CATEGORIFICATION OF QUASI-SPLIT iQUANTUM GROUPS

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ABSTRACT. We introduce a new family of graded 2-categories generalizing the 2-quantum groups introduced by Khovanov, Lauda and Rouquier. We use them to categorify quasi-split iquantum groups in all symmetric types.

CONTENTS

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1. Introduction

Drinfeld-Jimbo quantum groups were categorified by Khovanov and Lauda [KL10] and Rouquier [Rou08]. They introduced a family of graded 2-categories whose Grothendieck rings are isomorphic to Lusztig's modified integral form for the underlying quantum groups. This was proved for $\mathfrak{sl}_n$ and conjectured for other types in [KL10], and proved in general in [Web15, Web24]. The "2-quantum groups" arising from this work play an important role in higher representation theory and the related program to categorify Reshetikhin-Turaev invariants in low-dimensional topology.

In this paper, we introduce a more general family of graded 2-categories which are expected to categorify *iquantum groups* arising from quasi-split quantum symmetric pairs. We prove this whenever the underlying Cartan matrix is symmetric. For simplicity in the rest of the introduction, we restrict our attention just to this symmetric case.

Let Q be a loop-free quiver with vertex set I and $\#(i \to j)$ arrows from vertex i to vertex j. The corresponding Cartan matrix $A = (a_{i,j})_{i,j\in I}$ is defined by

$$a_{i,j} := \begin{cases} 2 & \text{if } i = j \\ -\#(i \to j) - \#(j \to i) & \text{if } i \neq j. \end{cases}$$

Also fix a realization of this Cartan matrix, that is, free Abelian groups X and Y with a perfect pairing $Y \times X \to \mathbb{Z}, (h, \lambda) \mapsto h(\lambda)$, and choices of *simple coroots* $h_i \in Y$ and *simple roots* $\alpha_j \in X$ such that $h_i(\alpha_j) = a_{i,j}$ for $i, j \in I$. We assume the simple coroots are linearly independent. Let $\tau : X \to X$ be an involution with dual $\tau^* : Y \to Y$, such that $\tau(\alpha_i) = \alpha_{\tau i}$ and $\tau^*(h_i) = h_{\tau i}$ for an induced involution $\tau : I \to I$. It follows that $a_{\tau i,\tau j} = a_{i,j}$ for all $i, j \in I$. Let $Y^\imath := \ker(\mathrm{id} + \tau^*)$. Also define $\varsigma \in \mathbb{Z}^I$ by setting

$$\varsigma_i := \begin{cases} -1 & \text{if } i = \tau i \\ \#(i \to \tau i) & \text{if } i \neq \tau i. \end{cases}$$

To this data, there is associated a *quasi-split iquantum group* $\mathrm{U}^\imath$. It is one of a general family of $\mathbb{Q}(q)$-algebras introduced by Kolb [Kol14], following Letzter's ground-breaking work [Let99] in finite type. Letzter and Kolb constructed $\mathrm{U}^\imath$ as a certain coideal subalgebra of the quantized enveloping algebra $\mathrm{U} = \langle q^h, e_i, f_i \mid h \in Y, i \in I\rangle$ of this Cartan type generated by q^h $(h \in Y^\imath)$ and

$$b_i := f_i + q^{\varsigma_i} e_{\tau i} k_i^{-1}$$

for $i \in I$, where $k_i := q^{h_i}$. The pair $(\mathrm{U}, \mathrm{U}^\imath)$ is called a *quantum symmetric pair* associated with the Satake diagram (I, τ). Ordinary quantum groups U can be viewed as a special case: they are the quasi-split iquantum groups arising from quantum symmetric pairs of *diagonal type* associated to the Satake diagram $(I^+ \sqcup I^-, \boldsymbol{\tau})$, where $I^\pm$ are copies of I interchanged by $\boldsymbol{\tau}$ (see [BW18a, Rem. 4.10]).

We need here the modified form $\dot{\mathrm{U}}^\imath$ of the iquantum group $\mathrm{U}^\imath$ introduced in [BW18a, BW21]. This is a locally unital algebra with mutually orthogonal distinguished idempotents 1_λ indexed by *iweights* λ, that is, elements of the quotient Abelian group $X^\imath := X/\mathrm{im}(\mathrm{id} + \tau)$. Denoting a pre-image of $\lambda \in X^\imath$ in X by $\hat{\lambda}$, we use $\lambda \pm \alpha_i$ to denote the image of $\hat{\lambda} \pm \alpha_i$ in $X^\imath$, and set

$$\lambda_i := (h_i - h_{\tau i})(\hat{\lambda}) \in \mathbb{Z}.$$

Also let $b_i^{(n)} 1_\lambda$ be the usual divided power $b_i^n 1_\lambda / [n]_q^!$ if $i \neq \tau i$, or the *idivided power* from [BW18b, BW18c] if $i = \tau i$ (see (3.6) below). Then the algebra $\dot{\mathrm{U}}^\imath$ can be defined by generators and relations as the locally unital $\mathbb{Q}(q)$-algebra with distinguished idempotents $1_\lambda\,(\lambda \in X^\imath)$ and generators $b_i 1_\lambda = 1_{\lambda-\alpha_i} b$ for all $i \in I$ and $\lambda \in X^\imath$, subject to the relation

$$\sum_{n=0}^{1-a_{i,j}} (-1)^n b_i^{(n)} b_j b_i^{(1-a_{i,j}-n)} 1_\lambda = \delta_{i,\tau j} \prod_{r=1}^{-a_{i,j}} (q^r - q^{-r}) \times \frac{(-1)^{a_{i,j}} q^{\lambda_i - \varsigma_i - \binom{a_{i,j}}{2}} - q^{\binom{a_{i,j}}{2} + \varsigma_i - \lambda_i}}{q - q^{-1}} b_i^{(-a_{i,j})} 1_\lambda$$

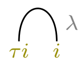
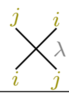
Generator	Degree
 $: B_i\mathbb{1}_\lambda \Rightarrow B_i\mathbb{1}_\lambda$	2
 $: B_{\tau i}B_i\mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$	$1+\varsigma_i-\lambda_i$
 $: \mathbb{1}_\lambda \Rightarrow B_{\tau i}B_i\mathbb{1}_\lambda$	$1+\varsigma_i-\lambda_i$
 $: \mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$ for $0 \leq n \leq \varsigma_i-\lambda_i$	$2n$
 $: B_iB_j\mathbb{1}_\lambda \Rightarrow B_jB_i\mathbb{1}_\lambda$	$-a_{i,j}$

TABLE 1. Generating 2-morphisms of $\mathfrak{U}^\imath$

for all $i \neq j$ in I and $\lambda \in X^\imath$. The mysterious expression on the right hand side of this relation in the case $i = \tau j$ was worked out originally in works of Letzer [Let03] and Balagović and Kolb [BK15]. The case $i = \tau i$ is also difficult as idivided powers are more complicated; the relation in this situation first appeared in [CLW21].

Now we come to the new definition of the 2-iquantum group $\mathfrak{U}^\imath$. The full formulation of this given in Definition 3.4 applies to symmetrizable (not merely symmetric) Cartan matrices but requires a mild additional hypothesis: *if $i, j \in I$ are τ-fixed points then $a_{i,j} \equiv a_{j,i}$ (mod 2)* (see Remark 3.3). The definition is a bit tidier if we make the following stronger assumption: *if $i, j \in I$ are τ-fixed points then $a_{i,j}$ is even.* This makes it possible to choose the orientation of edges in Q in such a way that $\#(i \to j) = \#(\tau j \to \tau i)$; in other words, τ defines an isomorphism from Q to the opposite quiver. We assume this is the case for the remainder of the introduction. We also fix an algebraically closed field $\mathbb{k}$, which should not be of characteristic 2 if $a_{i,\tau i} \neq 0$ for some i. For each i, let $c_i : X \to \mathbb{k}^\times$ be a group homomorphism such that

- $c_i(\alpha_j) = (-1)^{\#(j\to i)}$ for all $i, j \in I$;
- $c_{\tau i}(\tau(\lambda)) = (-1)^{h_i(\lambda)} c_i(\lambda)$ for all $\lambda \in X$ and $i \in I$ with $i \neq \tau i$.

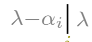
Assuming that the simple roots are linearly independent, such a family of what we call *normalization homomorphisms* always exists. For $i, j \in I$ and $\lambda \in X^\imath$ with pre-image $\hat\lambda \in X$, let

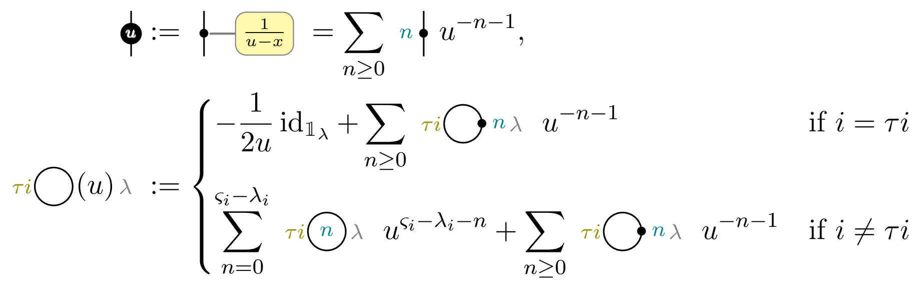
$$\gamma_i(\lambda) := \begin{cases} c_i\big(\hat\lambda - \tau(\hat\lambda)\big) & \text{if } i \neq \tau i\\ (-1)^{h_i(\hat\lambda)} & \text{if } i = \tau i,\end{cases} \qquad \zeta_i := \begin{cases} \pm 2^{\varsigma_i} & \text{if } i \neq \tau i\\ -\frac{1}{2} & \text{if } i = \tau i,\end{cases}$$

choosing the signs of ζ_i and $\zeta_{\tau i}$ when $i \neq \tau i$ so that one is positive and the other is negative. Let

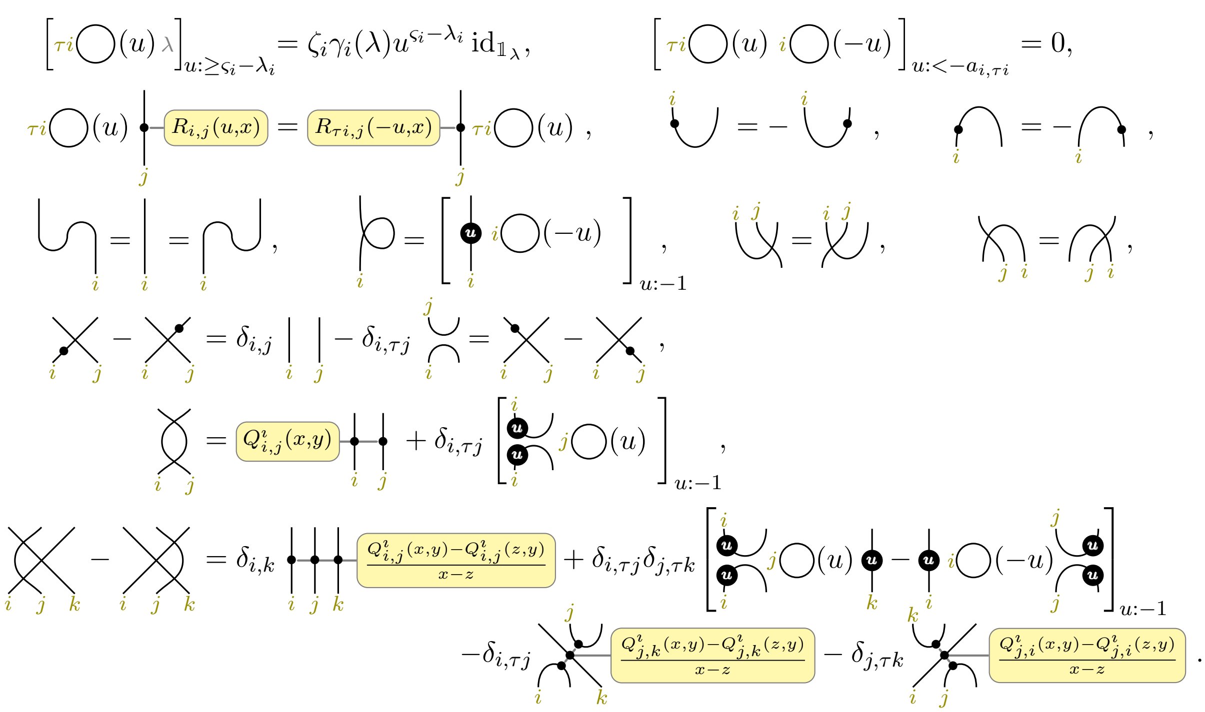
$$Q_{i,j}(x,y) := \begin{cases} (x-y)^{\#(i\to j)}(y-x)^{\#(j\to i)} & \text{if } i \neq j\\ 0 & \text{if } i = j,\end{cases} \qquad R_{i,j}(x,y) := \begin{cases} Q_{i,j}(x,y) & \text{if } i \neq j\\ 1/(x-y)^2 & \text{if } i = j,\end{cases}$$

$$Q^\imath_{i,j}(x,y) := (-1)^{\delta_{i,\tau j}} Q_{i,j}(x,y).$$

The *2-iquantum group* $\mathfrak{U}^\imath$ with these parameters is the graded $\mathbb{k}$-linear 2-category with object set $X^\imath$, generating 1-morphisms $B_i\mathbb{1}_\lambda = \mathbb{1}_{\lambda-\alpha_i}B_i : \lambda \to \lambda - \alpha_i$ for $\lambda \in X^\imath$ and $i \in I$, with identity 2-endomorphisms denoted by unoriented strings , and the generating 2-morphisms listed in Table 1. The generating 2-morphisms are subject to certain relations. We write these down using a generating function formalism which has proven very useful in previous work, for example, [BSW20, BSW23]; it will be explained fully later in the text. Briefly, our generating functions are formal Laurent series in a variable u^{-1}, and the variables x, y, z denote dots on strings in order from left to right. We omit 1- and 2-cell labels which can be inferred from context, and we omit them altogether if we are writing something which is true for all possible labels. There are also a couple of shorthands for important generating functions:



Here are the defining relations:



The 2-quantum groups of Khovanov, Lauda and Rouquier are isomorphic to 2-iquantum groups of diagonal type. There are two other special cases already in the literature: the categorification of iquantum groups of quasi-split type AIII with an even number of nodes from [BSWW18], and the nil-Brauer category which categorifies the split iquantum group of rank one from [BWW23, BWW24]. The dictionaries to all of these special cases are explained in Subsections 3.3 to 3.5.

The main results proved about this new family of 2-categories are as follows.

- **Theorem 5.10**: We prove that $\mathfrak{U}^\imath$ is non-degenerate. By this, we mean that the spanning sets for its 2-morphism spaces which are obtained by an obvious straightening rule are actually bases. This result generalizes the non-degeneracy of 2-quantum groups conjectured and proved for $\mathfrak{sl}_n$ by Khovanov and Lauda in [KL10], and proved in general in [Web24].
- **Theorem 5.17**: Then we explain how the graded ranks of 2-morphism spaces in $\mathfrak{U}^\imath$ can be computed using the non-degenerate symmetric bilinear form $(\cdot,\cdot)^\imath$ on the iquantum group $\dot{\mathrm{U}}^\imath$ defined in [BW18a, BW21]. This extends [KL10, Th. 2.7].
- **Theorem 6.5**: We classify indecomposables in the appropriate idempotent completions of the morphism categories in $\mathfrak{U}^\imath$. This is done by equipping the path algebras of these categories with *graded triangular bases* in the sense of [Bru25], with underlying Cartan algebras that are certain quiver Hecke algebras (see Definition 6.2). These bases also lead to the

definition of *standard modules* for the morphism categories of $\mathfrak{U}^\imath$, which are standardizations of the indecomposable projective modules for these Cartan algebras.

- **Theorem 6.18**: We prove that the split Grothendieck ring of $\mathfrak{U}^\imath$ is isomorphic to the $\mathbb{Z}[q,q^{-1}]$-form $\dot{\mathrm{U}}^\imath_{\mathbb{Z}}$ of $\dot{\mathrm{U}}^\imath$ generated by the divided/idivided powers $b_i^{(n)}1_\lambda$. This generalizes the known result for 2-quantum groups which was proved assuming non-degeneracy by Khovanov and Lauda in [KL10]. A consequence is that isomorphism classes of indecomposable objects in the categorification give rise to a new basis for $\dot{\mathrm{U}}^\imath_{\mathbb{Z}}$ which we call the *iorthodox basis*. Standard modules correspond to another basis, but over $\mathbb{Z}(\!(q)\!)$ rather than over $\mathbb{Z}[q,q^{-1}]$, which we call the *standardized orthodox basis*.

In the main body of the text, we work in a more general setup than in this introduction, allowing Cartan matrices that are merely symmetrizable, and more general choices of $Q_{i,j}(x,y)$. We conjecture that $\mathfrak{U}^\imath$ is non-degenerate in the general setup. However, at present, we are only able to prove it for "geometric" parameters as in Example 2.3; this is more general than the situation described in the introduction but still only applies to symmetric Cartan matrices. The other results referenced above are proved in complete generality but depend on non-degeneracy.

The main technical tool at the heart of the paper is a 2-functor $\Xi^\imath$ from $\mathfrak{U}^\imath$ to a localization of the 2-quantum group $\mathfrak{U}$ associated to U; see Theorem 4.13. We view this as a categorification of the standard embedding $\mathrm{U}^\imath \hookrightarrow \mathrm{U}$. The insights gained from the calculations establishing its existence were essential when we were working out some of finer details in the defining relations of $\mathfrak{U}^\imath$. The 2-functor $\Xi^\imath$ also plays a key role in the proof of non-degeneracy. We point out a special case of independent interest, which is a variant of this functor for ordinary 2-quantum groups related to the comultiplication $\Delta : \mathrm{U} \to \mathrm{U} \otimes \mathrm{U}$; see Theorem 4.14. We use this in Subsection 5.2 to give a self-contained proof of the non-degeneracy of 2-quantum groups for any symmetrizable Cartan matrix and any choice of parameters, making this article independent of [Web24].

Another important ingredient in our approach is an isometry $\jmath$ between $\dot{\mathrm{U}}^\imath 1_\lambda$ and the algebra $\mathbf{f}$ that is the negative (or positive) half of U; see Theorem 5.12 which extends [Wan25, Th. 2.8] and [BWW23, Th. 2.1]. The iSerre relation is equivalent to the relation obtained by applying $\jmath^{-1}$ to the ordinary Serre relation in $\mathbf{f}$. This point of view leads to a new proof of Balagović-Kolb-Letzter's relation; see Subsection 5.5. In the categorification, the role of the isometry $\jmath$ is played by the *standardization functor* arising from the theory of graded triangular bases. We use this to prove a categorical analog of the BKL relation; see Theorem 6.16. Our proof is quite different and much shorter than the one in [BSWW18] which treated the special case $a_{i,\tau i} = -1$. The other cases of the iSerre relation are categorified by a more explicit split exact complex, which is the same as in [KL11, Cor. 7] and [Rou08, Prop. 4.2] when $i \neq \tau i$, but there are additional subtleties involving the nil-Brauer category in the difficult case $i = \tau i$; see Theorem 6.13.

Moving forward, a basic open problem is to find situations in which the iorthodox basis coincides with the icanonical basis for $\dot{\mathrm{U}}^\imath_{\mathbb{Z}}$ from [BW18a, BW21]. This is closely related to the positivity of icanonical basis, which was proved already for finite and affine Satake types AIII in [LW18, FLL$^+$20], and conjectured for all locally finite symmetric quasi-split types in [Wan23, §9]. In view of this, it seems reasonable to conjecture that the iorthodox basis in characteristic 0 coincides with the icanonical basis for these types. This is already known to be the case in finite ADE diagonal types [Web15, Th. 8.7], for the iquantum group of split rank one [BWW23], and for the iquantum group of quasi-split type AIII with even number of nodes [BSWW18, Th. A].

The icanonical bases arising from iquantum groups of quasi-split AIII types are intimately connected to Kazhdan-Lusztig theory in types B/C/D. In view of this, the 2-iquantum groups of these types introduced in [BSWW18] and the present paper are widely expected to be related to the affine Brauer category introduced in [RS19], and the affine Brauer algebras appearing in many previous works such as [Naz96, AMR06, ES18]. It would be interesting to find a conceptual explanation of this which is similar to the way that 2-quantum groups of types $A^{(1)}_{p-1}$ and A_∞ are related to Heisenberg categories in [BSW20].

Conventions. We work over an $\mathbb{N}$-graded commutative ring $\Bbbk = \bigoplus_{n\geq 0} \Bbbk_n$; for Theorems 6.5 and 6.18 we need $\Bbbk_0$ to be a field. We use $\Bbbk^\times$ to denote the homogeneous units in $\Bbbk_0$. Let $[n]_q$ denote the quantum integer $\frac{q^n-q^{-n}}{q-q^{-1}}$, $[n]^!_q$ be the quantum factorial, and $\begin{bmatrix} m \\ n \end{bmatrix}_q$ be the quantum binomial coefficient. The bar involution on $\mathbb{Q}(q)$ is the field automorphism $f(q) \mapsto \overline{f(q)} := f(q^{-1})$. We also use q to denote the *upward* grading shift functor on the monoidal category of graded $\Bbbk$-modules, i.e., $(qV)_n = V_{n-1}$. A free graded $\Bbbk$-module V is *locally finite-dimensional and bounded below* if it is isomorphic to $\Bbbk^{\oplus f(q)} := \bigoplus_{i\in\mathbb{Z}} (q^i\Bbbk)^{\oplus n_i}$ for a formal Laurent series $f(q) = \sum_{i\in\mathbb{Z}} n_i q^i \in \mathbb{N}(\!(q)\!)$. In this situation, we denote $f(q)$ by $\operatorname{rank}_q V$ (or $\dim_q V$ if $\Bbbk$ is a field).

By a *graded category*, we mean a category enriched in graded $\Bbbk$-modules. A *graded 2-category* is a category enriched in graded categories. If $\mathbf{C}$ is a graded category, the dual graded category $\mathbf{C}^{\mathrm{op}}$ has the same objects as $\mathbf{C}$. For objects X and Y in $\mathbf{C}$, the morphism space $\operatorname{Hom}_{\mathbf{C}^{\mathrm{op}}}(X,Y)$ is a copy $\{f^{\mathrm{op}} \mid f \in \operatorname{Hom}_{\mathbf{C}}(Y,X)\}$ of $\operatorname{Hom}_{\mathbf{C}}(Y,X)$ with $\deg(f^{\mathrm{op}}) = \deg(f)$ and $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g\circ f)^{\mathrm{op}}$. The *opposite* $\mathfrak{C}^{\mathrm{op}}$ of a graded 2-category $\mathfrak{C}$ is the graded 2-category defined by reversing vertical composition (its morphism categories are the duals of the morphism categories in $\mathfrak{C}$). The *reverse* $\mathfrak{C}^{\mathrm{rev}}$ is defined by reversing horizontal composition (its morphism categories are the same as in $\mathfrak{C}$).

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2. Reminders about the categorification of quantum groups

In this section, we fix a symmetrizable Cartan datum, briefly recall the definition of the corresponding quantized enveloping algebra and Lusztig's modified integral form. Then we write down the definition of the 2-quantum group introduced by Khovanov, Lauda and Rouquier [KL10, Rou08], and record some further relations working in terms of generating functions.

2.1. **The quantum group** U. Let $A = (a_{i,j})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix. This means that $a_{i,i} = 2$ for all $i \in I$, $a_{i,j} \in -\mathbb{N}$ for $i \neq j$ in I, $a_{i,j} = 0 \Leftrightarrow a_{j,i} = 0$, and there are given positive integers d_i ($i \in I$) such that $d_i a_{i,j} = d_j a_{j,i}$ for all $i,j \in I$. We do not insist that the set I is finite, but the number of non-zero entries in each row and column of the Cartan matrix should be finite. Let X, the *weight lattice*, and Y, the *coweight lattice*, be free Abelian groups with a given perfect pairing $Y \times X \to \mathbb{Z}, (h,\lambda) \mapsto h(\lambda)$. We assume that X contains elements α_i ($i \in I$), called *simple roots*, and Y contains elements h_i ($i \in I$), called *simple coroots*, such that $h_i(\alpha_j) = a_{i,j}$ for all $i,j \in I$. We will assume that the simple coroots are linearly independent ("Y-regular") but do not require linear independence of the simple roots. Let

$$X^+ := \{\lambda \in X \mid h_i(\lambda) \geq 0 \text{ for all } i \in I \text{ with } h_i(\lambda) = 0 \text{ for all but finitely many } i\} \tag{2.1}$$

be the set of *dominant weights*.

The *quantized enveloping algebra* U attached to this root datum is a $\mathbb{Q}(q)$-algebra generated by elements $(q^h)_{h\in Y}$ and $(e_i, f_i)_{i\in I}$ satisfying the usual relations. In particular, $q^h q^{h'} = q^{h+h'}$, and

$$[e_i, f_j] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$$

where $k_i := q^{d_i h_i}$ and $q_i := q^{d_i}$. Our notational choices for U follow [Lus10] closely, the only departure being the use of e_i, f_i, k_i for its generators, which Lusztig denotes by $E_i, F_i, \widetilde{K}_i$. We do this so that we can use the upper case letters to denote corresponding *functors* when we pass to the categorification of U. We view U as a Hopf algebra with the comultiplication Δ from [Lus10]:

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \qquad \Delta(f_i) = 1 \otimes f_i + f_i \otimes k_i^{-1}, \qquad \Delta(q^h) = q^h \otimes q^h. \tag{2.2}$$

Actually, more natural for categorification is Lusztig's *modified form* $\dot{\mathrm{U}}$. This is the locally unital $\mathbb{Q}(q)$-algebra $\bigoplus_{\lambda,\mu\in X} 1_\lambda \mathrm{U} 1_\mu$ with the (mutually orthogonal) distinguished idempotents $\{1_\lambda \,|\, \lambda \in X\}$ and generators

$$e_i 1_\lambda = 1_{\lambda+\alpha_i} e_i, \qquad\qquad f_i 1_\lambda = 1_{\lambda-\alpha_i} f_i$$

for $\lambda \in X$ and $i \in I$. The relations in $\dot{\mathrm{U}}$ are derived from the ones in U using the rule that $q^h 1_\lambda = 1_\lambda q^h = q^{h(\lambda)} 1_\lambda$ for $\lambda \in X, h \in Y$. In particular, $k_i 1_\lambda = q_i^{h_i(\lambda)} 1_\lambda$. For example, the relation displayed in the previous paragraph becomes $[e_i, f_j]1_\lambda = \delta_{i,j}[h_i(\lambda)]_{q_i} 1_\lambda$, interpreting the commutator $[e_i, f_j]1_\lambda$ as $e_i f_j 1_\lambda - f_j e_i 1_\lambda$. There is also a natural completion $\widehat{\mathrm{U}}$ of $\dot{\mathrm{U}}$ consisting of matrices $(u_{\lambda,\mu})_{\lambda,\mu\in X}$ with $u_{\lambda,\mu} \in 1_\lambda \mathrm{U} 1_\mu$ such that there are only finitely many non-zero entries in each row and column. We identify $\dot{\mathrm{U}}$ with a subalgebra of $\widehat{\mathrm{U}}$ so that $1_\lambda u 1_\mu$ is the matrix with this as its (λ,μ)-entry and 0 in all other positions. A general element $(u_{\lambda,\mu})_{\lambda,\mu\in X} \in \widehat{\mathrm{U}}$ may also be denoted as the infinite sum $\sum_{\lambda,\mu\in X} u_{\lambda,\mu}$.

Finally, we pass to $\mathbb{Z}[q,q^{-1}]$-forms. Let $\dot{\mathrm{U}}_\mathbb{Z}$ be the integral form for $\dot{\mathrm{U}}$ generated by the divided powers $e_i^{(n)} 1_\lambda := e_i^n 1_\lambda \,/\, [n]^!_{q_i}$ and $f_i^{(n)} 1_\lambda := f_i^n 1_\lambda \,/\, [n]^!_{q_i}$ for all $\lambda \in X, i \in I$ and $n \geq 0$. There are some useful symmetries:

$$\text{(linear involution)}\quad \omega : \dot{\mathrm{U}}_\mathbb{Z} \xrightarrow{\sim} \dot{\mathrm{U}}_\mathbb{Z}, \qquad e_i^{(n)} 1_\lambda \mapsto f_i^{(n)} 1_{-\lambda}, \qquad f_i^{(n)} 1_\lambda \mapsto e_i^{(n)} 1_{-\lambda}, \tag{2.3}$$

$$\text{(anti-linear involution)}\quad \psi : \dot{\mathrm{U}}_\mathbb{Z} \xrightarrow{\sim} \dot{\mathrm{U}}_\mathbb{Z}, \qquad e_i^{(n)} 1_\lambda \mapsto e_i^{(n)} 1_\lambda, \qquad f_i^{(n)} 1_\lambda \mapsto f_i^{(n)} 1_\lambda, \tag{2.4}$$

$$\text{(linear anti-involution)}\quad \sigma : \dot{\mathrm{U}}_\mathbb{Z} \xrightarrow{\sim} \dot{\mathrm{U}}_\mathbb{Z}, \qquad e_i^{(n)} 1_\lambda \mapsto 1_{-\lambda} e_i^{(n)}, \qquad f_i^{(n)} 1_\lambda \mapsto 1_{-\lambda} f_i^{(n)}. \tag{2.5}$$

Let $\widehat{\mathrm{U}}_\mathbb{Z}$ be the completion of $\dot{\mathrm{U}}_\mathbb{Z}$ defined like in the previous paragraph.

2.2. **The 2-quantum group $\mathfrak{U}$.** In addition to the root datum just chosen, we need some additional parameters. Assume that $\Bbbk$ is an $\mathbb{N}$-graded commutative ring as in *Conventions*. For i, j in I, let $Q_{i,j}(x,y) \in \Bbbk[x,y]$ be a polynomial which is 0 if $i = j$, and which is homogeneous of degree $-2d_i a_{i,j}$ if x and y are assigned the degrees $2d_i$ and $2d_j$, respectively. We assume that

$$Q_{i,j}(x,y) = Q_{j,i}(y,x). \tag{2.6}$$

When $i \neq j$, we have that

$$Q_{i,j}(x,y) = t_{i,j} x^{-a_{i,j}} + \sum_{\substack{0\leq r<-a_{i,j}\\ 0\leq s<-a_{j,i}}} t_{i,j;r,s} x^r y^s + t_{j,i} y^{-a_{j,i}} \tag{2.7}$$

for $t_{i,j}, t_{j,i} \in \Bbbk_0$ and $t_{i,j;r,s} = t_{j,i;s,r} \in \Bbbk_{-2d_i a_{i,j} - 2d_i r - 2d_j s}$. We assume moreover that $t_{i,j}$ is invertible, so it is an element of $\Bbbk^\times$.

Definition 2.1. The *2-quantum group* $\mathfrak{U}$ with these parameters is the graded 2-category with object set X, generating 1-morphisms

$$E_i \mathbb{1}_\lambda = \mathbb{1}_{\lambda+\alpha_i} E_i \colon \lambda \to \lambda + \alpha_i, \qquad\qquad \mathbb{1}_{\lambda-\alpha_i} F_i = F_i \mathbb{1}_\lambda \colon \lambda \to \lambda - \alpha_i \tag{2.8}$$

for $\lambda \in X$ and $i \in I$, with identity 2-endomorphisms represented by oriented strings $\underset{i}{\uparrow}\,\lambda$ and $\underset{i}{\downarrow}\,\lambda$, and the four families of generating 2-morphisms displayed in the top half of Table 2. The degrees of the generating 2-morphisms are also recorded in this table. The 2-morphisms in the bottom part of Table 2 are not needed for the initial definition; they will be introduced in the next subsection. The generating 2-morphisms are subject to relations still to be explained. Before writing these down, we explain some conventions.

- We will usually only label one of the 2-cells in a string diagram with a weight – the others can then be worked out implicitly. When we omit *all* labels in 2-cells, it should be understood that we are discussing something that holds for all possible labels.

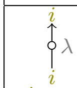
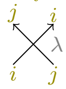
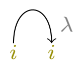
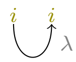
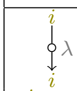
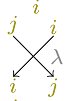
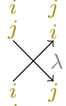
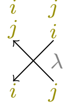
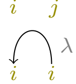
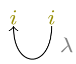
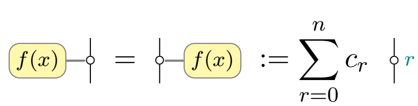
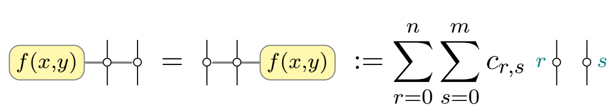
Generator	Degree
 $: E_i\mathbb{1}_\lambda \Rightarrow E_i\mathbb{1}_\lambda$	$2d_i$
 $: E_iE_j\mathbb{1}_\lambda \Rightarrow E_jE_i\mathbb{1}_\lambda$	$-d_ia_{i,j}$
 $: E_iF_i\mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$	$d_i(1-h_i(\lambda))$
 $: \mathbb{1}_\lambda \Rightarrow F_iE_i\mathbb{1}_\lambda$	$d_i(1+h_i(\lambda))$
 $: F_i\mathbb{1}_\lambda \Rightarrow F_i\mathbb{1}_\lambda$	$2d_i$
 $: F_iF_j\mathbb{1}_\lambda \Rightarrow F_jF_i\mathbb{1}_\lambda$	$-d_ia_{i,j}$
 $: E_iF_j\mathbb{1}_\lambda \Rightarrow F_jE_i\mathbb{1}_\lambda$	0
 $: F_iE_j\mathbb{1}_\lambda \Rightarrow E_jF_i\mathbb{1}_\lambda$	0
 $: F_iE_i\mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$	$d_i(1+h_i(\lambda))$
 $: \mathbb{1}_\lambda \Rightarrow E_iF_i\mathbb{1}_\lambda$	$d_i(1-h_i(\lambda))$

TABLE 2. Generating 2-morphisms of $\mathfrak{U}$

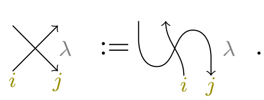
- We will label strings just at one end. If we omit a label or orientation, it means that we are discussing something that holds for all possibilities.
- When a dot is labelled by a multiplicity, we mean to take its power under vertical composition. For a polynomial $f(x) = \sum_{r=0}^n c_r x^r$, we use the shorthand



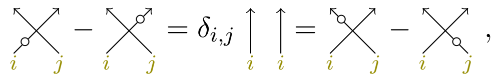
to "pin" $f(x)$ to a dot on a string. Similarly, for $f(x,y) = \sum_{r=0}^n \sum_{s=0}^m c_{r,s} x^r y^s$, we use

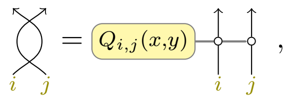


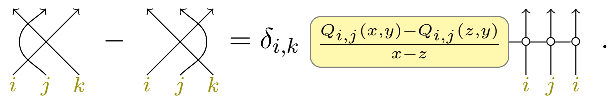
This notation extends to polynomials $f(x,y,z)$ in three variables pinned to three dots, with the convention that the variables in alphabetic order correspond to the dots ordered by the lexicographic order on their Cartesian coordinates. Thus, x corresponds to the leftmost dot, and the lowest one if there are several such dots in the same vertical line.
- We use the following shorthand to denote the composite 2-morphism obtained by "rotating" the generating 2-morphism:

 (2.9)

Now for the relations. There are three families. First, we have the *quiver Hecke algebra* relations:

 (2.10)

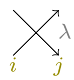
 (2.11)

 (2.12)

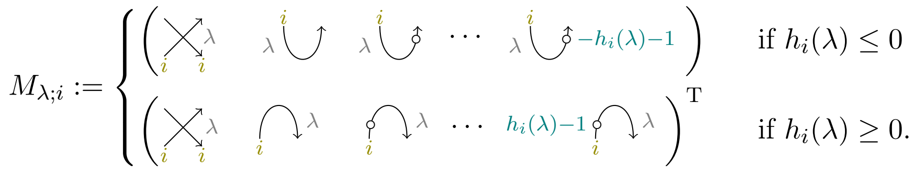
Next, the *right adjunction relations*:

 (2.13)

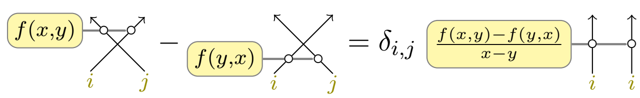
Finally, we have the *inversion relations* which assert that

 $: E_i F_j \mathbb{1}_\lambda \Rightarrow F_j E_i \mathbb{1}_\lambda$ (2.14)

is an isomorphism for all $\lambda \in X$ and $i \neq j$, as are the following matrices for all λ and i:

 (2.15)

Remark 2.2. The relations (2.10) are equivalent to the formula

 (2.16)

for any $f(x, y) \in \Bbbk[x, y]$. The fraction on the right hand side is the image of $f(x, y)$ under the Demazure operator.

2.3. **More generators and generating functions.** The presentation for $\mathfrak{U}$ just explained is due to Rouquier [Rou08]. In [KL10], Khovanov and Lauda found a different presentation with more generators and relations. The two approaches were reconciled in [Bru16]. We will explain this here in a form which also incorporates a renormalization of the extra generators. This depends on the additional choice of homomorphisms of Abelian groups

$$c_i : X \to \Bbbk^\times \tag{2.17}$$

with $c_i(\alpha_i) = 1$ for all $i \in I$, which we call *normalization homomorphisms*. Let

$$r_{i,j} := \begin{cases} \dfrac{c_i(\alpha_j)}{t_{i,j}} & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{2.18}$$

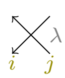
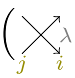
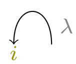
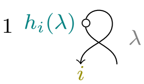
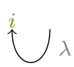
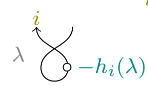
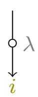
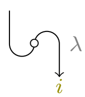
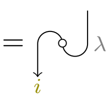
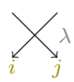
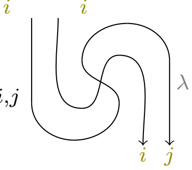
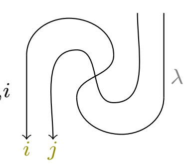
for $i, j \in I$. To match the conventions of [Bru16], take $c_i = 1$ for all i. Another choice proposed in [BHLW16] is to choose c_i so that $c_i(\alpha_j) = t_{i,j}$ for all $i \neq j$. It is very convenient when possible to find such functions because it implies that $r_{i,j} = 1$ for all $i, j \in I$, hence, by further relations recorded shortly, string diagrams are invariant under planar isotopy. Note we are requiring that c_i is a group homomorphism, whereas [BHLW16] only assumes that $c_i(\lambda + \alpha) = c_i(\lambda)c_i(\alpha)$ for any $\lambda \in X$ and α in the root lattice. It seems to us to be reasonable to impose this extra requirement, and it will be helpful later in the article.

Example 2.3 (Geometric parameters)**.** Suppose that $d_i = 1$ for all $i \in I$. The Cartan matrix is symmetric. Let Q be a quiver with vertex set I, no loops, and $\#(i \to j)$ directed edges from i to j such that $\#(i \to j) + \#(j \to i) = -a_{i,j}$ for all $i \neq j$. Then, for $i \neq j$, one can take

$$Q_{i,j}(x, y) := (x - y)^{\#(i \to j)}(y - x)^{\#(j \to i)}.$$

We call this a *geometric choice* of parameters for a symmetric Cartan matrix. It is particularly nice if one can also find normalization functions with $c_i(\alpha_j) = (-1)^{\#(j \to i)}$, for then we have that $r_{i,j} = 1$ for all $i, j \in I$; this is the case for the special situation discussed in the introduction.

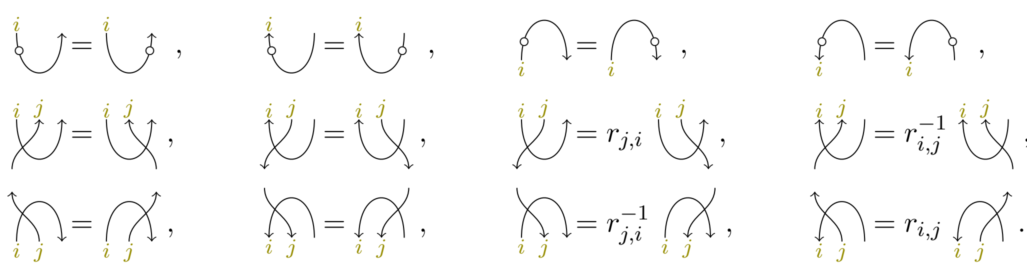
Now we define the additional 2-morphisms:

- Let  be $\Big($  $\Big)^{-1}$ if $i \neq j$, or the first entry of the matrix $-M_{\lambda;i}^{-1}$ if $i = j$.
- Let  be the last entry of $c_i(\lambda)^{-1} M_{\lambda;i}^{-1}$ if $h_i(\lambda) < 0$ or $-c_i(\lambda)^{-1}$  if $h_i(\lambda) \geq 0$.
- Let  be the last entry of $c_i(\lambda) M_{\lambda;i}^{-1}$ if $h_i(\lambda) > 0$ or $c_i(\lambda)$  if $h_i(\lambda) \leq 0$.
- Let  :=  =  .
- Let  $:= r_{i,j}$  $= r_{j,i}$  .

The equalities in the definitions of the downward dot and crossing just given are by no means obvious; they follow from the next relations (2.19) to (2.22). By [Bru16, Th. 4.3], the leftward cups and caps satisfy the zig-zag relations:

 (2.19)

We also have that


(2.20)
(2.21)
(2.22)

These follow from [Bru16, Lem. 2.1, Cor. 2.4, Lem. 5.1, Th. 5.3].

Remark 2.4. We emphasize that the 2-category $\mathfrak{U}$ depends only on the Cartan datum and the parameters $Q_{i,j}(x, y)$ chosen earlier. It is merely the normalization of the leftward cups and caps and the downward crossings that depends on the choice of the normalization homomorphisms c_i.

