambda)$ is of maximal Gelfand-Kirillov dimension amongst all supermodules in $\mathscr{O}$.

Then, fixing $\xi \in \mathfrak{t}^*_{\mathbb{Z}}/\approx$, the appropriate analog of the coinvariant algebra for the block $\mathscr{O}_\xi$ is the locally unital algebra

$$C_\xi := \bigoplus_{\text{Antidominant } \lambda,\mu \in \xi} \operatorname{Hom}_{\mathfrak{g}}(P(\lambda), P(\mu)).$$

For atypical blocks this algebra is infinite dimensional and no longer commutative. Still there is an analog of the Struktursatz:

Theorem 4.5 (Brundan, Losev, Webster) *The functor $\mathbb{V}_\xi : \mathscr{O}_\xi \to$ mof-C_ξ sending $M \in \mathscr{O}_\xi$ to*

$$\mathbb{V}_\xi M := \bigoplus_{\text{Antidominant } \lambda \in \xi} \operatorname{Hom}_{\mathfrak{g}}(P(\lambda), M)$$

is fully faithful on projectives.

However we do not at present know of any explicit description of the algebra C_ξ. Instead the proof of Theorem 3.6 involves another abelian category mod-H_ξ. This notation is strange because actually there is no single algebra H_ξ here, rather, there is an infinite *tower of cyclotomic quiver Hecke algebras* $H^1_\xi \subset H^2_\xi \subset H^3_\xi \subset \cdots$, which arise as the endomorphism algebras of larger and larger finite direct sums of indecomposable prinjective supermodules (with multiplicities). Then the category mod-H_ξ consists of sequences of finite dimensional modules over this tower of Hecke algebras subject to some stability condition. Moreover there is an explicitly constructed exact functor $\mathbb{U}_\xi : \mathscr{O}_\xi \to$ mod-H_ξ. The connection between this and the functor $\mathbb{V}_\xi$ comes from the following lemma.

Lemma 4.6 *There is a unique (up to isomorphism) equivalence of categories*

$$\mathbb{I}_\xi : \text{mod-}H_\xi \xrightarrow{\sim} \text{mof-}C_\xi$$

such that $\mathbb{V}_\xi \cong \mathbb{I}_\xi \circ \mathbb{U}_\xi$.

Proof. This follows because both of the functors $\mathbb{U}_\xi$ and $\mathbb{V}_\xi$ are quotient functors, i.e. they satisfy the universal property of the Serre quotient of $\mathscr{O}_\xi$ by the subcategory generated by $\{L(\lambda) \mid \lambda \in \xi$ such that λ is *not* antidominant$\}$. For $\mathbb{U}_\xi$ this universal property is established in [15, Theorem 4.9]. It is automatic for $\mathbb{V}_\xi$.

Each of the algebras H^r_ξ in the tower of Hecke algebras is naturally graded, so that we are able to define a corresponding graded category grmod-H_ξ. Then we prove that the modules $Y(\lambda) := \mathbb{U}_\xi P(\lambda) \in$ mod-H_ξ admit unique graded lifts $\dot{Y}(\lambda) \in$ grmod-H_ξ which are self-dual in an appropriate sense. Since the functor $\mathbb{U}_\xi$ is also fully faithful on projectives (e.g. by Theorem 4.5 and Lemma 4.6), we thus obtain a $\mathbb{Z}$-grading on the basic algebra

$$A_\xi := \bigoplus_{\lambda,\mu \in \xi} \operatorname{Hom}_{\mathfrak{g}}(P(\lambda), P(\mu)) \cong \bigoplus_{\lambda,\mu \in \xi} \operatorname{Hom}_{H_\xi}(\dot{Y}(\lambda), \dot{Y}(\mu)).$$

that is Morita equivalent to $\mathscr{O}_\xi$. This grading turns out to be Koszul, and grmof-A_ξ gives the desired graded lift $\dot{\mathscr{O}}_\xi$ of the block $\mathscr{O}_\xi$ from Theorem 3.6.