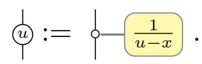
In most subsequent calculations, we will work systematically with generating functions, which in general will be formal Laurent series in auxiliary variables $u^{-1}, v^{-1}, \dots$. For such a generating function $f(u)$, we use the notation $[f(u)]_{u:n}$ to denote its u^n-coefficient, $[f(u)]_{u;\geq 0}$ for its polynomial part, and so on. For any polynomial $f(x)$, we have that

$$\left[\frac{f(u)}{u - x}\right]_{u:-1} = f(x), \qquad \left[\frac{f(u)}{u - x}\right]_{u:<0} = \frac{f(x)}{u - x}. \tag{2.23}$$

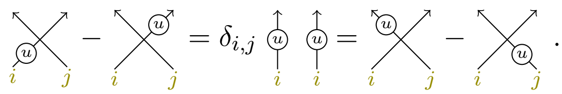
We view the series

$$\frac{1}{u-x} = \sum_{r\geq 0} x^r u^{-r-1}. \tag{2.24}$$

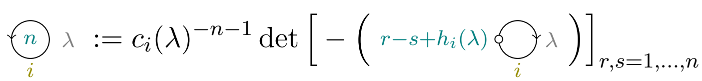
as a generating function for multiple dots on a string, introducing the shorthand


(2.25)

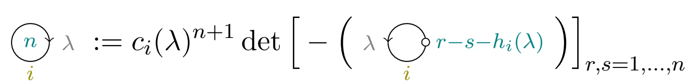
The following is a consequence of (2.10):


(2.26)

The most interesting relations in $\mathfrak{U}$ involve bubbles. To formulate them, we first introduce the *fake bubbles*[1]


(2.27)

for $0 \leq n \leq h_i(\lambda)$, and


(2.28)

for $0 \leq n \leq -h_i(\lambda)$. When $n = 0$, these are 0×0 matrices, whose determinants should be interpreted as $\mathrm{id}_{\mathbb{1}_\lambda}$. We put the fake bubbles together to form the *fake bubble polynomials*

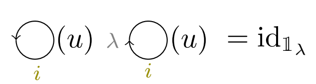

(2.29)

then define the *bubble generating functions*


(2.30)

The determinants in (2.27) and (2.28) arise from the solution of equations implied by the relations

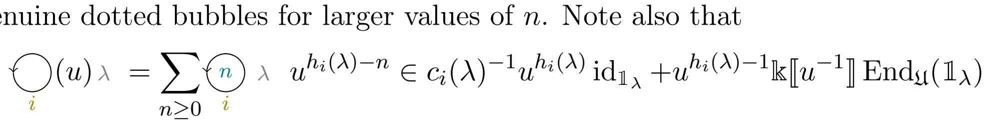

(2.31)


(2.32)

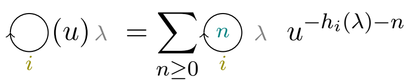
Again, these are derived from the defining relations in [Bru16]. More generally, for any $n \geq 0$, we define the *bubbles of degree* $2d_in$


(2.33)

These are just the originally defined fake bubbles if $n \leq h_i(\lambda)$ or $n \leq -h_i(\lambda)$, respectively, and they are genuine dotted bubbles for larger values of n. Note also that


(2.34)

[1] In the literature, our fake bubbles are usually denoted by negatively dotted bubbles, a convention introduced by Lauda [Lau10]. This would be too confusing in this context—later on we will invert dots on strings labelled by $i = \tau i$, so that negatively dotted bubbles will have a natural meaning.



$$\in c_i(\lambda)u^{-h_i(\lambda)}\,\mathrm{id}_{\mathbb{1}_\lambda} + u^{-h_i(\lambda)-1}\Bbbk[\![u^{-1}]\!]\,\mathrm{End}_{\mathfrak{U}}(\mathbb{1}_\lambda). \tag{2.35}$$

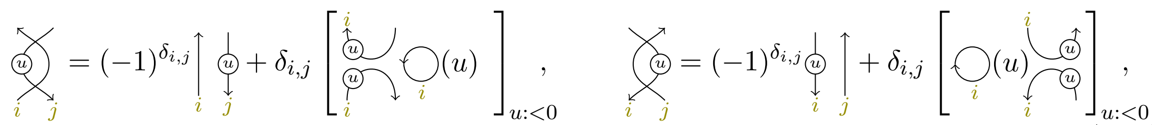
To further illustrate the usefulness of the generating functions, we next record the remaining relations from [KL10] written using them. They all follow from relations derived in [KL10, Bru16] by equating coefficients, but this takes some effort. Most of these generating function forms were originally worked out in [BSW20, Web24]. Let

$$R_{i,j}(x,y) := \begin{cases} r_{i,j}Q_{i,j}(x,y) & \text{if } i \neq j \\ \dfrac{1}{(x-y)^2} & \text{if } i = j. \end{cases} \tag{2.36}$$

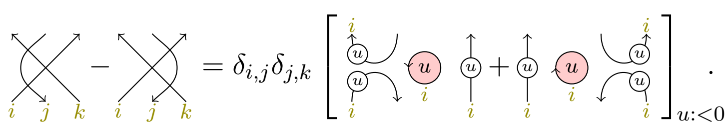
Then:

(2.37)

(2.38)



(2.39)

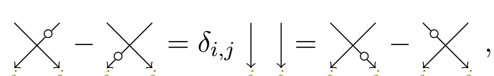


(2.40)

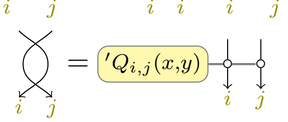
The following theorem is a version of the main result of [Bru16]. It gives another presentation for $\mathfrak{U}$ which is more symmetric but less efficient than Rouquier's presentation formulated above. It is essentially the presentation of Khovanov and Lauda from [KL10].

Theorem 2.5. *The 2-category $\mathfrak{U}$ can be defined equivalently as the graded 2-category with objects X, generating 1-morphisms $E_i\mathbb{1}_\lambda = \mathbb{1}_{\lambda+\alpha_i}E_i, F_i\mathbb{1}_\lambda = \mathbb{1}_{\lambda-\alpha_i}F_i$ ($\lambda \in X, i \in I$), and all of the generating 2-morphism listed in Table 2 (both halves), subject to the relations* (2.10) *to* (2.13)*,* (2.19) *to* (2.22)*,* (2.31)*,* (2.32)*,* (2.38) *and* (2.39) *(interpreted using the shorthands* (2.27) *to* (2.30)*).*

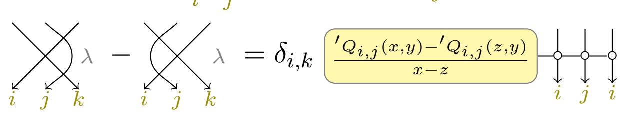
Remark 2.6. Yet more relations can be obtained by partially or fully rotating (2.10) to (2.12) and (2.37) to (2.40), i.e., attaching some cups to the bottom and some caps to the top of these relations then simplifying using (2.19) to (2.22). For example, the full rotations of (2.10) to (2.12) produce



(2.41)



(2.42)



(2.43)

where

$$'Q_{i,j}(x,y) := r_{i,j}r_{j,i}Q_{i,j}(x,y). \tag{2.44}$$

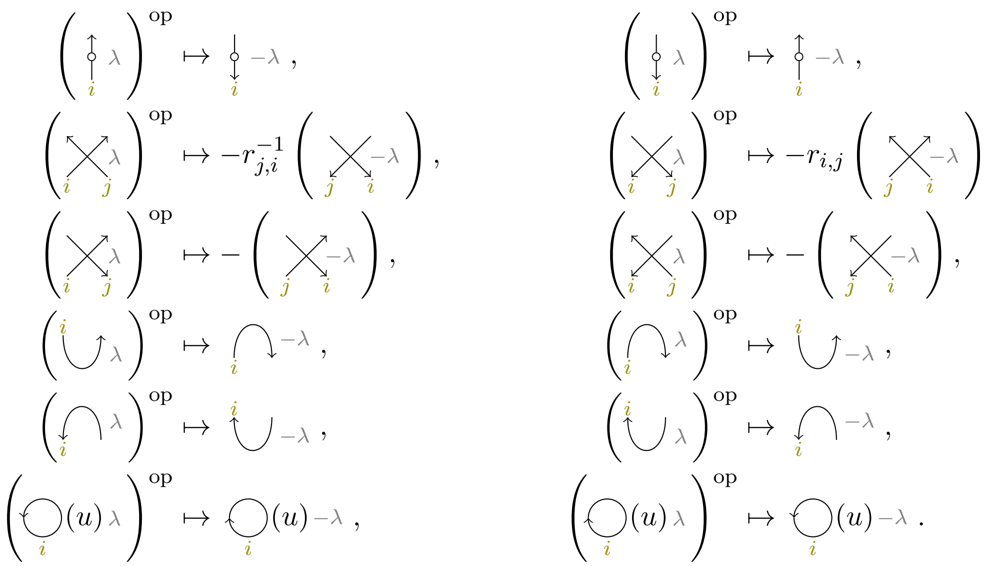
Thus, the downward dots and crossings also satisfy quiver Hecke algebra relations, but for a different matrix of parameters and some different sign conventions.

2.4. **Symmetries.** Next we explain how to lift the symmetries ω, ψ and σ from (2.3) to (2.5) to the 2-quantum group.

There is an isomorphism of graded 2-categories

$$\bar{\Omega} : \mathfrak{U}^{\mathrm{op}} \xrightarrow{\sim} \mathfrak{U} \tag{2.45}$$

defined on objects by $\lambda \mapsto -\lambda$, on generating 1-morphisms by $E_i \mathbb{1}_\lambda \mapsto F_i \mathbb{1}_{-\lambda}$ and $F_i \mathbb{1}_\lambda \mapsto E_i \mathbb{1}_{-\lambda}$, and on generating 2-morphisms by

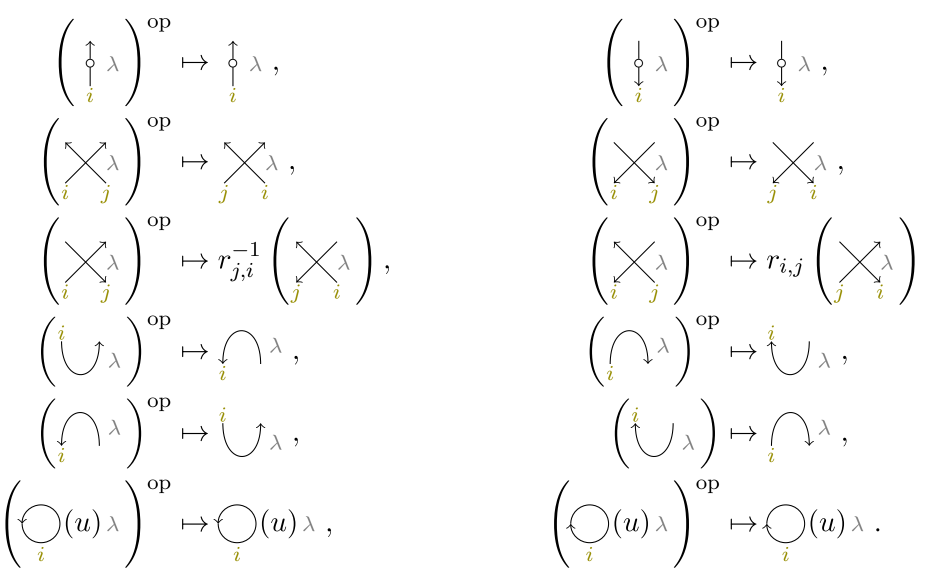


Roughly, this reflects string diagrams in a horizontal axis and negates all 2-cell labels (there are some additional scalars arising from crossings). The proof of existence of $\bar{\Omega}$ is an easy relations check, using the original "minimal" presentation for $\mathfrak{U}$; see [Bru16, Th. 2.3]. It is the 2-categorical analog not of ω but of the anti-linear involution $\bar{\omega} := \psi \circ \omega$.

Corresponding to the bar involution ψ, there is an isomorphism of graded 2-categories

$$\Psi : \mathfrak{U}^{\mathrm{op}} \xrightarrow{\sim} \mathfrak{U} \tag{2.46}$$

defined on objects by $\lambda \mapsto \lambda$, on generating 1-morphisms by $E_i \mathbb{1}_\lambda \mapsto E_i \mathbb{1}_\lambda$ and $F_i \mathbb{1}_\lambda \mapsto F_i \mathbb{1}_\lambda$, and on generating 2-morphisms by

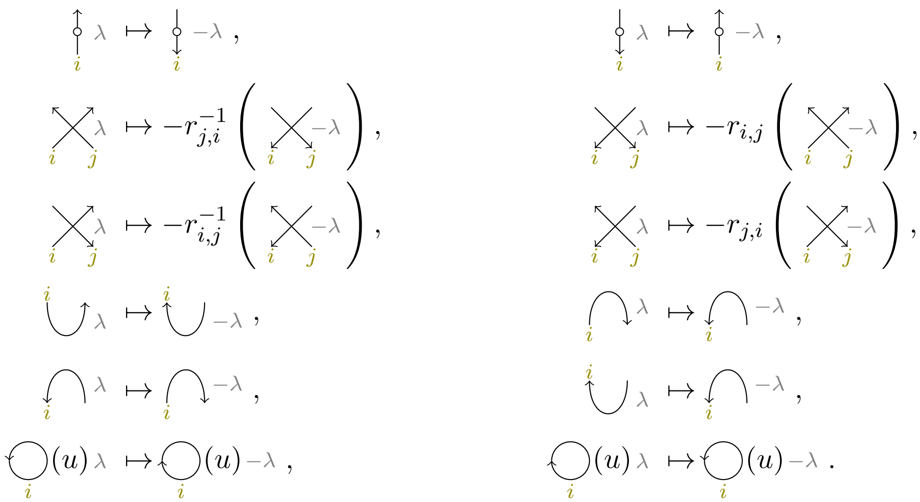


The proof of existence of Ψ is again a relations check, but one needs to use the more symmetric presentation from Theorem 2.5.

The composition

$$\Omega := \Psi \circ \bar{\Omega}^{\mathrm{op}} : \mathfrak{U} \xrightarrow{\sim} \mathfrak{U} \tag{2.47}$$

is a categorical analog of the Chevalley involution ω. It maps $\lambda \mapsto -\lambda, E_i \mathbb{1}_\lambda \mapsto F_i \mathbb{1}_{-\lambda}, F_i \mathbb{1}_\lambda \mapsto E_i \mathbb{1}_{-\lambda}$, and

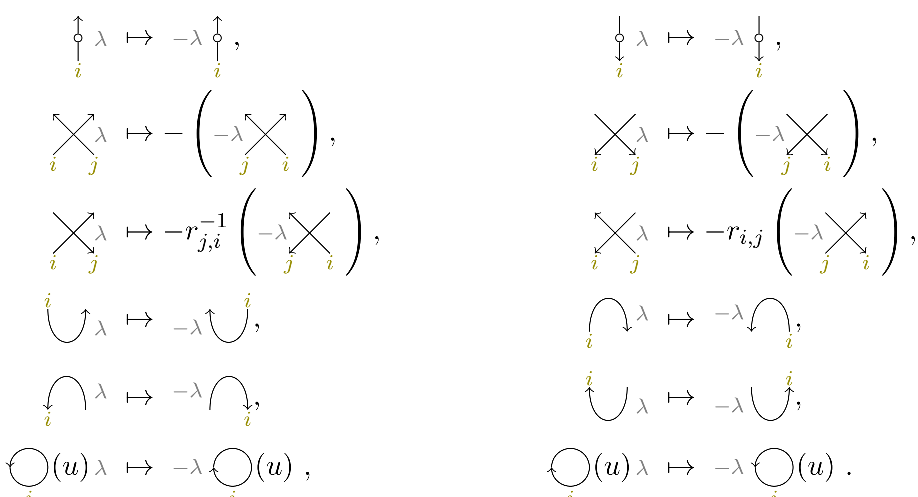


Roughly, this reverses orientations of strings and negates all 2-cell labels (with some additional scalars arising from crossings). Beware that $\Omega \neq \Omega^{-1}$ unless $r_{i,j} = r_{j,i}$ for all $i, j \in I$.

There is also an isomorphism

$$\Sigma : \mathfrak{U}^{\mathrm{rev}} \xrightarrow{\sim} \mathfrak{U} \tag{2.48}$$

which is the 2-categorical analog of the symmetry σ from (2.5). This is defined on objects by $\lambda \mapsto -\lambda$, on generating 1-morphisms by $E_i \mathbb{1}_\lambda \mapsto \mathbb{1}_{-\lambda} E_i$ and $F_i \mathbb{1}_\lambda \mapsto \mathbb{1}_{-\lambda} F_i$, and on generating 2-morphisms by

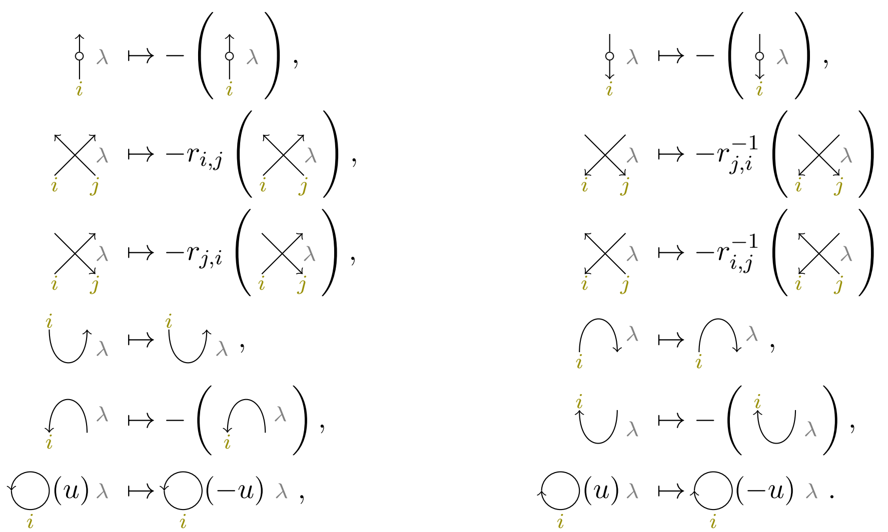


Roughly, this reflects string diagrams in a vertical axis (with some additional scalars arising from crossings).

Finally, let $\bar{\mathfrak{U}}$ be the 2-quantum group defined from the same Cartan datum as $\mathfrak{U}$ but replacing $Q_{i,j}(x, y)$ with $\bar{Q}_{i,j}(x, y) := r_{i,j} r_{j,i} Q_{i,j}(-x, -y)$ and c_i with $\bar{c}_i : X \to \Bbbk^\times, \lambda \mapsto (-1)^{h_i(\lambda)} c_i(\lambda)$. The scalars $r_{i,j}$ become $\bar{r}_{i,j} := r_{j,i}^{-1}$. Then there is an isomorphism of graded 2-categories

$$J : \bar{\mathfrak{U}} \xrightarrow{\sim} \mathfrak{U} \tag{2.49}$$

which is the identity on objects and 1-morphisms, and is defined on 2-morphisms by



3. New 2-categories for quasi-split iquantum groups

We continue with the Cartan datum and parameters fixed in the previous section. We are now going to propose an analogous framework for categorification of quasi-split iquantum groups arising from quantum symmetric pairs.

3.1. **The iquantum group** $\mathrm{U}^\imath(\varsigma)$**.** From a quasi-split Satake diagram of the same Cartan type as U, we obtain an involution τ of the set I with $a_{i,j} = a_{\tau i,\tau j}$ and $d_i = d_{\tau i}$. We allow $\tau = \mathrm{id}$. We pick $\varsigma = (\varsigma_i)_{i\in I} \in \mathbb{Z}^I$ so that

$$\varsigma_i + \varsigma_{\tau i} = -a_{i,\tau i} \qquad \text{and} \qquad \varsigma_i \geq 0 \text{ if } i \neq \tau i. \tag{3.1}$$

If $i = \tau i$ then $\varsigma_i = -1$, and if $a_{i,\tau i} = 0$ then $\varsigma_i = 0$. We also assume there are involutions $\tau : X \to X$ and $\tau^* : Y \to Y$ such that $\tau^*(h)(\lambda) = h(\tau(\lambda))$ for all $\lambda \in X$ and $h \in Y$, and also $\tau(\alpha_i) = \alpha_{\tau i}$ and $\tau^*(h_i) = h_{\tau i}$ for each $i \in I$. The *iweight lattice* and *icoweight lattice* are the Abelian groups

$$X^\imath := X \big/ \mathrm{im}(\mathrm{id} + \tau), \qquad\qquad Y^\imath := \ker(\mathrm{id} + \tau^*).$$

When $\lambda, \mu, \ldots$ are iweights in the quotient $X^\imath$, we will use the notation $\hat{\lambda}, \hat{\mu}, \ldots$ to denote a pre-image in X. For $\lambda \in X^\imath$ and $\pi \in X$, we write $\lambda + \pi$ for the sum of λ and the canonical image of π in $X^\imath$. For $\lambda \in X^\imath$ with pre-image $\hat{\lambda} \in X$ and $i \in I$, we define

$$\lambda_i := (h_i - h_{\tau i})(\hat{\lambda}) \in \mathbb{Z}. \tag{3.2}$$

This is well defined independent of the choice of the pre-image $\hat{\lambda}$. It is also useful to note that

$$(\lambda + \alpha_j)_i = \lambda_i + a_{i,j} - a_{i,\tau j}, \tag{3.3}$$

$$\varsigma_{\tau i} - \lambda_{\tau i} = \lambda_i - \varsigma_i - a_{i,\tau i}. \tag{3.4}$$

If $\tau i = i$ then $\lambda_i = 0$. In this case, the *parity* of $h_i(\hat{\lambda})$ is well defined independent of the choice of the pre-image $\hat{\lambda}$.

The *iquantum group* associated to the above data is the subalgebra $\mathrm{U}^\imath = \mathrm{U}^\imath(\varsigma)$ of U generated by the elements q^h ($h \in Y^\imath$) and

$$b_i := f_i + q_i^{\varsigma_i} e_{\tau i} k_i^{-1} \tag{3.5}$$

for each $i \in I$. It is a (right) coideal subalgebra of U, and the pair $(\mathrm{U}, \mathrm{U}^\imath)$ is a *quantum symmetric pair* of quasi-split type.

Remark 3.1. The formula (3.5) is essentially the "standard embedding" from [Kol14], but it is not the most general embedding $\mathrm{U}^\imath(\varsigma) \hookrightarrow \mathrm{U}$: one can take

$$b_i := f_i + q_i^{\varsigma_i} e_{\tau i} k_i^{-1} + \tfrac{q_i^{\mu_i} - q_i^{-\mu_i}}{q_i - q_i^{-1}} k_i^{-1}$$

for some $\mu = (\mu_i)_{i\in I} \in \mathbb{Z}^I$ such that $\mu_i = 0$ either if $i \neq \tau i$, or if there exists $j \in I$ with $j = \tau j$ and $a_{i,j} \equiv 1 \pmod 2$.

We will need the *modified form* $\dot{\mathrm{U}}^\imath = \dot{\mathrm{U}}^\imath(\varsigma)$ for $\mathrm{U}^\imath$, which was introduced in [BW18a] (see also [BW21, Sec. 3.5]). This can be viewed as a subalgebra of the completion $\hat{\mathrm{U}}$ of the modified form of U. Given $\lambda \in X^\imath$, we let $1_\lambda := \sum_{\hat\lambda} 1_{\hat\lambda} \in \hat{\mathrm{U}}$ summing over all pre-images $\hat\lambda \in X$ of λ. Then

$$\dot{\mathrm{U}}^\imath := \bigoplus_{\lambda,\mu \in X^\imath} 1_\lambda \mathrm{U}^\imath 1_\mu \subset \hat{\mathrm{U}}.$$

Although rarely a unital subalgebra of $\hat{\mathrm{U}}$, it is a locally unital algebra in its own right with distinguished idempotents $\{1_\lambda \mid \lambda \in X^\imath\}$. As such, it is generated by the elements $b_i 1_\lambda = 1_{\lambda-\alpha_i} b_i$ for all $i \in I$ and $\lambda \in X^\imath$; this depends on the observation that $-\alpha_i$ and $\alpha_{\tau i}$ have the same image in $X^\imath$.

Following [BW18b, BW18c], we define elements $b_i^{(n)} 1_\lambda = 1_{\lambda - n\alpha_i} b_i^{(n)} \in \dot{\mathrm{U}}^\imath$, which we call *divided powers* if $i \neq \tau i$ or *idivided powers* if $i = \tau i$:

$$b_i^{(n)} 1_\lambda := \begin{cases} \dfrac{1}{[n]^!_{q_i}}\, b_i^n 1_\lambda & \text{if } i \neq \tau i \\ \dfrac{1}{[n]^!_{q_i}} \displaystyle\prod_{\substack{m=0 \\ m \equiv h_i(\hat\lambda) \,(\mathrm{mod}\ 2)}}^{n-1} \left(b_i^2 - [m]_{q_i}^2\right) 1_\lambda & \text{if } i = \tau i \text{ and } n \text{ is even} \\ \dfrac{1}{[n]^!_{q_i}}\, b_i \displaystyle\prod_{\substack{m=1 \\ m \equiv h_i(\hat\lambda) \,(\mathrm{mod}\ 2)}}^{n-1} \left(b_i^2 - [m]_{q_i}^2\right) 1_\lambda & \text{if } i = \tau i \text{ and } n \text{ is odd.} \end{cases} \tag{3.6}$$

Let $\dot{\mathrm{U}}^\imath_{\mathbb{Z}} = \dot{\mathrm{U}}^\imath_{\mathbb{Z}}(\varsigma) \subset \widehat{\mathrm{U}}_{\mathbb{Z}}$ be the $\mathbb{Z}[q,q^{-1}]$-form for $\dot{\mathrm{U}}^\imath$ generated by $b_i^{(n)} 1_\lambda\, (\lambda \in X^\imath, i \in I, n \geq 0)$. Again, this was introduced originally in [BW18a].

There is an explicit presentation for $\dot{\mathrm{U}}^\imath$, which is implied by the presentation for $\mathrm{U}^\imath$ derived in [CLW21, Th. 3.1] (see also [BK15, Th. 3.6]): it is the locally unital algebra with (mutually orthogonal) distinguished idempotents $\{1_\lambda \mid \lambda \in X^\imath\}$ and generators $b_i 1_\lambda = 1_{\lambda-\alpha_i} b_i\ (\lambda \in X^\imath, i \in I)$ subject just to the *iSerre relations*

$$\sum_{n=0}^{1-a_{i,j}} (-1)^n b_i^{(n)} b_j b_i^{(1-a_{i,j}-n)} 1_\lambda = \delta_{i,\tau j} \prod_{r=1}^{-a_{i,j}} (q_i^r - q_i^{-r}) \times \frac{(-1)^{a_{i,j}} q_i^{\lambda_i - \varsigma_i - \binom{a_{i,j}}{2}} - q_i^{\binom{a_{i,j}}{2} + \varsigma_i - \lambda_i}}{q_i - q_i^{-1}} b_i^{(-a_{i,j})} 1_\lambda \tag{3.7}$$

for all $i \neq j$ in I and $\lambda \in X^\imath$, where $b_i^{(n)} 1_\lambda$ is as defined by (3.6). If $a_{i,\tau i} = 0$ (when $\varsigma_i = 0$) then this relation implies that $[b_{\tau i}, b_i] 1_\lambda = [\lambda_i]_{q_i} 1_\lambda$ If $a_{i,\tau i} = -1$ then it gives that $(b_{\tau i} b_i^{(2)} - b_i b_{\tau i} b_i + b_i^{(2)} b_{\tau i}) 1_\lambda = -(q_i^{\lambda_i - \varsigma_i - 1} + q_i^{1+\varsigma_i - \lambda_i}) b_i 1_\lambda$.

Next, we introduce some symmetries. Define ${}'\varsigma = ({}'\varsigma_i)_{i\in I}$ by ${}'\varsigma_i := \varsigma_{\tau i}$. Then there are maps

$$\text{(linear isomorphism)} \quad \omega^\imath : \dot{\mathrm{U}}^\imath_{\mathbb{Z}}(\varsigma) \xrightarrow{\sim} \dot{\mathrm{U}}^\imath_{\mathbb{Z}}({}'\varsigma), \qquad b_i^{(n)} 1_\lambda \mapsto b_{\tau i}^{(n)} 1_{-\lambda}, \tag{3.8}$$

$$\text{(anti-linear involution)} \quad \psi^\imath : \dot{\mathrm{U}}^\imath_{\mathbb{Z}}(\varsigma) \xrightarrow{\sim} \dot{\mathrm{U}}^\imath_{\mathbb{Z}}(\varsigma), \qquad b_i^{(n)} 1_\lambda \mapsto b_i^{(n)} 1_\lambda, \tag{3.9}$$

$$\text{(linear anti-isomorphism)} \quad \sigma^\imath : \dot{\mathrm{U}}^\imath_{\mathbb{Z}}(\varsigma) \xrightarrow{\sim} \dot{\mathrm{U}}^\imath_{\mathbb{Z}}({}'\varsigma), \qquad b_i^{(n)} 1_\lambda \mapsto 1_{-\lambda} b_i^{(n)}. \tag{3.10}$$

The existence of these is a relations check using (3.7).

Lemma 3.2. *Suppose we are given two families of parameters $(\varsigma_i)_{i\in I}$ and $(\varsigma_i^\dagger)_{i\in I}$ satisfying* (3.1) *and an element $\pi \in X^\imath$ such that $\pi_i = \varsigma_i^\dagger - \varsigma_i$ for each $i \in I$. Then there is a $\mathbb{Q}(q)$-algebra isomorphism*

$$s : \dot{\mathrm{U}}^\imath(\varsigma) \xrightarrow{\sim} \dot{\mathrm{U}}^\imath(\varsigma^\dagger), \qquad b_i 1_\lambda \mapsto b_i 1_{\lambda+\pi}.$$

It restricts to an isomorphism $\dot{\mathrm{U}}^\imath_{\mathbb{Z}}(\varsigma) \xrightarrow{\sim} \dot{\mathrm{U}}^\imath_{\mathbb{Z}}(\varsigma^\dagger)$ between the integral forms.

Proof. Note that such an iweight π exists providing $\varsigma_i = \varsigma_i^\dagger$ for all but finitely many $i \in I$; for example, we could take π to be the image in $X^\imath$ of $\sum_{i\in I_1}(\varsigma_i^\dagger - \varsigma_i)\varpi_i$ where I_1 is a set of representatives for the τ-orbits of size 2 in I. To construct the homomorphism s, we use the description by generators and relations. There is only the one relation (3.7), and the argument reduces easily to checking that

$$\lambda_i - \varsigma_i = (\lambda+\pi)_i - \varsigma_i^\dagger, \tag{3.11}$$

which is clear. The homomorphism is an isomorphism because it has a two-sided inverse mapping $b_i 1_\lambda \mapsto b_i 1_{\lambda-\pi}$. It restricts to an isomorphism between integral forms because it takes $b_i^{(n)} 1_\lambda \mapsto b_i^{(n)} 1_{\lambda+\pi}$ for all $n \geq 1$. □

Lemma 3.2 justifies omitting ς from the notation $\dot{\mathrm{U}}^\imath_{\mathbb{Z}}(\varsigma)$, although its embedding into $\widehat{\mathrm{U}}_{\mathbb{Z}}$ does depend on this choice.

3.2. **Definition of the 2-iquantum group $\mathfrak{U}^\imath(\varsigma, \zeta)$.** As usual, $\Bbbk$ is an $\mathbb{N}$-graded commutative ring. If $a_{i,\tau i} \neq 0$ for some $i \in I$ then we also require that 2 is invertible in $\Bbbk$. The parameters $\varsigma = (\varsigma_i)_{i\in I} \in \mathbb{Z}^I$ of the iquantum group $\mathrm{U}^\imath$ are as in (3.1). Other notation is the same as in Section 2: we have fixed the parameters $Q_{i,j}(x,y) = Q_{j,i}(y,x) \in \Bbbk[x,y]$ with $Q_{i,i}(x,y) = 0$ and normalization homomorphisms $c_i : X \to \Bbbk^\times$, leading to the definitions of $r_{i,j} \in \Bbbk^\times$ in (2.18) and the rational functions $R_{i,j}(x,y) \in \Bbbk(x,y)$ in (2.36).

We need to impose some additional hypotheses giving compatibility of the parameters with τ. We assume that

$$c_{\tau i}(\tau\lambda) = (-1)^{h_i(\lambda)} c_i(\lambda) \tag{3.12}$$

for all $\lambda \in X$ and $i \in I$ with $i \neq \tau i$. A more irksome assumption is that

$$Q_{\tau i,\tau j}(x,y) = r_{i,j} r_{j,i} r_{i,\tau j} r_{j,\tau i} Q_{i,j}(-x,-y) \tag{3.13}$$

for all $i, j \in I$; the need for this is explained in Remark 3.5. We impose a stronger assumption on $Q_{i,\tau i}(x,y)$ when $i \neq \tau i$, namely, that it is the product of linear factors

$$Q_{i,\tau i}(x,y) = \begin{cases} c_i(\alpha_{\tau i}) \displaystyle\prod_{r=1}^{-a_{i,\tau i}/2} \big(x - \xi_{i,r} y\big)\big(x - \xi_{i,r}^{-1} y\big) & \text{if } a_{i,\tau i} \text{ is even} \\ c_i(\alpha_{\tau i})\ (x-y) \displaystyle\prod_{r=1}^{-(a_{i,\tau i}+1)/2} \big(x - \xi_{i,r} y\big)\big(x - \xi_{i,r}^{-1} y\big) & \text{if } a_{i,\tau i} \text{ is odd} \end{cases} \tag{3.14}$$

for $\xi_{i,r} = \xi_{\tau i,r} \in \Bbbk^\times$ (or a unit in some field extension of $\Bbbk_0$) such that $1 + \xi_{i,r}$ is invertible. The most important situation is when all $\xi_{i,r}$ are 1 so $Q_{i,\tau i}(x,y) = c_i(\alpha_{\tau i})(x-y)^{-a_{i,\tau i}}$; another valid choice is $Q_{i,\tau i}(x,y) = c_i(\alpha_{\tau i})(x^{-a_{i,\tau i}} - y^{a_{i,\tau i}})$. In view of (2.18) and (2.36), the assumption (3.14) implies for all $i \in I$ that

$$r_{i,\tau i} = 1, \tag{3.15}$$

$$R_{i,\tau i}(x,-x) = R_{i,\tau i}(1,-1) x^{\varsigma_i + \varsigma_{\tau i}} \text{ with } R_{i,\tau i}(1,-1) \in \Bbbk^\times. \tag{3.16}$$

If $\Bbbk$ is an algebraically closed field then (3.14) is equivalent to assuming that the equations (3.13), (3.15) and (3.16) hold.

Remark 3.3. The assumption (3.13) imposes some restrictions on the root datum. Suppose that there exist $i \neq j$ in I with $i = \tau i$ and $j = \tau j$. Taking $x^{-a_{i,j}}$-coefficients in (3.13) gives

$$(r_{i,j} r_{j,i})^2 = (-1)^{a_{i,j}}. \tag{3.17}$$

Since the left hand side is symmetric in i and j, this implies that $(-1)^{a_{i,j}} = (-1)^{a_{j,i}}$, hence, we must have that $a_{i,j} \equiv a_{j,i} \pmod 2$.

There is one more choice to be made which is not as important as the ones above (see Lemma 3.6): let $\zeta = (\zeta_i)_{i \in I}$ be a choice of scalars in $\Bbbk^\times$ such that

$$\zeta_i \zeta_{\tau i} = (-1)^{\varsigma_{\tau i}+1} R_{i,\tau i}(1,-1) \qquad \text{with} \qquad \zeta_i := -1/2 \text{ if } i = \tau i. \tag{3.18}$$

Finally, let $\gamma_i : X^\imath \to \Bbbk^\times$ be the group homomorphism defined by

$$\gamma_i(\lambda) := \begin{cases} c_i\big(\hat\lambda - \tau(\hat\lambda)\big) & \text{if } i \neq \tau i \\ (-1)^{h_i(\hat\lambda)} & \text{if } i = \tau i \end{cases} \tag{3.19}$$

for $\lambda \in X^\imath$ and any pre-image $\hat\lambda \in X$. This is well defined[2] independent of the choice of $\hat\lambda$. The assumption (3.12) implies that

$$\gamma_i(\lambda)\gamma_{\tau i}(\lambda) = (-1)^{\lambda_i} \tag{3.20}$$

for $i \in I$ and $\lambda \in X^\imath$.

Now comes the main definition. We emphasize that the 2-category $\mathfrak{U}^\imath(\varsigma,\zeta)$ that it introduces depends not only on ς and ζ, but also on the Satake datum, the choices of $Q_{i,j}(x,y)$ and the normalization homomorphisms c_i. In the statement of the relations that follows, we use the additional shorthand

$$Q^\imath_{i,j}(x,y) := (-1)^{\delta_{i,\tau i}} Q_{i,j}(x,y) \tag{3.21}$$

in order to draw attention to an important but annoying sign.

Definition 3.4. The *2-iquantum group* $\mathfrak{U}^\imath = \mathfrak{U}^\imath(\varsigma,\zeta)$ with the parameters chosen above is the graded 2-category with object set $X^\imath$, generating 1-morphisms

$$B_i \mathbb{1}_\lambda = \mathbb{1}_{\lambda-\alpha_i} B_i : \lambda \to \lambda - \alpha_i$$

for $\lambda \in X^\imath$ and $i \in I$, with identity 2-endomorphisms denoted by unoriented strings $\lambda-\alpha_i \,\big|_i\, \lambda$, and generating 2-morphisms for all $i, j \in I$ and $\lambda \in X^\imath$ as Table 3. The degrees of the generating 2-morphisms are also listed in this table. The generating 2-morphisms are subject to certain relations. Before writing these down, we explain some further conventions.

- As in the previous section, we will usually label strings just at one end, but note for unoriented cups and caps that the string label switches from i to τi at the critical point.
- As before, we will only label one of the 2-cells by the iweight λ, and if we omit all labels we are writing something that is true for all possible labels.
- We will now be using black (closed) dots rather than the white (open) dots used in the previous section. Pins are defined in just the same way as before. In place of (2.25), we now need *two* shorthands:

(3.22)

Recall also our convention when multi-variable polynomials are pinned to several dots—the alphabetical ordering of the variables x, y, z corresponds to the lexicographical ordering of the Cartesian coordinates of the dots.

[2] This is the first place in which we are using the assumption that c_i is a homomorphism—otherwise, to have the properties required later on, $\gamma_i(\lambda)$ would need to be defined as $c_i(\hat\lambda)/c_i(\tau(\hat\lambda))$, assuming that this is well defined.

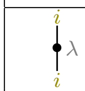
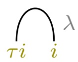
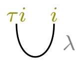
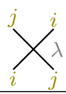
Generator	Degree
 $: B_i \mathbb{1}_\lambda \Rightarrow B_i \mathbb{1}_\lambda$	$2d_i$
 $: B_{\tau i} B_i \mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$	$d_i(1+\varsigma_i-\lambda_i)$
 $: \mathbb{1}_\lambda \Rightarrow B_{\tau i} B_i \mathbb{1}_\lambda$	$d_i(1+\varsigma_i-\lambda_i)$
$\tau i\,\text{\textcircled{n}}\,\lambda \; : \mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$ for $0 \le n \le \varsigma_i - \lambda_i$	$2d_i n$
 $: B_i B_j \mathbb{1}_\lambda \Rightarrow B_j B_i \mathbb{1}_\lambda$	$-d_i a_{i,j}$

TABLE 3. Generating 2-morphisms of $\mathfrak{U}^\imath$

- We use the following generating functions for dotted bubbles:

$$\tau i\,\text{\textcircled{u}}\,\lambda \;:= \sum_{n=0}^{\varsigma_i-\lambda_i} \tau i\,\text{\textcircled{n}}\,\lambda \;\; u^{\varsigma_i-\lambda_i-n} (\text{which is 0 if } i=\tau i), \tag{3.23}$$

$$\tau i \bigcirc(u)\,\lambda \;:= \begin{cases} \tau i \bigcirc\!\!\bullet_u\,\lambda \;+\; \tau i\,\text{\textcircled{u}}\,\lambda & \text{if } i \neq \tau i \\ \tau i \bigcirc\!\!\bullet_u\,\lambda \;-\; \dfrac{1}{2u}\,\mathrm{id}_{\mathbb{1}_\lambda} & \text{if } i = \tau i. \end{cases} \tag{3.24}$$

As we did in the previous section, we call these the *fake bubble polynomial* and the *bubble generating function*, respectively. The relation (3.27) below implies that

$$\tau i \bigcirc(u)\;\lambda \;\in \zeta_i\gamma_i(\lambda) u^{\varsigma_i-\lambda_i}\,\mathrm{id}_{\mathbb{1}_\lambda} + u^{\varsigma_i-\lambda_i-1}\Bbbk[\![u^{-1}]\!]\,\mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda). \tag{3.25}$$

For any $n \ge 0$, we let

$$\tau i\,\text{\textcircled{n}}\,\lambda \;:= \Big[\, \tau i \bigcirc(u)\,\lambda \,\Big]_{u:\varsigma_i-\lambda_i-n}. \tag{3.26}$$

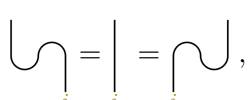
This is one of the generating 2-morphisms when $n \le \varsigma_i - \lambda_i$, or a genuine dotted bubble for larger values of n. By a *fake bubble* we mean $\tau i\,\text{\textcircled{n}}\,\lambda$ for $1 \le n \le \varsigma_i - \lambda_i$. We do not count the bubble $\tau i\,\text{\textcircled{0}}\,\lambda$ of lowest degree as being a fake bubble because it equals $\zeta_i\gamma_i(\lambda)\,\mathrm{id}_{\mathbb{1}_\lambda}$.

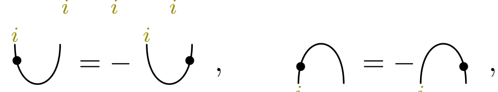
The defining relations are as follows:

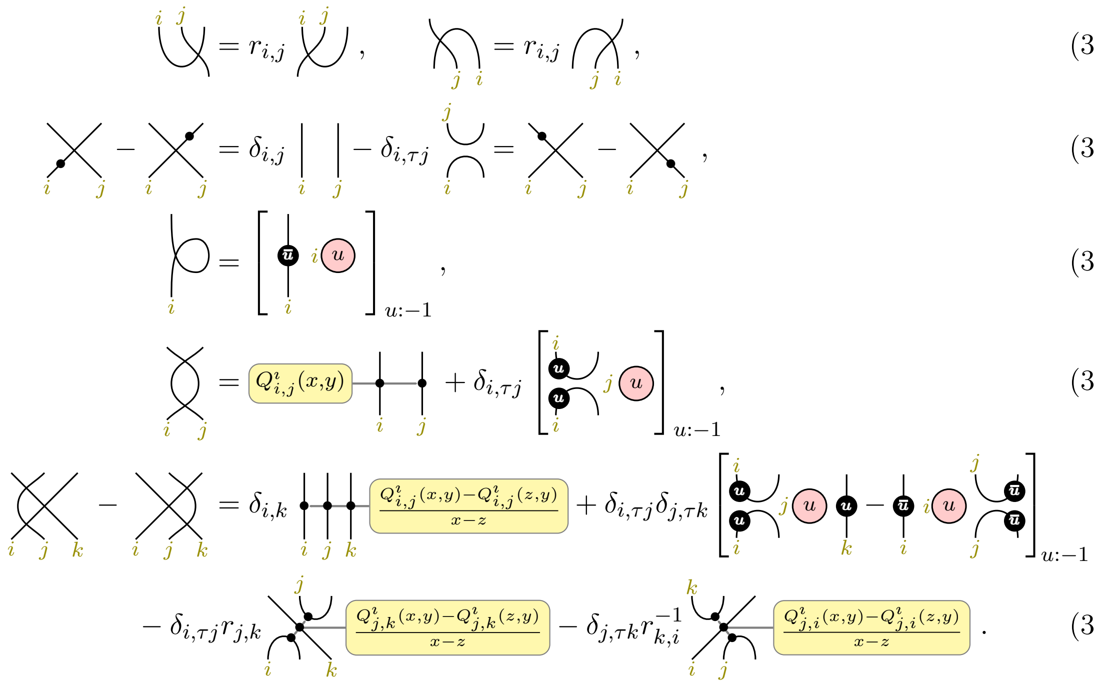
$$\Big[\tau i \bigcirc(u)\,\lambda\Big]_{u:\ge \varsigma_i-\lambda_i} = \zeta_i\gamma_i(\lambda) u^{\varsigma_i-\lambda_i}\,\mathrm{id}_{\mathbb{1}_\lambda}, \tag{3.27}$$

$$\tau i \bigcirc(u)\,\lambda\;\, i \bigcirc(-u) \;=\; -R_{i,\tau i}(u,-u)\,\mathrm{id}_{\mathbb{1}_\lambda}, \tag{3.28}$$

$$\tau i \bigcirc(u)\;\big|_j\!\bullet\!-\!\boxed{R_{i,j}(u,x)} \;=\; \boxed{R_{\tau i,j}(-u,x)}\!-\!\bullet\big|_j\;\tau i \bigcirc(u)\,, \tag{3.29}$$

 (3.30)

 (3.31)



(3.32)

(3.33)

(3.34)

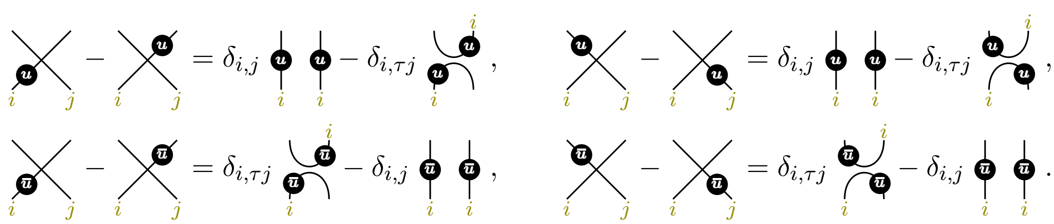
(3.35)

(3.36)

The additional generating function in (3.22) is needed because we now have that

(3.37)

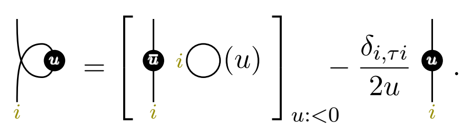
The following are consequences of (3.33):



(3.38)

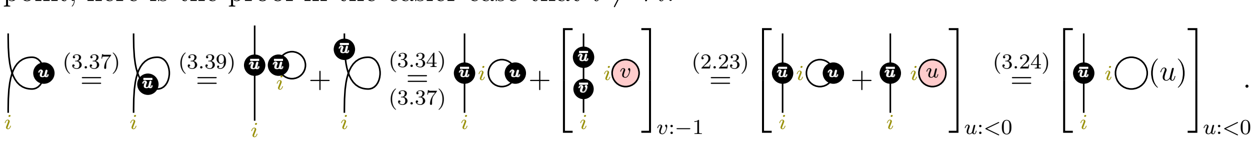
(3.39)

Using one of these, the following general form of the curl relation (3.34) can be deduced:

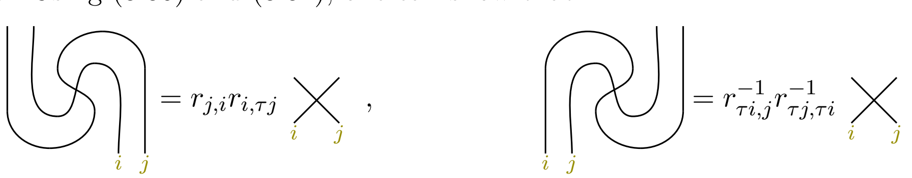


(3.40)

In the case $i = \tau i$, the calculation proving this is explained in the proof of [BWW24, Th. 2.5]; this depends on the connection to nil-Brauer in Subsection 3.3. Since it is instructive to see at this point, here is the proof in the easier case that $i \neq \tau i$:

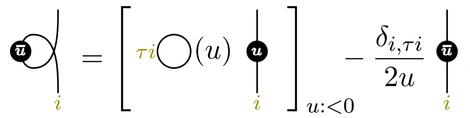


Remark 3.5. Using (3.30) and (3.32), one can show that

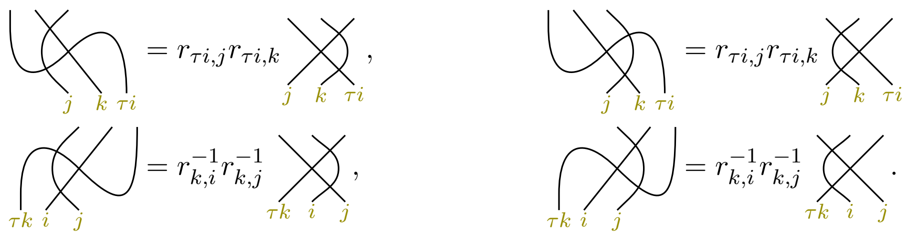


(3.41)

Now the relation (3.35) can be rotated through 180° by attaching a nested pair of cups and caps, using (3.41) to simplify the result, to obtain the identity (3.13). This is the reason for this assumption—it ensures that the quadratic relation is invariant under rotation. Rotating the curl relation (3.40) in a similar way gives the new relation

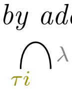
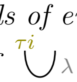

(3.42)

We leave it to the reader to check that the other defining relations of $\mathfrak{U}^\imath$ are invariant under rotation. When checking the rotation invariance of (3.36), the following are helpful:


(3.43)
(3.44)

Lemma 3.6. *Let ς and ζ be as above, and suppose that $\varsigma^\dagger = (\varsigma_i^\dagger)_{i\in I}$ and $\zeta^\dagger = (\zeta_i^\dagger)_{i\in I}$ is another choice for these parameters satisfying* (3.1) *and* (3.18)*. Let $\mathfrak{U}^\imath(\varsigma,\zeta)$ and $\mathfrak{U}^\imath(\varsigma^\dagger,\zeta^\dagger)$ be the corresponding 2-iquantum groups defined using the same choices of $Q_{i,j}(x,y)$ and c_i. Let π be an iweight such that $\pi_i = \varsigma_i^\dagger - \varsigma_i$ for each i, and choose scalars $\kappa_i \in \Bbbk^\times$ such that $\zeta_i = \kappa_i^2\zeta_i^\dagger\gamma_i(\pi)$ and $\kappa_i\kappa_{\tau i} = 1$ for each $i \in I$. Then there is an isomorphism of graded 2-categories*

$$S : \mathfrak{U}^\imath(\varsigma,\zeta) \xrightarrow{\sim} \mathfrak{U}^\imath(\varsigma^\dagger,\zeta^\dagger)$$

taking the object λ to $\lambda+\pi$, the 1-morphism $B_i\mathbb{1}_\lambda$ to $B_i\mathbb{1}_{\lambda+\pi}$, and defined on a string diagram representing a 2-morphism by adding π to the labels of every 2-cell then multiplying by κ_i for each occurrence of the generator  *or the generator*  *in the diagram, and by κ_i^2 for each fake bubble*  *.*

Proof. It is well defined by a straightforward verification of the defining relations. One needs to use (3.11) to see that everything makes sense. It is an isomorphism because it has a two-sided inverse defined in a similar way, replacing each κ_i by κ_i^{-1}. ☐

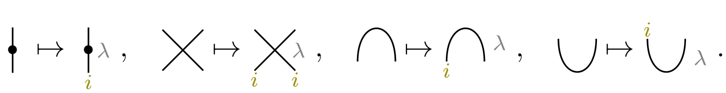
Lemma 3.6 justifies omitting ς and ζ from the notation $\mathfrak{U}^\imath(\varsigma,\zeta)$. Note though that the degrees of cups, caps and bubbles depend on the choice of ς.

3.3. **Example: split rank one.** Suppose that $i \in I$ is fixed by τ. Fix also $\lambda \in X^\imath$ and consider the strict graded monoidal category $\mathbf{End}_{\mathfrak{U}^\imath}(\lambda)$. Recall from (3.19) that $\gamma_i(\lambda) = (-1)^{h_i(\hat\lambda)}$. The relation (3.27) is equivalent to

$$\textstyle{}_i\bigcirc_\lambda = t\,\mathrm{id}_{\mathbb{1}_\lambda} \qquad \text{where} \qquad t := \frac{1-\gamma_i(\lambda)}{2}. \tag{3.45}$$

This together with the other relations in (3.30) to (3.36) (taking all strings to be labelled by i) are the defining relations of the *nil-Brauer category* $\mathbf{NB}_t$ from [BWW24, Def. 2.1]. Hence, there is a strict graded monoidal functor $\Upsilon : \mathbf{NB}_t \to \mathbf{End}_{\mathfrak{U}^\imath}(\lambda)$ such that

$$B \mapsto B_i\mathbb{1}_\lambda, \tag{3.46}$$



3.4. **Example: diagonal embedding.** In this subsection, we go back to the setup of Section 2, dropping the additional hypotheses on parameters introduced in Subsections 3.1 and 3.2. Let U be the quantum group from Subsection 2.1. Let $\boldsymbol{I} := I^+ \sqcup I^-$ be the disjoint union of two copies $I^+ := \{i^+ \mid i \in I\}$ and $I^- := \{i^- \mid i \in I\}$ of I. This is the index set for the following "doubled" Cartan datum:

- The Cartan matrix is defined by $a_{i^+,j^+} = a_{i^-,j^-} := a_{i,j}$ and $a_{i^+,j^-} = a_{i^-,j^+} := 0$, and it is symmetrized by $d_{i^+} = d_{i^-} := d_i$ for each $i, j \in I$.
- The weight lattice $\boldsymbol{X}$ is $X \oplus X$, and the coweight lattice $\boldsymbol{Y}$ is $Y \oplus Y$ paired with $\boldsymbol{X}$ in the obvious way, i.e., $(h^+, h^-) \in \boldsymbol{Y}$ maps $(\lambda^+, \lambda^-) \in \boldsymbol{X}$ to $h^+(\lambda^+) + h^-(\lambda^-)$.
- The simple roots are defined by $\alpha_{i^+} := (\alpha_i, 0)$ and $\alpha_{i^-} := (0, \alpha_i)$ for $i \in I$.
- The simple coroots are defined by $h_{i^+} := (h_i, 0)$ and $h_{i^-} := (0, h_i)$ for $i \in I$.

We denote the resulting quantum group by $\mathbf{U}$. It is identified with the tensor product $\mathrm{U} \otimes_{\mathbb{Q}(q)} \mathrm{U}$ so that $q^{(h^+,h^-)}$, $e_{i^+}, e_{i^-}, f_{i^+}$ and f_{i^-} correspond to $q^{h^+} \otimes q^{h^-}$, $e_i \otimes 1, 1 \otimes e_i, f_i \otimes 1$ and $1 \otimes f_i$, respectively. The *diagonal embedding* of U into $\mathbf{U}$ is the $\mathbb{Q}(q)$-algebra homomorphism

$$
\begin{gathered}
(\omega \otimes \mathrm{id}) \circ \Delta : \mathrm{U} \hookrightarrow \mathbf{U}, \\
q^h \mapsto q^{-h} \otimes q^h, \qquad e_i \mapsto b_{i^+} := f_i \otimes 1 + k_i^{-1} \otimes e_i, \qquad f_i \mapsto b_{i^-} := 1 \otimes f_i + e_i \otimes k_i^{-1}
\end{gathered}
\tag{3.47}
$$

for $h \in Y, i \in I$.

Let $\boldsymbol{\tau} : \boldsymbol{I} \to \boldsymbol{I}$ be the involution which switches i^+ and i^- for each $i \in I$. There are also involutions $\boldsymbol{\tau} : \boldsymbol{X} \to \boldsymbol{X}$ and $\boldsymbol{\tau}^* : \boldsymbol{Y} \to \boldsymbol{Y}$ switching the summands in the obvious way. Let $\mathbf{U}^\imath$ be the corresponding iquantum group with iweight lattice $\boldsymbol{X}^\imath$ and icoweight lattice $\boldsymbol{Y}^\imath$. By definition, $\mathbf{U}^\imath$ is the subalgebra of $\mathbf{U}$ generated by $q^{-h} \otimes q^h$ $(h \in Y)$ and b_{i^+}, b_{i^-} for $i \in I$. Comparing these formulae with (3.47), we see that the diagonal embedding restricts to an algebra isomorphism

$$
d : \mathrm{U} \xrightarrow{\sim} \mathbf{U}^\imath, \qquad q^h \mapsto q^{-h} \otimes q^h, \qquad e_i \mapsto b_{i^+}, \qquad f_i \mapsto b_{i^-}
\tag{3.48}
$$

for $h \in Y, i \in I$. Thus, d identifies U with $\mathbf{U}^\imath$, which is a quasi-split iquantum group of diagonal type. Note also that there is an isomorphism $X \xrightarrow{\sim} \boldsymbol{X}^\imath, \lambda \mapsto \bar{\lambda}$, where $\bar{\lambda}$ is the canonical image of $(0, \lambda) \in \boldsymbol{X}$ in $\boldsymbol{X}^\imath$. This satisfies $\overline{\lambda + \alpha_i} = \bar{\lambda} - \alpha_{i^+} = \bar{\lambda} + \alpha_{i^-}$ for $i \in I$.

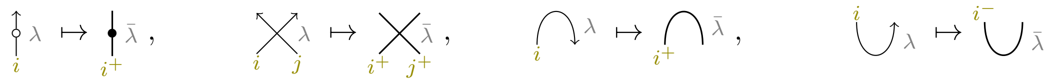
Our next theorem is a categorification of the isomorphism (3.48). Let $\mathfrak{U}$ be the 2-quantum group from Subsection 2.2 corresponding to U. So $\mathfrak{U}$ has object set X and generating 1-morphisms $E_i \mathbb{1}_\lambda, F_i \mathbb{1}_\lambda$ $(\lambda \in X, i \in I)$. Its definition involves parameters $Q_{i,j}(x,y)$. We also assume c_i and $r_{i,j}$ have been chosen as in (2.17) and (2.18). Let $\boldsymbol{\mathfrak{U}}^\imath$ be the 2-iquantum group corresponding to $\mathbf{U}^\imath$ from Subsection 3.2 defined from the parameters

$$
\begin{aligned}
Q_{i^+,j^+}(x,y) &:= Q_{i,j}(x,y), & Q_{i^-,j^-}(x,y) &:= r_{i,j} r_{j,i} Q_{i,j}(-x,-y), && (3.49)\\
Q_{i^+,j^-}(x,y) &:= 1, & Q_{i^-,j^+}(x,y) &:= 1, && (3.50)\\
c_{i^+}\big((\lambda^+,\lambda^-)\big) &:= c_i(\lambda^+), & c_{i^-}\big((\lambda^+,\lambda^-)\big) &:= (-1)^{h_i(\lambda^-)} c_i(\lambda^-), && (3.51)\\
r_{i^+,j^+} &:= r_{i,j}, & r_{i^-,j^-} &:= r_{j,i}^{-1}, && (3.52)\\
r_{i^+,j^-} &:= 1, & r_{i^-,j^+} &:= 1, && (3.53)\\
\zeta_{i^+} &:= 1, & \zeta_{i^-} &:= -1, && (3.54)\\
\varsigma_{i^+} &:= 0, & \varsigma_{i^-} &:= 0, && (3.55)
\end{aligned}
$$

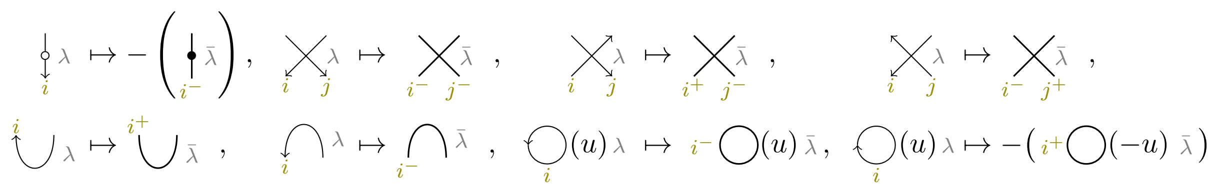
for $i, j \in I$ and $\lambda^+, \lambda^- \in X$. All of the hypotheses in Subsection 3.2 are satisfied. Also, recalling (3.19), we have that $\gamma_{i^+}\big(\bar{\lambda}\big) = c_i(\lambda)^{-1}$ and $\gamma_{i^-}\big(\bar{\lambda}\big) = (-1)^{h_i(\lambda)} c_i(\lambda)$ for $i \in I$ and $\lambda \in X$.

Theorem 3.7. *There is an isomorphism of graded 2-categories* $D : \mathfrak{U} \xrightarrow{\sim} \mathbf{\mathfrak{U}}^\imath$ *such that*

$$\lambda \mapsto \bar\lambda, \qquad E_i\mathbb{1}_\lambda \mapsto B_{i^+}\mathbb{1}_{\bar\lambda}, \qquad F_i\mathbb{1}_\lambda \mapsto B_{i^-}\mathbb{1}_{\bar\lambda},$$

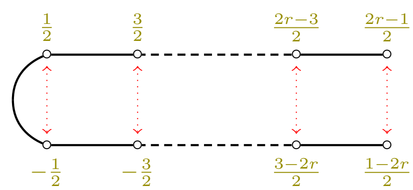


for $\lambda \in X, i, j \in I$. *It maps*



Proof. One first checks from Tables 2 and 3 that the degrees of 2-morphisms on either side are the same. Then the theorem follows by comparing the presentation of $\mathbf{\mathfrak{U}}^\imath$ with the one for $\mathfrak{U}$ from Theorem 2.5, using also the additional relations mentioned in Remark 2.6. □

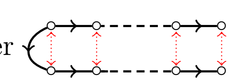
3.5. **Example: quasi-split type AIII with an even number of nodes.** There is one more special case which already appears in the existing literature: the categorification of the quasi-split $\imath$-quantum group of type AIII that was introduced in [BSWW18]. The Satake diagram is



for $r \geq 1$. So $I = \left\{\frac{1-2r}{2}, \dots, -\frac12, \frac12, \dots, \frac{2r-1}{2}\right\}$ and $\tau i = -i$. Let $I^+ := \{i \in I \mid i > 0\}$. As in [BSWW18], we realize the weight lattice X of the root system of type A_{2r} as the quotient of the free abelian group $\bigoplus_{i=-r}^r \mathbb{Z}\varepsilon_i$ by the relation $\sum_{i=-r}^r \varepsilon_i = 0$, then define α_i to be the image of $\varepsilon_{i-\frac12} - \varepsilon_{i+\frac12}$ for each $i \in I$. Let Y be the dual lattice spanned by the coroots h_i ($i \in I$) such that $h_i(\alpha_j) = 2\delta_{i,j} - \delta_{i+1,j} - \delta_{i-1,j}$ (the usual type A_{2r} Cartan matrix). The involution $\tau : X \to X$ is induced by the map $\varepsilon_i \mapsto \varepsilon_{-i}$; it takes $\alpha_i \mapsto \alpha_{-i}$. Similarly, $\tau^* : Y \to Y$ takes $h_i \mapsto h_{-i}$. Let $X^\imath$ and $Y^\imath$ be as before. Assuming that 2 is invertible in $\Bbbk$, $\mathfrak{U}^{\mathtt{BSWW}}$ is the graded 2-category with object set $X^\imath$, and generating 1-morphisms $E_i\mathbb{1}_\lambda = \mathbb{1}_{\lambda+\alpha_i}E_i$ and $F_i\mathbb{1}_\lambda = \mathbb{1}_{\lambda-\alpha_i}F_i$ for $i \in I^+$ and $\lambda \in X^\imath$. The identity 2-endomorphisms of $E_i\mathbb{1}_\lambda$ and $F_i\mathbb{1}_\lambda$ are denoted by an upward and a downward string labelled by i, respectively, just like we did for the ordinary 2-quantum group $\mathfrak{U}$. Then there are generating 2-morphisms denoted by dots, crossings, cups and caps subject to relations which can be found in [BSWW18, Def. 3.1, Def. 3.3].

Also let $\mathfrak{U}^\imath$ be the 2-iquantum group from Subsection 3.2 associated to the Satake diagram (I, τ). We choose the parameters by setting

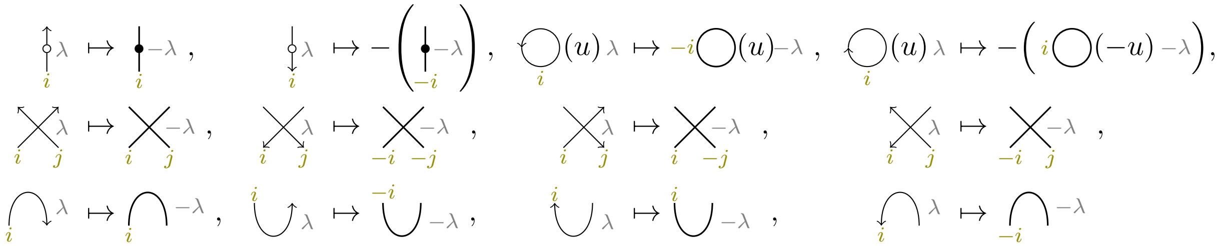
$$Q_{i,j}(x,y) = \begin{cases} 0 & \text{if } i = j \\ x - y & \text{if } j = i+1 \text{ and } i > 0, \text{ or } j = i-1 \text{ and } j < 0 \\ y - x & \text{if } j = i+1 \text{ and } i < 0, \text{ or } j = i-1 \text{ and } j > 0 \\ 1 & \text{otherwise} \end{cases} \tag{3.56}$$

for $i, j \in I$. These are the geometric parameters arising from the quiver . Let $c_i(\lambda) := 1$ if $i > 0$ and $c_i(\lambda) := (-1)^{h_i(\lambda)}$ if $i < 0$. We have that $r_{i,j} = 1$ unless $j = i - 1$ and $i \neq \frac12$, in which case $r_{i,j} = -1$. We also take $\varsigma_{\frac12} := 1$ and $\varsigma_i := 0$ for all other $i \in I$, and let $\zeta_i := 1$ if $i > 0$,

$\zeta_{-\frac{1}{2}} := -2$, and $\zeta_i := -1$ if $i < -1$. All of the hypotheses (3.1) and (3.12) to (3.14) hold. We have that $\gamma_i(\lambda) = 1$ if $i > 0$ and $\gamma_i(\lambda) = (-1)^{\lambda_i}$ if $i < 0$.

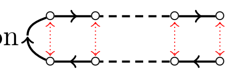
Theorem 3.8. *There is an isomorphism of graded 2-categories* $\Phi : \mathfrak{U}^{\mathtt{BSWW}} \xrightarrow{\sim} \mathfrak{U}^\imath$ *such that*

$$\lambda \mapsto -\lambda, \qquad E_i\mathbb{1}_\lambda \mapsto B_i\mathbb{1}_{-\lambda}, \qquad F_i\mathbb{1}_\lambda \mapsto B_{-i}\mathbb{1}_{-\lambda},$$



for $\lambda \in X^\imath, i, j \in I^+$.

Proof. This is another elementary comparison of relations—all of the defining relations for $\mathfrak{U}^{\mathtt{BSWW}}$ hold in $\mathfrak{U}^\imath$ so that the 2-functor makes sense, and there is also a 2-functor $\mathfrak{U}^\imath \rightarrow \mathfrak{U}^{\mathtt{BSWW}}$ that is the 2-sided inverse of Φ defined by checking that all of the defining relations for $\mathfrak{U}^\imath$ hold in $\mathfrak{U}^{\mathtt{BSWW}}$. $\square$

Remark 3.9. Having developed this general framework, it now seems more elegant to take the geometric parameters arising from the orientation  . Then we define $c_i(\lambda) := (-1)^{t_{i-1}(\lambda)}$ where $t_i : X \rightarrow \mathbb{Z}$ is the unique homomorphism with $t_i(\alpha_j) = (2r+1)\delta_{i,j}$. These normalization functions satisfy (3.12). Moreover, we have that $c_i(\alpha_j) = (-1)^{\#(j\rightarrow i)}$ for this new quiver, so that $r_{i,j} = 1$ for all $i, j \in I$. We leave it to the reader to check that $\mathfrak{U}^\imath$ defined with these new parameters choices is isomorphic to the one in Theorem 3.8 via an isomorphism which rescales caps, cups and crossings.

3.6. **The ibraid relation.** We refer to the relation (3.36) in the case that the string labels (i, j, k) are $(i, \tau i, i)$ for $i \neq \tau i$ as the *ibraid relation*.

Definition 3.10. The *weak 2-iquantum group* $\widetilde{\mathfrak{U}}^\imath(\varsigma, \zeta)$ is the graded 2-category defined in almost the same way as in Definition 3.4 but omitting all instances of the ibraid relation.

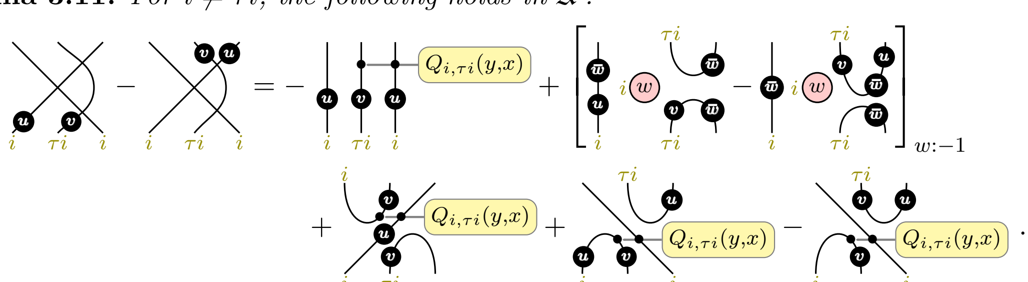
In the remainder of the section, we work in $\widetilde{\mathfrak{U}}^\imath$. We want to show that the ibraid relation holds automatically if $a_{i,\tau i} = 0$, and it holds up to multiplication by a polynomial when $a_{i,\tau i} < 0$. Consequently, although it cannot be omitted in the definition of $\mathfrak{U}^\imath$, the precise form of the ibraid relation is determined by the other relations.

We will soon need the following for $f(x, y) \in \Bbbk[x, y]$:

$$\left[\frac{f(u,v)}{(u-x)(v-y)(u-z)}\right]_{\substack{u:-1\\ v:-1}} = \left[\frac{f(u,y)}{(u-x)(u-z)}\right]_{u:-1} = \frac{f(x,y)-f(z,y)}{x-z}. \tag{3.57}$$

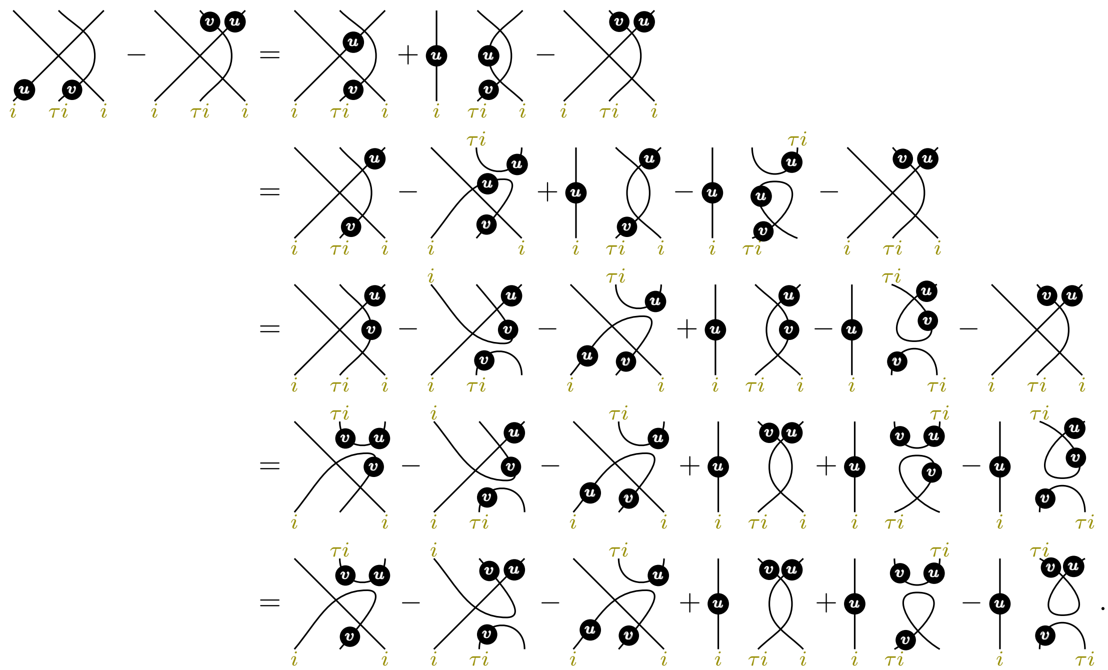
This follows from (2.23) and partial fractions.

Lemma 3.11. *For* $i \neq \tau i$, *the following holds in* $\widetilde{\mathfrak{U}}^\imath$:

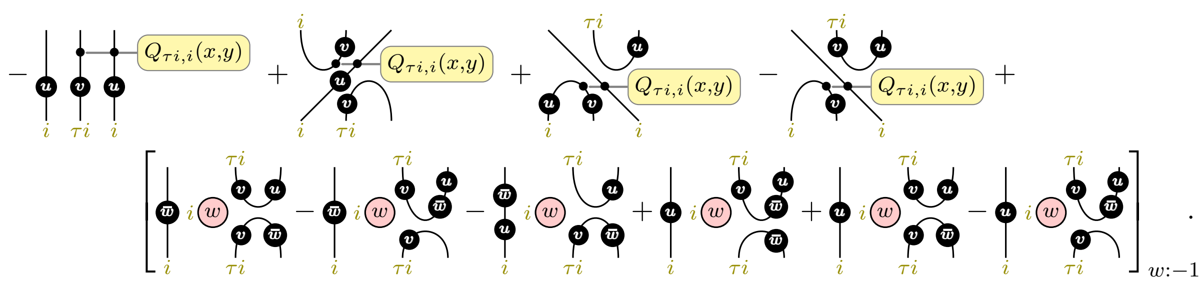


(3.58)

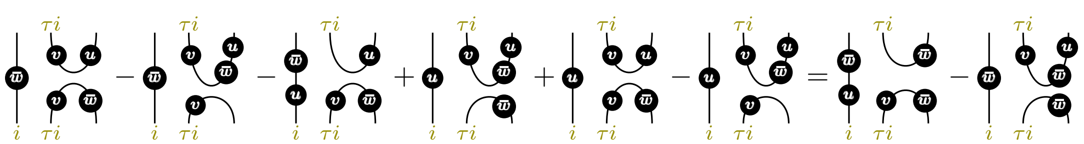
Proof. We apply (3.38) several times:



Then we expand the final expression thus obtained using (3.32), (3.34) and (3.35) to see that it equals



It remains to observe that

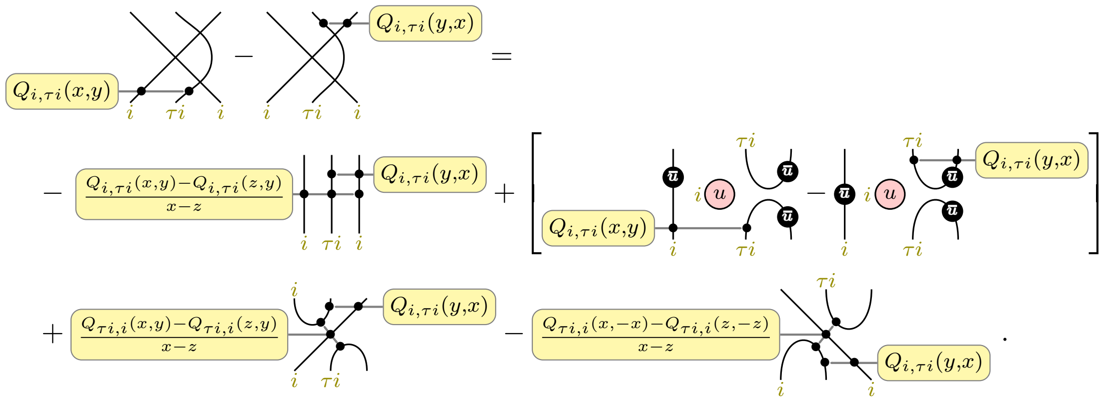


Indeed, if one denotes a dot on the left hand string by x and dots on the bottom right or top right strings by y and z, this can be rewritten as the elementary algebraic expression

$$
\begin{aligned}
&\frac{1}{(w+x)(v+y)(w+y)(u-z)(v+z)} - \frac{1}{(w+x)(v+y)(u-z)(v+z)(w+z)} \\
&\qquad - \frac{1}{(w+x)(u-x)(v+y)(w+y)(u-z)} + \frac{1}{(u-x)(w+y)(u-z)(v+z)(w+z)} \\
&\qquad + \frac{1}{(u-x)(v+y)(w+y)(u-z)(v+z)} - \frac{1}{(u-x)(v+y)(u-z)(v+z)(w+z)} \\
&\qquad\quad = \frac{1}{(u-x)(w+x)(v+y)(w+y)(w+z)} - \frac{1}{(w+x)(w+y)(u-z)(v+z)(w+z)}.
\end{aligned}
$$

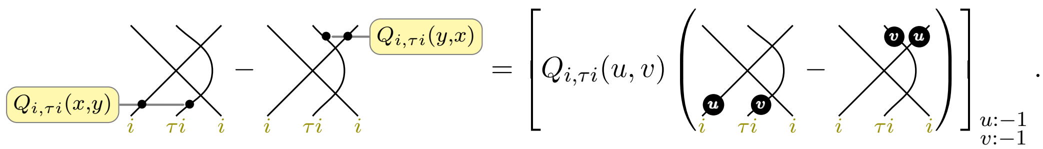
This may be checked by putting the fractions over a common denominator. $\square$

Corollary 3.12. *If $i \neq \tau i$ then*



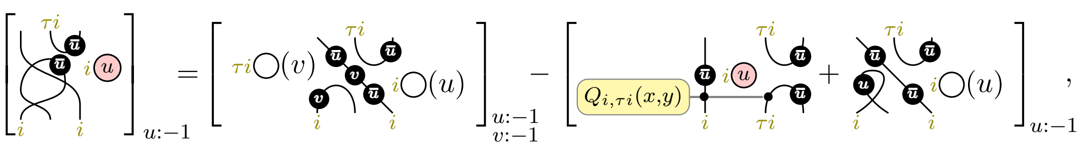
(3.59)

Proof. By (2.23), we have that

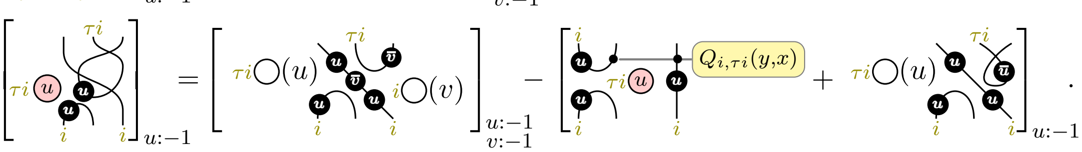


The corollary follows from this, (3.57) and (3.58). ☐

Lemma 3.13. *Assuming that $i \neq \tau i$, the following hold in $\widetilde{\mathfrak{U}}^\imath$:*

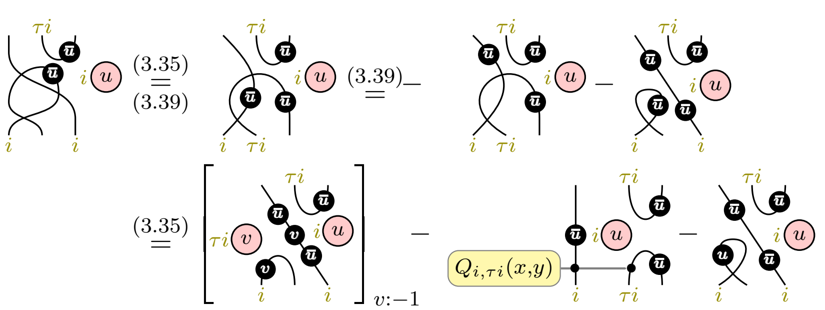


(3.60)



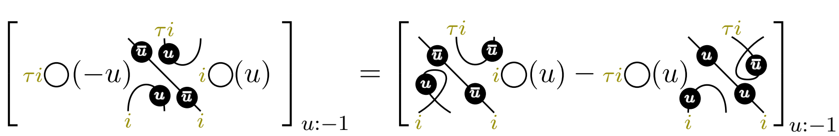
(3.61)

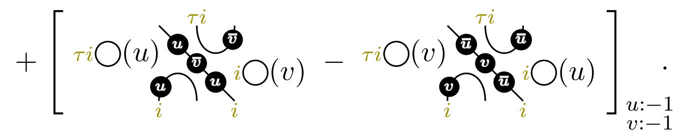
Proof. We have that



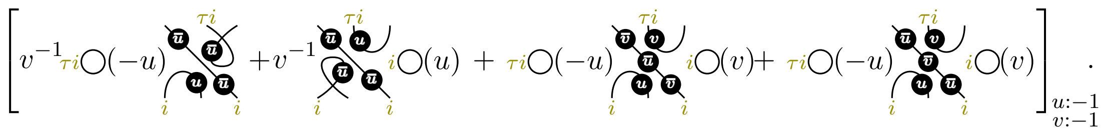
When evaluated at $u = -1$, the fake bubble polynomials can be replaced by bubble generating functions, yielding (3.60). The identity (3.61) follows by rotating (3.60). ☐

Lemma 3.14. *Again assuming that $i \neq \tau i$, we have that*



 (3.62)

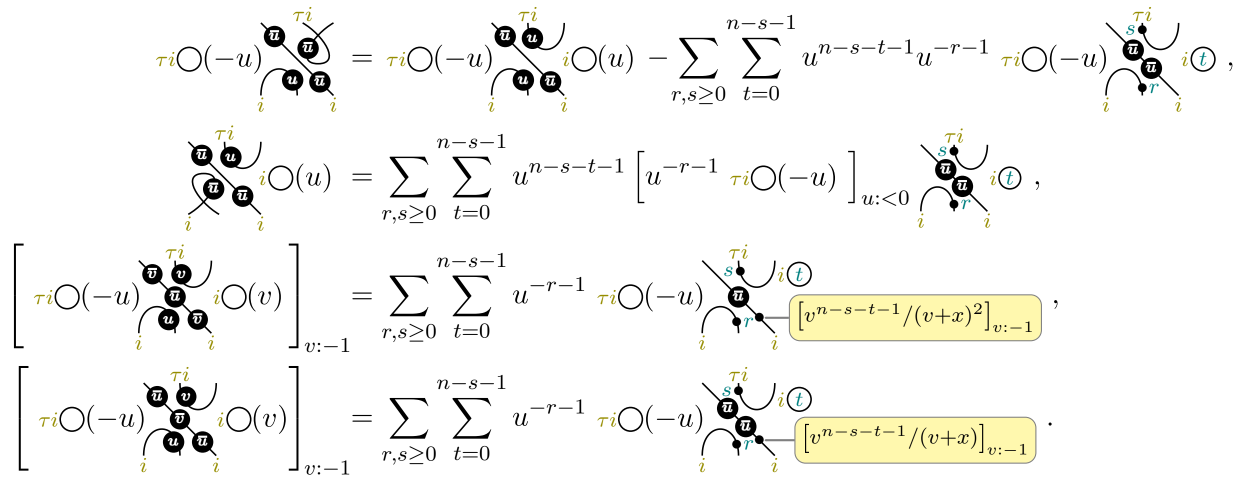
Proof. After some easy manipulations, the right hand side of the identity to be proved is

 (3.63)

Now we assume that the weight labelling the right hand 2-cell is $\lambda \in X^\imath$ and let $n := \varsigma_{\tau i} - \lambda_{\tau i}$. Recall from (3.26) that

$$ {}_i\bigcirc(u)_\lambda = \sum_{t \geq 0} u^{n-t}\ {}_i\textcircled{\scriptsize t}_\lambda . $$

We use this, together with (3.40) for the first two, to expand the four terms on the right hand side of (3.63) (we will take the u^{-1}-coefficients shortly):



The u^{-1}-coefficient of the first term on the right hand side of the first equation here is the left hand side of (3.62). It remains to show that the u^{-1}-coefficient of the sum of the four summations in the above equations is 0. This follows because for $r, s \geq 0$ and $0 \leq t \leq n-s-1$ we have that

$$ \left[\frac{u^{n-s-t-1}}{(u+x)^2}\left(u^{-r-1}{}_{\tau i}\bigcirc(-u)_{\lambda-\alpha_i}\right) - \frac{u^{n-s-t-1}}{(u+x)^2}\left[u^{-r-1}{}_{\tau i}\bigcirc(-u)_{\lambda-\alpha_i}\right]_{u:<0}\right]_{u:-1} = $$
$$ \left[\left(\frac{1}{u+x}\left[\frac{v^{n-s-t-1}}{(v+x)^2}\right]_{v:-1} + \frac{1}{(u+x)^2}\left[\frac{v^{n-s-t-1}}{v+x}\right]_{v:-1}\right)\left(u^{-r-1}\ {}_{\tau i}\bigcirc(-u)_{\lambda-\alpha_i}\right)\right]_{u:-1}. $$

To prove this identity, note that the left hand side simplifies to

$$ \left[\frac{u^{n-s-t-1}}{(u+x)^2}\left[u^{-r-1}{}_{\tau i}\bigcirc(-u)_{\lambda-\alpha_i}\right]_{u:\geq 0}\right]_{u:-1}. $$

It is easy to see that this is equal to the right hand side when $t = n-s-1$. The case $t = n-s-2$ is only slightly harder; it follows because