The results just described provide a substitute for Soergel's Endomorphismensatz for $\mathfrak{gl}_{n|m}(\mathbb{C})$, with the tower of cyclotomic quiver Hecke algebras replacing the coinvariant algebra. However we still do not find this completely satisfactory, and actually believe that it should be possible to give an explicit (graded!) description of the basic algebra C_ξ itself.

This seems like a tractable problem whose solution could suggest some more satisfactory geometric picture underpinning the rich structure of super category $\mathscr{O}$.

Example 4.7 Here we give explicit generators and relations for the algebra C_0 for the principal block of $\mathscr{O}$ for $\mathfrak{gl}_{2|1}(\mathbb{C})$. The prinjectives are indexed by $\mathbb{Z}$ and their Verma flags are as displayed on the bottom row of the diagram in Example 3.4. The algebra C_0 is isomorphic to the path algebra of the same infinite linear quiver as in Example 2.8 modulo the relations

$$\begin{aligned} e_{i+1}e_i = f_i f_{i+1} &= 0 && \text{for all } i \in \mathbb{Z}, \\ f_{i+1}e_{i+1}f_{i+1}e_{i+1} + e_i f_i e_i f_i &= 0 && \text{for } i \leq -2 \text{ or } i \geq 1, \\ f_0 e_0 + e_{-1}f_{-1}e_{-1}f_{-1} &= 0, \\ f_1 e_1 f_1 e_1 + e_0 f_0 &= 0. \end{aligned}$$

Moreover the appropriate grading on C_0 is defined by setting $\deg(e_i) = \deg(f_i) = 1 + \delta_{i,0}$. Here is a brief sketch of how one can see this. The main point is to exploit Theorem 3.6: the grading on $\dot{\mathscr{O}}_0$ induces a positive grading on C_0 with degree zero component $\bigoplus_{i\in\mathbb{Z}} \mathbb{C}1_i$. Let $D(i)$ be the one-dimensional irreducible C_0-module corresponding to $i \in \mathbb{Z}$ and let $Q(i)$ be its projective cover (equivalently, injective hull). The proof of Theorem 3.6 implies further that these modules possess self-dual graded lifts $\dot{D}(i)$ and $\dot{Q}(i)$. A straightforward calculation using the graded version of BGG reciprocity and the information in Example 3.4 gives the graded composition multiplicities of each $\dot{Q}(i)$. From this one deduces for each $i \in \mathbb{Z}$ that there are unique (up to scalars) non-zero homomorphisms $e_i : \dot{Q}(i-1) \to \dot{Q}(i)$ and $f_i : \dot{Q}(i) \to \dot{Q}(i-1)$ that are homogeneous of degree $1 + \delta_{i,0}$. By considering images and kernels of these homomorphisms and using self-duality, it follows that each $\dot{Q}(i)$ has irreducible head $q^{-2}\dot{D}(i)$, irreducible socle $q^2\dot{D}(i)$, and heart $\operatorname{rad}\dot{Q}(i)/\operatorname{soc}\dot{Q}(i) \cong \dot{Q}_-(i) \oplus \dot{Q}_+(i)$, where $\dot{Q}_-(0) := \dot{D}(-1)$, $\dot{Q}_+(-1) := \dot{D}(0)$ and all other $\dot{Q}_\pm(i)$ are uniserial with layers $q^{-1}\dot{D}(i \pm 1), \dot{D}(i), q\dot{D}(i \pm 1)$ in order from top to bottom. Hence $(e_i f_i)^{2-\delta_{i,0}} \neq 0 \neq (f_{i+1}e_{i+1})^{2-\delta_{i+1,0}}$ for each $i \in \mathbb{Z}$. Since each $\operatorname{End}_{C_0}(\dot{Q}(i))$ is one-dimensional in degree 4, it is then elementary to see that e_i and f_i can be scaled to ensure that the given relations hold, and the result follows.