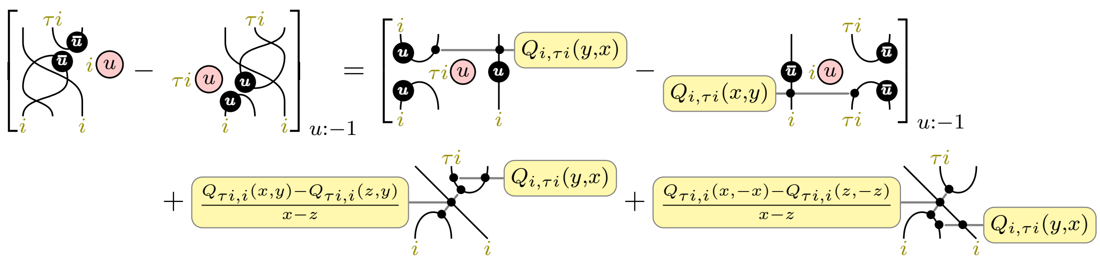
$$ \frac{u}{(u+x)^2} = \frac{1}{u+x}\left[\frac{v}{(v+x)^2}\right]_{v:-1} + \frac{1}{(u+x)^2}\left[\frac{v}{v+x}\right]_{v:-1}. $$

Finally suppose that $0 \leq t \leq n-s-3$ and set $k := n-s-t-1$, so $k \geq 2$. The identity to be proved is equivalent to

$$\sum_{m\geq 0}\left[\frac{1}{(u+x)^2}\right]_{u:-m-k-1}\left[u^{-r-1}{}_{\tau i}\bigcirc(-u)\ {\scriptstyle\lambda-\alpha_i}\right]_{u:m} =$$
$$\sum_{m\geq 0}\left[\frac{1}{u+x}\frac{1}{(v+x)^2}+\frac{1}{(u+x)^2}\frac{1}{v+x}\right]_{\substack{u:-m-1\\ v:-k-1}}\left[u^{-r-1}{}_{\tau i}\bigcirc(-u)\ {\scriptstyle\lambda-\alpha_i}\right]_{u:m}.$$

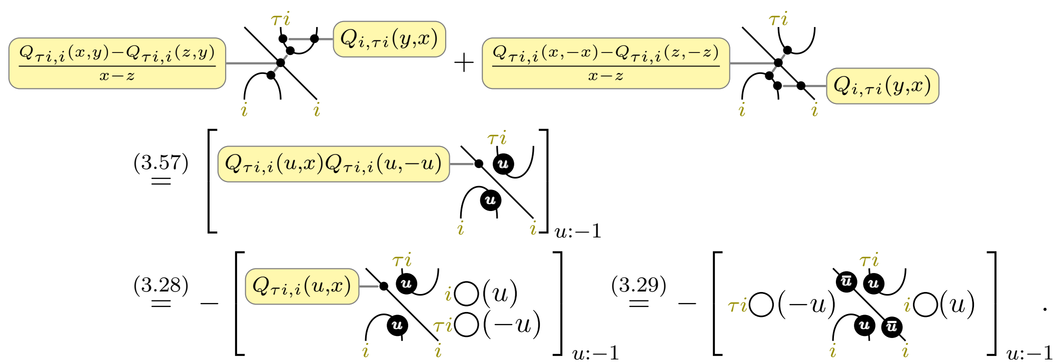
This follows since by binomial expansions the coefficient of $\left[u^{-r-1}{}_{\tau i}\bigcirc(-u)\ {\scriptstyle\lambda-\alpha_i}\right]_{u:m}$ on the left hand side is $(m+k)(-x)^{m+k-1}$, and on the right hand side it is $(-x)^m\cdot k(-x)^{k-1}+m(-x)^{m-1}\cdot(-x)^k$. □

Lemma 3.15. *If $i\neq \tau i$ then*



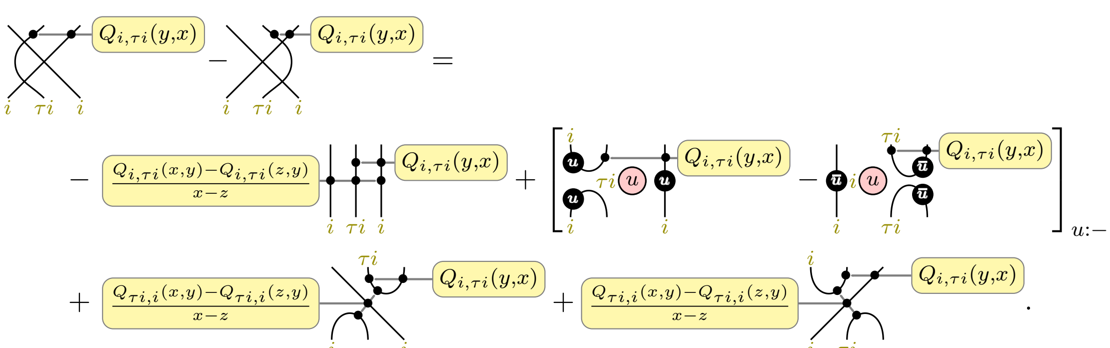
(3.64)

Proof. We have that

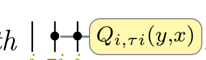


Then we expand the final expression just obtained as a sum of four terms using (3.62). Now the lemma follows by taking the difference of the equations (3.60) and (3.61), using the identity just established to rewrite the right hand side. □

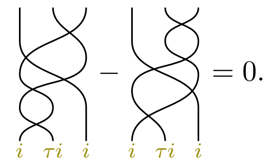
Theorem 3.16. *In the weak 2-iquantum group $\widetilde{\mathfrak{U}}^\imath$, we have for $i\neq\tau i$ that*



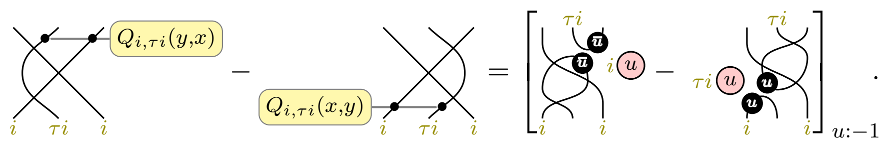
(3.65)

(This is the ibraid relation vertically composed on top with *.)*

Proof. We start from the ordinary braid relation (3.36) with strings labelled $(\tau i, i, i)$ (its right hand side is zero). Adding crossings to the bottom left pair of strings and to the top right pair of strings produces the relation



Then we use (3.35) to expand the double crossings in this equation to deduce that



The identity (3.65) follows by adding (3.59) to this equation then expanding the right hand side of the result using (3.64). $\square$

Corollary 3.17. *When the string labels (i, j, k) are equal to $(i, \tau i, i)$ for $i \in I$ with $a_{i,\tau i} = 0$, the ibraid relation* (3.36) *holds in the weak 2-iquantum group $\widetilde{\mathfrak{U}}^\imath$.*

Proof. This case of the relation (3.65) is equivalent to the relation (3.36) because $Q_{i,\tau i}(y, x) \in \Bbbk^\times$ when $a_{i,\tau i} = 0$. $\square$

3.7. **Symmetries.** Suppose we are given parameters satisfying all of the required properties (3.1) and (3.12) to (3.14). Another admissible choice of parameters is obtained by replacing $Q_{i,j}(x, y)$ with ${}'Q_{i,j}(x, y) := r_{i,j} r_{j,i} Q_{i,j}(x, y)$, hence $r_{i,j}$ with ${}'r_{i,j} := r_{j,i}^{-1}$, keeping the other parameters unchanged. Let ${}'\mathfrak{U}^\imath(\varsigma, \zeta)$ be the 2-iquantum group defined using these primed parameters. Also define ${}'\varsigma = ({}'\varsigma_i)_{i\in I}$ and ${}'\zeta = ({}'\zeta_i)_{i\in I}$ so that

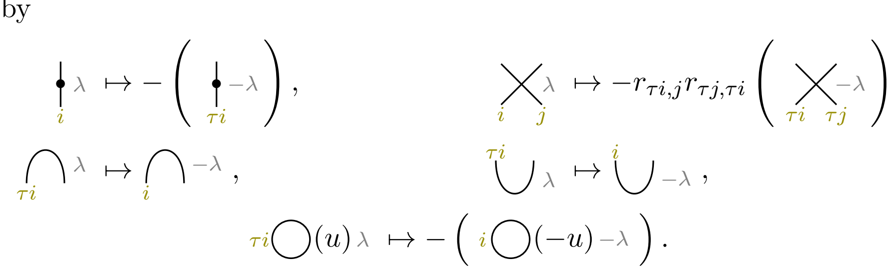
$$ {}'\varsigma_i = \varsigma_{\tau i}, \qquad \varsigma_i = {}'\varsigma_{\tau i}, \qquad {}'\zeta_i = (-1)^{\varsigma_{\tau i}+1}\zeta_{\tau i}, \qquad \zeta_i = (-1)^{{}'\varsigma_{\tau i}+1}\, {}'\zeta_{\tau i}. \tag{3.66} $$

These also satisfy (3.18), so there are two more 2-iquantum groups $\mathfrak{U}^\imath({}'\varsigma, {}'\zeta)$ and ${}'\mathfrak{U}^\imath({}'\varsigma, {}'\zeta)$.

There is an isomorphism of graded 2-categories

$$ \Omega^\imath : \mathfrak{U}^\imath(\varsigma, \zeta) \xrightarrow{\sim} \mathfrak{U}^\imath({}'\varsigma, {}'\zeta) \tag{3.67} $$

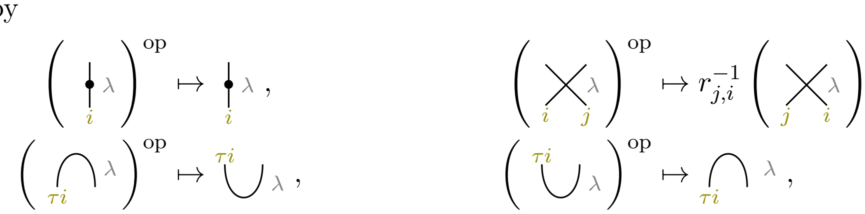
defined on objects by $\lambda \mapsto -\lambda$, on generating 1-morphisms by $B_i \mathbb{1}_\lambda \mapsto B_{\tau i} \mathbb{1}_{-\lambda}$, and on generating 2-morphisms by

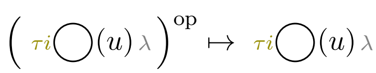


Also there is an isomorphism of graded 2-categories

$$ \Psi^\imath : \mathfrak{U}^\imath(\varsigma, \zeta)^{\mathrm{op}} \xrightarrow{\sim} {}'\mathfrak{U}^\imath(\varsigma, \zeta) \tag{3.68} $$

defined on objects by $\lambda \mapsto \lambda$, on generating 1-morphisms by $B_i \mathbb{1}_\lambda \mapsto B_i \mathbb{1}_\lambda$, and on generating 2-morphisms by

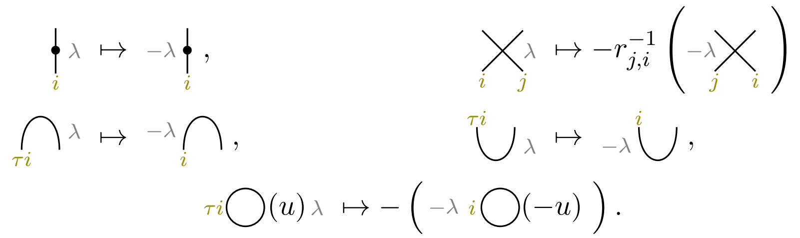




Finally, there is an isomorphism

$$\Sigma^\imath : \mathfrak{U}^\imath(\varsigma,\zeta)^{\mathrm{rev}} \xrightarrow{\sim} {}'\mathfrak{U}^\imath({}'\varsigma,{}'\zeta) \tag{3.69}$$

taking $\lambda \mapsto -\lambda$, $B_i 1\!\!1_\lambda \mapsto 1\!\!1_{-\lambda} B_i$, and



All of these things follow by elementary calculations with the defining relations. We regard $\Omega^\imath$, $\Psi^\imath$ and $\Sigma^\imath$ as categorical analogs of the maps $\omega^\imath$, $\psi^\imath$ and $\sigma^\imath$ from (3.8) to (3.10).

4. Categorification of the standard embedding

In this section, we construct a 2-functor from the 2-iquantum group $\mathfrak{U}^\imath$ to a 2-category obtained by localizing the 2-quantum group $\mathfrak{U}$. One can view this as a categorification of the standard embedding of $\dot{\mathrm{U}}^\imath_{\mathbb{Z}}$ into $\widehat{\mathrm{U}}_{\mathbb{Z}}$, although this statement should be taken with a pinch of salt since we are only able to do this after some localization—our formulae require certain polynomials in the dots to be invertible. This 2-functor will be used in a critical way to prove the main results of the article in the final section.

4.1. **Adapted grading.** We begin with a few reminders about q-envelopes; our conventions are the same as in [BE17, Def. 6.10]. For a graded category $\mathbf{C}$, its *q-envelope* (also known as *graded envelope*) is the graded category $\mathbf{C}_q$ with objects that are formal symbols $q^n X$ for $n \in \mathbb{Z}$ and all objects X of $\mathbf{C}$, with $\mathrm{Hom}_{\mathbf{C}_q}(q^n X, q^m Y)$ being the graded vector space $q^{m-n}\,\mathrm{Hom}_{\mathbf{C}}(X,Y)$. Thus, a morphism $f : X \Rightarrow Y$ of degree d in $\mathbf{C}$ can also be viewed as a 2-morphism $f : q^n X \Rightarrow q^m Y$ in $\mathbf{C}_q$ of degree $d+m-n$. Composition in $\mathbf{C}_q$ is defined by $(q^n f)\circ(q^m g) := q^{m+n}(f\circ g)$.

The category $\mathbf{C}_q$ admits an invertible *grading shift functor* $q : \mathbf{C}_q \to \mathbf{C}_q$ sending $q^n X$ to $q^{n+1}X$ and defined in the obvious way on morphisms. The original category $\mathbf{C}$ is naturally identified with a full subcategory of $\mathbf{C}_q$ so that object X of $\mathbf{C}$ is $q^0 X$ in $\mathbf{C}_q$, and the inclusion $\mathbf{C} \to \mathbf{C}_q$ is universal amongst all graded functors from $\mathbf{C}$ to graded categories admitting a grading shift functor. We also note that

$$(\mathbf{C}^{\mathrm{op}})_q = \left(\mathbf{C}_{q^{-1}}\right)^{\mathrm{op}}, \tag{4.1}$$

that is, the grading shift functor on $(\mathbf{C}^{\mathrm{op}})_q$ is the inverse of the grading shift functor of $\mathbf{C}_q$ viewed as an endofunctor of $(\mathbf{C}_q)^{\mathrm{op}}$.

If $\mathfrak{C}$ is a graded 2-category, its q-envelope $\mathfrak{C}_q$ is the 2-category with the same objects as $\mathfrak{C}$ and morphism categories that are the q-envelopes of the morphism categories of $\mathbf{C}$. Horizontal composition of 1-morphisms in $\mathfrak{C}_q$ is defined by $(q^n X)(q^m Y) := q^{n+m}(XY)$, and the definition of horizontal composition of 2-morphisms is also obvious. The original graded 2-category $\mathfrak{C}$ is naturally identified with a wide (= all objects) and full (= all 2-morphisms) sub-2-category of $\mathfrak{C}_q$.

Now we return to the setup of Section 3. In particular, $\tau : I \to I$ is an involution and we have chosen $\varsigma = (\varsigma_i)_{i\in I} \in \mathbb{Z}^I$ satisfying (3.1) and $\zeta = (\zeta_i)_{i\in I}$ satisfying (3.18). We are going to work with a slightly different version $\mathfrak{U}(\varsigma,\zeta)$ of the 2-quantum group $\mathfrak{U}$ which is graded in a way that is adapted to the standard embedding of $\dot{\mathrm{U}}^\imath$ into $\widehat{\mathrm{U}}$. By definition, $\mathfrak{U}(\varsigma,\zeta)$ is the wide and full sub-2-category of $\mathfrak{U}_q$ generated by the 1-morphisms $F_i 1\!\!1_\lambda$ and $q_i^{\varsigma_{\tau i} - h_{\tau i}(\lambda)} E_i 1\!\!1_\lambda$ for $i \in I$ and $\lambda \in X$. The grading

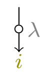
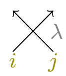
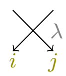
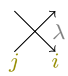
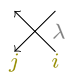
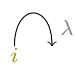
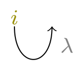
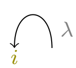
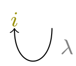
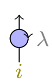
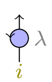
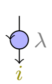
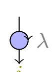
Generator	Degree
 $: q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda)} E_i \mathbb{1}_\lambda \Rightarrow q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda)} E_i \mathbb{1}_\lambda$	$2d_i$
 $: F_i \mathbb{1}_\lambda \Rightarrow F_i \mathbb{1}_\lambda$	$2d_i$
 $: q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda+\alpha_j)} q_j^{\varsigma_{\tau j}-h_{\tau j}(\lambda)} E_i E_j \mathbb{1}_\lambda \Rightarrow q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda)} q_j^{\varsigma_{\tau j}-h_{\tau j}(\lambda+\alpha_i)} E_j E_i \mathbb{1}_\lambda$	$-d_i a_{i,j}$
 $: F_i F_j \mathbb{1}_\lambda \Rightarrow F_j F_i \mathbb{1}_\lambda$	$-d_i a_{i,j}$
 $: q_j^{\varsigma_{\tau j}-h_{\tau j}(\lambda-\alpha_i)} E_j F_i \mathbb{1}_\lambda \Rightarrow q_j^{\varsigma_{\tau j}-h_{\tau j}(\lambda)} F_i E_j \mathbb{1}_\lambda$	$-d_i a_{i,\tau j}$
 $: q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda)} F_j E_i \mathbb{1}_\lambda \Rightarrow q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda-\alpha_j)} E_i F_j \mathbb{1}_\lambda$	$d_i a_{i,\tau j}$
 $: q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda-\alpha_i)} E_i F_i \mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$	$d_i(1+\varsigma_i-h_i(\lambda)+h_{\tau i}(\lambda))$
 $: \mathbb{1}_\lambda \Rightarrow q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda)} F_i E_i \mathbb{1}_\lambda$	$d_i(1+\varsigma_{\tau i}+h_i(\lambda)-h_{\tau i}(\lambda))$
 $: q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda)} F_i E_i \mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$	$d_i(1-\varsigma_{\tau i}+h_i(\lambda)+h_{\tau i}(\lambda))$
 $: \mathbb{1}_\lambda \Rightarrow q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda-\alpha_j)} E_i F_i \mathbb{1}_\lambda$	$d_i(1-\varsigma_i-h_i(\lambda)-h_{\tau i}(\lambda))$
 $: q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda)} E_i \mathbb{1}_\lambda \Rightarrow q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda)} E_i \mathbb{1}_\lambda$	$2d_i(\varsigma_{\tau i} - h_{\tau i}(\lambda))$
 $: q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda)} E_i \mathbb{1}_\lambda \Rightarrow q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda)} E_i \mathbb{1}_\lambda$	$-2d_i(\varsigma_{\tau i} - h_{\tau i}(\lambda))$
 $: F_i \mathbb{1}_\lambda \Rightarrow F_i \mathbb{1}_\lambda$	$2d_i(\varsigma_i + h_{\tau i}(\lambda))$
 $: F_i \mathbb{1}_\lambda \Rightarrow F_i \mathbb{1}_\lambda$	$-2d_i(\varsigma_i + h_{\tau i}(\lambda))$

Table 4. Adapted degrees of 2-morphisms in $\mathfrak{U}(\varsigma, \zeta)$

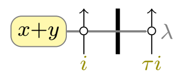
shift $q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda)}$ matches the second term in the standard embedding (3.5). The q-envelopes of $\mathfrak{U}$ and $\mathfrak{U}(\varsigma, \zeta)$ are equal. Also if we forget the grading, i.e., we view $\mathfrak{U}$ as a $\Bbbk$-linear 2-category rather than a graded 2-category, then $\mathfrak{U}(\varsigma, \zeta)$ is simply equal to $\mathfrak{U}$.

All string diagrams in the remainder of the section will represent 2-morphisms in $\mathfrak{U}(\varsigma, \zeta)$ rather than $\mathfrak{U}$, that is, we will be using the upward string labelled i to denote the identity 2-endomorphism of $q_i^{\varsigma_{\tau i}-h_{\tau i}(\lambda)} E_i \mathbb{1}_\lambda$ (whereas before it was the identity 2-endomorphism of $E_i \mathbb{1}_\lambda$). The effect of this new convention is that the degrees of the 2-morphisms represented by cups and caps in this section are different to the degrees of the 2-morphisms represented by the same cups and caps in Section 2. We have listed the new degrees of all of the generating 2-morphisms of $\mathfrak{U}(\varsigma, \zeta)$ in Table 4.

4.2. **Internal bubbles.** Next we pass to a localization $\underline{\mathfrak{U}}(\varsigma,\zeta)$ of $\mathfrak{U}(\varsigma,\zeta)$ which is obtained by adjoining two-sided inverses to the 2-morphisms[3]

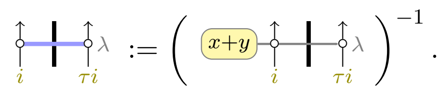
(4.2)

for all $\lambda \in X$, $i \in I$ with $i \neq \tau i$, and two-sided inverses to the 2-morphisms



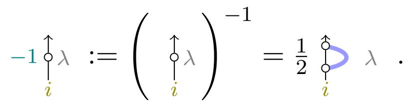
(4.3)

for all $\lambda \in X$ and $i \in I$, where the thick string in the middle denotes some number (possibly zero) of other strings separating the outer strings. We denote the inverse of (4.3) by a solid line which we refer to as a *teleporter*:



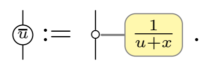
(4.4)

We extend the teleporter notation further so that the endpoints do not need to be at the same horizontal level, and the strings these points lie on do not need to be oriented in the same direction. There is no ambiguity because of the relations (2.20). We refer to [BWW24, §3] for further explanations in the special case that $i = \tau i$, in which case the dot is invertible in $\underline{\mathfrak{U}}(\varsigma,\zeta)$ with inverse



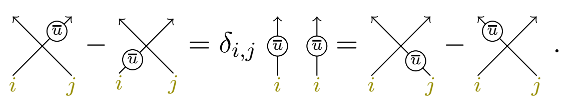
(4.5)

To work efficiently with teleporters, we need the following variation on (2.25):



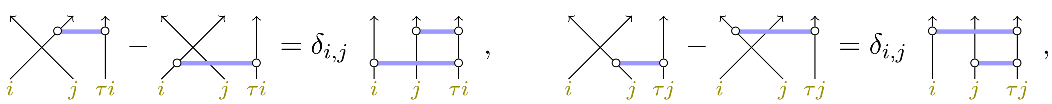
(4.6)

In fact, this can be obtained from (2.25) by replacing u by $-u$ then multiplying by -1. With this trick, the following is a consequence of (2.26):



(4.7)

Here are a couple of useful identities involving teleporters:

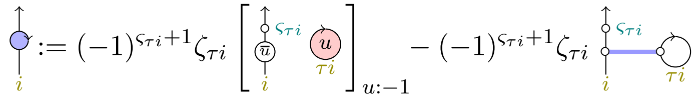


(4.8)

(4.9)

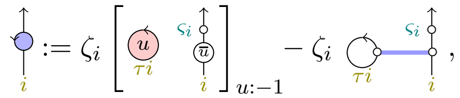
The first equality in (4.9) follows because $\frac{1}{u-x} - \frac{1}{u+y} = \frac{x+y}{(u-x)(u+y)}$, and the second equality is similar. There are plenty of variations of (4.8) and (4.9) obtained by rotating or adding additional strings passed over by the teleporter. We will apply these as needed later on.

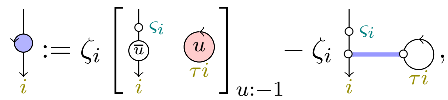
We introduce one more shorthand for the following 2-morphisms in $\underline{\mathfrak{U}}(\varsigma,\zeta)$ which we refer to as *internal bubbles*. The definition of these involves $\zeta = (\zeta_i)_{i\in I}$ for the first time. In the special case that $i = \tau i$, the internal bubbles were defined already in [BWW24, (3.38)–(3.39)] by exactly the same formula as below. In general, for $i \in I$ and $\lambda \in X$, we let

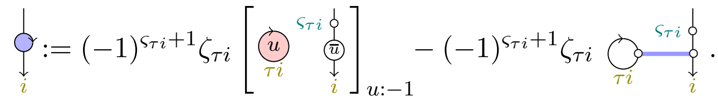


(4.10)

[3]The invertibility of the morphisms in (4.2) is used in one place only—see the penultimate paragraph of the proof of Theorem 4.13.

 (4.11)

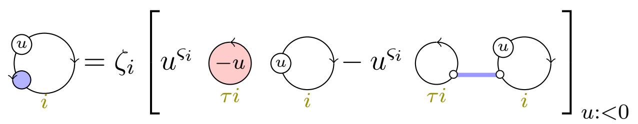
 (4.12)

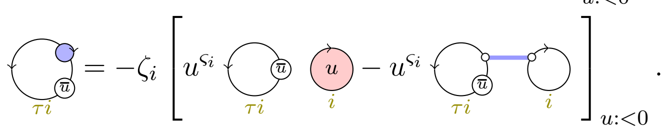
 (4.13)

These definitions apply for all possible weights λ labelling the rightmost 2-cells—the definitions are independent of the weight so we have omitted it, following our usual practice. However, the degrees of internal bubbles depend on the choice of the weight λ; they are listed in the bottom part of Table 4 in Subsection 4.1.

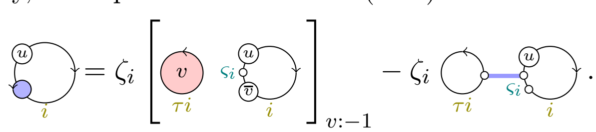
It is easy to see that internal bubbles slide over cups and caps in the obvious way. Also, they commute with dots and with other internal bubbles on the same string. The rest of this subsection is taken up with proving some rather techical identities involving internal bubbles, which are needed in the proof of the important Theorem 4.13 below.

Lemma 4.1. *For $i \in I$ with $i \neq \tau i$ (equivalently, $\varsigma_i \geq 0$), the following hold in $\mathfrak{U}(\varsigma, \zeta)$:*

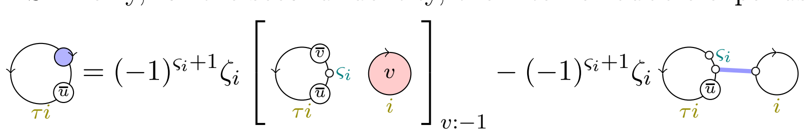
 (4.14)

 (4.15)

Proof. For the first identity, we expand the definition (4.11) of the internal bubble to see that

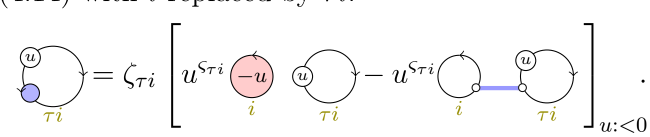


The second term is equal to the second term in (4.14) thanks to (2.23), and the first terms are equal by a similar trick. Similarly, for the second identity, the internal bubble expands to give

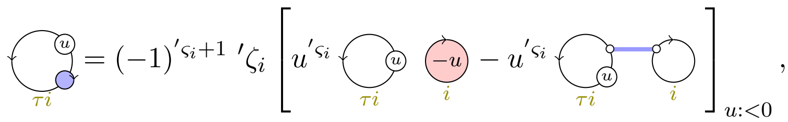


which is equal to the right hand side of (4.15). □

Remark 4.2. It is useful to be able to exploit the symmetries $\bar{\Omega}$ and Σ from (2.45) and (2.48) extended to the localization to prove relations involving internal bubbles. We forget the grading for this since $\bar{\Omega}$ and Σ do not respect the adapted grading. More problematic is that $\bar{\Omega}$ and Σ do not interchange clockwise and counterclockwise internal bubbles in the obvious way. However, the internal bubbles depend implicitly on ς and ζ, and we can view $\bar{\Omega}$ and Σ instead as $\Bbbk$-linear 2-functors $\bar{\Omega} : \mathfrak{U}(\varsigma, \zeta)^{\mathrm{op}} \to \mathfrak{U}({}'\varsigma, {}'\zeta)$ and $\Sigma : \mathfrak{U}(\varsigma, \zeta)^{\mathrm{rev}} \to \mathfrak{U}({}'\varsigma, {}'\zeta)$ where ${}'\varsigma, {}'\zeta$ are the primed parameters from (3.66). Then both $\bar{\Omega}$ and Σ take the clockwise and counterclockwise internal bubbles in $\mathfrak{U}(\varsigma, \zeta)$ to the counterclockwise and clockwise ones in $\mathfrak{U}({}'\varsigma, {}'\zeta)$ defined using the primed parameters. To illustrate, we can take (4.14) with i replaced by τi:

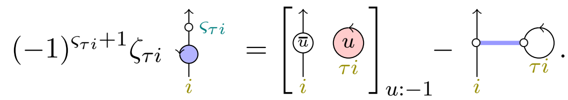


Applying Σ gives the identity



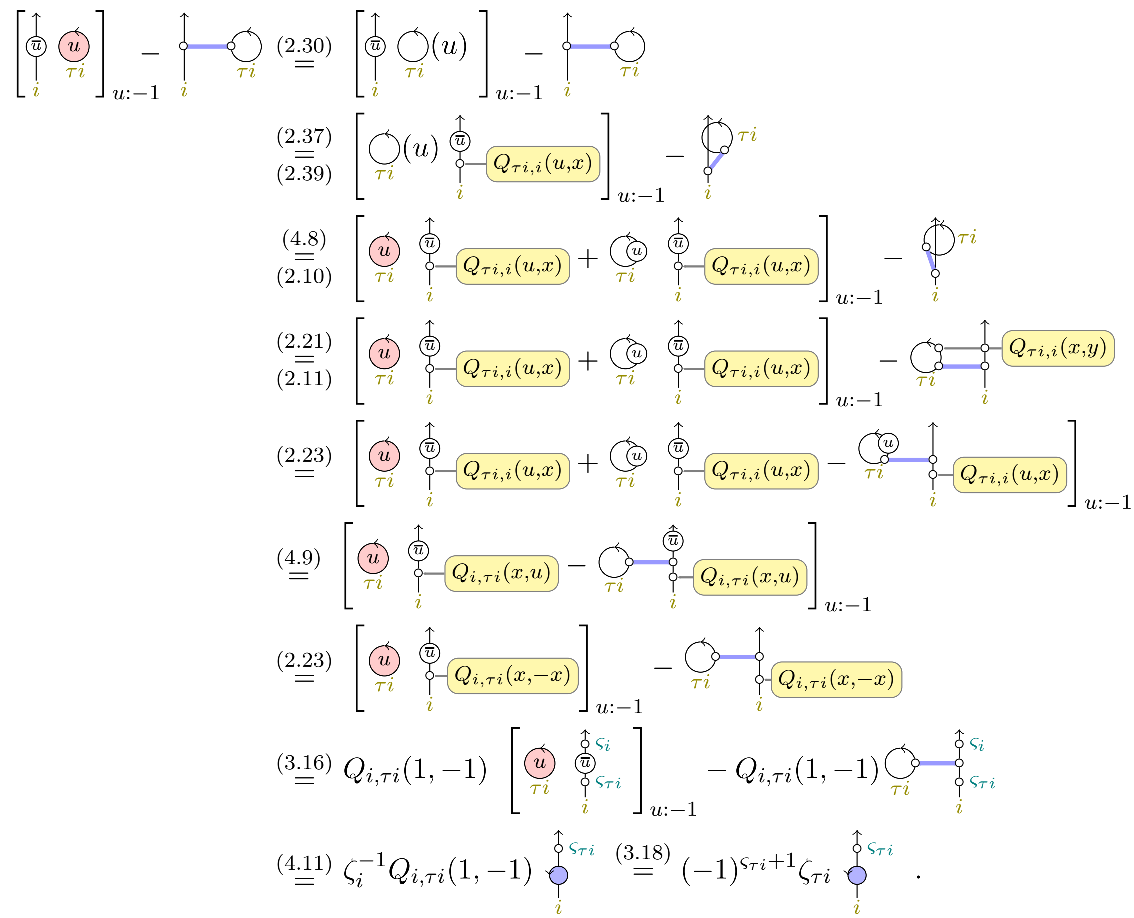
in $\mathfrak{U}({'\varsigma}, {'\zeta})$. Replacing u by $-u$, ${'\varsigma_i}$ by ς_i and ${'\zeta_i}$ by ζ_i in this identity recovers (4.15).

Lemma 4.3. *If $i \neq \tau i$ then the following hold in $\mathfrak{U}(\varsigma, \zeta)$:*



(4.16)

Proof. We start from the right hand side and calculate:

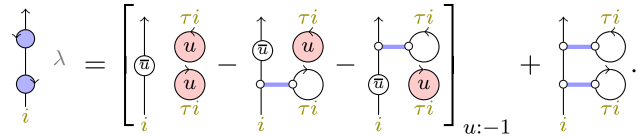


□

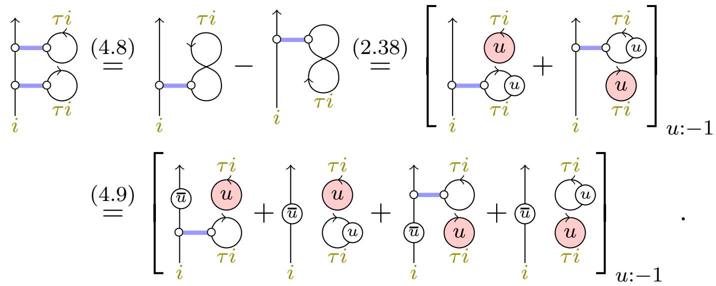
Lemma 4.4. *Clockwise and counterclockwise internal bubbles are inverses of each other:*

(4.17)

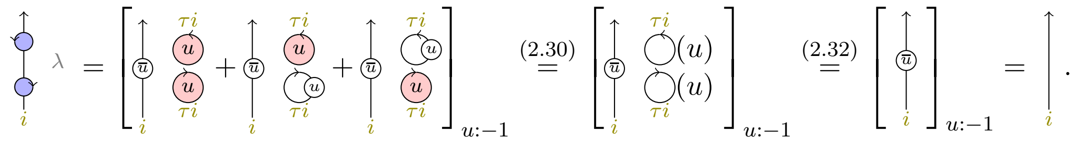
Proof. This is proved in the case that $i = \tau i$ in [BWW24, Lem. 3.5]. Now suppose that $i \neq \tau i$. We just prove the first equality. The others then follow since internal bubbles commute and we can rotate through 180° (or apply $\bar{\Omega}$). Using the definition (4.10) and Lemma 4.3, we have that



Now we use

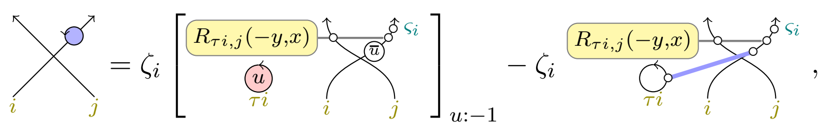


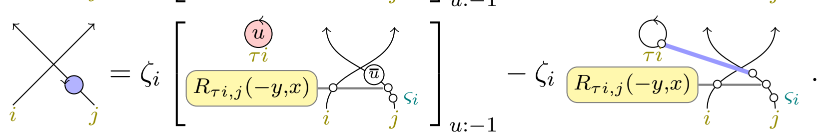
Putting these two things together, we deduce that



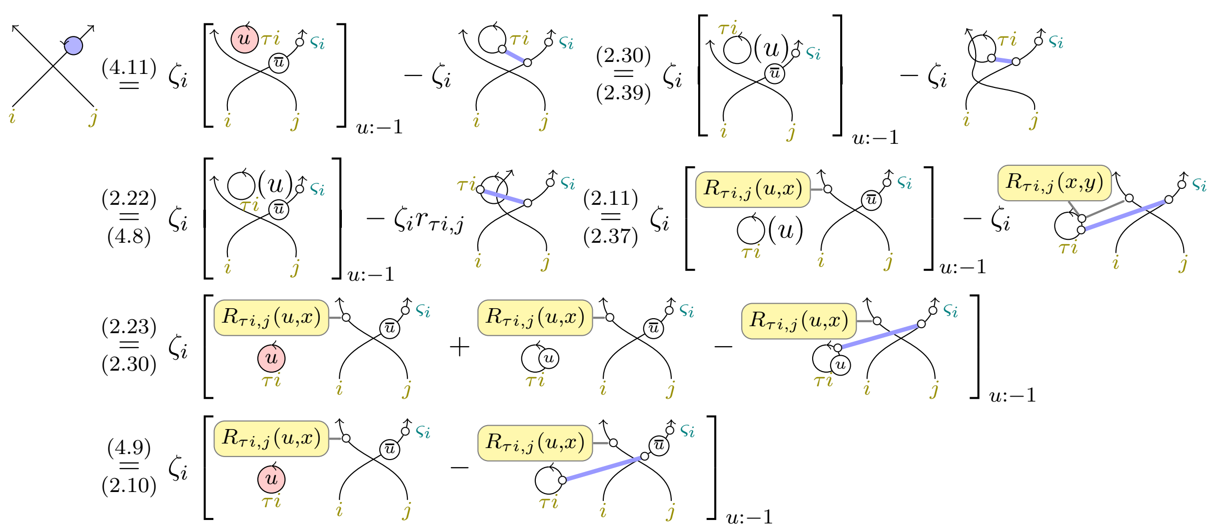
□

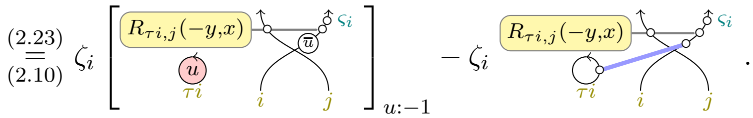
Lemma 4.5. *Assuming $\tau i \neq j$, the following hold in $\mathfrak{U}(\varsigma, \zeta)$:*

 (4.18)

 (4.19)

Proof. The following proves (4.18):



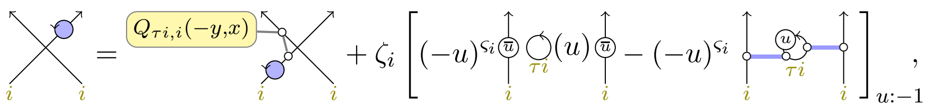


The proof of (4.19) is similar. □

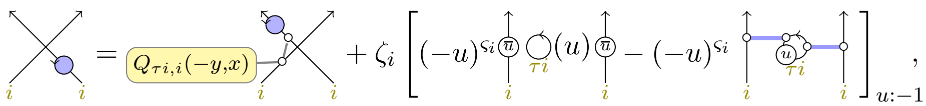
Lemma 4.6. *Assuming $i \neq j$ and $i \neq \tau j$, the following hold in $\mathfrak{U}(\varsigma, \zeta)$:*

(4.20)

Also if $i \neq \tau i$ then



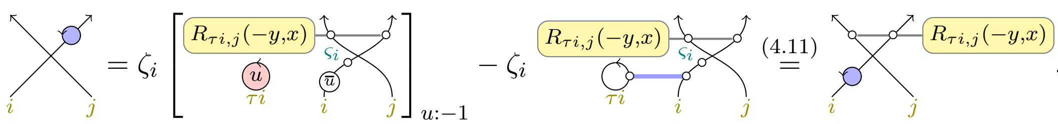
(4.21)



(4.22)

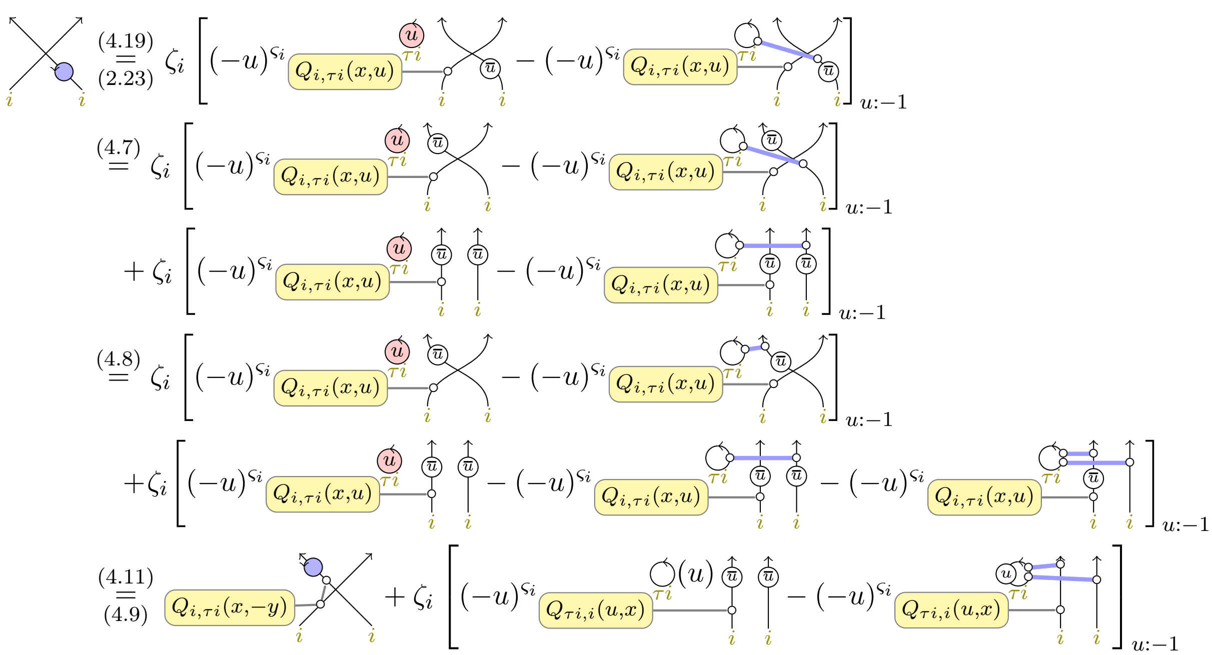
(4.23)

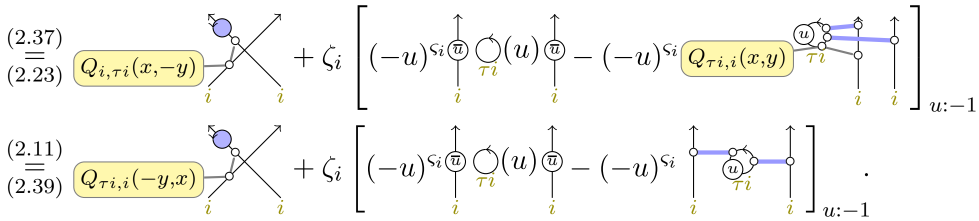
Proof. Suppose first that $i \neq j$ and $i \neq \tau j$. We use (2.10) and (4.8) to slide the dots and teleporter downward past the crossing in (4.18) to obtain



This proves the first equality from (4.20). The proof of the second one is similar, or it can be deduced from the first equality by applying Σ using the technique of Remark 4.2 plus Lemma 4.4.

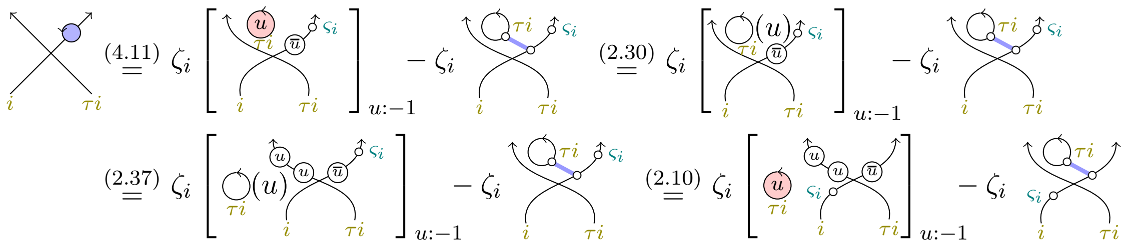
The following calculation proves (4.22):



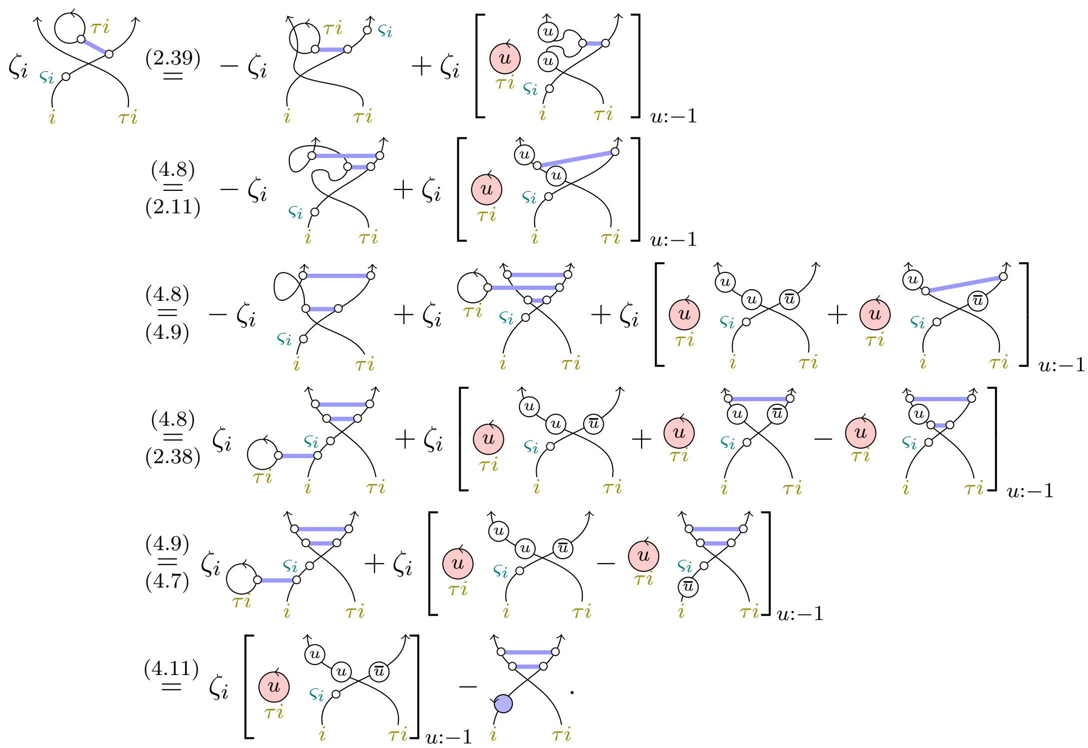


The proof of (4.21) is similar, starting from (4.18) instead of (4.19).

It remains to prove (4.23). We start as in the first line of the proof of Lemma 4.5:

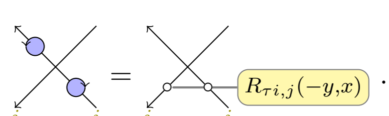


The second term is now more complicated to simplify since there is an extra term:



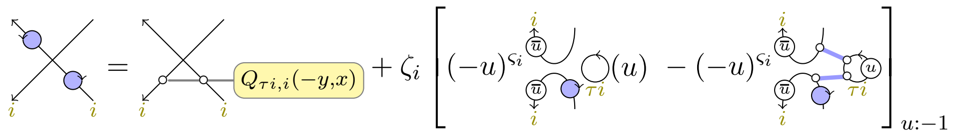
Substituting this into our previous equation yields the first equality in (4.23). The second follows on applying Σ and using Lemma 4.4. $\square$

Corollary 4.7. *If $i \neq j$ and $i \neq \tau j$ then*

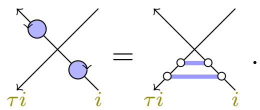


(4.24)

If $i \neq \tau i$ then

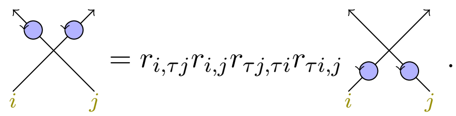


(4.25)



(4.26)

Lemma 4.8. *For all $i, j \in I$, we have that*



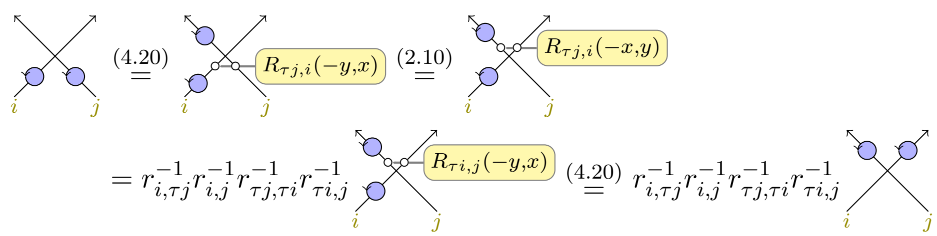
(4.27)

Proof. If $i = j = \tau i$ then this is proved in [BWW24, Lem. 3.11]. For the other cases, suppose to start with just that $i \neq \tau j$. Then

$$R_{\tau j,i}(-x,y) \overset{(2.36)}{=} r_{\tau j,i} Q_{\tau j,i}(-x,y) = r_{\tau j,i} Q_{i,\tau j}(y,-x)$$
$$\overset{(3.13)}{=} r_{i,\tau j}^{-1} r_{i,j}^{-1} r_{\tau j,\tau i}^{-1} Q_{\tau i,j}(-y,x) \overset{(2.36)}{=} r_{i,\tau j}^{-1} r_{i,j}^{-1} r_{\tau j,\tau i}^{-1} r_{\tau i,j}^{-1} R_{\tau i,j}(-y,x).$$

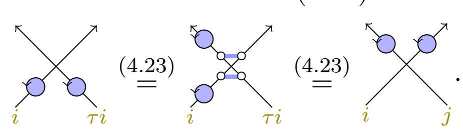
When $i = j \neq \tau i$, this shows that $Q_{\tau i,i}(-x,y) = Q_{\tau i,i}(-y,x)$, so that it is symmetric in x and y.

Now suppose that $i \neq j$ and $i \neq \tau j$. Then, using the identity proved in the previous paragraph for the unlabelled equality, we have that

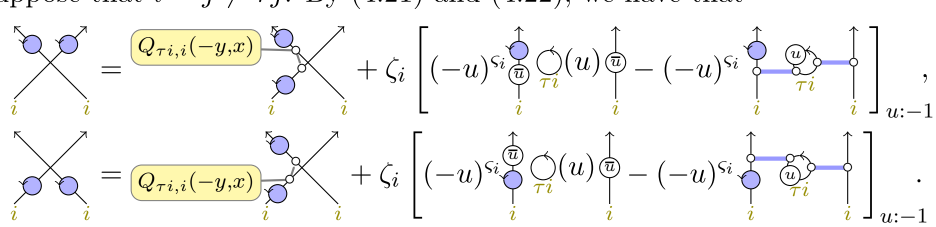


This proves (4.27) when $i \neq j$ and $i \neq \tau j$.

If $i = \tau j \neq \tau i$, we instead use the two identities in (4.23):

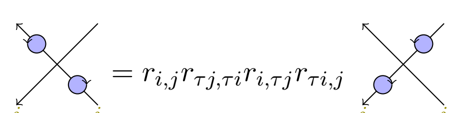


Finally suppose that $i = j \neq \tau j$. By (4.21) and (4.22), we have that



The right hand sides of these equations are equal because $Q_{\tau i,i}(-x,y) = Q_{\tau i,i}(-y,x)$. ∎

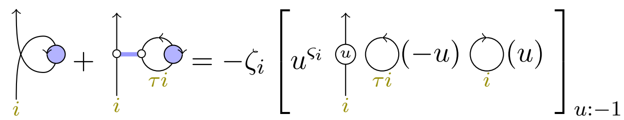
Corollary 4.9. *We have that*



(4.28)

for all $i, j \in I$.

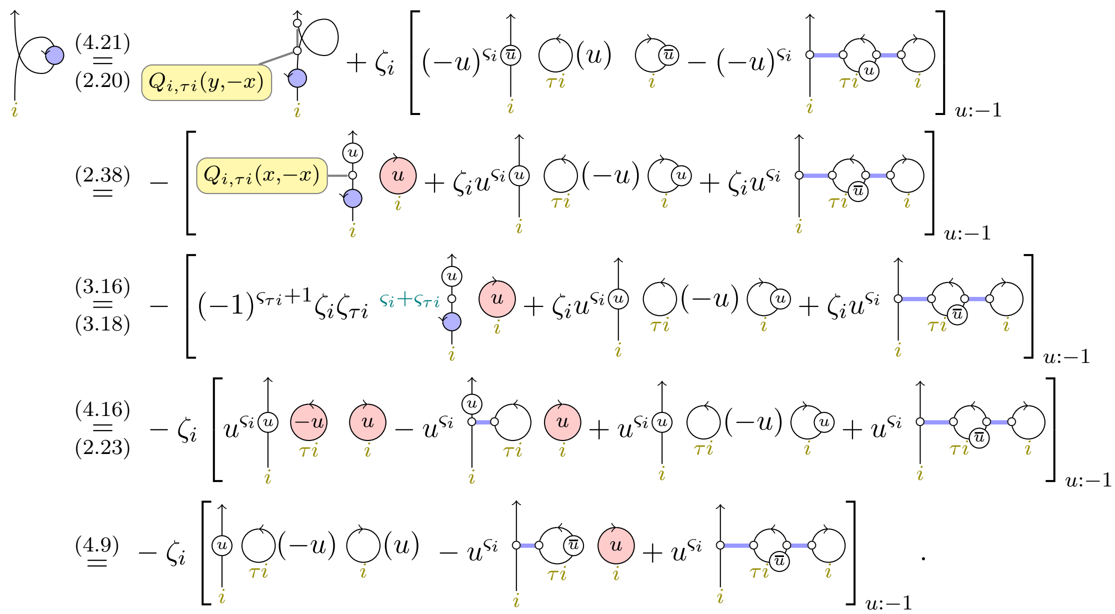
Lemma 4.10. *If $i \neq \tau i$ then*



(4.29)

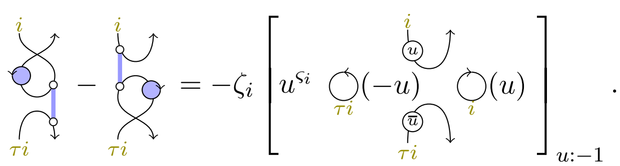
in $\mathfrak{U}(\varsigma, \zeta)$.

Proof. We have that



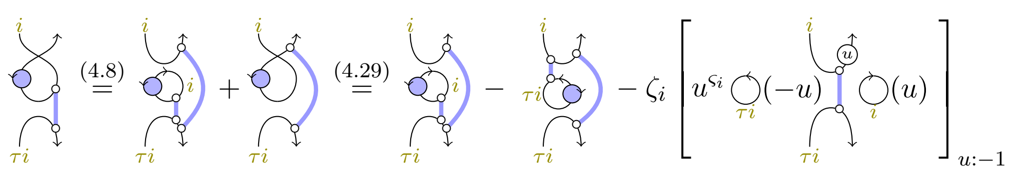
The identity (4.29) follows from this and (4.10). □

Lemma 4.11. *If $i \neq \tau i$ then*

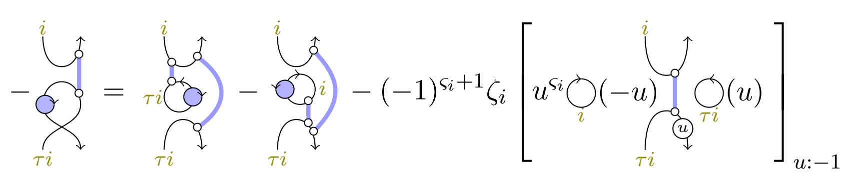


(4.30)

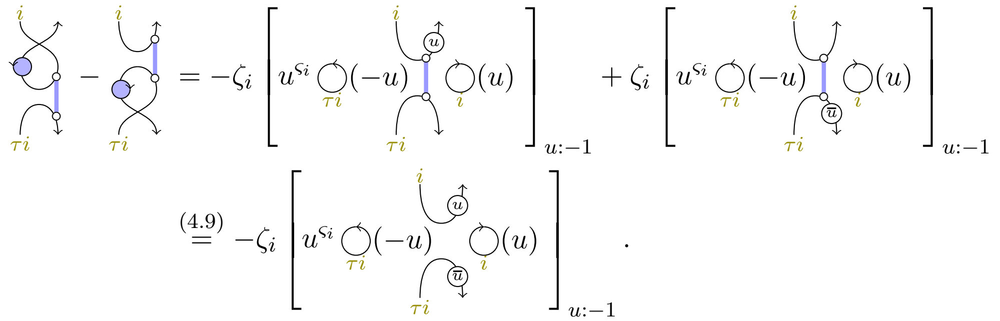
Proof. We begin with



Applying $\bar{\Omega}$ to this identity with i replaced by τi gives also that

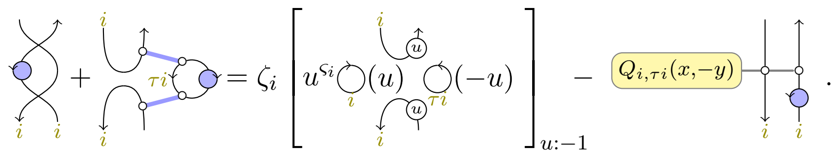


Now we replace u by $-u$ in the final term of this last expression then add it to the previous equation to obtain



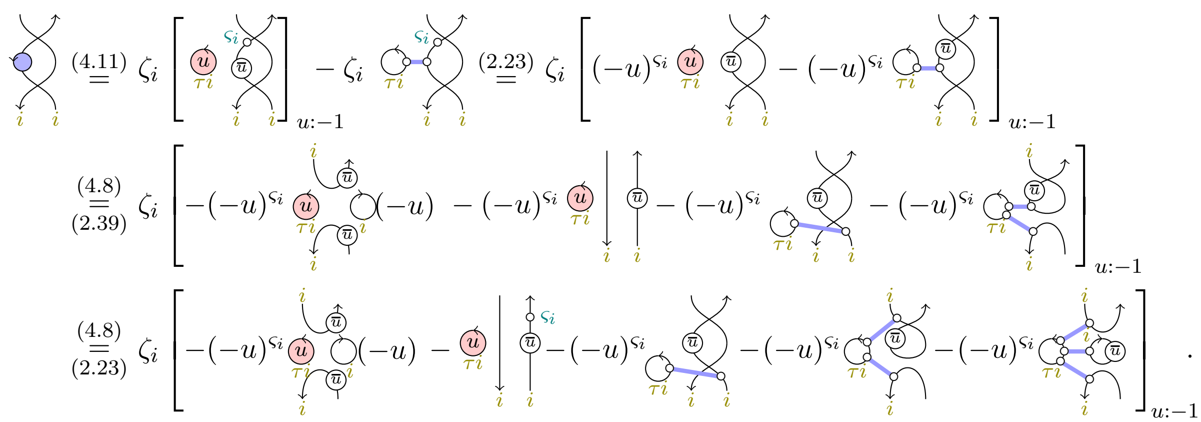
□

Lemma 4.12. *For $i \in I$ with $i \neq \tau i$, we have that*

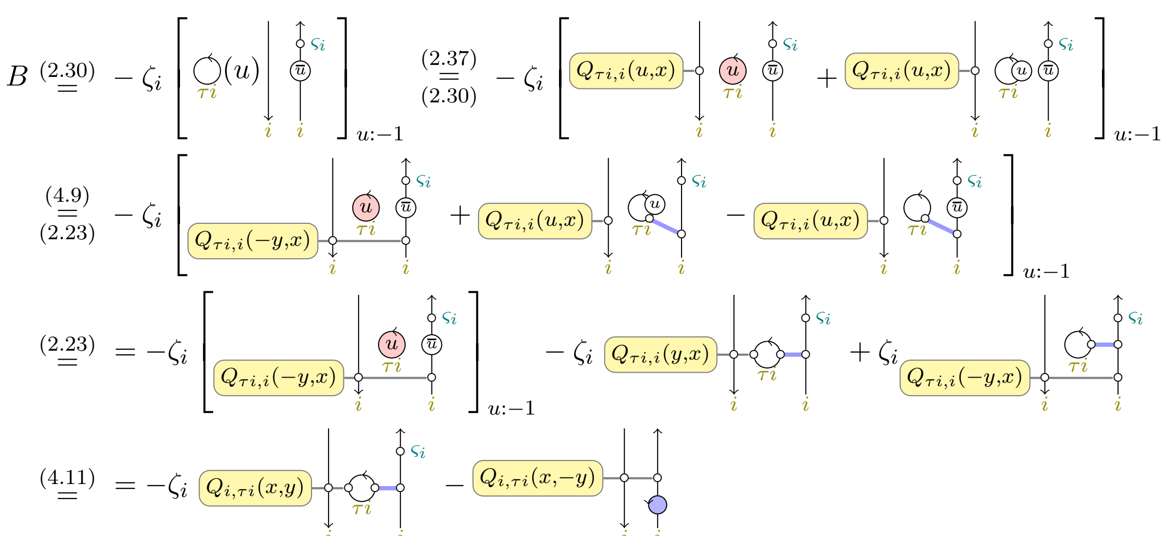


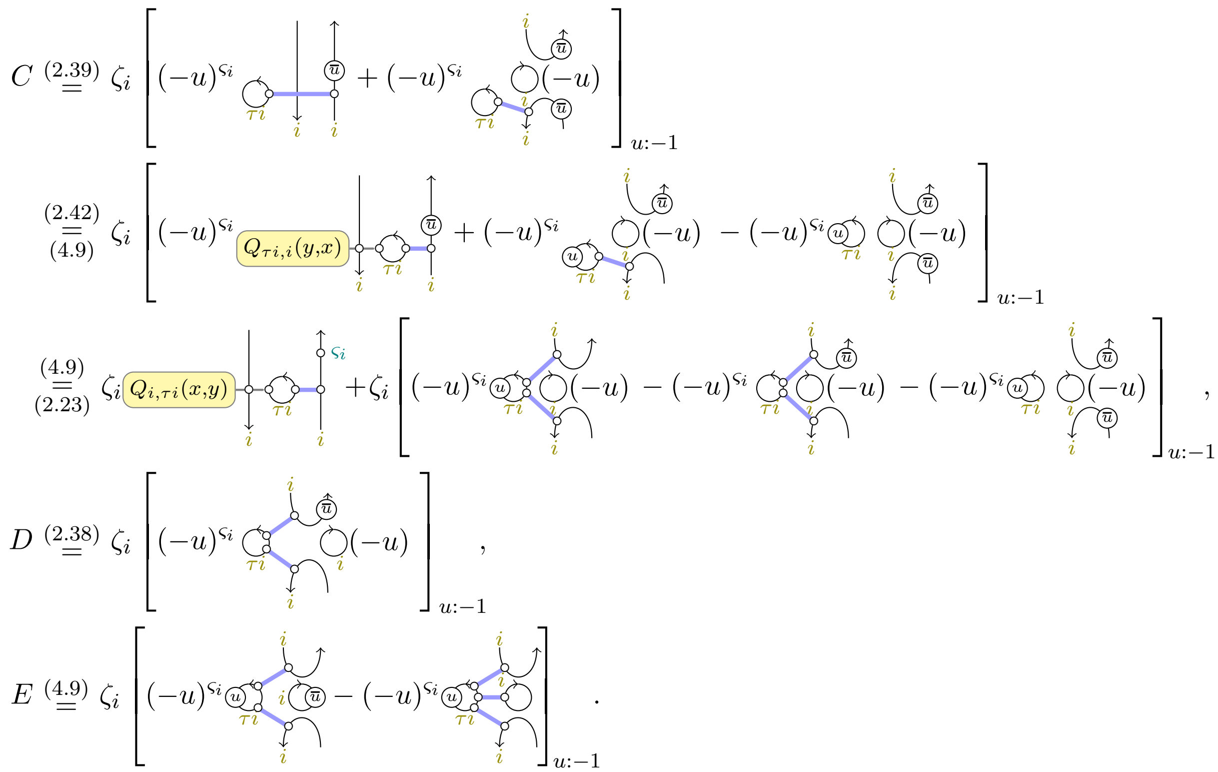
(4.31)

Proof. We start by calculating

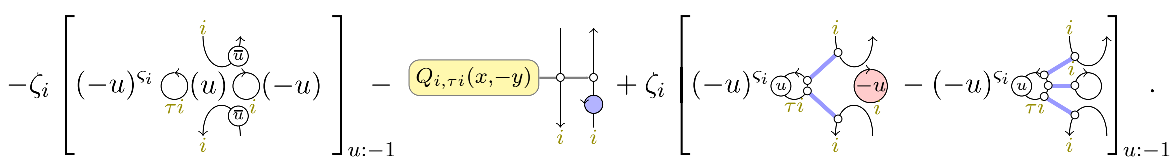


Let us call the four terms in this final expression A, B, C, D and E in order from left to right, so it is $A + B + C + D + E$. We have that

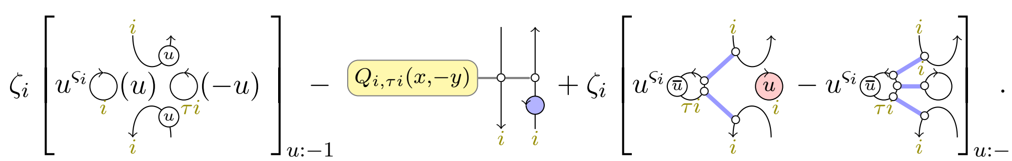




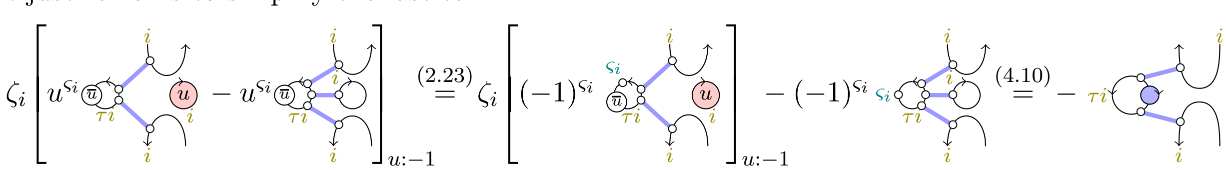
The first term in the expansion of B cancels with the first term in the expansion of C, D cancels with the third term in the expansion of C, the final term in the expansion of C can be combined with A, and by (2.30) the first term of E cancels with some of the second term of C leaving just some fake bubble polynomial. We deduce that $A+B+C+D+E$ equals



Now we replace u by $-u$ (remembering the overall minus sign as we are taking the u^{-1}-coefficient) to deduce that our expression equals



It just remains to simplify the last term:



□

4.3. **Construction of the 2-functor.** Let $\mathfrak{U}^\imath(\varsigma,\zeta)$ be as in Definition 3.4. Now we would like to construct a 2-functor from $\mathfrak{U}^\imath(\varsigma,\zeta)$ to $\underline{\mathfrak{U}}(\varsigma,\zeta)$. However, this does not quite make sense since the object set of $\underline{\mathfrak{U}}(\varsigma,\zeta)$ is X, whereas the object set of $\mathfrak{U}^\imath$ is $X^\imath$. So first we must modify $\underline{\mathfrak{U}}(\varsigma,\zeta)$, contracting its object set along the quotient map $X \twoheadrightarrow X^\imath$. We denote the result by $\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta)$. By definition, this has object set $X^\imath$, and a 1-morphism $G\mathbb{1}_\lambda = \mathbb{1}_\mu G : \lambda \to \mu$ in $\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta)$ is a word G in the generators E_i and F_i whose total weight $\varpi \in X$ obtained by adding α_i for each E_i and $-\alpha_i$ for each F_i satisfies $\mu = \lambda + \varpi$. For two such words $G\mathbb{1}_\lambda$ and $H\mathbb{1}_\lambda$, the 2-morphism space $\mathrm{Hom}_{\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta)}(G\mathbb{1}_\lambda, H\mathbb{1}_\lambda)$ is the graded vector space $\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta)}(G\mathbb{1}_\lambda, H\mathbb{1}_\lambda)_n$ where

$$\mathrm{Hom}_{\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta)}(G\mathbb{1}_\lambda, H\mathbb{1}_\lambda)_n := \prod_{\hat\lambda} \mathrm{Hom}_{\underline{\mathfrak{U}}(\varsigma,\zeta)}(G\mathbb{1}_{\hat\lambda}, H\mathbb{1}_{\hat\lambda})_n$$

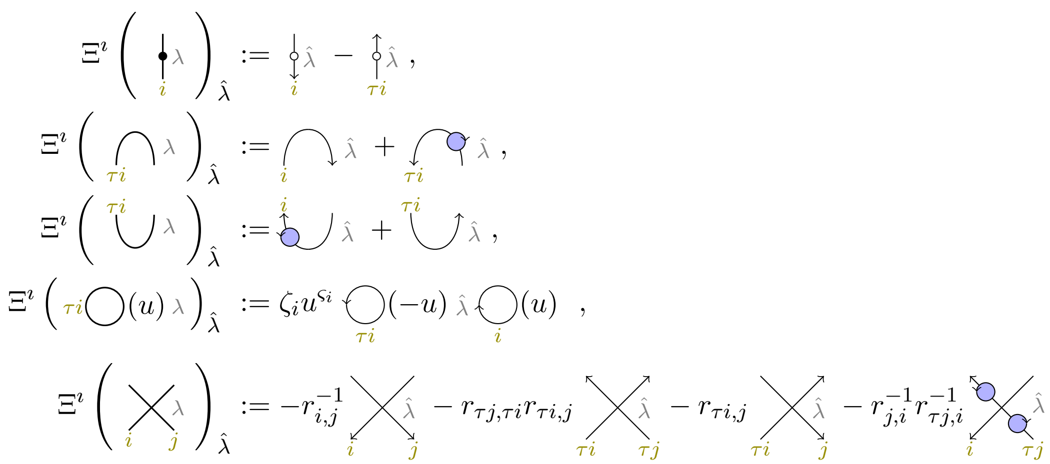
taking the product over all pre-images $\hat\lambda \in X$ of $\lambda \in X^\imath$. Any 2-morphism in this space can be represented as a tuple $f = (f_{\hat\lambda})$ with components indexed by the pre-images $\hat\lambda$ of λ. Horizontal and vertical composition in $\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta)$ are induced by the ones in $\underline{\mathfrak{U}}(\varsigma,\zeta)$ in an obvious way.

Let $\mathrm{Add}(\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta))$ be the additive envelope of $\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta)$. It has the same objects as $\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta)$, but its 1-morphisms are finite direct sums of 1-morphisms in $\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta)$, and 2-morphisms are matrices of 2-morphisms in $\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta)$. Vertical composition of 2-morphisms is by matrix multiplication. In the sequel, we will suppress the matrices, denoting 2-morphisms in $\mathrm{Add}(\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta))$ simply as finite sums of 2-morphisms. The string labels at the top and bottom of each string diagram in such a sum determines the matrix entry that it belongs to in an unambiguous way.

Now we are ready to state and prove the main theorem. This generalizes [BWW24, Th. 4.2] which is essentially the same result for the special case of the nil-Brauer category. The monoidal functor constructed in [BWW24, Th. 4.2] composed with the monoidal automorphism of the nil-Brauer category that negates crossings and dots is equal to the 2-functor here for the split iquantum group of rank one. We will refer this often in the proof of the next theorem since we need all of the calculations made in [BWW24] for strings labelled by $i = \tau i$.

For $i \in I$ with $i \neq \tau i$, we choose $\mathrm{sgn}(i) \in \{\pm 1\}$ such that $\mathrm{sgn}(\tau i) = -\mathrm{sgn}(i)$. In other words, we pick a decomposition $I = I_1 \sqcup I_0 \sqcup I_{-1}$ such that I_1 is a set of representatives for the τ-orbits of size 2, $I_{-1} = \tau(I_1)$, and I_0 consists of the τ-fixed points. Then, for $i \in I$ with $i \neq \tau i$, we define $\mathrm{sgn}(i) := \pm 1$ according to whether $i \in I_{\pm 1}$. These signs are used in an essential way in the next theorem in order to break some symmetry.

Theorem 4.13. *There is a strict graded 2-functor $\Xi^\imath : \mathfrak{U}^\imath(\varsigma,\zeta) \to \mathrm{Add}(\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta))$ which is the identity on objects, takes $B_i\mathbb{1}_\lambda \mapsto F_i\mathbb{1}_\lambda \oplus q_i^{\varsigma_i - h_i(\lambda)} E_{\tau i}\mathbb{1}_\lambda$, and is defined on 2-morphisms by*



(4.32)

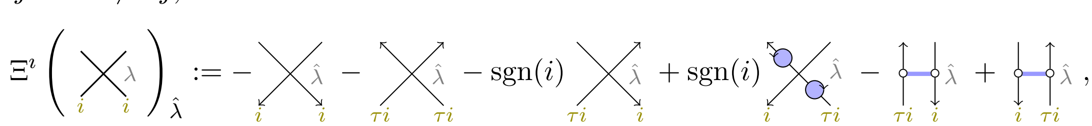
(4.33)

(4.34)

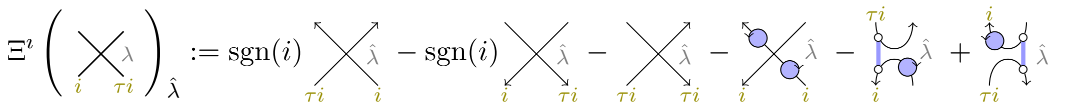
(4.35)

(4.36)

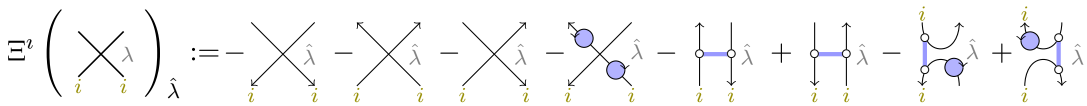
if $i \neq j$ and $i \neq \tau j$,



(4.37)

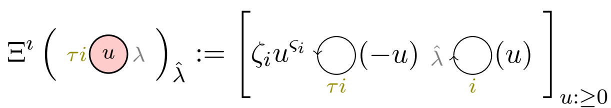
 (4.38)

if $i \neq \tau i$, and

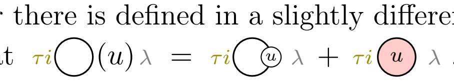
 (4.39)

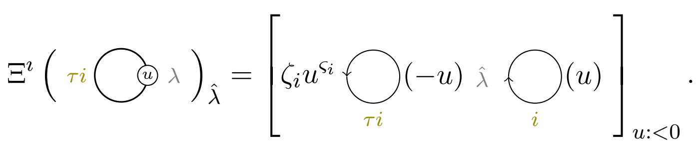
if $i = \tau i$.

Proof. First, one checks that the degrees are consistent. This follows by inspecting Tables 3 and 4 in Subsections 3.2 and 4.1. To construct the functor, we define it on objects and generating 1-morphisms as in the statement of the theorem, and on generating 2-morphisms according to (4.32) to (4.34) and (4.36) to (4.39) together with the following which specifies $\Xi^\imath$ on the fake bubble generators:

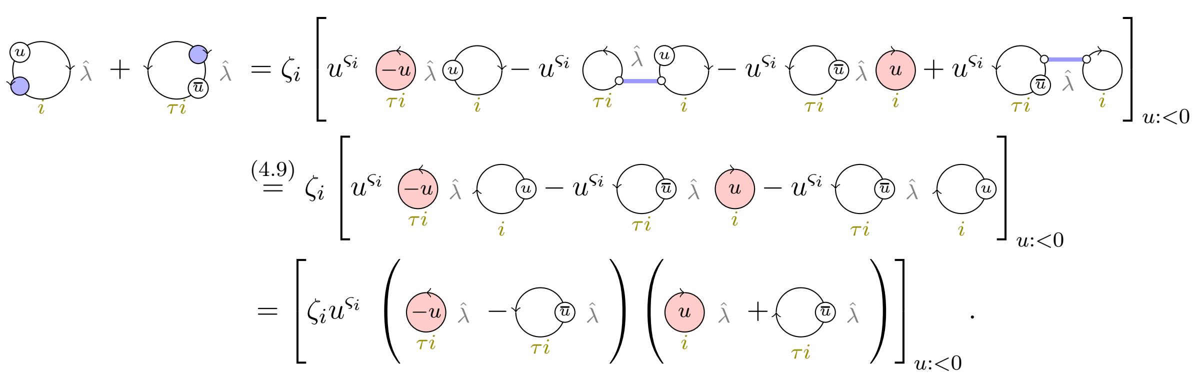
 (4.40)

To show that it is well defined, we need to check that the images of the defining relations of $\mathfrak{U}^\imath(\varsigma, \zeta)$ are satisfied in $\widehat{\underline{\mathfrak{U}}}(\varsigma, \zeta)$. This will take up most of the rest of the proof.

Before we verify any further relations, we check that (4.35) holds. If $i = \tau i$ then it follows from [BWW24, (4.2)]; one needs to compose with the automorphism of the nil-Brauer category that negates dots and crossings because the monoidal functor there is defined in a slightly different way to the 2-functor $\Xi^\imath$ here. Now suppose that $i \neq \tau i$, so that . Since (4.40) is true by the definition of $\Xi^\imath$, to establish (4.35) it suffices to show that



Using Lemma 4.1, the left hand side is



This is equal to the right hand side of the identity we are trying to prove thanks to (2.30).

To complete the proof, it remains to check the defining relations (3.27) to (3.36). This is done in Appendix A. The calculation requires all the relations established in the previous subsection, and we also use Theorem 3.16 when checking the difficult ibraid relation. ☐

4.4. **Special case: categorification of comultiplication.** Let $\mathfrak{U}$ be any 2-quantum group with parameters $(Q_{i,j}(x, y))_{i,j \in I}$ and normalization homomorphisms c_i, leading to $r_{i,j}$ as in (2.18). We also need the 2-quantum group we denote by $\mathfrak{U} \odot \mathfrak{U}$, which is defined from the doubled Cartan datum from Subsection 3.4 using the parameters

$$Q_{i^+,j^+}(x,y) = Q_{i^-,i^-}(x,y) := Q_{i,j}(x,y), \qquad Q_{i^+,j^-}(x,y) = Q_{i^-,j^+}(x,y) := 1, \tag{4.41}$$

$$c_{i^+}\big((\lambda^+,\lambda^-)\big) := c_i(\lambda^+), \qquad c_{i^-}\big((\lambda^+,\lambda^-)\big) := c_i(\lambda^-), \tag{4.42}$$

$$r_{i^+,j^+} = r_{i^-,j^-} := r_{i,j}, \qquad r_{i^+,j^-} = r_{i^-,j^+} := 1. \tag{4.43}$$

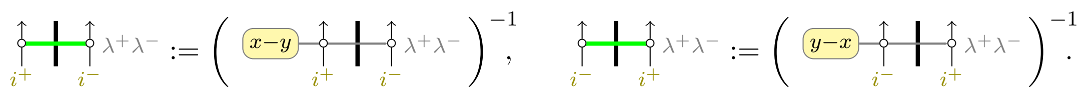
We think of $\mathfrak{U} \odot \mathfrak{U}$ as a commuting pair of copies of $\mathfrak{U}$, with generating 1-morphisms $E_{i^+}\mathbb{1}_{(\lambda^+,\lambda^-)}$, $F_{i^+}\mathbb{1}_{(\lambda^+,\lambda^-)}$, $E_{i^-}\mathbb{1}_{(\lambda^+,\lambda^-)}$ and $F_{i^-}\mathbb{1}_{(\lambda^+,\lambda^-)}$. In fact, we prefer to work with a variant $\mathfrak{U} \circledast \mathfrak{U}$, which is the wide and full sub-2-category of $(\mathfrak{U} \odot \mathfrak{U})_q$ generated by the 1-morphisms

$$(E_i \otimes 1)\mathbb{1}_{(\lambda^+,\lambda^-)} := E_{i^+}\mathbb{1}_{(\lambda^+,\lambda^-)}, \qquad (K_i \otimes E_i)\mathbb{1}_{(\lambda^+,\lambda^-)} := q_i^{h_i(\lambda^+)} E_{i^-}\mathbb{1}_{(\lambda^+,\lambda^-)}, \tag{4.44}$$

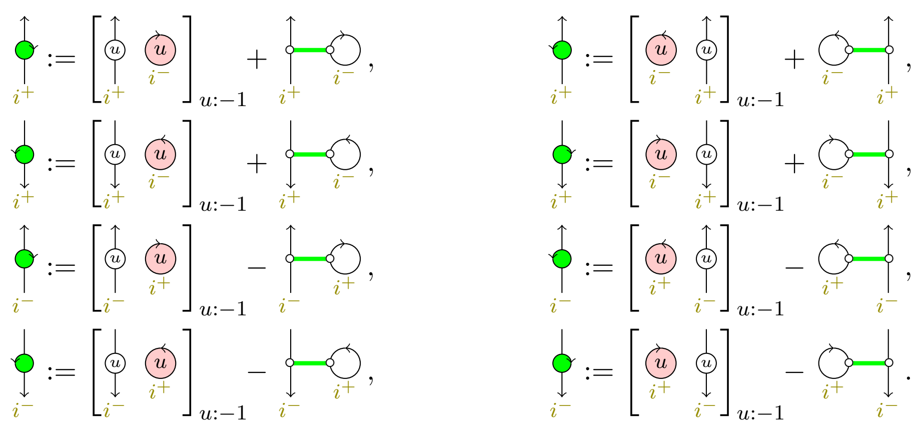
$$(F_i \otimes K_i^{-1})\mathbb{1}_{(\lambda^+,\lambda^-)} := q_i^{-h_i(\lambda^-)} F_{i^+}\mathbb{1}_{(\lambda^+,\lambda^-)}, \qquad (1 \otimes F_i)\mathbb{1}_{(\lambda^+,\lambda^-)} := F_{i^-}\mathbb{1}_{(\lambda^+,\lambda^-)}, \tag{4.45}$$

for all $i \in I$ and $\lambda^+, \lambda^- \in X$, We will be drawing string diagrams to represent 2-morphisms in $\mathfrak{U} \circledast \mathfrak{U}$ rather than in $\mathfrak{U} \odot \mathfrak{U}$, so that upward strings labelled i^+ and i^- denote the identity 2-endomorphisms of $E_i \otimes 1$ and $K_i \otimes E_i$, and downward strings labelled i^+ and i^- denote the identity 2-endomorphisms of $F_i \otimes K_i^{-1}$ and $1 \otimes F_i$, respectively. This just means that we are using another grading convention for string diagrams. In the ungrading setting, $\mathfrak{U} \circledast \mathfrak{U}$ and $\mathfrak{U} \odot \mathfrak{U}$ are the same $\Bbbk$-linear 2-category.

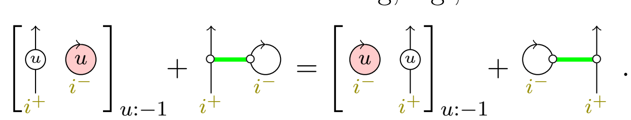
Next, we pass to a localization $\mathfrak{U} \underline{\circledast} \mathfrak{U}$ of $\mathfrak{U} \circledast \mathfrak{U}$ by adjoining two-sided inverses to all 2-morphisms given by a dot on an upward string labelled i^+ *minus* a dot on an upward string labelled i^-. We denote these two-sided inverses by


(4.46)

They are similar to the teleporters defined earlier in the section but with a different sign (which is why we are using a new color). As we did before, we extend the notation so that we can also attach teleporters to downward strings. Then we define internal bubbles in $\mathfrak{U} \underline{\circledast} \mathfrak{U}$ by


(4.47)
(4.48)
(4.49)
(4.50)

These have similar properties to the internal bubbles defined before. For example, the clockwise and counterclockwise internal bubbles are two-sided inverses. Note also that the bubbles in the above definitions can be drawn on the other side of the string, e.g., we have that



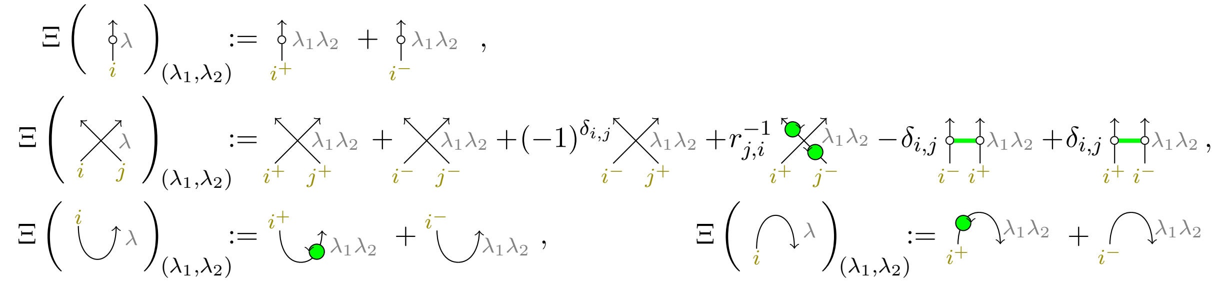
These statements follow from Lemmas 4.3 and 4.4 since the new internal bubbles are images of (a special case of) the old ones under an automorphism $\Omega \otimes I$; see the last sentence of the proof of the next theorem. We also point out that the formulae defining these internal bubbles are the same as the formulae defining analogous morphisms in the Heisenberg category from [BSW23, (5.27)–(5.28)] and they appeared already for $\mathfrak{sl}_2$ in [Web24, (6.3)–(6.4)].