Remark 4.8 With a similar analysis, one can show for any $n \geq 1$ that the algebra C_ξ associated to the block ξ of $\mathfrak{gl}_{n|1}(\mathbb{C})$ containing the weight $-\rho = (0,\dots,0|0)$ is described by the same quiver as in Examples 2.8 and 4.7 subject instead to the relations

$$e_{i+1}e_i = f_i f_{i+1} = (f_{i+1}e_{i+1})^{n-\delta_{i+1,0}(n-1)} + (e_i f_i)^{n-\delta_{i,0}(n-1)} = 0$$

for all $i \in \mathbb{Z}$. This time $\deg(e_i) = \deg(f_i) = 1 + \delta_{i,0}(n-1)$.

To finish the article we draw attention to one more piece of this puzzle. First we need to introduce the *principal W-superalgebra* $W_{n|m}$ associated to $\mathfrak{g} = \mathfrak{gl}_{n|m}(\mathbb{C})$. Let π be a two-rowed array of boxes with a connected strip of $\min(n,m)$ boxes in its first (top) row and a connected strip of $\max(n,m)$ boxes in its second (bottom) row; each box in the first row should be immediately above a box in the second row but the boxes in the rows need not be left-justified. We write the numbers $1,\dots,n$ in order into the boxes on a row of length n and the numbers $n+1,\dots,n+m$ in order into the boxes on the other row. Also let s_- (resp. s_+) be the number of boxes overhanging on the left hand side (resp. the right hand side) of this diagram. For example here is a choice of the diagram π for $\mathfrak{gl}_{5,2}(\mathbb{C})$:

	6	7		
1	2	3	4	5

For this $s_- = 1$ and $s_+ = 2$. Numbering the columns of π by $1,2,\dots$ from left to right, we let $\mathrm{col}(i)$ be the column number of the box containing the entry i. Then define a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{d\in\mathbb{Z}} \mathfrak{g}(d)$ by declaring that the matrix unit $e_{i,j}$ is of degree $\mathrm{col}(j) - \mathrm{col}(i)$, and let

$$\mathfrak{p} := \bigoplus_{d\geq 0} \mathfrak{g}(d), \qquad \mathfrak{h} := \mathfrak{g}(0), \qquad \mathfrak{m} := \bigoplus_{d<0} \mathfrak{g}(d).$$

Let $e := e_{1,2} + \cdots + e_{n-1,n} + e_{n+1,n+2} + \cdots + e_{n+m-1,n+m} \in \mathfrak{g}(1)$. This is a representative for the principal nilpotent orbit in $\mathfrak{g}$. Let $\chi : \mathfrak{m} \to \mathbb{C}$ be the one-dimensional representation with $\chi(x) := \mathrm{str}(xe)$. Finally set

$$W_{n|m} := \{u \in U(\mathfrak{p}) \,|\, u\mathfrak{m}_\chi \subseteq \mathfrak{m}_\chi U(\mathfrak{g})\}$$

where $\mathfrak{m}_\chi := \{x - \chi(x) \,|\, x \in \mathfrak{m}\} \subset U(\mathfrak{m})$. It is easy to check that $W_{n|m}$ is a subalgebra of $U(\mathfrak{p})$.

Theorem 4.9 (Brown, Brundan, Goodwin) *The superalgebra $W_{n|m}$ contains some explicit even elements $\{d_i^{(r)} \,|\, i = 1,2, r > 0\}$ and odd elements $\{f^{(r)} \,|\, r > s_-\} \cup \{e^{(r)} \,|\, r > s_+\}$. These elements generate $W_{n|m}$ subject only to the following relations:*

$$\begin{aligned}
d_1^{(r)} &= 0 \qquad \textit{if } r > \min(m,n),\\
[d_i^{(r)}, d_j^{(s)}] &= 0,\\
[e^{(r)}, e^{(s)}] &= 0,\\
[f^{(r)}, f^{(s)}] &= 0,\\
[d_i^{(r)}, e^{(s)}] &= (-1)^p \sum_{a=0}^{r-1} d_i^{(a)} e^{(r+s-1-a)},\\
[d_i^{(r)}, f^{(s)}] &= -(-1)^p \sum_{a=0}^{r-1} f^{(r+s-1-a)} d_i^{(a)},\\
[e^{(r)}, f^{(s)}] &= (-1)^p \sum_{a=0}^{r+s-1} \tilde{d}_1^{(a)} d_2^{(r+s-1-a)}.
\end{aligned}$$

Here $d_i^{(0)} = 1$ and $\tilde{d}_i^{(r)}$ is defined recursively from $\sum_{a=0}^{r} \tilde{d}_i^{(a)} d_i^{(r-a)} = \delta_{r,0}$. Also $p := \bar{0}$ if the numbers $1,\dots,n$ appear on the first row of π, and $p := \bar{1}$ otherwise.

The relations in Theorem 4.9 arise from the defining relations for the Yangian $Y(\mathfrak{gl}_{1|1})$. In fact the structure of the superalgebra $W_{n|m}$ is quite interesting. To start with there is an explicit description of its center $Z(W_{n|m})$, which is canonically isomorphic to the center $Z(\mathfrak{g})$ of the universal enveloping superalgebra of $\mathfrak{g}$ itself. All of the supercommutators $[e^{(r)}, f^{(s)}]$ are central. Then $W_{n|m}$ possesses a triangular decomposition $W_{n|m} = W_{n|m}^- W_{n|m}^0 W_{n|m}^+$, i.e. the multiplication map $W_{n|m}^- \otimes W_{n|m}^0 \otimes W_{n|m}^+ \to W_{n|m}$ is a vector space isomorphism where

- $W_{n|m}^-$ is a Grassmann algebra generated freely by $\{f^{(r)} \,|\, s_- < r \leq s_- + \min(m,n)\}$;
- $W_{n|m}^+$ is a Grassmann algebra generated freely by $\{e^{(r)} \,|\, s_+ < r \leq s_+ + \min(m,n)\}$;
- $W_{n|m}^0$ is the polynomial algebra on $\{d_1^{(r)}, d_2^{(s)} \,|\, 0 < r \leq \min(m,n), 0 < s \leq \max(m,n)\}$.

Using this one can label its irreducible representations by mimicking the usual arguments of highest weight theory. They are all finite dimensional, in fact of dimension $2^{\min(m,n)-k}$ where

k is the degree of atypicality of the corresponding central character. Moreover the irreducible representations of integral central character are parametrized by the antidominant weights in $\mathfrak{t}^*_{\mathbb{Z}}$, i.e. the same weights that index the prinjective supermodules in $\mathscr{O}_{\mathbb{Z}}$.

The principal W-superalgebra is relevant to our earlier discussion because of the existence of the *Whittaker coinvariants functor*

$$\mathbb{W} := H_0(\mathfrak{m}_\chi, -) : \mathscr{O} \to W_{n|m}\text{-smod}.$$

This is an exact functor sending $M \in \mathscr{O}$ to the vector superspace $H_0(\mathfrak{m}_\chi, M) := M/\mathfrak{m}_\chi M$ of $\mathfrak{m}_\chi$-coinvariants in M. The definition of $W_{n|m}$ ensures that this is a (finite dimensional) left $W_{n|m}$-supermodule in the natural way. Then it turns out that the functor $\mathbb{W}$ sends the irreducible $L(\lambda) \in \mathscr{O}$ to an irreducible $W_{n|m}$-supermodule if λ is antidominant or to zero otherwise, and every irreducible $W_{n|m}$-supermodule arises in this way (up to isomorphism and parity switch).

Theorem 4.10 (Brown, Brundan, Goodwin) *For $\xi \in \mathfrak{t}^*_{\mathbb{Z}}/\approx$, let $\mathbb{W}_\xi : \mathscr{O}_\xi \to W_{n|m}$-smod be the restriction of the Whittaker coinvariants functor. As $\mathbb{V}_\xi$ is a quotient functor, there exists a unique (up to isomorphism) exact functor*

$$\mathbb{J}_\xi : \text{mof-}C_\xi \to W_{n|m}\text{-smod}$$

such that $\mathbb{W}_\xi \cong \mathbb{J}_\xi \circ \mathbb{V}_\xi$. This defines an equivalence of categories between mof-C_ξ *and a certain full subcategory $\mathscr{R}_\xi$ of $W_{n|m}$*-smod *which is closed under taking submodules, quotients and finite direct sums.*