Finally, we contract the object set of $\mathfrak{U}\underline{\circledast}\mathfrak{U}$ along the quotient map $\boldsymbol{X} \twoheadrightarrow X, (\lambda^+, \lambda^-) \mapsto \lambda^+ + \lambda^-$ to obtain a graded 2-category $\mathfrak{U}\,\widehat{\underline{\circledast}}\,\mathfrak{U}$. Its object set is X. A 1-morphism $G\mathbb{1}_\lambda = \mathbb{1}_\mu G : \lambda \to \mu$ in $\mathfrak{U}\,\widehat{\underline{\circledast}}\,\mathfrak{U}$ for G that is a monomial in $E_i \otimes 1, K_i \otimes E_i, F_i \otimes K_i^{-1}, 1 \otimes F_i$ $(i \in I)$ whose total weight obtained by adding α_i for each $E_i \otimes 1$ or $K_i \otimes E_i$ and $-\alpha_i$ for each $F_i \otimes K_i^{-1}$ or $1 \otimes F_i$ is equal to $\mu - \lambda$. Then

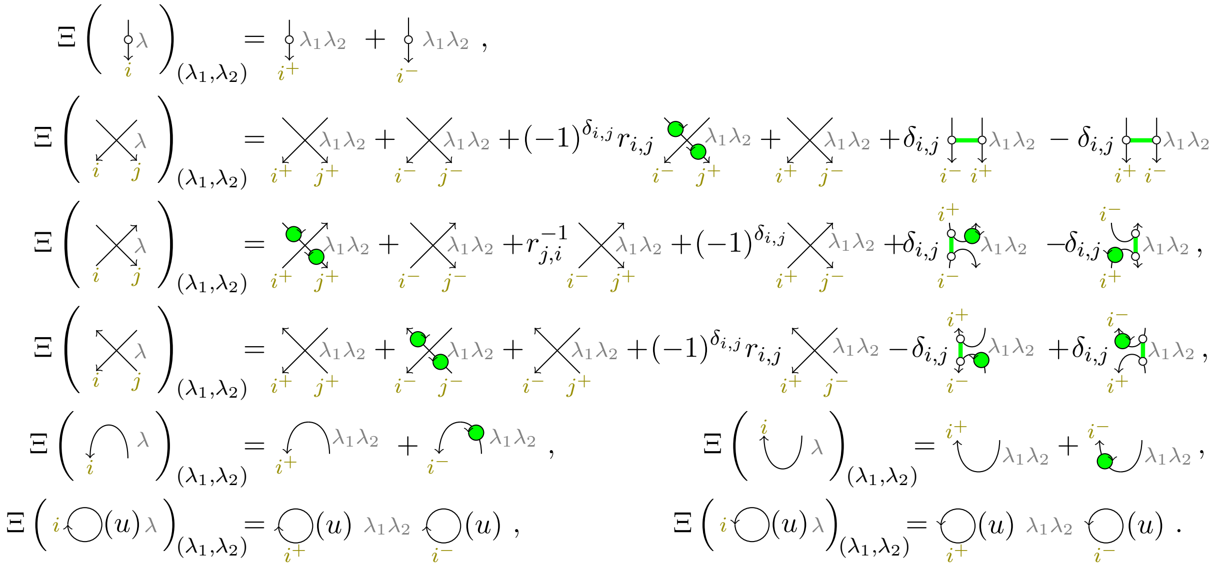
$$\mathrm{Hom}_{\mathfrak{U}\,\widehat{\underline{\circledast}}\,\mathfrak{U}}(G\mathbb{1}_\lambda, H\mathbb{1}_\lambda)_n := \prod_{\lambda^+ + \lambda^- = \lambda} \mathrm{Hom}_{\mathfrak{U}\underline{\circledast}\mathfrak{U}}\left(G\mathbb{1}_{(\lambda^+,\lambda^-)}, H\mathbb{1}_{(\lambda^+,\lambda^-)}\right)_n.$$

Now we can state a reformulation of Theorem 4.13 in this special case. The 2-functor Ξ that it defines is a categorical version of the comultiplication $\Delta : \mathrm{U} \to \mathrm{U} \otimes_{\mathbb{Q}(q)} \mathrm{U}$. The devoted reader could reprove this theorem from scratch by checking relations directly—it is easier than the proof of Theorem 4.13 but not by much.

Theorem 4.14. *There is a unique strict graded 2-functor* $\Xi : \mathfrak{U} \to \mathrm{Add}(\mathfrak{U}\,\widehat{\underline{\circledast}}\,\mathfrak{U})$ *which is the identity on objects, takes* $E_i\mathbb{1}_\lambda \mapsto (E_i \otimes 1)\mathbb{1}_\lambda \oplus (K_i \otimes E_i)\mathbb{1}_\lambda$ *and* $F_i\mathbb{1}_\lambda \mapsto (1 \otimes F_i)\mathbb{1}_\lambda \oplus (F_i \otimes K_i^{-1})\mathbb{1}_\lambda$*, and is defined on 2-morphisms by*



for $i, j \in I$ *and* $\lambda^+, \lambda^- \in X$ *with* $\lambda^+ + \lambda^- = \lambda$*. We also have that*



Proof. We will deduce this from Theorem 4.13 by composing with some isomorphisms, paralleling the identity $\Delta = (\omega \otimes \mathrm{id}) \circ d$ for the underlying quantum groups. First, we need the isomorphism

$$D : \mathfrak{U} \xrightarrow{\sim} \boldsymbol{\mathfrak{U}}^\imath$$

from Theorem 3.7 from the 2-quantum group $\mathfrak{U}$ to the quasi-split 2-iquantum $\boldsymbol{\mathfrak{U}}^\imath$ of diagonal type. Let $\boldsymbol{\mathfrak{U}}$ be the 2-quantum group associated to the doubled Cartan datum with parameters as in (3.49) to (3.53). Theorem 4.13 in this special case gives us a strict graded 2-functor

$$\boldsymbol{\Xi}^\imath : \boldsymbol{\mathfrak{U}}^\imath \to \widehat{\underline{\boldsymbol{\mathfrak{U}}}}(\varsigma, \zeta),$$

where $\boldsymbol{\mathfrak{U}}(\varsigma,\zeta)$ is the version of $\boldsymbol{\mathfrak{U}}$ with adapted grading from Subsection 4.1, and ς, ζ are as in (3.54) and (3.55). Finally, we need the isomorphism of graded 2-categories

$$\Omega \otimes J : \boldsymbol{\mathfrak{U}} \to \mathfrak{U} \odot \mathfrak{U} \tag{4.51}$$

defined on objects by $(\lambda^+,\lambda^-) \mapsto (-\lambda^+,\lambda^-)$, and on 1-morphisms by

$$E_{i^+}\mathbb{1}_{(\lambda^+,\lambda^-)} \mapsto F_{i^+}\mathbb{1}_{(-\lambda^+,\lambda^-)}, \qquad F_{i^+}\mathbb{1}_{(\lambda^+,\lambda^-)} \mapsto E_{i^+}\mathbb{1}_{(-\lambda^+,\lambda^-)},$$
$$E_{i^-}\mathbb{1}_{(\lambda^+,\lambda^-)} \mapsto E_{i^-}\mathbb{1}_{(-\lambda^+,\lambda^-)}, \qquad F_{i^-}\mathbb{1}_{(\lambda^+,\lambda^-)} \mapsto F_{i^-}\mathbb{1}_{(-\lambda^+,\lambda^-)}.$$

On a string diagram representing a 2-morphism, the 2-functor $\Omega \otimes J$ applies the map $(\lambda^+,\lambda^-) \mapsto (-\lambda^+,\lambda^-)$ to 2-cell labels, reverses the orientation of all strings labelled by I^+, and multiplies by

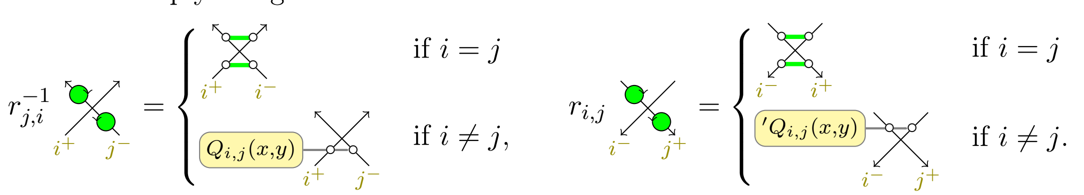
- -1 for each i^-, i^-, i^- or i^- for $i \in I$,
- $-r_{i,j}$ for each crossing of the form $i^+\ j^+$, $j^+\ i^+$, $i^-\ j^-$ or $j^-\ i^-$ for $i, j \in I$, and
- $-r_{i,j}^{-1}$ for each crossing of the form $i^+\ j^+$, $j^+\ i^+$, $j^-\ i^-$ or $i^-\ j^-$ for $i, j \in I$

appearing in the original string diagram. In fact, the 2-category $\boldsymbol{\mathfrak{U}}$ can be viewed as combining copies of $\mathfrak{U}$ and of $\bar{\mathfrak{U}}$ which commute with each other; from this perspective, the automorphism $\Omega \otimes J$ agrees with Ω from (2.47) on the copy of $\mathfrak{U}$, and it is the isomorphism J from (2.49) on the copy of $\bar{\mathfrak{U}}$. The 2-category $\mathfrak{U} \circledast \mathfrak{U}$ is the image of $\boldsymbol{\mathfrak{U}}(\varsigma,\zeta)$ under the canonical extension $\Omega \otimes J : \boldsymbol{\mathfrak{U}}_q \to (\mathfrak{U} \odot \mathfrak{U})_q$ to the q-envelopes, with the 1-morphisms (4.44) and (4.45) being the images of $F_{i^+}\mathbb{1}_{(-\lambda^+,\lambda^-)}$, $q_i^{h_i(\lambda^+)}E_{i^-}\mathbb{1}_{(-\lambda^+,\lambda^-)}$, $q_i^{-h_i(\lambda^-)}E_{i^+}\mathbb{1}_{(-\lambda^+,\lambda^-)}$, and $F_{i^-}\mathbb{1}_{(-\lambda^+,\lambda^-)}$, respectively. This functor also extends further to additive envelopes. Now everything is set up, and we can simply define

$$\Xi := (\Omega \otimes I) \circ \boldsymbol{\Xi}^\imath \circ D : \mathfrak{U} \to \mathrm{Add}(\mathfrak{U} \mathbin{\widehat{\circledast}} \mathfrak{U}). \tag{4.52}$$

It remains to calculate the images of the generating 2-morphisms to obtain the formulas in the statement of the theorem, using the observation that the new internal bubbles (4.47) to (4.50) are the images under $\Omega \otimes I$ of the internal bubbles i^+, i^+, i^+, i^+, $-i^-$, $-i^-$, $-i^-$, $-i^-$, respectively. $\square$

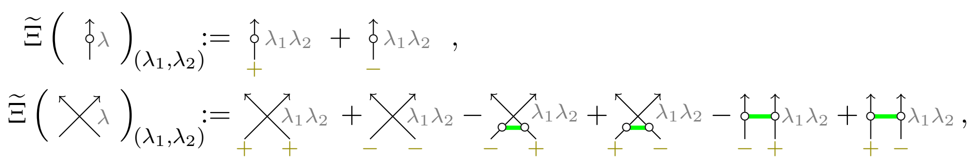
Remark 4.15. The upward and downward crossings with internal bubbles in Theorem 4.14 can be written more simply using the observation that



(4.53)

This follows using Corollary 4.7 (actually, its image under $\Omega \otimes I$).

Remark 4.16. In the $\mathfrak{sl}_2$ case, a very similar 2-functor to Ξ was constructed in [Web24, Th. 3.1]. Denoting the one in [Web24] by $\widetilde{\Xi}$ and using $+$ and $-$ instead of the red and blue strings there, it is defined by



However, unlike Ξ, $\widetilde{\Xi}$ does not preserve degrees of 2-morphisms. Although rightward cups and caps are simpler, the leftward ones are more complicated:

For general $\mathfrak{g}$, there is also another 2-functor of a similar nature constructed in [Web24, Lem. 3.5] whose codomain is another 2-quantum group obtained by "unfurling" the Cartan datum.

5. Non-degeneracy and related combinatorics

For any symmetrizable Cartan datum and any choice of parameters, the 2-quantum group $\mathfrak{U}$ is non-degenerate, that is, the 2-morphism spaces in $\mathfrak{U}$ have explicit diagrammatic bases. This was conjectured originally in [KL10], where it was proved for $\mathfrak{sl}_n$. The conjecture was subsequently proved first for all finite types, and then in general in [Web24, Th. 3.6]. By [KL10, Th. 2.7, Prop. 3.12], the validity of the Non-Degeneracy Conjecture implies that graded ranks of 2-morphism spaces in $\mathfrak{U}$ can be computed using Lusztig's symmetric bilinear form

$$(\cdot,\cdot) : \dot{\mathrm{U}}_{\mathbb{Z}} \times \dot{\mathrm{U}}_{\mathbb{Z}} \to \mathbb{Z}[q, q^{-1}] \tag{5.1}$$

from [Lus10, Th. 26.1.2] (we recall its definition in Remark 5.14). In this section, we are going to use the 2-functor $\Xi^\imath$ from Theorem 4.13 to deduce analogous results for 2-iquantum groups with geometric parameters.

5.1. **Notation for words and monomials.** Let $\langle I\rangle$ be the set of words $\boldsymbol{i} = i_1 \cdots i_l$ for $l \geq 0$ and $i_1, \dots, i_l \in I$. We denote the length l of the word by $l(\boldsymbol{i})$ and let $\mathrm{wt}^\imath(\boldsymbol{i})$ be the image of $\alpha_{i_1} + \cdots + \alpha_{i_l}$ in $X^\imath$. Words of this form will be used to index various monomials:

$$b_{\boldsymbol{i}} := b_{i_1} \cdots b_{i_l}, \qquad B_{\boldsymbol{i}} := B_{i_1} \cdots B_{i_l}, \tag{5.2}$$

$$\theta_{\boldsymbol{i}} := \theta_{i_1} \cdots \theta_{i_l}, \qquad \Theta_{\boldsymbol{i}} := \Theta_{i_1} \otimes \cdots \otimes \Theta_{i_l}. \tag{5.3}$$

The first of these is an element of $\mathrm{U}^\imath$. The second has no meaning by itself but, given an iweight $\lambda \in X^\imath$, the notation $B_{\boldsymbol{i}}\mathbb{1}_\lambda$ denotes the 1-morphism in the 2-iquantum group $\mathfrak{U}^\imath$ that is the horizontal composition of the generating 1-morphisms according to the sequence. The third is a monomial in Lusztig's algebra $\mathbf{f}$, which will be introduced in Subsection 5.4. The fourth denotes an object in a certan quiver Hecke category which is TBD ("to be defined"—see Definition 6.2) but it seems convenient to include the definition $\Theta_{\boldsymbol{i}}$ here already.

More generally, we will eventually need *divided power words* of the form $\boldsymbol{i} = i_1^{(n_1)} \cdots i_l^{(n_l)}$ for $l \geq 0, i_1, \dots, i_l \in I$ and $n_1, \dots, n_l \geq 1$. We denote the set of all such by $\langle\!\langle I\rangle\!\rangle$. The set $\langle I\rangle$ is a subset of $\langle\!\langle I\rangle\!\rangle$ in the obvious way. For such a divided power word, we define $\mathrm{wt}^\imath(\boldsymbol{i})$ to be the image of $n_1\alpha_{i_1} + \cdots + n_l\alpha_{i_l}$ in $X^\imath$, and let

$$b_{\boldsymbol{i}} := b_{i_1}^{(n_1)} \cdots b_{i_l}^{(n_l)}, \qquad B_{\boldsymbol{i}} := B_{i_1}^{(n_1)} \cdots B_{i_l}^{(n_l)}, \tag{5.4}$$

$$\theta_{\boldsymbol{i}} := \theta_{i_1}^{(n_1)} \cdots \theta_{i_l}^{(n_l)}, \qquad \Theta_{\boldsymbol{i}} := \Theta_{i_1}^{(n_1)} \otimes \cdots \otimes \Theta_{i_l}^{(n_l)}. \tag{5.5}$$

Given $\boldsymbol{i} \in \langle\!\langle I\rangle\!\rangle$ and an iweight λ, the notation $b_{\boldsymbol{i}}1_\lambda$ denotes the corresponding product of divided/idivided powers in $\dot{\mathfrak{U}}^\imath$ from (3.6). The other monomial introduced in (5.4) involves $B_i^{(n)}$ which is TBD. The monomials (5.5) involve $\theta_i^{(n)} := \theta_i^n/[n]^!_{q_i}$ and $\Theta_i^{(n)}$ which is TBD (see Subsection 6.2).

There is also the set $\langle I^+, I^-\rangle$ of words in the alphabet $\boldsymbol{I} = I^+ \sqcup I^- = \{i^+, i^- \mid i \in I\}$. We define the weight $\mathrm{wt}(\boldsymbol{i}) \in X$ of such a word to be the sum of the weights of its letters, with the understanding that $\mathrm{wt}(i^+) = \alpha_i$ and $\mathrm{wt}(i^-) = -\alpha_i$. We let $g_{i^+} := e_i, g_{i^-} := f_i, G_{i^+} := E_i$ and $G_{i^-} := F_i$, then for $\boldsymbol{i} = i_1 \cdots i_l \in \langle I^+, I^- \rangle$ we define

$$g_{\boldsymbol{i}} := g_{i_1} \cdots g_{i_l}, \qquad\qquad G_{\boldsymbol{i}} := G_{i_1} \cdots G_{i_l}. \tag{5.6}$$

The monomial $g_{\boldsymbol{i}}$ is an element of U, and $G_{\boldsymbol{i}}\mathbb{1}_\lambda$ is a 1-morphism in $\mathfrak{U}$ for any $\lambda \in X$. One could also introduce the set $\langle\langle I^+, I^- \rangle\rangle$ of divided power words which index monomials in divided powers of e_i, f_i or E_i, F_i, but actually we will not need these subsequently.

5.2. **Non-degeneracy of 2-quantum groups.** We begin by giving a self-contained proof of the non-degeneracy of 2-quantum groups for all symmetrizable Kac-Moody types. Although similar in spirit to the proof given in [Web24], we will prove it using the comultiplication 2-functor Ξ from Theorem 4.14 instead of the unfurling 2-functor from [Web24, Lem. 3.5]. Let $\mathfrak{U}$ be as in Subsection 2.2. We will forget the grading on $\mathfrak{U}$, viewing it as a $\Bbbk$-linear 2-category.

A *2-representation* $\mathbf{R}$ of $\mathfrak{U}$ is the data of a strict $\Bbbk$-linear 2-functor from $\mathfrak{U}$ to the 2-category of $\Bbbk$-linear categories. It means that we are given $\Bbbk$-linear categories $\mathbb{1}_\lambda\mathbf{R}$ for each $\lambda \in X$ and $\Bbbk$-linear functors $E_i : \mathbb{1}_\lambda\mathbf{R} \to \mathbb{1}_{\lambda+\alpha_i}\mathbf{R}$ and $F_i : \mathbb{1}_\lambda\mathbf{R} \to \mathbb{1}_{\lambda-\alpha_i}\mathbf{R}$ for each $\lambda \in X$ and $i \in I$. Moreover, every 2-morphism in $\mathfrak{U}$ induces a natural transformation between the appropriate compositions of these functors in a way that respects horizontal and vertical composition. We abuse notation by writing $\mathbf{R}$ also for the $\Bbbk$-linear category $\coprod_{\lambda\in X} \mathbb{1}_\lambda\mathbf{R}$ or, if we are in a setting in which each $\mathbb{1}_\lambda\mathbf{R}$ is additive, for the additive $\Bbbk$-linear category $\bigoplus_{\lambda\in X} \mathbb{1}_\lambda\mathbf{R}$. Then we refer to $\mathbb{1}_\lambda\mathbf{R}$ as the *λ-weight subcategory.*

There is a natural notion of morphism $M : \mathbf{R} \to \mathbf{S}$ between two 2-representations. It is the data of a $\Bbbk$-linear functor which restricts to $\Bbbk$-linear functors $M_\lambda : \mathbb{1}_\lambda\mathbf{R} \to \mathbb{1}_\lambda\mathbf{S}$ for each $\lambda \in X$, plus natural isomorphisms $E_i^{\mathbf{S}} \circ M_\lambda \cong M_{\lambda+\alpha_i} \circ E_i^{\mathbf{R}}$ and $F_i^{\mathbf{S}} \circ M_\lambda \cong M_{\lambda-\alpha_i} \circ F_i^{\mathbf{R}}$ which are compatible with the natural transformations arising from 2-morphisms in $\mathfrak{U}$; see [BD17, Def. 4.6] for more details. We use the term *equivariant functor* rather than "morphism of 2-representations", and call it an *equivariant equivalence* if each M_λ is an equivalence of categories.

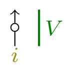
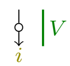
One can also take some commutative $\Bbbk$-algebra $\mathbb{K}$ and consider $\mathbb{K}$-*linear 2-representations* and $\mathbb{K}$-*linear equivariant functors* between them. These are just 2-representations and equivariant functors as defined above for the base change $\mathfrak{U} \otimes_\Bbbk \mathbb{K}$. We say that $\mathbf{R}$ is a $\mathbb{K}$-*finite 2-representation* to indicate that the $\Bbbk$-algebra $\mathbb{K}$ is an algebraically closed field, $\mathbf{R}$ is a $\mathbb{K}$-linear 2-representation, and all morphism spaces of $\mathbf{R}$ are finite-dimensional as vector spaces over $\mathbb{K}$.

Assume from now on that $\mathbf{R}$ is a $\mathbb{K}$-finite 2-representation. There is a canonically induced structure of $\mathbb{K}$-finite 2-representation on the additive Karoubi envelope (=the idempotent completion of the additive envelope) of $\mathbf{R}$. The Yoneda embedding induces a contravariant equivariant between this and another $\mathbb{K}$-finite 2-representation we denote by $\mathbf{R}$-proj, whose λ-weight subcategory is the category $\mathbb{1}_\lambda\mathbf{R}$-proj of finitely generated projective left $\mathbb{1}_\lambda\mathbf{R}$-modules. We will work here with the latter construction, the advantage being that $\mathbb{1}_\lambda\mathbf{R}$-proj is a full subcategory of the *Abelian* category $\mathbb{1}_\lambda\mathbf{R}$-mod of locally finite-dimensional left $\mathbb{1}_\lambda\mathbf{R}$-modules, that is, the category of $\mathbb{K}$-linear functors from $\mathbb{1}_\lambda\mathbf{R}$ to the category of finite-dimensional vector spaces over $\mathbb{K}$. A disadvantage[4] is that the Yoneda equivalence being used is contravariant, so that an application of the anti-involution Ψ from (2.46) is needed in the construction the required natural transformations.

One can think about $\mathbb{1}_\lambda\mathbf{R}$-mod in the more traditional language of modules by passing to the path algebra of $\mathbb{1}_\lambda\mathbf{R}$. This is spelled out in [BD17], where Abelian categories of this sort are called *Schurian categories.* In a Schurian category, the endomorphism algebra of any finitely generated object is finite-dimensional, and the endomorphism algebra of an irreducible object is one-dimensional. Also, it makes sense to talk about composition multiplicities of any object, which are finite. However, objects (even finitely generated ones) are not necessarily of finite length, and morphism spaces between objects that are not finitely generated can be infinite-dimensional.

[4]As usual, one could avoid this by working with right modules, but that creates its own problems in a different place—one has to reverse the tensor product of bimodules.

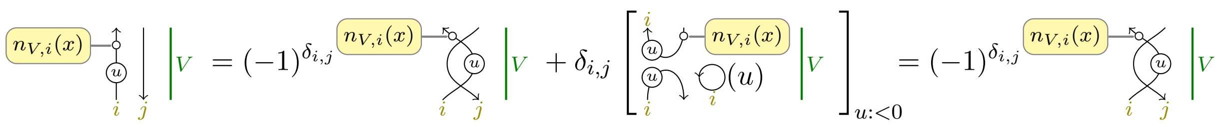
Let V be a finitely generated $\mathbb{1}_\lambda\mathbf{R}$-module for $\lambda \in X$. Since $\mathrm{End}_{\mathbb{1}_\lambda\mathbf{R}}(V)$ is finite-dimensional, we can talk about the (monic) minimal polynomial $m_f(x) \in \mathbb{K}[x]$ of any morphism $f : V \to V$. By adjunction properties, both of E_iV and F_iV are finitely generated, so it makes sense to define $m_{V,i}(x)$ and $n_{V,i}(x)$ be the minimal polynomials of

 $: E_iV \to E_iV,$  $: F_iV \to F_iV,$ (5.7)

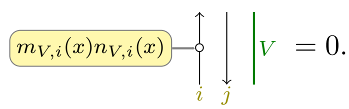
respectively. The *spectrum* of $\mathbf{R}$ is the set of roots of the minimal polynomials $m_{V,i}(x)$ for all $i \in I$, all finitely generated $\mathbb{1}_\lambda\mathbf{R}$-modules V and all $\lambda \in X$. Equivalently, by an argument with adjunctions, it is the set of roots of the minimal polynomials $n_{V,i}(x)$ for all i and V.

Lemma 5.1. *Let V be a finitely generated left $\mathbb{1}_\lambda\mathbf{R}$-module and $i,j \in I$. All roots of the minimal polynomials $m_{E_jV,i}(x), m_{F_jV,i}$, $n_{E_jV,i}(x)$ and $n_{F_jV,i}(x)$ are roots of $m_{V,i}(x)$ or $n_{V,i}(x)$ or, when $j \neq i$, $Q_{i,j}(x,\beta)$ for roots β of $m_{V,j}(x)$ or $n_{V,j}(x)$.*

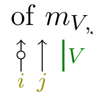
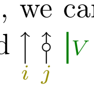
Proof. First, we look at $m_{F_jV,i}(x)$. From (2.39), we have that

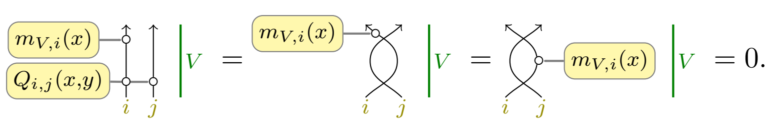


Multiplying by $m_{V,i}(u)$ and taking the u^{-1}-coefficient using (2.23) gives that

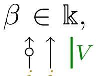
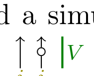


It follows that $m_{F_jV,i}(x)$ divides $m_{V,i}(x)n_{V,i}(x)$. This shows that roots of $m_{F_jV,i}(x)$ are roots of $m_{V,i}(x)$ or $n_{V,i}(x)$. A similar argument shows that roots of $n_{E_jV,i}(x)$ are roots of $m_{V,i}(x)$ or $n_{V,i}(x)$.

Next suppose that α is a root of $m_{E_jV,i}(x)$ for $j \neq i$. For some root β of $m_{V,j}(x)$, we can find a simultaneous eigenvector $v \in E_iE_jV$ for the commuting endomorphisms  and  with eigenvalues α and β, respectively. We have that



Acting on v, we deduce that $m_{V,i}(\alpha)Q_{i,j}(\alpha,\beta) = 0$. So α is a root of $m_{V,i}(x)Q_{i,j}(x,\beta)$. This shows for $j \neq i$ that roots of $m_{E_jV,i}(x)$ are roots of $m_{V,i}(x)$ or $Q_{i,j}(x,\beta)$ for roots β of $m_{V,j}(x)$. A similar argument shows that roots of $n_{F_jV,i}(x)$ are roots of $n_{V,i}(x)$ or $Q_{i,j}(x,\beta)$ for roots β of $n_{V,j}(x)$, again assuming $j \neq i$.

Finally, we show that any root of $m_{E_iV,i}(x)$ is a root of $m_{V,i}(x)$. A similar argument shows that any root of $n_{F_iV,i}(x)$ is a root of $n_{V,i}(x)$ to finish the proof of the lemma. Let α be a root of $m_{E_iV,i}(x)$. For some $\beta \in \Bbbk$, we can find a simultaneous eigenvector $v \in E_i^2V$ for the commuting endomorphisms $x :=$  and $y :=$  with eigenvalues α and β, respectively. Also let $s :=$ . We have that $ysv = sxv - v = \alpha sv - v$. If sv is a multiple of v then $sv = 0$ since $s^2 = 0$, hence, the identity in the previous sentence gives that $v = 0$. This is not the case. So v and sv are linearly independent. On the subspace of E_i^2V spanned by v and sv, the matrix of y is $\left(\begin{smallmatrix} \beta & -1 \\ 0 & \alpha \end{smallmatrix}\right)$. This shows that α is an eigenvalue of y, that is, it is a root of $m_{V,i}(x)$. $\square$

Consider the $\Bbbk$-linear 2-category $\mathfrak{U} \circledast \mathfrak{U}$ from Subsection 4.4 with weight lattice $X \oplus X$. We are ignoring the grading, so it is just the same as the 2-category $\mathfrak{U} \odot \mathfrak{U}$ generated by a commuting pair of copies of $\mathfrak{U}$. Also recall the 2-categories $\mathfrak{U} \underline{\circledast} \mathfrak{U}$ and $\mathfrak{U} \widehat{\underline{\circledast}} \mathfrak{U}$. If $\mathbf{R}$ and $\mathbf{S}$ are $\mathbb{K}$-finite 2-representations of $\mathfrak{U}$, there is a $\mathbb{K}$-finite 2-representation $\mathbf{R} \boxtimes \mathbf{S}$ of $\mathfrak{U} \circledast \mathfrak{U}$ defined by letting $\mathbb{1}_{(\lambda^+,\lambda^-)}(\mathbf{R} \boxtimes \mathbf{S})$ be the $\mathbb{K}$-linearized Cartesian product $(\mathbb{1}_{\lambda^+}\mathbf{R}) \boxtimes (\mathbb{1}_{\lambda^-}\mathbf{S})$. The functors E_{i^+} and F_{i^+} are $E_i \boxtimes \mathrm{id}$ and $F_i \boxtimes \mathrm{id}$, while E_{j^-} and F_{j^-} are $\mathrm{id} \boxtimes E_j$ and $\mathrm{id} \boxtimes F_j$, which commute strictly with E_{i^+} and F_{i^+}. Crossings with one string labelled i^+ ($i \in I$) and the other labelled j^- ($j \in I$) act as identity natural transformations. The natural transformations defining the actions of the other generating 2-morphisms of $\mathfrak{U} \circledast \mathfrak{U}$ are the obvious ones from the actions of $\mathfrak{U}$ on $\mathbf{R}$ and $\mathbf{S}$.

Suppose in addition that $\mathbf{R}$ and $\mathbf{S}$ have disjoint spectra. Then the action of $\mathfrak{U} \circledast \mathfrak{U}$ on $\mathbf{R} \boxtimes \mathbf{S}$ extends to an action of the localization $\mathfrak{U} \underline{\circledast} \mathfrak{U}$. Using the 2-functor Ξ from Theorem 4.14, we obtain from this a $\mathbb{K}$-linear 2-representation $\mathbf{R} \otimes \mathbf{S}$ of $\mathfrak{U}$ itself with

$$\mathbb{1}_\lambda(\mathbf{R} \otimes \mathbf{S}) := \bigoplus_{\lambda^+ + \lambda^- = \lambda} (\mathbb{1}_{\lambda^+}\mathbf{R}) \boxtimes (\mathbb{1}_{\lambda^-}\mathbf{S}).$$

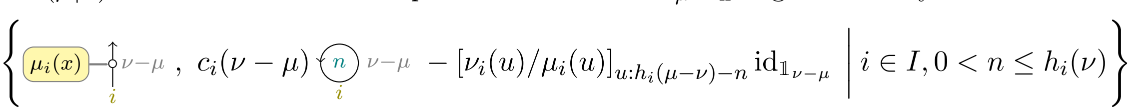
It might not be $\mathbb{K}$-finite.

For this construction to be useful, we need a supply of $\mathbb{K}$-finite 2-representations. For each $\lambda \in X$, there is a corresponding *left regular 2-representation* $\mathfrak{U}\mathbb{1}_\lambda$ of $\mathfrak{U}$. This is the 2-representation with $\mathbb{1}_\kappa \mathfrak{U} \mathbb{1}_\lambda := \mathbf{Hom}_{\mathfrak{U}}(\lambda, \kappa)$. The functors E_i and F_i, and the natural transformations associated to 2-morphisms, are defined by composing horizontally on the left. The *generalized cyclotomic quotients* (GCQ for short) in the next definition first appeared in [Web15, Prop. 5.6]. The *cyclotomic quotients*, which are a special case, have a much longer history.

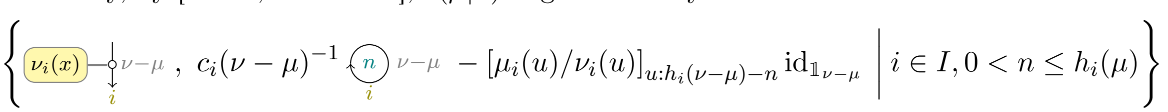
Definition 5.2. Suppose that we are given a pair $\mu, \nu \in X^+$ of dominant weights, a commutative $\Bbbk$-algebra $\mathbb{K}$, and monic polynomials $\mu_i(x), \nu_i(x) \in \mathbb{K}[x]$ for each $i \in I$ with $\deg(\mu_i(x)) = h_i(\mu)$ and $\deg(\nu_i(x)) = h_i(\nu)$. The *generalized cyclotomic quotient* of $\mathfrak{U}$ associated to this data is the $\mathbb{K}$-linear 2-representation

$$\mathbf{H}(\mu|\nu) := (\mathfrak{U}\mathbb{1}_{\nu-\mu} \otimes_{\Bbbk} \mathbb{K}) / \mathbf{I}(\mu|\nu)$$

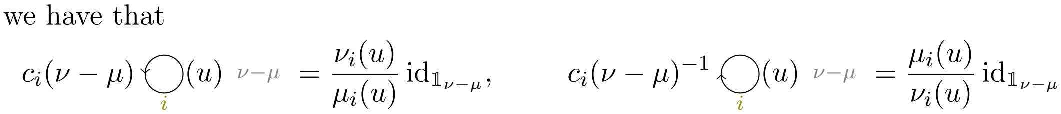
where $\mathbf{I}(\mu|\nu)$ is the $\mathbb{K}$-linear sub-2-representation of $\mathfrak{U}\mathbb{1}_{\nu-\mu} \otimes_{\Bbbk} \mathbb{K}$ generated by



Equivalently, by [BD17, Lem. 4.14], $\mathbf{I}(\mu|\nu)$ is generated by



and we have that


(5.8)

in $\mathbf{H}(\mu|\nu)$. We denote the object $\mathbb{1}_{\nu-\mu}$ of $\mathbb{1}_{\nu-\mu}\mathbf{H}(\mu|\nu)$ by $V(\mu|\nu)$. It is a generating object in the sense that any other object can be obtained from $V(\mu|\nu)$ by applying a sequence of the functors E_i, F_i ($i \in I$). If either $\mu = 0$ or $\nu = 0$, we call $\mathbf{H}(\mu|\nu)$ simply a *cyclotomic quotient*.

We will soon need a theorem proved in the graded setting in [Web17, Cor. 3.20] giving a Morita equivalent realization of cyclotomic quotients in terms of cyclotomic quiver Hecke algebras. Another proof of the result valid also when the grading is forgotten is given in [Rou12, Th. 4.25]; this depends on [Rou12, Th. 4.24], for which Rouquier cites [KK12, Web17] noting that the arguments from [KK12] are also valid in the ungraded setting. We state the theorem shortly, after some more definitions.

Definition 5.3. The *quiver Hecke category* $\mathbf{QH}$ is the strict graded monoidal category with generating objects E_i ($i \in I$) and generating morphisms

of degrees $2d_i$ and $-d_i a_{i,j}$, respectively, subject to the relations (2.10) to (2.12). For μ and $\mu_i(x) \in \mathbb{K}[x]$ as in Definition 5.2, let $\mathbf{I}(\mu)$ be the left tensor ideal of $\mathbf{QH} \otimes_{\Bbbk} \mathbb{K}$ generated by $\left\{ \boxed{\mu_i(x)}_{\,i} \,\middle|\, i \in I \right\}$. Then we pass to the quotient category

$$\mathbf{QH}(\mu) := (\mathbf{QH} \otimes_{\Bbbk} \mathbb{K}) / \mathbf{I}(\mu).$$

This $\mathbb{K}$-linear category is a *cyclotomic quiver Hecke category*. If $\mathbb{K}$ is a graded $\Bbbk$-algebra and each $\mu_i(x)$ is a homogeneous polynomial then $\mathbf{QH}(\mu)$ is also a graded category.

For $l \geq 0$, the locally unital endomorphism algebra

$$\mathrm{QH}_l := \bigoplus_{\substack{i_1,\dots,i_l \in I \\ j_1,\dots,j_l \in I}} \mathrm{Hom}_{\mathbf{QH}}(E_{j_1} \otimes \cdots \otimes E_{j_l}, E_{i_1} \otimes \cdots \otimes E_{i_l}) \tag{5.9}$$

is the *quiver Hecke algebra* introduced in [Rou08, KL09]. If I is finite then QH_l is a unital algebra, but in general it is merely locally unital. It has a well-known basis as a free $\Bbbk$-module with elements are the composition of a monomial $x_1^{n_1} \cdots x_l^{n_l}$ $(n_1, \dots, n_l \geq 0)$, a string diagram representing a reduced expression for a permutation $w \in S_l$, and an idempotent indexed by a sequence $(i_1, \dots, i_l) \in I^l$. Since there are no non-zero morphisms $E_{j_1} \otimes \cdots \otimes E_{j_l} \to E_{i_1} \otimes \cdots \otimes E_{i_{l'}}$ for $l \neq l'$, we obtain from this a basis for each morphism space in $\mathbf{QH}$.

Switching attention to the quotient category $\mathbf{QH}(\mu)$, the locally unital endomorphism algebra

$$\mathrm{QH}_l(\mu) := \bigoplus_{\substack{i_1,\dots,i_l \in I \\ j_1,\dots,j_l \in I}} \mathrm{Hom}_{\mathbf{QH}(\mu)}(E_{j_1} \otimes \cdots \otimes E_{j_l}, E_{i_1} \otimes \cdots \otimes E_{i_l}) \tag{5.10}$$

is a *cyclotomic quiver Hecke algebra*. By [Web17, Cor. 3.26], it is free of finite rank as a $\mathbb{K}$-module; in particular, it is a unital algebra. It follows that each morphism space in $\mathbf{QH}(\mu)$ is free of finite rank as a $\mathbb{K}$-module. There is a $\mathbb{K}$-linear functor

$$F : \mathbf{QH}(\mu) \to \mathbf{H}(\mu|\nu) \tag{5.11}$$

defined by acting on the generating object $V(\mu|\nu)$.

Theorem 5.4 (Kang-Kashiwara, Rouquier, Webster)**.** *Assuming that $\nu = 0$, the functor F is fully faithful, and it induces an equivalence* $\mathbf{QH}(\mu)$-proj $\xrightarrow{\sim}$ $\mathbf{H}(\mu|0)$-proj.

The next lemma is a partial extension of Theorem 5.4 to GCQs. This can also be deduced from results in [Web15, Web24], but the point is to give an independent proof of it using Theorem 4.14. See also Corollary 5.7 for a stronger version.

Lemma 5.5. *In the setup of Definition 5.2, suppose that the ground ring $\Bbbk$ is an integral domain and that $\mathbb{K}$ is an algebraically closed field. Assume that the polynomials $\mu_i(x), \nu_i(x)$ for $i \in I$ are chosen so that all roots of the polynomials $\mu_i(x)$ (resp., $\nu_i(x)$) are algebraic (resp., transcendental) over the field of fractions of $\Bbbk$. Then the functor $F : \mathbf{QH}(\mu) \to \mathbf{H}(\mu|\nu)$ from* (5.11) *is also fully faithful when the dominant weight ν is non-zero.*

Proof. We claim that the spectrum of $\mathbf{H}(\mu|0)$ consists of elements of $\mathbb{K}$ that are algebraic over $\mathrm{Frac}(\Bbbk)$, and the spectrum of $\mathbf{H}(0|\nu)$ consists of elements of $\mathbb{K}$ that are transcendental over $\mathrm{Frac}(\Bbbk)$. To justify this, the minimal polynomials $m_{V(\mu|0),i}(x)$ divide $\mu_i(x)$ and $n_{V(\mu|0),i}(x) = 1$, so the roots of these polynomials are algebraic over $\mathrm{Frac}(\Bbbk)$ by assumption. The minimal polynomials $n_{V(0|\nu),i}(x)$ divide $\nu_i(x)$ and $m_{V(0|\nu),i}(x) = 1$, so the roots of these polynomials are transcendental over $\mathrm{Frac}(\Bbbk)$ by assumption. The other objects of $\mathbf{H}(\mu|0)$ and $\mathbf{H}(0|\nu)$ are of the form $G_{\boldsymbol{i}} V(\mu|0)$ or $G_{\boldsymbol{i}} V(0|\nu)$ for some word $\boldsymbol{i} \in \langle I^+, I^- \rangle$. For $\alpha, \beta \in \mathbb{K}$ with $Q_{i,j}(\alpha, \beta) = 0$ for some $i \neq j$, α is algebraic over $\mathrm{Frac}(\Bbbk)$ if and only if β is algebraic over $\mathrm{Frac}(\Bbbk)$. Using this, Lemma 5.1 and induction on the length of $\boldsymbol{i}$, it follows that the roots of $m_{G_{\boldsymbol{i}}V(\mu|0),i}(x)$ and $n_{G_{\boldsymbol{i}}V(\mu|0),i}(x)$ are algebraic over $\operatorname{Frac}(\Bbbk)$, and the roots of $m_{G_{\boldsymbol{i}}V(0|\nu),i}(x)$ and $n_{G_{\boldsymbol{i}}V(0|\nu),i}(x)$ are transcendental over $\operatorname{Frac}(\Bbbk)$. The claim follows.