Thus $\mathbb{W}_\xi : \mathscr{O}_\xi \to \mathscr{R}_\xi$ is another quotient functor which is fully faithful on projectives. One intriguing consequence is that for blocks ξ of maximal atypicality (in which all the irreducible $W_{n|m}$-supermodules are one-dimensional), the algebra C_ξ can be realized also as an "idempotented quotient" of $W_{n|m}$. We already saw a very special case of this in Example 2.8, and describe the next easiest case in Example 4.11 below; hopefully this gives a rough idea of what we mean by "idempotented quotient." Unfortunately in general we still have no idea of the precise form that the relations realizing C_ξ as a quotient of $W_{n|m}$ should take.

Example 4.11 The generators and relations for the principal W-superalgebra $W_{2|1}$ from Theorem 4.9 collapse to just requiring generators $c := d_2^{(1)} - d_1^{(1)}, d := -d_1^{(1)}, e := e^{(1+s_+)}$ and $f := f^{(1+s_-)}$, subject to the relations $[c,d] = [c,e] = [c,f] = 0$ (i.e. c is central), $[d,e] = e, [d,f] = -f$ and $e^2 = f^2 = 0$. Let C_0 be the algebra described explicitly in Example 4.7 and let $\widehat{C}_0$ be its completion consisting of (possibly infinite) formal sums $\{\sum_{i,j\in\mathbb{Z}} a_{i,j} \,|\, a_{i,j} \in 1_i C_0 1_j\}$. The relations imply that there is a homomorphism

$$\phi : W_{2|1} \to \widehat{C}_0, \qquad c \mapsto 0, \quad d \mapsto \sum_{i\in\mathbb{Z}} i 1_i, \quad e \mapsto \sum_{i\in\mathbb{Z}} e_i, \quad f \mapsto \sum_{i\in\mathbb{Z}} f_i.$$

Then we have that $C_0 = \bigoplus_{i,j\in\mathbb{Z}} 1_i \phi(W_{2|1}) 1_j$.

Notes. Soergel's Theorems 4.1–4.2 were proved originally in [45]. Soergel's proof of the Endomorphismensatz goes via deformed category $\mathscr{O}$; Bernstein subsequently gave a more elementary proof in [6]. We also mention [46] which contains a generalization of the Struktursatz to parabolic category $\mathscr{O}$; our Theorem 4.5 is a close relative of that. For the formal definition of the category mod-H_ξ stable modules over the tower of Hecke algebras mentioned briefly here we refer to [15, §4]. Theorem 4.5 follows from [15, Theorem 4.10] and Lemma 4.6. Example 4.7 was computed with help from Catharina Stroppel.

The definition of principal W-algebras for semisimple Lie algebras is due to Kostant [32], although of course the language is much more recent. Kostant showed in the classical case that the principal W-algebra is canonically isomorphic to the center $Z(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$. The explicit presentation for the principal W-superalgebra for $\mathfrak{gl}_{n|m}(\mathbb{C})$ from Theorem 4.9 is proved in [8, Theorem 4.5]. For the classification of irreducible $W_{n|m}$-supermodules we refer to [8, Theorems 7.2–7.3].

The Whittaker coinvariants functor $\mathbb{W}$ for the principal nilpotent orbit of $\mathfrak{gl}_{n|m}(\mathbb{C})$ is studied in detail in [9], culminating in the proof of Theorem 4.10. The identification of C_ξ as an idempotented quotient of $W_{n|m}$ is also explained more fully there. The idea that Soergel's functor $\mathbb{V}$ is related to $\mathbb{W}$ goes back to the work of Backelin [1] in the classical case. In [35, Theorem 4.7], Losev has developed a remarkably general theory of Whittaker coinvariant functors associated to arbitrary nilpotent orbits in semisimple Lie algebras; see also the brief discussion of Lie superalgebras in [35, §6.3.2].

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