The claim implies that the spectra of $\mathbf{H}(\mu|0)$ and $\mathbf{H}(0|\nu)$ are disjoint. So we obtain a $\mathbb{K}$-linear 2-representation $\mathbf{H}(\mu|0) \otimes \mathbf{H}(0|\nu)$ of $\mathfrak{U}$ via the construction with Theorem 4.14 explained before Definition 5.2. Denoting its object $(V(\mu|0), V(0|\nu))$ simply by V, we have that

$$\boxed{\mu_i(x)}\!\!-\!\!\uparrow_{i^+}\ \Big|_V = \uparrow_{i^-}\ \Big|_V = 0, \qquad \downarrow_{i^+}\ \Big|_V = \boxed{\nu_i(x)}\!\!-\!\!\downarrow_{i^-}\ \Big|_V = 0,$$

$$c_i(-\mu) \circlearrowleft_{i^+}(u)\ \Big|_V = \frac{1}{\mu_i(u)}\operatorname{id}_V, \qquad c_i(\nu) \circlearrowleft_{i^-}(u)\ \Big|_V = \nu_i(u)\operatorname{id}_V.$$

The faithfulness of the functor in Theorem 5.4 implies that there is a faithful functor

$$\widetilde{F} : \mathbf{QH}(\mu) \to \mathbf{H}(\mu|0) \otimes \mathbf{H}(0|\nu)$$

defined by acting on V. There is a canonical equivariant functor $C : \mathfrak{U}\mathbb{1}_{\nu-\mu} \to \mathbf{H}(\mu|0) \otimes \mathbf{H}(0|\nu)$ taking $\mathbb{1}_{\nu-\mu}$ to V. The image of $\boxed{\mu_i(x)}\!\!-\!\!\uparrow_{i}\ {\scriptstyle \nu-\mu}$ is $\boxed{\mu_i(x)}\!\!-\!\!\uparrow_{i^+}\ \Big|_V + \boxed{\mu_i(x)}\!\!-\!\!\uparrow_{i^-}\ \Big|_V = 0$, and the image of $c_i(\nu-\mu) \circlearrowleft_{i}(u)\ {\scriptstyle \nu-\mu}$ is $c_i(\nu-\mu) \circlearrowleft_{i^+}(u) \circlearrowleft_{i^-}(u)\ \Big|_V = \nu_i(u)/\mu_i(u)\operatorname{id}_V$. So C factors through the quotient to induce an equivariant functor $\bar{C} : \mathbf{H}(\mu|\nu) \to \mathbf{H}(\mu|0) \otimes \mathbf{H}(0|\nu)$. Finally we observe that the following diagram of functors commutes:

$$\begin{tikzcd}
\mathbf{H}(\mu|\nu) \arrow[rr, "\bar{C}"] & & \mathbf{H}(\mu|0) \otimes \mathbf{H}(0|\nu) \\
& \mathbf{QH}(\mu) \arrow[ul, "F"] \arrow[ur, "\widetilde{F}"'] &
\end{tikzcd}$$

We deduce that F is faithful because $\widetilde{F}$ is faithful. It is obvious that F is full. □

Now the tools are in place, and we can state and prove Khovanov-Lauda's Non-Degeneracy Conjecture for 2-quantum groups. Using the conventions from Subsection 5.1, for $\boldsymbol{i}, \boldsymbol{j} \in \langle I^+, I^- \rangle$ with $\operatorname{wt}(\boldsymbol{i}) = \operatorname{wt}(\boldsymbol{j})$, $G_{\boldsymbol{i}}\mathbb{1}_\lambda$ and $G_{\boldsymbol{j}}\mathbb{1}_\lambda$ are 1-morphisms belonging the same morphism category of $\mathfrak{U}$. By an $\boldsymbol{i} \times \boldsymbol{j}$ *shape*, we mean an oriented string diagram $\vec{D}$ with no dots or closed components ("bubbles") such that $\vec{D}\ {\scriptstyle\lambda}$ represents a 2-morphism $G_{\boldsymbol{j}}\mathbb{1}_\lambda \Rightarrow G_{\boldsymbol{i}}\mathbb{1}_\lambda$ in $\mathfrak{U}$. Such a shape $\vec{D}$ is *reduced* if no string has more than one critical point (i.e., no zig-zags), no pair of strings cross each other twice (i.e., no bigons), and there are no self-intersections of a string with itself (i.e., no loops). Note in a reduced shape that the boundary points of any cup are on the top edge and the boundary points of any cap are on the bottom edge; there are also *propagating strings* which have one boundary point on the top edge and one on the bottom edge. We say that two shapes are *equivalent* if they define the same matching between the boundary points. Then we fix a set $\vec{\operatorname{sh}}(\boldsymbol{i} \times \boldsymbol{j})$ of representatives for the equivalence classes of reduced $\boldsymbol{i} \times \boldsymbol{j}$ shapes. Assuming this set is non-empty, we have that $l(\boldsymbol{i}) \equiv l(\boldsymbol{j}) \pmod 2$, so it makes sense to consider $l := (|l(\boldsymbol{i})| + |l(\boldsymbol{j})|)/2$; this is the number of connected components in any $\boldsymbol{i} \times \boldsymbol{j}$ shape. Then, for each $\vec{D} \in \vec{\operatorname{sh}}(\boldsymbol{i} \times \boldsymbol{j})$, we fix a choice of points $p_1, \dots, p_l$ away from crossings and critical points, one on each connected component of the shape, ordering them according to the lexicographic ordering of their Cartesian coordinates. For $f \in \Bbbk[x_1, \dots, x_l]$, we let $\vec{D}(f)\ {\scriptstyle\lambda} : G_{\boldsymbol{j}}\mathbb{1}_\lambda \Rightarrow G_{\boldsymbol{i}}\mathbb{1}_\lambda$ be the 2-morphism obtained from $\vec{D}$ by pinning f to the points $p_1, \dots, p_l$.

Theorem 5.6. *For $\lambda \in X$, the $\Bbbk$-algebra $\operatorname{End}_{\mathfrak{U}}(\mathbb{1}_\lambda)$ is freely generated by either of the sets*

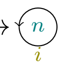
$$\left\{ \circlearrowleft_i^{\,n}\ {\scriptstyle\lambda} \ \middle|\ i \in I, n \geq 1 \right\} \quad \text{or} \quad \left\{ \circlearrowright_i^{\,n}\ {\scriptstyle\lambda} \ \middle|\ i \in I, n \geq 1 \right\}. \tag{5.12}$$

Moreover, for words $\boldsymbol{i}, \boldsymbol{j} \in \langle I^+, I^- \rangle$ *with* $\mathrm{wt}(\boldsymbol{i}) = \mathrm{wt}(\boldsymbol{j}), l(\boldsymbol{i}) \equiv l(\boldsymbol{j}) \pmod 2$ *and* $l := (l(\boldsymbol{i}) + l(\boldsymbol{j}))/2$, *the 2-morphism space* $\mathrm{Hom}_{\mathfrak{U}}(G_{\boldsymbol{j}} \mathbb{1}_\lambda, G_{\boldsymbol{i}} \mathbb{1}_\lambda)$ *is free as a right* $\mathrm{End}_{\mathfrak{U}}(\mathbb{1}_\lambda)$*-module with basis given by the morphisms*

$$\left\{ \vec{D}(x_1^{n_1} \cdots x_l^{n_l}) \ \lambda \ \middle| \ \vec{D} \in \vec{\mathrm{sh}}(\boldsymbol{i} \times \boldsymbol{j}), n_1, \dots, n_l \geq 0 \right\}. \tag{5.13}$$

Proof. We assume for the proof that the normalization functions c_i are all 1. This just amounts to rescaling diagrams by units (recall Remark 2.4), so it is permitted. Note also that $\mathrm{End}_{\mathfrak{U}}(\mathbb{1}_\lambda)$ is freely generated by the first set in (5.12) if and only if it is freely generated by the second set. This follows from the infinite Grassmannian relation (2.32). We will prove the result for the first set.

Let $\mathrm{R} := \mathbb{k}\big[\beta_i^{(n)} \mid i \in I, n \geq 1\big]$ and $e : \mathrm{R} \to \mathrm{End}_{\mathfrak{U}}(\mathbb{1}_\lambda)$ be the homomorphism mapping

$$\beta_i^{(n)} \mapsto$$  λ .

We view $\mathrm{Hom}_{\mathfrak{U}}(G_{\boldsymbol{j}} \mathbb{1}_\lambda, G_{\boldsymbol{i}} \mathbb{1}_\lambda)$ as a right R-module with action of $\beta \in \mathrm{R}$ defined by horizontally composing with $e(\beta)$ on the right of a string diagram. We need to show that the set (5.13) generates $\mathrm{Hom}_{\mathfrak{U}}(G_{\boldsymbol{j}} \mathbb{1}_\lambda, G_{\boldsymbol{i}} \mathbb{1}_\lambda)$ as a right R-module, and that this set is R-linearly independent. The spanning part follows because there is a straightening algorithm to take any string diagram representing a 2-morphism $G_{\boldsymbol{j}} \mathbb{1}_\lambda \Rightarrow G_{\boldsymbol{i}} \mathbb{1}_\lambda$ to an R-linear combination of elements of (5.13). This is explained in [KL10, Sec. 3.2.3] and [Lau10, Sec. 8]; the algorithm proceeds by induction on the number of crossings in the diagram.

It remains to prove that the set (5.13) is R-linearly independent. For this, we make two more reductions:

- We can assume that the ground ring $\mathbb{k}$ is the quotient of the polynomial algebra
$$\mathbb{Z}[t_{i,j}, t_{i,j;r,s} \mid i \neq j \text{ in } I, 0 \leq r < -a_{i,j}, 0 \leq s < -a_{j,i}]$$
by relations asserting that each $t_{i,j}$ is invertible and $t_{i,j;r,s} = t_{j,i;s,r}$, taking $Q_{i,j}(x,y)$ to be the generic polynomial defined by (2.7). The grading on $\mathbb{k}$ is defined so that $t_{i,j}$ is of degree 0 and $t_{i,j;r,s}$ is of degree $-2d_i a_{i,j} - 2d_i r - 2d_j s$. The 2-category $\mathfrak{U}$ over any other ground ring for any choice of parameters can be obtained from this by base change. So if we can prove the linear independence in this special case, it is true in general.
- As noted in [KL10, Rem. 3.16], the proof of linear independence reduces further to the case that $\boldsymbol{i} = i_1^+ \cdots i_l^+$ and $\boldsymbol{j} = j_1^+ \cdots i_l^+$ for $l \geq 0$ and $(i_1, \dots, i_l), (j_1, \dots, j_l) \in I^l$ in the same S_l-orbit. So now we have that $G_{\boldsymbol{i}} = E_{i_1} \cdots E_{i_l}$ and $G_{\boldsymbol{j}} = E_{j_1} \cdots E_{j_l}$.

Having made these reductions, consider a linear relation

$$\sum_{\substack{\vec{D} \in \vec{\mathrm{sh}}(\boldsymbol{i} \times \boldsymbol{j}) \\ n_1, \dots, n_l \geq 0}} \vec{D}(x_1^{n_1} \cdots x_l^{n_l}) \ \lambda \ e(\beta(\vec{D}; n_1, \dots, n_l)) = 0 \tag{5.14}$$

in $\mathrm{Hom}_{\mathfrak{U}}(G_{\boldsymbol{j}} \mathbb{1}_\lambda, G_{\boldsymbol{i}} \mathbb{1}_\lambda)$ for $\beta(\vec{D}; n_1, \dots, n_l) \in \mathrm{R}$. Now we pick sufficiently large dominant weights $\mu, \nu \in X^+$ so that

(1) $\lambda = \nu - \mu$;
(2) every $\beta(\vec{D}; n_1, \dots, n_l)$ lies in the subalgebra $\mathbb{k}\big[\beta_i^{(n)} \mid i \in I, 1 \leq n \leq h_i(\nu)\big]$ of R;
(3) $\beta(\vec{D}; n_1, \dots, n_l) \neq 0 \ \Rightarrow \deg\left(\vec{D}(x_1^{n_1} \cdots x_l^{n_l})\right) < \min(d_{i_1} h_{i_1}(\mu), \dots, d_{i_l} h_{i_l})$.

Consider the GCQ $\mathbf{H}(\mu|\nu)$ of $\mathfrak{U}$ defined over the algebraic closure $\mathbb{K}$ of the field extension of $\mathrm{Frac}(\mathbb{k})$ obtained by adjoining indeterminates $\{z_{i,n} \mid i \in I, 1 \leq n \leq h_i(\nu)\}$, taking the monic polynomials $\mu_i(x) := x^{h_i(\mu)}$ and $\nu_i(x) := (x - z_{i,1}) \cdots (x - z_{i,h_i(\nu)})$. Acting with (5.14) on the generating object, we deduce using (5.8) that

$$\sum_{\substack{\vec{D}\in\vec{\mathrm{sh}}(\boldsymbol{i}\times\boldsymbol{j})\\ n_1,\dots,n_l\geq 0}} f(\beta(\vec{D};n_1,\dots,n_l))\vec{D}(x_1^{n_1}\cdots x_l^{n_l})\ \Big|_{V(\mu|\nu)}\ = 0 \tag{5.15}$$

where $f:\mathrm{R}\to\mathbb{K}$ is the $\Bbbk$-algebra homomorphism mapping $\beta_i^{(n)}$ to the $x^{h_i(\nu)-n}$-coefficient of $\nu_i(x)$ if $1\leq n\leq h_i(\nu)$ and to 0 if $n>h_i(\nu)$. Up to some signs, $f\big(\beta_i^{(1)}\big),\dots,f\big(\beta_i^{(h_i(\nu))}\big)$ are elementary symmetric polynomials in $z_{i,1},\dots,z_{i,h_i(\nu)}$. So the elements $f\big(\beta_i^{(n)}\big)$ for $i\in I$ and $1\leq n\leq h_i(\nu)$ are algebraically independent over $\mathrm{Frac}(\Bbbk)$.

The objects $G_{\boldsymbol{i}}V(\mu|\nu)$ and $G_{\boldsymbol{j}}V(\mu|\nu)$ are in the image of the functor F from Lemma 5.5. As F is fully faithful by that lemma, we can rewrite (5.15) equivalently as

$$\sum_{\substack{\vec{D}\in\vec{\mathrm{sh}}(\boldsymbol{i}\times\boldsymbol{j})\\ n_1,\dots,n_l\geq 0}} f(\beta(\vec{D};n_1,\dots,n_l))\vec{D}(x_1^{n_1}\cdots x_l^{n_l})\ = 0 \tag{5.16}$$

in the morphism space $\mathrm{Hom}_{\mathbf{QH}(\mu)}(E_{j_1}\otimes\cdots\otimes E_{j_l}, E_{i_1}\otimes\cdots\otimes E_{i_l})$ of the cyclotomic quiver Hecke category. This morphism space is naturally identified with a morphism space in the $\mathbb{K}$-linear category $(\mathbf{QH}^0/\mathbf{J})\otimes_\Bbbk\mathbb{K}$, where $\mathbf{QH}^0$ is the quiver Hecke category that is the full monoidal subcategory of $\mathbf{QH}$ generated by the morphisms $E_{i_1},\dots,E_{i_l}$ (cf. [Rou08, Cor. 3.8]), and $\mathbf{J}$ is the left tensor ideal of $\mathbf{QH}^0$ generated by the homogeneous morphisms $\underset{i_1}{\big|}\!\bullet\, h_{i_1}(\mu)$, . . . , $\underset{i_l}{\big|}\!\bullet\, h_{i_l}(\mu)$. Note that $\mathbf{QH}^0/\mathbf{J}$ is a *graded* $\Bbbk$-linear category. The hypothesis (3) means that all $\vec{D}(x_1^{n_1}\cdots x_l^{n_l})$ such that $\beta(\vec{D};n_1,\dots,n_l)\neq 0$ are of degree strictly smaller than the degrees of any of the generating morphisms of $\mathring{\mathbf{I}}(\mu)$. Morphism spaces of $\mathbf{QH}^0/\mathbf{J}$, hence, of $(\mathbf{QH}^0/\mathbf{J})\otimes_\Bbbk\mathbb{K}$ in such small degrees have the same bases as the corresponding morphism spaces of $\mathbf{QH}^0$. From this, we deduce that all of these $\vec{D}(x_1^{n_1}\cdots x_l^{n_l})$ are linearly independent. Hence, $f(\beta(\vec{D};n_1,\dots,n_l))=0$ for all $\vec{D}\in\vec{\mathrm{sh}}(\boldsymbol{i}\times\boldsymbol{j})$ and $n_1,\dots,n_l\geq 0$. The restriction of $f:\mathrm{R}\to\mathbb{K}$ to $\Bbbk\big[\beta_i^{(n)}\,\big|\,i\in I, 1\leq n\leq h_i(\nu)\big]$ is injective since the images of its generators are algebraically independent over $\Bbbk$. Each $\beta(\vec{D};n_1,\dots,n_l)$ lies in this subalgebra by the hypothesis (2). So we have that $\beta(\vec{D};n_1,\dots,n_l)=0$ for all $\vec{D}$ and $n_1,\dots,n_l$. ☐

We have already discussed some of the well-known consequences of Theorem 5.6. Here is one more application, which strengthens Lemma 5.5.

Corollary 5.7. *The functor $F:\mathbf{QH}(\mu)\to\mathbf{H}(\mu|\nu)$ from* (5.11) *is fully faithful for all $\mu,\nu\in X^+$ and all choices of the commutative $\Bbbk$-algebra $\mathbb{K}$ and the polynomials $\mu_i(x),\nu_i(x)\in\mathbb{K}[x]$.*

Proof. This follows from Theorem 5.6 by the argument from the proof of [BSW20, Lem. 5.6]. ☐

5.3. **Non-degeneracy of 2-iquantum groups with geometric parameters.** Now we would like to upgrade Theorem 5.6 from 2-quantum groups to 2-iquantum groups. Let notation be as in Subsection 3.2. We remind that the existence of an admissible choice of parameters implicitly puts a restriction on the Cartan matrix, as discussed in Remark 3.3.

Take words $\boldsymbol{i},\boldsymbol{j}\in\langle I\rangle$ with $\mathrm{wt}^\imath(\boldsymbol{i})=\mathrm{wt}^\imath(\boldsymbol{j})$. For any $\lambda\in X^\imath$, $B_{\boldsymbol{i}}\mathbb{1}_\lambda$ and $B_{\boldsymbol{j}}\mathbb{1}_\lambda$ are objects of the same 1-morphism category of $\mathfrak{U}^\imath$. An $\boldsymbol{i}\times\boldsymbol{j}$*-shape* now means an unoriented string diagram D with no dots and no bubbles so that $D\ {\scriptstyle\lambda}: B_{\boldsymbol{j}}\mathbb{1}_\lambda\Rightarrow B_{\boldsymbol{i}}\mathbb{1}_\lambda$ is a 2-morphism in $\mathfrak{U}^\imath$ for any $\lambda\in X^\imath$. As before, such a shape is *reduced* if there are no zig-zags, bigons or loops, and two shapes are *equivalent* if they define the same matching between the boundary points. Then we fix a set $\mathrm{sh}(\boldsymbol{i}\times\boldsymbol{j})$ of representatives for the equivalence classes of reduced $\boldsymbol{i}\times\boldsymbol{j}$ shapes and, for $D\in\mathrm{sh}(\boldsymbol{i}\times\boldsymbol{j})$ with l connected components, we fix a normally-ordered choice of points $p_1,\dots,p_l$ away from crossings and critical points, one on each connected component of the shape. For $f\in\Bbbk[x_1,\dots,x_l]$, we let $D(f)\ {\scriptstyle\lambda}: B_{\boldsymbol{j}}\mathbb{1}_\lambda\Rightarrow B_{\boldsymbol{i}}\mathbb{1}_\lambda$ be the 2-morphism in $\mathfrak{U}^\imath$ obtained from D by pinning f to the points

$p_1, \dots, p_l$. Also let I_1 be a set of representatives for the τ-orbits on I of size 2, and let I_0 be the set of τ-fixed points.

Definition 5.8. We say that the 2-iquantum group $\mathfrak{U}^\imath$ is *non-degenerate* if, for each $\lambda \in X^\imath$, the $\Bbbk$-algebra $\mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$ is freely generated by the set

$$\left\{ \tau i \; \textcircled{\scriptsize n} \; \lambda \;\middle|\; i \in I_1, n \geq 1 \right\} \bigsqcup \left\{ i \; \textcircled{\scriptsize n} \; \lambda \;\middle|\; i \in I_0, n \geq 1, n \text{ odd} \right\} \tag{5.17}$$

and moreover, for $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle$ with $\mathrm{wt}^\imath(\boldsymbol{i}) = \mathrm{wt}^\imath(\boldsymbol{j}), l(\boldsymbol{i}) \equiv l(\boldsymbol{j}) \pmod 2$ and $l := (l(\boldsymbol{i}) + l(\boldsymbol{j}))/2$, the 2-morphism space $\mathrm{Hom}_{\mathfrak{U}^\imath}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda)$ is free as a right $\mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$-module with basis

$$\left\{ D(x_1^{n_1} \cdots x_l^{n_l}) \; \lambda \;\middle|\; D \in \mathrm{sh}(\boldsymbol{i} \times \boldsymbol{j}), n_1, \dots, n_l \geq 0 \right\}. \tag{5.18}$$

The spanning part of non-degeneracy is relatively straightforward:

Lemma 5.9. *For any $\lambda \in X^\imath$, the commutative $\Bbbk$-algebra $\mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$ is generated by the set (5.17). For any $\lambda \in X^\imath$ and $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle$ with $\mathrm{wt}^\imath(\boldsymbol{i}) = \mathrm{wt}^\imath(\boldsymbol{j})$, the 2-morphism space $\mathrm{Hom}_{\mathfrak{U}^\imath}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda)$ is generated as a right $\mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$-module by the set (5.18).*

Proof. Any dotted bubble representing a 2-endomorphism of $\mathbb{1}_\lambda$ can be expressed as a polynomial in elements of the set (5.17). This is clear from the relation (3.28) for bubbles $i \; \textcircled{\scriptsize n} \; \lambda$ for $i \in I_1$. For bubbles with $i \in I_0$, it is not such an obvious consequence of (3.28), but using this relation any $i \; \textcircled{\scriptsize n} \; \lambda$ with $i \in I_0$ and n even can be written as a quadratic expression involving these bubbles for smaller odd values of n; e.g., see the paragraph before [BWW24, Cor. 2.6] where this is deduced from an identity involving Schur's q-functions. Also the elements in (5.17) generate $\mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$, and the 2-morphisms (5.18) generate $\mathrm{Hom}_{\mathfrak{U}^\imath}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda)$ as a right $\mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$-module, because there is a straightening algorithm similar to the Khovanov-Lauda straightening algorithm for 2-quantum groups mentioned in the proof of Theorem 5.6. This proceeds by induction on the number of crossings, using all of the defining relations of $\mathfrak{U}^\imath$. ∎

The difficult part of establishing non-degeneracy of 2-iquantum groups is the linear independence. The following treats the case of geometric parameters as in Example 2.3.

Theorem 5.10. *Assuming that the Cartan matrix is symmetric and that the parameters satisfy $Q_{i,j}(x,y) = t_{i,j}(x-y)^{-a_{i,j}}$ for all $i \neq j$ (with $t_{i,j} \in \Bbbk^\times$ as usual), the 2-iquantum group $\mathfrak{U}^\imath$ is non-degenerate.*

Proof. Let $\mathrm{R} := \Bbbk\left[\beta_i^{(m)}, \beta_j^{(n)} \;\middle|\; i \in I_1, j \in I_0, m, n \geq 1, m \text{ odd}\right]$ and $e : \mathrm{R} \to \mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$ be the homomorphism mapping

$$\beta_i^{(m)} \mapsto \zeta_i^{-1}\gamma_i(\lambda)^{-1} \; \tau i \; \textcircled{\scriptsize m} \; \lambda, \qquad \beta_j^{(n)} \mapsto \zeta_j^{-1}\gamma_j(\lambda)^{-1} \; j \; \textcircled{\scriptsize n} \; \lambda, \tag{5.19}$$

for $i \in I_1$, $j \in I_0$ and $m, n \geq 1$ with n odd. For $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle$ of the same iweight, we view $\mathrm{Hom}_{\mathfrak{U}^\imath}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda)$ as a right R-module so that $\beta \in \mathrm{R}$ acts by horizontally composing on the right with $e(\beta)$. In view of Lemma 5.9, we must show that the 2-morphisms (5.18) are R-linearly independent.

Let $\mathfrak{U}(\varsigma, \zeta)$, $\underline{\mathfrak{U}}(\varsigma, \zeta)$ and $\widehat{\underline{\mathfrak{U}}}(\varsigma, \zeta)$ be as in Subsections 4.1 to 4.3. We are going to work with the $\Bbbk$-linear 2-category $\mathfrak{U}_c$ obtained by completing $\mathfrak{U}(\varsigma, \varepsilon)$ with respect to the grading. It has the same objects and 1-morphisms as $\mathfrak{U}(\varsigma, \varepsilon)$ and, for words $\boldsymbol{k}, \boldsymbol{\ell} \in \langle I^+, I^- \rangle$ of the same weight, its 2-morphism space $\mathrm{Hom}_{\mathfrak{U}_c}(G_{\boldsymbol{\ell}}\mathbb{1}_\lambda, G_{\boldsymbol{k}}\mathbb{1}_\lambda)$ is the product $\prod_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathfrak{U}(\varsigma,\varepsilon)}(G_{\boldsymbol{\ell}}\mathbb{1}_\lambda, G_{\boldsymbol{k}}\mathbb{1}_\lambda)_n$. The horizontal and vertical compositions of 2-morphisms in $\mathfrak{U}(\varsigma, \varepsilon)$ extend to the grading completion because $\mathrm{Hom}_{\mathfrak{U}(\varsigma,\varepsilon)}(G_{\boldsymbol{\ell}}\mathbb{1}_\lambda, G_{\boldsymbol{k}}\mathbb{1}_\lambda)_n = \{0\}$ for $n \ll 0$. The 2-morphism spaces in $\mathfrak{U}_c$ have topological bases arising from Theorem 5.6: given $\lambda \in X$, $\mathrm{End}_{\mathfrak{U}_c}(\mathbb{1}_\lambda)$ is the grading completion of $\mathrm{End}_{\mathfrak{U}(\varsigma,\zeta)}(\mathbb{1}_\lambda)$ and

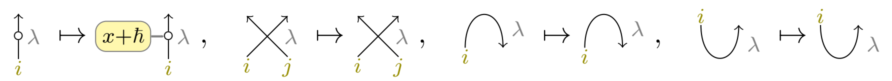
$\boldsymbol{k}, \boldsymbol{\ell}$ as before with $l(\boldsymbol{k}) \equiv l(\boldsymbol{\ell}) \pmod 2$ and $l := (l(\boldsymbol{k}) + l(\boldsymbol{\ell}))/2$, the space $\mathrm{Hom}_{\mathfrak{U}_c}(G_{\boldsymbol{\ell}} \mathbb{1}_\lambda, G_{\boldsymbol{k}} \mathbb{1}_\lambda)$ is free as a topological $\mathrm{End}_{\mathfrak{U}_c}(\mathbb{1}_\lambda)$-module with basis

$$\left\{ \vec{D}(x_1^{n_1}, \dots, x_l^{n_l}) \;\middle|\; \vec{D} \in \vec{\mathrm{sh}}(\boldsymbol{k} \times \boldsymbol{\ell}), n_1, \dots, n_l \geq 0 \right\}.$$

This makes $\mathfrak{U}_c$ better for the present purpose than the localization $\underline{\mathfrak{U}}(\varsigma, \zeta)$, where we do not know any reasonable basis.

Fix $\hbar \in \Bbbk^\times$. There is an automorphism (not graded)

$$\eta : \mathfrak{U}(\varsigma, \varepsilon) \to \mathfrak{U}(\varsigma, \varepsilon),$$



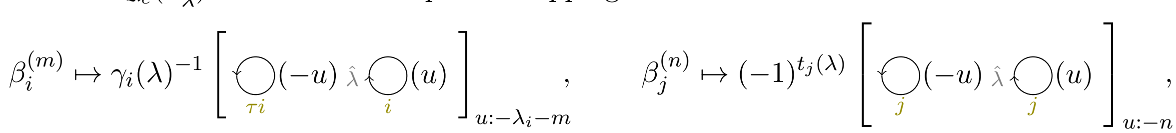
This follows by an easy relations check; the key point is that $Q_{i,j}(x+\hbar, y+\hbar) = Q_{i,j}(x, y)$. We also have that

The proof of this is explained in [BWW24, Lem. 5.3]. The 2-morphisms obtained by applying η to (4.2) and (4.3) have non-zero constant terms (for (4.2) this uses the assumption about the parameters again). So they are invertible in $\mathfrak{U}_c$. Hence, $\eta : \mathfrak{U}(\varsigma, \zeta) \to \mathfrak{U}_c$ induces a strict $\Bbbk$-linear 2-functor $\underline{\eta} : \underline{\mathfrak{U}}(\varsigma, \zeta) \to \mathfrak{U}_c$. Finally, we collapse the object set of $\mathfrak{U}_c$ along the fibers of the quotient map $X \twoheadrightarrow X^\imath$ in the same way that $\widehat{\underline{\mathfrak{U}}}(\varsigma, \zeta)$ was constructed from $\underline{\mathfrak{U}}(\varsigma, \zeta)$, to obtain a $\Bbbk$-linear 2-category $\widehat{\mathfrak{U}}_c$ with object set $X^\imath$, and a strict $\Bbbk$-linear 2-functor $\widehat{\underline{\eta}} : \widehat{\underline{\mathfrak{U}}}(\varsigma, \zeta) \to \widehat{\mathfrak{U}}_c$. We will use the composition $\widehat{\underline{\eta}} \circ \Xi^\imath : \mathfrak{U}^\imath \to \widehat{\mathfrak{U}}_c$ for $\Xi^\imath$ from Theorem 4.13 to complete the proof of the theorem.

Suppose for a contradiction that there is a non-trivial linear relation

$$\sum_{\substack{D \in \mathrm{sh}(\boldsymbol{i} \times \boldsymbol{j}) \\ n_1, \dots, n_l \geq 0}} D(x_1^{n_1} \cdots x_l^{n_l}) \;\lambda\; e(\beta(D; n_1, \dots, n_l)) = 0 \tag{5.20}$$

for $\beta(D; n_1, \dots, n_l) \in \mathrm{R}$. Choose a lift $\hat{\lambda} \in X$ of λ and a shape $D \in \mathrm{sh}(\boldsymbol{i} \times \boldsymbol{j})$ which has a maximal number of crossings amongst all D' such that $\beta(D'; n_1, \dots, n_l) \neq 0$ for some $n_1, \dots, n_l$. Let $\vec{D}$ be the string diagram obtained from D by orienting strings so that each cap and each cup is directed from left to right and each propagating string is directed from bottom to top, labelling endpoints by the same label i as in D if downward and by τi if upward. This produces words $\boldsymbol{k}, \boldsymbol{\ell} \in \langle I^+, I^- \rangle$ such that $\vec{D}$ is an $\boldsymbol{k} \times \boldsymbol{\ell}$ shape. Since D is reduced, $\vec{D}$ is reduced, and we may assume that $\vec{D} \in \vec{\mathrm{sh}}(\boldsymbol{k} \times \boldsymbol{\ell})$, taking the same choice of distinguished points for $\vec{D}$ as for D. We observe since it is reduced that $\vec{D}$ only involves upward, downward and rightward crossings and rightward cups and caps. Let $f : \mathrm{R} \to \mathrm{End}_{\mathfrak{U}_c}(\mathbb{1}_{\hat{\lambda}})$ be the homomorphism mapping



for $i \in I_1$, $j \in I_0$ and $m, n \geq 1$ with n odd. These are algebraically independent elements of $\mathrm{End}_{\mathfrak{U}_c}(\mathbb{1}_{\hat{\lambda}})$ (see [BWW24, (5.9)] which explains this for $i \in I_0$). So f is injective. Now we apply the 2-functor $\widehat{\underline{\eta}} \circ \Xi^\imath$ to (5.20) then project onto the $G_{\boldsymbol{k}} \mathbb{1}_{\hat{\lambda}} \times G_{\boldsymbol{\ell}} \mathbb{1}_{\hat{\lambda}}$-matrix entry of the result to obtain the equality

$$\sum_{n_1, \dots, n_l \geq 0} c_D D(x_1^{n_1} \cdots x_l^{n_l}) \;\hat{\lambda}\; f(\beta(D; n_1, \dots, n_l)) + (*) = 0 \tag{5.21}$$

in $\mathrm{Hom}_{\mathfrak{U}_c}\left(G_{\boldsymbol{\ell}} \mathbb{1}_{\hat{\lambda}}, G_{\boldsymbol{k}} \mathbb{1}_{\hat{\lambda}}\right)$, where c_D is a non-zero scalar depending only on D arising as a product of the coefficients in (4.32) to (4.34) and (4.36) to (4.39), and $(*)$ is an $\mathrm{End}_{\mathfrak{U}_c}(\mathbb{1}_{\hat{\lambda}})$-linear combination of diagrams of shapes that are not equivalent to $\vec{D}$ with the same number as or fewer crossings than D. The terms in $(*)$ may involve teleporters, and leftward cups, caps and crossings decorated with internal bubbles. All terms in $(*)$ can be "straightened" to rewrite them as $\mathrm{End}_{\mathfrak{U}_c}(\mathbb{1}_{\hat{\lambda}})$-linear combination of $\vec{D}'(x_1^{n_1}\cdots x_l^{n_l})$ for $\vec{D}' \in \vec{\mathrm{sh}}(\boldsymbol{k}\times\boldsymbol{\ell})$ different from $\vec{D}$. Using what we know about the topological basis of $\mathrm{Hom}_{\mathfrak{U}_c}(G_{\boldsymbol{\ell}}\mathbb{1}_{\hat{\lambda}}, G_{\boldsymbol{k}}\mathbb{1}_{\hat{\lambda}})$, we deduce that from (5.21) that $f(\beta(D;n_1,\dots,n_l)) = 0$ for all $n_1,\dots,n_l \geq 0$. As f is injective, this shows that all $\beta(D;n_1,\dots,n_l)$ are zero, which is a contradiction. □

We have some hope that the general approach in the proofs of Theorems 5.6 and 5.10 can also be used to prove non-degeneracy for non-geometric values of parameters. However, it seems to be hard to understand the spectrum when one makes some sufficiently generic deformation of the polynomials $Q_{i,j}(x,y)$, so we have been unsuccessful in our attempts to prove the following:

Conjecture 5.11. *$\mathfrak{U}^\imath$ is non-degenerate in all cases.*

5.4. **Graphical interpretation of bilinear forms.** In this subsection, we go back to the general quasi-split iquantum group setup of Subsection 3.1. We will not be using 2-iquantum groups so we do not need any choice of parameters $Q_{i,j}(x,y)$ or normalization functions c_i to have been made. This means that the results in this subsection (except for Theorem 5.17) are valid for *all* quasi-split iquantum groups, even the ones excluded by Remark 3.3. When there is some underlying quasi-split iquantum group $\mathfrak{U}^\imath$ which is non-degenerate in the sense of Definition 5.8, the bilinear form on $\dot{\mathrm{U}}^\imath$ described combinatorially here computes graded dimensions of 2-morphism spaces in $\mathfrak{U}^\imath$ (this is the content of Theorem 5.17).

We need to make a few more reminders about quantum groups. Let $\mathbf{f}$ be Lusztig's $\mathbb{Q}(q)$-algebra generated by θ_i $(i\in I)$. It is isomorphic both to the subalgebra U^+ of U generated by all e_i $(i\in I)$ and to the subalgebra U^- of U generated by all f_i $(i\in I)$ via the $\mathbb{Q}(q)$-algebra homomorphisms

$$\mathbf{f} \xrightarrow{\sim} \mathrm{U}^+,\quad x\mapsto x^+, \qquad\qquad \mathbf{f} \xrightarrow{\sim} \mathrm{U}^-,\quad x\mapsto x^-, \tag{5.22}$$

defined by setting $\theta_i^+ := e_i$ and $\theta_i^- := f_i$. The algebra $\mathbf{f}$ is naturally graded by

$$\Lambda := \sum_{i\in I}\mathbb{N}\alpha_i \tag{5.23}$$

with θ_i being of degree α_i (Lusztig denotes this monoid by $\mathbb{N}[I]$). Let ${}_iR$ and R_i be the $\mathbb{Q}(q)$-linear endomorphisms of $\mathbf{f}$ denoted ${}_ir/(1-q_i^{-2})$ and $r_i/(1-q_i^{-2})$ in [Lus10, §1.2.13]. They are uniquely determined by the properties that ${}_iR(1) = R_i(1) = 0$, ${}_iR(\theta_j) = R_i(\theta_j) = \delta_{i,j}/(1-q_i^{-2})$, and

$${}_iR(xy) = {}_iR(x)y + q_i^{h_i(\alpha)}x\,{}_iR(y), \qquad\qquad R_i(xy) = q_i^{h_i(\beta)}R_i(x)y + xR_i(y) \tag{5.24}$$

for $x\in\mathbf{f}_\alpha$ and $y\in\mathbf{f}_\beta$. By [Lus10, Prop. 3.1.6(b)], for $i\in I$ and $x\in\mathbf{f}$, we have that

$$[e_i, x^-] = q_i^{-1}\big(k_i\,{}_iR(x)^- - R_i(x)^-k_i^{-1}\big). \tag{5.25}$$

Lusztig's non-degenerate symmetric bilinear form $(\cdot,\cdot) : \mathbf{f}\times\mathbf{f}\to\mathbb{Q}(q)$ is characterized by the properties $(1,1) = 1$ and

$$(\theta_i x, y) = (x, {}_iR(y)), \qquad\qquad (x\theta_i, y) = (x, R_i(y)) \tag{5.26}$$

for $x,y\in\mathbf{f}$ and $i\in I$; see [Lus10, 1.2.13(a)].

Recall from [Lus10, §19.1.1] the linear anti-involution ρ of the algebra U such that

$$\rho(e_i) = q_ik_if_i, \qquad\qquad \rho(f_i) = q_i^{-1}e_ik_i^{-1}, \qquad\qquad \rho(q^h) = q^h \tag{5.27}$$

for all $i\in I$ and $h\in Y$. It induces an anti-automorphism of $\dot{\mathrm{U}}$ which fixes each 1_λ $(\lambda\in X)$ and satisfies

$$\rho(e_i1_\lambda) = q_i^{1+h_i(\lambda)}1_\lambda f_i, \qquad\qquad \rho(f_i1_\lambda) = q_i^{1-h_i(\lambda)}1_\lambda e_i. \tag{5.28}$$

By [BW18a, Prop. 4.6], ρ preserves the subalgebra $\mathrm{U}^\imath$, and

$$\rho(b_i) = q_i^{-1-\varsigma_{\tau i}} b_{\tau i} k_{\tau i} k_i^{-1}. \tag{5.29}$$

This induces an anti-automorphism of $\dot{\mathrm{U}}^\imath$ which fixes each 1_λ ($\lambda \in X^\imath$) and satisfies

$$\rho(b_i 1_\lambda) = q_i^{1+\varsigma_i-\lambda_i} 1_\lambda b_{\tau i}. \tag{5.30}$$

The powers of q in (5.28) and (5.30) are degrees of cups and caps in $\mathfrak{U}$ and $\mathfrak{U}^\imath$; cf. Tables 2 and 3.

Denote by $L(\lambda)$ the irreducible U-module of highest weight $\lambda \in X^+$, with η_λ being a highest weight vector. There is a unique symmetric bilinear form $(\cdot,\cdot)_\lambda$ on $L(\lambda)$ such that $(\eta_\lambda,\eta_\lambda)_\lambda = 1$ and

$$(uv, w)_\lambda = (v, \rho(u)w)_\lambda \tag{5.31}$$

for all $v, w \in L(\lambda)$ and $u \in \mathrm{U}$. Moreover, $(v,w)_\lambda = 0$ for v and w from different weight spaces of $L(\lambda)$. By [Lus10, Prop. 19.3.7], the form $(\cdot,\cdot)$ on $\mathbf{f}$ is related to these forms by

$$(x, y) = \lim_{\lambda\to\infty} (x^- \eta_\lambda, y^- \eta_\lambda)_\lambda \tag{5.32}$$

for $x, y \in \mathbf{f}$, where the limit is taken over $\lambda \in X$ with $h_i(\lambda) \to \infty$ for all $i \in I$, and the convergence is in $\mathbb{Q}(\!(q^{-1})\!)$.

From now on, λ denotes an iweight in $X^\imath$, and $\hat\lambda \in X$ is a pre-image. According to [BW18a, Def. 6.25], there is a symmetric bilinear form $(\cdot,\cdot)^\imath : \dot{\mathrm{U}}^\imath \times \dot{\mathrm{U}}^\imath \to \mathbb{Q}(q)$ such that $(x1_\lambda, y1_{\lambda'})^\imath = 0$ for $x, y \in \mathrm{U}^\imath$ and $\lambda \neq \lambda'$ in $X^\imath$ and

$$(x1_\lambda, y1_\lambda)^\imath = \lim_{\hat\lambda\to\infty} (x\eta_{\hat\lambda}, y\eta_{\hat\lambda})_{\hat\lambda} \tag{5.33}$$

for $x, y \in \mathrm{U}^\imath$, where the limit is taken over pre-images $\hat\lambda \in X$ of λ with $h_i(\hat\lambda) \to \infty$ for all $i \in I$, and convergence is in $\mathbb{Q}(\!(q^{-1})\!)$. By [BW18a, Cor. 6.26], we have that

$$(ux1_\lambda, y1_\lambda)^\imath = (x1_\lambda, \rho(u)y1_\lambda)^\imath \tag{5.34}$$

for $u, x, y \in \mathrm{U}^\imath$ and $\lambda \in X^\imath$. Also $(\cdot,\cdot)^\imath$ is non-degenerate by [BW18a, Th. 6.27] and [BW21, Th. 7.6]. The following theorem shows that $\mathrm{U}^\imath 1_\lambda$ equipped with this bilinear form is isometric to $\mathbf{f}$ with its bilinear form $(\cdot,\cdot)$. It extends [Wan25, Th. 2.8] and [BWW23, Th. 2.1].

Theorem 5.12. *Fix $\lambda \in X^\imath$. There is a unique $\mathbb{Q}(q)$-linear isomorphism $\jmath : \mathrm{U}^\imath 1_\lambda \xrightarrow{\sim} \mathbf{f}$ such that*

$$\lim_{\hat\lambda\to\infty} \big(x\eta_{\hat\lambda}, y^-\eta_{\hat\lambda}\big)_{\hat\lambda} = \big(\jmath(x1_\lambda), y\big) \tag{5.35}$$

for $x \in \mathrm{U}^\imath$ and $y \in \mathbf{f}$, where the limit is taken over pre-images $\hat\lambda$ of λ with $h_i(\hat\lambda) \to \infty$ for all $i \in I$, and convergence is in $\mathbb{Q}(\!(u^{-1})\!)$. Moreover:

(1) $\jmath(1_\lambda) = 1$;
(2) $\jmath(b_i x 1_\lambda) = \theta_i\, \jmath(x1_\lambda) + q_i^{1+\varsigma_i-\kappa_i}\, {}_{\tau i}R\big(\jmath(x1_\lambda)\big)$ for any $x \in \mathrm{U}^\imath$ which is homogeneous in the sense that $x1_\lambda = 1_\kappa x$ for $\kappa \in X^\imath$;
(3) $(x1_\lambda, y1_\lambda)^\imath = \big(\jmath(x1_\lambda), \jmath(y1_\lambda)\big)$ for all $x, y \in \mathrm{U}^\imath$.

Proof. Suppose that x is some element of $\mathrm{U}^\imath$ with $1_\kappa x = x1_\lambda$ for $\kappa \in X^\imath$, and that we are given an element $\jmath(x1_\lambda) \in \mathbf{f}$ such that (5.35) is true for all $y \in \mathbf{f}$. Take some $i \in I$. We claim that there is a unique element $\jmath(b_i x 1_\lambda)$ of $\mathbf{f}$ such that

$$\lim_{\hat\lambda\to\infty} \big(b_i x\eta_{\hat\lambda}, y^-\eta_{\hat\lambda}\big)_{\hat\lambda} = \big(\jmath(b_i x1_\lambda), y\big) \tag{5.36}$$

for all $y \in \mathbf{f}$, namely, the element $\jmath(b_i x 1_\lambda)$ defined by the formula in (2). The uniqueness of $\jmath(b_i x 1_\lambda)$ is clear from the non-degeneracy of the form $(\cdot,\cdot)$. For existence, we have for $y \in \mathbf{f}$ and $\jmath(b_i x 1_\lambda)$ defined by this formula that

$$\big(\jmath(b_i x1_\lambda), y\big) = \Big(\theta_i \jmath(x1_\lambda) + q_i^{1+\varsigma_i-\kappa_i}\, {}_{\tau i}R\big(\jmath(x1_\lambda)\big), y\Big)$$

$$
\begin{aligned}
&\overset{(5.26)}{=} \Big(\jmath(x1_\lambda), {}_iR(y) + q_i^{1+\varsigma_i-\kappa_i}\theta_{\tau i} y\Big)\\
&\overset{(5.35)}{=} \lim_{\hat\lambda\to\infty} \Big(x\eta_{\hat\lambda}, \big({}_iR(y)^- + q_i^{1+\varsigma_i-\kappa_i} f_{\tau i} y^-\big)\eta_{\hat\lambda}\Big)_{\hat\lambda}\\
&= \lim_{\hat\lambda\to\infty} \Big(x\eta_{\hat\lambda}, \big({}_iR(y)^- - k_i^{-1}R_i(y)^- k_i^{-1} + q_i^{1+\varsigma_i-\kappa_i} f_{\tau i} y^-\big)\eta_{\hat\lambda}\Big)_{\hat\lambda}\\
&\overset{(5.25)}{=} \lim_{\hat\lambda\to\infty} \Big(x\eta_{\hat\lambda}, \big(q_i k_i^{-1} e_i y^- + q_i^{1+\varsigma_i-\kappa_i} f_{\tau i} y^-\big)\eta_{\hat\lambda}\Big)_{\hat\lambda}\\
&\underset{(5.27)}{\overset{(5.26)}{=}} \lim_{\hat\lambda\to\infty} \Big(\big(f_i + q_i^{2+\varsigma_i-\kappa_i} k_{\tau i}^{-1} e_{\tau i}\big)x\eta_{\hat\lambda}, y^-\eta_{\hat\lambda}\Big)_{\hat\lambda}\\
&= \lim_{\hat\lambda\to\infty} \big((f_i + q_i^{\varsigma_i} e_{\tau i} k_i^{-1})x\eta_{\hat\lambda}, y^-\eta_{\hat\lambda}\big)_{\hat\lambda} = \lim_{\hat\lambda\to\infty} \big(b_i x\eta_{\hat\lambda}, y^-\eta_{\hat\lambda}\big)_{\hat\lambda},
\end{aligned}
$$

which checks (5.36).

Now we can prove existence of a linear map $\jmath$ satisfying (5.35). We set $\jmath(1_\lambda) := 1$. Then (5.35) holds for $x = 1$ and all $y \in \mathbf{f}$ by the definition (5.32). For any monomial x in $\{b_i \mid i \in I\}$, we can use the claim in the previous paragraph plus induction on the length of the monomial to construct a vector $\jmath(x1_\lambda) \in \mathbf{f}$ such that (5.35) holds for all $y \in \mathbf{f}$. Then we pick a basis for $\mathrm{U}^\imath 1_\lambda$ consisting of some such monomials applied to 1_λ, and define $\jmath$ on these basis elements as just explained, extending linearly to obtain the desired $\mathbb{Q}(q)$-linear map $\jmath : \mathrm{U}^\imath 1_\lambda \to \mathbf{f}$ such that (5.35) is true for all $x \in U^\imath$. The uniqueness of $\jmath$ follows from the non-degeneracy of the form $(\cdot,\cdot)^-$.

Next, we establish the properties (1)–(3). The first is clear from (5.32). The second follows from the first paragraph of the proof. For the third, in view of (5.35), it suffices to show that

$$
\lim_{\hat\lambda\to\infty} \big(x\eta_{\hat\lambda}, \jmath(y1_\lambda)^-\eta_{\hat\lambda}\big)_{\hat\lambda} = (x1_\lambda, y1_\lambda)^\imath \tag{5.37}
$$

for $x, y \in \mathrm{U}^\imath$. To see this, we may assume that y is a monomial in $\{b_i \mid i \in I\}$, and proceed by induction on its length, the case $y = 1$ being clear from (5.33). For the induction step, we assume that (5.37) is true for a monomial y with $y1_\lambda = 1_\kappa y$ and all $x \in \mathrm{U}^\imath$, and prove it for $b_i y$, using (2) for the first equality:

$$
\begin{aligned}
\lim_{\hat\lambda\to\infty} \big(x\eta_{\hat\lambda}, \jmath(b_i y1_\lambda)^-\eta_{\hat\lambda}\big)_{\hat\lambda} &= \lim_{\hat\lambda\to\infty} \Big(x\eta_{\hat\lambda}, \big(f_i\jmath(y1_\lambda)^- + q_i^{1+\varsigma_i-\kappa_i}{}_{\tau i}R(\jmath(y1_\lambda))^-\big)\eta_{\hat\lambda}\Big)_{\hat\lambda}\\
&= \lim_{\hat\lambda\to\infty} \Big(x\eta_{\hat\lambda}, \big(f_i\jmath(y1_\lambda)^- + q_i^{-1-\varsigma_{\tau i}} k_i^{-1}(k_{\tau i}\,{}_{\tau i}R(\jmath(y1_\lambda))^- - R_{\tau i}(\jmath(y1_\lambda))^- k_{\tau i}^{-1})\big)\eta_{\hat\lambda}\Big)_{\hat\lambda}\\
&\overset{(5.25)}{=} \lim_{\hat\lambda\to\infty} \big(x\eta_{\hat\lambda}, \big(f_i\jmath(y1_\lambda)^- + q_i^{-\varsigma_{\tau i}} k_i^{-1} e_{\tau i}\jmath(y1_\lambda)^-\big)\eta_{\hat\lambda}\big)_{\hat\lambda} = \lim_{\hat\lambda\to\infty} \big(x\eta_{\hat\lambda}, b_i\,\jmath(y1_\lambda)^-\eta_{\hat\lambda}\big)_{\hat\lambda}\\
&\overset{(5.31)}{=} \lim_{\hat\lambda\to\infty} \big(\rho(b_i)x\eta_{\hat\lambda}, \jmath(y1_\lambda)^-\eta_{\hat\lambda}\big)_{\hat\lambda} \overset{(5.37)}{=} (\rho(b_i)x1_\lambda, y1_\lambda)^\imath \overset{(5.34)}{=} (x1_\lambda, b_i y1_\lambda)^\imath.
\end{aligned}
$$

Now (3) is proved.

Finally, we must show that $\jmath$ is an isomorphism. Using (1)–(2) and induction on length, it follows that any monomial in $\{\theta_i \mid i \in I\}$ lies in the image of $\jmath$. Hence, $\jmath$ is surjective. It is injective by (3) and the non-degeneracy of the form $(\cdot,\cdot)^\imath$. □

For $\boldsymbol{i}, \boldsymbol{j} \in \langle I\rangle$, we now need the set $\mathrm{sh}(\boldsymbol{i}\times\boldsymbol{j})$ of reduced $\boldsymbol{i}\times\boldsymbol{j}$ shapes introduced before Definition 5.8; this set is empty unless $l(\boldsymbol{i}) \equiv l(\boldsymbol{j}) \pmod 2$. We defined $\mathrm{sh}(\boldsymbol{i}\times\boldsymbol{j})$ in terms of string diagrams representing morphisms in $\mathfrak{U}^\imath$, but the definition can also be formulated in purely diagrammatic terms—it is just a diagram representing a matching between the words $\boldsymbol{i}$ and $\boldsymbol{j}$ which is reduced in the sense that the number of crossings is as small as possible. Recall the definitions of $b_{\boldsymbol{i}}$ and $\theta_{\boldsymbol{i}}$ from (5.2) and (5.3).

Theorem 5.13. *For $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle$ and $\lambda \in X^\imath$, we have that*

$$(\theta_{\boldsymbol{i}}, \theta_{\boldsymbol{j}}) = \sum_{\substack{D \in \mathrm{sh}(\boldsymbol{i} \times \boldsymbol{j}) \\ \textit{cup-cap-free}}} q^{-\deg(D\ \lambda)} \Big/ (1-q_{k_1}^{-2}) \cdots (1-q_{k_l}^{-2}), \tag{5.38}$$

$$(\jmath(b_{\boldsymbol{i}} 1_\lambda), \theta_{\boldsymbol{j}}) = \sum_{\substack{D \in \mathrm{sh}(\boldsymbol{i} \times \boldsymbol{j}) \\ \text{cap-free}}} q^{-\deg(D\ \lambda)} \Big/ (1-q_{k_1}^{-2}) \cdots (1-q_{k_l}^{-2}), \tag{5.39}$$

$$\big(b_{\boldsymbol{i}} 1_\lambda, b_{\boldsymbol{j}} 1_\lambda\big)^\imath = \sum_{D \in \mathrm{sh}(\boldsymbol{i} \times \boldsymbol{j})} q^{-\deg(D\ \lambda)} \Big/ (1-q_{k_1}^{-2}) \cdots (1-q_{k_l}^{-2}), \tag{5.40}$$

where $\deg\big(D\ \lambda\big)$ *is computed according to Table 3, and $k_1, \dots, k_l \in I$ are the labels at the distinguished points $p_1, \dots, p_l$ of the strings in D.*

Proof. The first formula (5.38) is well known, e.g., it was exploited already in [KL09].

Next we prove (5.39). We proceed by induction on $l(\boldsymbol{i})$. If $l(\boldsymbol{i}) = 0$ then both sides are 0 unless $l(\boldsymbol{j}) = 0$, when they are both 1. So the induction base is true. Now we assume (5.39) is true for some $\boldsymbol{i}$ and all $\boldsymbol{j}$, take $h \in I$, and prove the result for the slightly longer word $h\boldsymbol{i}$ and all $\boldsymbol{j}$. Let $\kappa := \lambda - \mathrm{wt}^\imath(\boldsymbol{i})$. By Theorem 5.12(2) and (5.26), we have that

$$(\jmath(b_h b_{\boldsymbol{i}} 1_\lambda), \theta_{\boldsymbol{j}}) = (\theta_h \jmath(b_{\boldsymbol{i}} 1_\lambda), \theta_{\boldsymbol{j}}) + q_h^{1+\varsigma_h - \kappa_h} (\jmath(b_{\boldsymbol{i}} 1_\lambda), \theta_{\tau h} \theta_{\boldsymbol{j}}). \tag{5.41}$$

Now consider a cup-free shape $D \in \mathrm{sh}(h\boldsymbol{i} \times \boldsymbol{j})$. Its vertices along the top edge are labelled according to the letters of the word $h\boldsymbol{i}$, and the vertices along the bottom edge are labelled according to $\boldsymbol{j}$. Let $k_1, \dots, k_l$ be the labels of its strings at the distinguished points. We say D is of type I if the string with one endpoint at the top left vertex labelled h is propagating, that is, its other endpoint is at the bottom of the diagram. We say it is of type II otherwise, in which case this string is a cup connecting the top left vertex to another vertex at the top of the diagram. The argument is completed using (5.41) on adding the equations established in the following two claims:

Claim 1: $$(\theta_h \jmath(\boldsymbol{i} 1_\lambda), \theta_{\boldsymbol{j}}) = \sum_{\substack{D \in \mathrm{sh}(h\boldsymbol{i} \times \boldsymbol{j}) \\ \text{cap-free of type I}}} q^{-\deg(D\ \lambda)} \Big/ (1-q_{k_1}^{-2}) \cdots (1-q_{k_l}^{-2}).$$

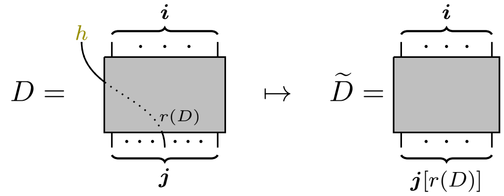
To prove this, suppose that $\boldsymbol{j} = j_1 \cdots j_n$. For $1 \le r \le n$, let $\boldsymbol{j}[r] := j_1 \cdots j_{r-1} j_{r+1} \cdots j_n$. By (5.24), we have that

$${}_h R(\theta_{\boldsymbol{j}}) = \sum_{\substack{1 \le r \le n \\ j_r = h}} q_h^{a_{h,j_1} + \cdots + a_{h,j_{r-1}}} \theta_{\boldsymbol{j}[r]} \Big/ (1 - q_h^{-2}).$$

Also using (5.26), we deduce that

$$(\theta_h \jmath(b_{\boldsymbol{i}} 1_\lambda), \theta_{\boldsymbol{j}}) = (\jmath(b_{\boldsymbol{i}} 1_\lambda), {}_h R(\theta_{\boldsymbol{j}})) = \sum_{\substack{1 \le r \le n \\ j_r = h}} q_h^{a_{h,j_1} + \cdots + a_{h,j_{r-1}}} \big(\jmath(b_{\boldsymbol{i}} 1_\lambda), \theta_{\boldsymbol{j}[r]}\big) \Big/ (1 - q_h^{-2}). \tag{5.42}$$

Now let $D \in \mathrm{sh}(h\boldsymbol{i} \times \boldsymbol{j})$ be a cup-free shape of type I. Consider the string in D which has one endpoint at the top left vertex. Define $r(D)$ to be the vertex number of its other endpoint, indexing vertices at the bottom of the diagram by $1, \dots, n$ from left to right. Let $\widetilde{D} \in \mathrm{sh}(\boldsymbol{i} \times \boldsymbol{j}[r(D)])$ be the cap-free shape obtained by removing this string from D. The function



is a bijection from the set of cup-free shapes $D \in \mathrm{sh}(h\boldsymbol{i} \times \boldsymbol{j})$ of type I to the disjoint union of the sets of cup-free shapes in $\mathrm{sh}(\boldsymbol{i} \times \boldsymbol{j}[r])$ for $1 \le r \le n$ with $j_r = h$; the inverse bijection inserts a string of color h in the obvious way. In passing from D to $\widetilde{D}$ we have removed crossings of a string labelled h with strings labelled $j_1, \dots, j_{r(D)-1}$, so $\deg(\widetilde{D}\ \lambda) = \deg(D\ \lambda) + d_h a_{h,j_1} + \cdots + d_h a_{h,j_{r(D)-1}}$. Putting these things together shows that

$$\sum_{\substack{D \in \mathrm{sh}(h\boldsymbol{i}\times\boldsymbol{j})\\ \text{cap-free of type I}}} q^{-\deg(D\ \lambda)} \ \big/\ (1-q_{k_1}^{-2})\cdots(1-q_{k_l}^{-2})$$

$$= \sum_{\substack{D \in \mathrm{sh}(h\boldsymbol{i}\times\boldsymbol{j})\\ \text{cap-free of type I}}} q_h^{a_{h,j_1}+\cdots+a_{h,j_{r(D)-1}}} q^{-\deg(\widetilde{D}\ \lambda)} \ \big/\ (1-q_{k_1}^{-2})\cdots(1-q_{k_l}^{-2})$$

$$= \sum_{\substack{1\le r\le n\\ j_r=h}} q_h^{a_{h,j_1}+\cdots+a_{h,j_{r-1}}} \sum_{\substack{\widetilde{D} \in \mathrm{sh}(\boldsymbol{i}\times\boldsymbol{j}[r])\\ \text{cap-free}}} q^{-\deg(\widetilde{D}\ \lambda)} \ \big/\ (1-q_{k_1}^{-2})\cdots(1-q_{k_l}^{-2}).$$

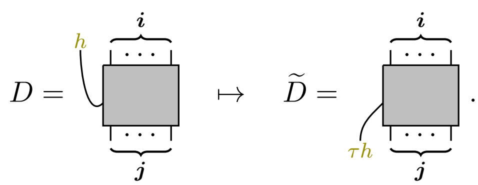
By the induction hypothesis, the final expression here is equal to

$$\sum_{\substack{1\le r\le n\\ j_r=h}} q_h^{a_{h,j_1}+\cdots+a_{h,j_{r-1}}} \big(\jmath(b_{\boldsymbol{i}}1_\lambda), \theta_{\boldsymbol{j}[r]}\big) \ \big/\ (1-q_h^{-2}),$$

which is the desired $(\theta_h \jmath(b_{\boldsymbol{i}}1_\lambda), \theta_{\boldsymbol{j}})$ by (5.42).

<u>Claim 2</u>: $q_h^{1+\varsigma_h-\kappa_h}\big(\jmath(b_{\boldsymbol{i}}1_\lambda), \theta_{\tau h}\theta_{\boldsymbol{j}}\big) = \displaystyle\sum_{\substack{D \in \mathrm{sh}(h\boldsymbol{i},\boldsymbol{j})\\ \text{cap-free of type II}}} q^{-\deg(D\ \lambda)} \ \big/\ (1-q_{k_1}^{-2})\cdots(1-q_{k_l}^{-2}).$

Consider the bijection from the set of cap-free shapes $D \in \mathrm{sh}(h\boldsymbol{i} \times \boldsymbol{j})$ of type II to the set of cap-free shapes $\widetilde{D} \in \mathrm{sh}(\boldsymbol{i} \times (\tau h)\boldsymbol{j})$ defined so that



This removes a cup of degree $d_h(1 + \varsigma_{\tau h} - (\kappa + \alpha_h)_{\tau h}) = -d_h(1 + \varsigma_h - \kappa_h)$. so we have that $q^{-\deg(D\ \lambda)} = q_h^{1+\varsigma_h-\kappa_h} q^{-\deg(\widetilde{D}\ \lambda)}$. Using the induction hypothesis for the first equality, we deduce:

$$q_h^{1+\varsigma_h-\kappa_h}(\jmath(b_{\boldsymbol{i}}1_\lambda), \theta_{\tau h}\theta_{\boldsymbol{j}}) = q_h^{1+\varsigma_h-\kappa_h} \sum_{\substack{\widetilde{D} \in \mathrm{sh}(\boldsymbol{i}\times(\tau h)\boldsymbol{j})\\ \text{cap-free}}} q^{-\deg(\widetilde{D}\ \lambda)} \ \big/\ (1-q_{k_1}^{-2})\cdots(1-q_{k_l}^{-2})$$

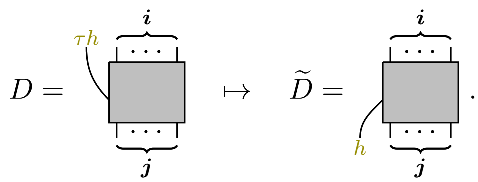
$$= \sum_{\substack{D \in \mathrm{sh}(h\boldsymbol{i}\times\boldsymbol{j})\\ \text{cap-free of type II}}} q^{-\deg(D\ \lambda)} \ \big/\ (1-q_{k_1}^{-2})\cdots(1-q_{k_l}^{-2}).$$

Now (5.39) is proved. We move on to (5.40), which we prove by induction on the length of the word $\boldsymbol{j}$. If $\boldsymbol{j}$ is of length 0 then $\jmath(b_{\boldsymbol{j}}1_\lambda) = 1$ and $(b_{\boldsymbol{i}}1_\lambda, b_{\boldsymbol{j}}1_\lambda)^\imath = (\jmath(b_{\boldsymbol{i}}1_\lambda), 1)$ for all $\boldsymbol{i} \in \langle I\rangle$ by Theorem 5.12. This is computed by the stated sum over shapes thanks to (5.39). This checks the induction base. For the induction step, we assume that (5.40) is true for some word $\boldsymbol{j} \in \langle I\rangle$ and all words $\boldsymbol{i}$, and prove it for $h\boldsymbol{j}$ for some $h \in I$. Let κ be the iweight with $b_{\boldsymbol{j}}\mathbb{1}_\lambda = \mathbb{1}_\kappa b_{\boldsymbol{j}}$. We have that

$$\big(b_{\boldsymbol{i}}1_\lambda, b_h b_{\boldsymbol{j}}1_\lambda\big)^\imath \overset{(5.34)}{\underset{(5.30)}{=}} q_h^{1+\varsigma_h-\kappa_h}\big(b_{\tau h}b_{\boldsymbol{i}}1_\lambda, b_{\boldsymbol{j}}1_\lambda\big)^\imath$$

$$= q_h^{1+\varsigma_h-\kappa_h} \sum_{D\in \mathrm{sh}((\tau h)\boldsymbol{i}\times\boldsymbol{j})} q^{-\deg(D\,\lambda)} \;\big/\; (1-q_{k_1}^{-2}\cdots(1-q_{k_l}^{-2})$$
$$= \sum_{\widetilde{D}\in \mathrm{sh}(\boldsymbol{i}\times h\boldsymbol{j})} q^{-\deg(\widetilde{D}\,\lambda)} \;\big/\; (1-q_{k_1}^{-2})\cdots(1-q_{k_l}^{-2}).$$

The second equality here follows by the induction hypothesis. The final one follows by a similar argument to the proof of Claim 2, using the bijection $\mathrm{sh}((\tau h)\boldsymbol{i}\times\boldsymbol{j}) \xrightarrow{\sim} \mathrm{sh}(\boldsymbol{i}\times h\boldsymbol{j}), D\mapsto \widetilde{D}$ defined by



This completes the proof of (5.40) and hence the theorem. $\square$

Remark 5.14. As well as the symmetric bilinear forms $(\cdot,\cdot)$ on $\mathbf{f}$ and $(\cdot,\cdot)^\imath$ on $\dot{\mathrm{U}}^\imath$, there is Lusztig's non-degenerate symmetric form $(\cdot,\cdot)$ on $\dot{\mathrm{U}}$, which was mentioned already in (5.1). This is uniquely determined by the following properties:

- $(1_\kappa x 1_\lambda, 1_{\kappa'} y 1_{\lambda'}) = 0$ for any $x, y \in \mathrm{U}$ and weights with $\kappa \neq \kappa'$ or $\lambda\neq\lambda'$;
- $(x^+1_\lambda, y^+1_\lambda) = (x^-1_\lambda, y^-1_\lambda) = (x,y)$ for any $\lambda\in X$ and $x,y\in\mathbf{f}$;
- $(ux1_\lambda, y1_\lambda) = (x1_\lambda, \rho(u)y1_\lambda)$ for any $\lambda \in X$ and $u,x,y\in\mathrm{U}$.

In [KL10, Th. 2.7], Khovanov and Lauda gave the following graphical interpretation of this bilinear form: for $\boldsymbol{i},\boldsymbol{j}\in\langle I^+, I^-\rangle$ and $\lambda\in X$ we have that

$$\big(g_{\boldsymbol{i}}1_\lambda, g_{\boldsymbol{j}}1_\lambda\big) = \sum_{\vec{D}\in \vec{\mathrm{sh}}(\boldsymbol{i}\times\boldsymbol{j})} q^{-\deg(\vec{D}\;\lambda)} \;\big/\; (1-q_{k_1}^{-2})\cdots(1-q_{k_l}^{-2}) \tag{5.43}$$

where $\deg(\vec{D}\;\lambda)$ is the degree of this 2-morphism $G_{\boldsymbol{j}}\mathbb{1}_\lambda \to G_{\boldsymbol{i}}\mathbb{1}_\lambda$ computed according to Table 2, and $k_1,\dots,k_l\in I$ are the labels of the strings in $\vec{D}$. In view of Theorem 3.7, this identity is a special case of the identity (5.40) in Theorem 5.13.

For the purposes of categorification, we prefer to replace the bilinear forms $(\cdot,\cdot)$ and $(\cdot,\cdot)^\imath$ with sesquilinear forms (anti-linear with respect to the bar involution in the first argument). To define these, let $\psi:\mathbf{f}\to\mathbf{f}$ be the usual bar involution, that is, the anti-linear algebra involution with $\psi(\theta_i)=\theta_i$ for all $i\in I$. Then we define

$$\langle\cdot,\cdot\rangle : \mathbf{f}\times\mathbf{f}\to\mathbb{Q}(q), \qquad \langle x,y\rangle := (\psi(x),y), \tag{5.44}$$
$$\langle\cdot,\cdot\rangle^\imath : \dot{\mathrm{U}}^\imath\times\dot{\mathrm{U}}^\imath\to\mathbb{Q}(q), \qquad \langle x1_\lambda, y1_{\lambda'}\rangle^\imath := (\psi^\imath(x1_\lambda), y1_{\lambda'})^\imath\,. \tag{5.45}$$

To write down the analog of Theorem 5.12 for these sesquilinear forms, we also need the linear maps ${}_i\widetilde{R} := \psi\circ {}_iR\circ\psi : \mathbf{f}\to\mathbf{f}$ and $\tilde{\jmath} := \psi\circ\jmath\circ\psi^\imath : \dot{\mathrm{U}}^\imath 1_\lambda \xrightarrow{\sim} \mathbf{f}$. From (5.24) and (5.26), we get that

$$ {}_i\widetilde{R}(1) = 0, \qquad {}_i\widetilde{R}(\theta_j) = \frac{\delta_{i,j}}{1-q_i^2}, \qquad {}_i\widetilde{R}(xy) = {}_i\widetilde{R}(x)y + q_i^{-h_i(\alpha)}x\,{}_i\widetilde{R}(y), \tag{5.46}$$
$$\big\langle\theta_i x, y\big\rangle = \big\langle x, {}_iR(y)\big\rangle, \qquad \big\langle x, \theta_i y\big\rangle = \big\langle {}_i\widetilde{R}(x), y\big\rangle, \tag{5.47}$$

for $x\in\mathbf{f}_\alpha, y\in\mathbf{f}_\beta$. From Theorem 5.12, we get that

$$\tilde{\jmath}(1) = 1, \qquad \tilde{\jmath}(b_i x1_\lambda) = \theta_i\tilde{\jmath}(x1_\lambda) + q_i^{\kappa_i-\varsigma_i-1}\,{}_{\tau i}\widetilde{R}\left(\tilde{\jmath}(x1_\lambda)\right), \tag{5.48}$$
$$\big\langle x1_\lambda, y1_\lambda\big\rangle^\imath = \big\langle\tilde{\jmath}(x1_\lambda), \jmath(y1_\lambda)\big\rangle \tag{5.49}$$

for $x, y \in \mathfrak{U}^\imath$ such that $x1_\lambda = 1_\kappa x$. For $\boldsymbol{i} \in \langle\!\langle I \rangle\!\rangle$, we define[5] $\delta_{\boldsymbol{i}}1_\lambda$ and $\wp_{\boldsymbol{i}}1_\lambda$ to be the unique elements of $\dot{U}^\imath 1_\lambda$ such that

$$\tilde{\jmath}(\delta_{\boldsymbol{i}}1_\lambda) = \theta_{\boldsymbol{i}}, \qquad\qquad \jmath(\wp_{\boldsymbol{i}}1_\lambda) = \theta_{\boldsymbol{i}}. \tag{5.50}$$

Note that $\delta_{\boldsymbol{i}}1_\lambda = \psi^\imath(\wp_{\boldsymbol{i}}1_\lambda)$. Now we state some corollaries to Theorem 5.13. The first simply restates the theorem using the new notation.

Corollary 5.15. *For $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle$, we have that*

$$\left\langle \delta_{\boldsymbol{i}}1_\lambda, \wp_{\boldsymbol{j}}1_\lambda \right\rangle^\imath = \sum_{\substack{D \in \mathrm{sh}(\boldsymbol{i}\times\boldsymbol{j}) \\ \textit{cup-cap-free}}} q^{-\deg(D\ \lambda)} \,/\, (1-q_{k_1}^{-2})\cdots(1-q_{k_l}^{-2}), \tag{5.51}$$

$$\left\langle b_{\boldsymbol{i}}1_\lambda, \wp_{\boldsymbol{j}}1_\lambda \right\rangle^\imath = \sum_{\substack{D \in \mathrm{sh}(\boldsymbol{i}\times\boldsymbol{j}) \\ \textit{cap-free}}} q^{-\deg(D\ \lambda)} \,/\, (1-q_{k_1}^{-2})\cdots(1-q_{k_l}^{-2}), \tag{5.52}$$

$$\left\langle b_{\boldsymbol{i}}1_\lambda, b_{\boldsymbol{j}}1_\lambda \right\rangle^\imath = \sum_{D \in \mathrm{sh}(\boldsymbol{i}\times\boldsymbol{j})} q^{-\deg(D\ \lambda)} \,/\, (1-q_{k_1}^{-2})\cdots(1-q_{k_l}^{-2}), \tag{5.53}$$

$k_1, \dots, k_l$ being the string labels of D as before.

Proof. This follows from (5.38) to (5.40) and the definitions (5.44) and (5.45) because $\theta_{\boldsymbol{i}}$ and $b_{\boldsymbol{i}}\mathbb{1}_\lambda$ are invariant under ψ and $\psi^\imath$, respectively. ☐

In the next corollary, the involution $\tau : I \to I$ enters into the combinatorics via the introduction of a lower finite partial order $\leq$ on the monoid Λ from (5.23) defined by

$$\alpha \leq \beta \Leftrightarrow (\beta - \alpha) \in \sum_{i \in I} \mathbb{N}(\alpha_i + \alpha_{\tau i}). \tag{5.54}$$

Note that $\alpha \leq \beta \Rightarrow \mathrm{wt}^\imath(\alpha) = \mathrm{wt}^\imath(\beta)$. We also define a function

$$|\cdot| : \langle\!\langle I \rangle\!\rangle \to \Lambda, \qquad\qquad |\boldsymbol{i}| := n_1\alpha_{i_1} + \cdots + n_l\alpha_{i_l} \tag{5.55}$$

for $\boldsymbol{i} = i_1^{(n_1)} \cdots i_l^{(n_l)}$.

Corollary 5.16. *Let $\boldsymbol{i}, \boldsymbol{j} \in \langle\!\langle I \rangle\!\rangle$ and $\lambda \in X^\imath$. If $\langle b_{\boldsymbol{i}}1_\lambda, \wp_{\boldsymbol{j}}1_\lambda\rangle^\imath \neq 0$ then we have that $|\boldsymbol{j}| \leq |\boldsymbol{i}|$.*

Proof. Since it just amounts to scaling by a non-zero scalar, we may as well assume that $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle$. Suppose that $\langle b_{\boldsymbol{i}}1_\lambda, \wp_{\boldsymbol{j}}1_\lambda\rangle^\imath \neq 0$. Then (5.52) implies that there exists a cap-free shape $D \in \mathrm{sh}(\boldsymbol{i} \times \boldsymbol{j})$. It follows that the word $\boldsymbol{j}$ can be obtained from $\boldsymbol{i}$ by removing pairs of letters of the form $(h, \tau h)$ (one such pair for each cup in D), then permuting the remaining letters. Each such removal makes $|\boldsymbol{i}|$ smaller in the partial order $\leq$. ☐

Finally in this subsection, we assume that we are once again given the additional data needed to define the 2-iquantum group $\mathfrak{U}^\imath$.

Theorem 5.17. *Assume that $\mathfrak{U}^\imath$ is non-degenerate in the sense of Definition 5.8. For $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle$ with $\mathrm{wt}^\imath(\boldsymbol{i}) = \mathrm{wt}^\imath(\boldsymbol{j})$, we have that*

$$\langle b_{\boldsymbol{i}}1_\lambda, b_{\boldsymbol{j}}1_\lambda\rangle^\imath = \mathrm{rank}_{q^{-1}} \mathrm{Hom}_{\mathfrak{U}^\imath}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda), \tag{5.56}$$

where the graded rank is as a free graded right $\mathrm{End}_{\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$*-module (see* Conventions*).*

Proof. Both sides are 0 unless $\mathrm{wt}^\imath(\boldsymbol{i}) = \mathrm{wt}^\imath(\boldsymbol{j})$. In that case, both sides are computed by the same sum over $\mathrm{sh}(\boldsymbol{i} \times \boldsymbol{j})$ thanks to (5.53) and the definition of non-degeneracy. ☐

[5] This notation will be justified in the next section when we relate $\delta_{\boldsymbol{i}}1_\lambda$ and $\wp_{\boldsymbol{i}}1_\lambda$ to certain standard and costandard modules $\Delta(\boldsymbol{i})$ and $\nabla(\boldsymbol{i})$.

5.5. **Application: the Balagović-Kolb-Letzter relation.** We end the section by using the isometry $\jmath$ from Theorem 5.12 to give a conceptual explanation of the Balagović-Kolb-Letzter case of the iSerre relation.

Lemma 5.18. *Suppose that $i \in I$ and $\lambda \in X^\imath$ with pre-image $\hat\lambda \in X$. For $n \geq 0$, we have that*

$$b_{i^{(n)}} 1_\lambda = \begin{cases} \delta_{i^{(n)}} 1_\lambda & \text{if } i \neq \tau i \\ \displaystyle\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{q_i^{m(2m-1)}}{(1-q_i^4)(1-q_i^8)\cdots(1-q_i^{4m})} \delta_{i^{(n-2m)}} 1_\lambda & \text{if } i = \tau i \text{ and } n \equiv h_i(\hat\lambda) \pmod 2 \\ \displaystyle\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{q_i^{m(2m+1)}}{(1-q_i^4)(1-q_i^8)\cdots(1-q_i^{4m})} \delta_{i^{(n-2m)}} 1_\lambda & \text{if } i = \tau i \text{ and } n \not\equiv h_i(\hat\lambda) \pmod 2. \end{cases}$$

Proof. In the case $i = \tau i$, this follows from [BWW23, Th. 2.7] (and we will not use this formula subsequently). To prove it in the (much easier!) case that $i \neq \tau i$, it suffices to show that $b_{i^n} 1_\lambda = \delta_{i^n} 1_\lambda$ (we simply multiplied both sides by $[n]^!_{q_i}$). To prove this, we show that both sides pair in the same way with the spanning set $\{\wp_{\boldsymbol{j}} 1_\lambda \mid \boldsymbol{j} \in \langle I \rangle\}$ for $\dot{\mathrm{U}}^\imath 1_\lambda$. Since $i \neq \tau i$, the weight $n\alpha_i$ is minimal in the poset Λ. From Corollary 5.16, it follows that the only $\boldsymbol{j} \in \langle I \rangle$ such that $\langle b_{i^n} 1_\lambda, \wp_{\boldsymbol{j}} 1_\lambda \rangle^\imath \neq 0$ is $\boldsymbol{j} = i^n$. This is clearly also the case for $\delta_{i^n} 1_\lambda$ by (5.51). Also by (5.51) and (5.52) we have that $\langle b_{i^n} 1_\lambda, \wp_{i^n} 1_\lambda \rangle^\imath = \langle \delta_{i^n} 1_\lambda, \wp_{i^n} 1_\lambda \rangle^\imath$. □

Lemma 5.19. *Suppose that $i, j \in I$ with $i \neq \tau i$ and $i \neq j$. For $m \geq 1$ and $0 \leq n \leq m$, we have that*

$$b_{i^{(n)} j i^{(m-n)}} 1_\lambda = \begin{cases} \delta_{i^{(n)} j i^{(m-n)}} 1_\lambda & \text{if } i \neq \tau j \\ \delta_{i^{(n)} j i^{(m-n)}} 1_\lambda + f^\lambda_{n,m;i}(q)\, \delta_{i^{(m-1)}} 1_\lambda & \text{if } i = \tau j \end{cases} \tag{5.57}$$

where

$$f^\lambda_{n,m;i}(q) := \frac{q_i^{1+\lambda_i - \varsigma_i - (m-n-1)(1-a_{i,\tau i}) - m} \begin{bmatrix} m-1 \\ n-1 \end{bmatrix}_{q_i}}{1-q_i^2} + \frac{q_i^{1+(m-n-1)(1-a_{i,\tau i}) + \varsigma_i - \lambda_i} \begin{bmatrix} m-1 \\ n \end{bmatrix}_{q_i}}{1-q_i^2}. \tag{5.58}$$

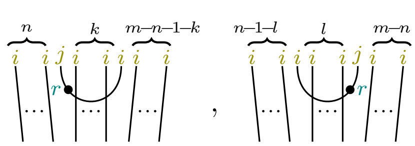
Proof. The strategy is the same as the proof of Lemma 5.18 just explained. The weight $m\alpha_i + \alpha_j$ is minimal in the poset Λ if $i \neq \tau j$, and if $i = \tau j$ then the only $\beta \in \Lambda$ with $\beta < m + \alpha_i + \alpha_j$ is $\beta = (m-1)\alpha_i$. Also (5.51) and (5.52) imply that $\left\langle b_{i^n j i^{m-n}} 1_\lambda, \wp_{\boldsymbol{j}} 1_\lambda \right\rangle = \left\langle \delta_{i^n j i^{m-n}} 1_\lambda, \wp_{\boldsymbol{j}} 1_\lambda \right\rangle$ for any $\boldsymbol{j} \in \langle I \rangle$ with $|\boldsymbol{j}| = m\alpha_i + \alpha_j$. It follows that

$$b_{i^n j i^{m-n}} 1_\lambda = \begin{cases} \delta_{i^n j i^{m-n}} 1_\lambda & \text{if } i \neq \tau j \\ \delta_{i^n j i^{m-n}} + f(q) \delta_{i^{m-1}} 1_\lambda & \text{if } i = \tau j \end{cases}$$

for some $f(q) \in \mathbb{Q}(q)$. This already proves the lemma in the case $i \neq \tau j$. When $i = \tau j$, we still need to compute $f(q)$. Applying $\langle \cdot, \wp_{i^{m-1}} 1_\lambda \rangle^\imath$ to the equation gives that

$$f(q^{-1}) = \frac{\left\langle b_{i^n j i^{m-n}} 1_\lambda, \wp_{i^{m-1}} 1_\lambda \right\rangle^\imath}{\left\langle \delta_{i^{m-1}} 1_\lambda, \wp_{i^{m-1}} 1_\lambda \right\rangle^\imath}.$$

In view of (5.51) and (5.52), it follows that $f(q) = \sum_D q^{\deg(D\ \lambda)}$ summing over

$$D \in \Bigg\{$$


$$\Bigg|\ 0 \leq k \leq m-n-1,\ 0 \leq l \leq n-1,\ r \geq 0 \Bigg\}. \tag{5.59}$$

We deduce that

$$f(q) = \sum_{k=0}^{m-n-1} \frac{q_i^{1+\varsigma_i - (\lambda - (m-n-1)\alpha_i)_i - 2k}}{1-q_i^2} + \sum_{l=0}^{n-1} \frac{q_i^{1+\varsigma_j - (\lambda - (m-n)\alpha_i)_j - 2l}}{1-q_i^2}$$

$$= \frac{q_i^{1+(m-n-1)(1-a_{i,j})+\varsigma_i-\lambda_i}[m-n]_{q_i}}{1-q_i^2} + \frac{q_i^{1+\lambda_i-\varsigma_i-(m-n-1)(1-a_{i,j})-m}[n]_{q_i}}{1-q_i^2}. \tag{5.60}$$

To deduce the formula for $f^\lambda_{n,m;i}(q)$ in the statement, multiply by $[m-1]^!_{q_i}/[m-n]^!_{q_i}[n]^!_{q_i}$. □

Now we can give a new proof of the relation (3.7) in the difficult case $i = \tau j$. Assume that $i \neq j = \tau i$ and set $m := 1 - a_{i,j}$. Applying the inverse of the isomorphism $\widetilde{\jmath}$ to the Serre relation in $\mathbf{f}$ gives that $\sum_{n=0}^m (-1)^n \delta_{i^{(n)}ji^{(m-n)}} 1_\lambda = 0$. Since $\delta_{i^{(n)}ji^{(m-n)}} 1_\lambda = b_{i^{(n)}ji^{(m-n)}} 1_\lambda - f^\lambda_{n,m;i}(q) b_{i^{(m-1)}} 1_\lambda$ by Lemmas 5.18 and 5.19, we deduce that

$$\sum_{n=0}^m (-1)^n b_i^{(n)} b_j b_i^{(m-n)} 1_\lambda = \sum_{n=0}^m (-1)^n f^\lambda_{n,m;i}(q) b_i^{(m-1)} 1_\lambda.$$

The iSerre relation (3.7) for $i = \tau j$, which was proved originally in [BK15, Th. 3.6], follows from this using the next lemma to simplify the right hand side.

Lemma 5.20. *Let $m := 1 - a_{i,\tau i}$ for $i \in I$ with $i \neq \tau i$. We have that*

$$\sum_{n=0}^m (-1)^n f^\lambda_{n,m;i}(q) = \prod_{r=1}^{m-1} (q_i^r - q_i^{-r}) \times \frac{(-1)^{m-1} q_i^{\lambda_i - \varsigma_i - \binom{m}{2}} - q_i^{\binom{m}{2} + \varsigma_i - \lambda_i}}{q_i - q_i^{-1}}.$$

Proof. We replace each $f^\lambda_{n,m;i}(q)$ by the sum of two fractions that is its definition (5.58) to rewrite the left hand side of the identity to be proved as the sum of two summations. Then we reindex the first summation to see that it is equal to

$$\frac{q_i^{\lambda_i-\varsigma_i-\binom{m}{2}}}{q_i - q_i^{-1}} \left(\sum_{n=0}^{m-1} (-1)^n q_i^{nm-\binom{m}{2}} {\textstyle\begin{bmatrix} m-1 \\ n \end{bmatrix}_{q_i}} \right) - \frac{q_i^{\binom{m}{2}+\varsigma_i-\lambda_i}}{q_i - q_i^{-1}} \left(\sum_{n=0}^{m-1} (-1)^n q_i^{\binom{m}{2}-nm} {\textstyle\begin{bmatrix} m-1 \\ n \end{bmatrix}_{q_i}} \right).$$

To see that this expression is equal to the desired right hand side, it remains to apply the elementary identity

$$\prod_{r=1}^{m-1} (q^r - q^{-r}) = \sum_{n=0}^{m-1} (-1)^n q^{\binom{m}{2}-nm} {\textstyle\begin{bmatrix} m-1 \\ n \end{bmatrix}_q} = (-1)^{m-1} \sum_{n=0}^{m-1} (-1)^n q^{nm-\binom{m}{2}} {\textstyle\begin{bmatrix} m-1 \\ n \end{bmatrix}_q}, \tag{5.61}$$

which follows from [Lus10, 1.3.1(c)] taking $z = -v^2$. □

6. Identification of the Grothendieck ring

Assuming the non-degeneracy of the 2-quantum group $\mathfrak{U}^\imath$ which we (re-)proved in Theorem 5.6, the arguments in [KL10, Sec. 3.8, Sec. 3.9] show that the Grothendieck ring of the appropriate completion of $\mathfrak{U}$ is isomorphic to $\dot{\mathrm{U}}_\mathbb{Z}$, i.e., $\mathfrak{U}$ categorifies $\dot{\mathrm{U}}_\mathbb{Z}$. The main goal in this section is to prove the analogous statement for all quasi-split iquantum groups which are non-degenerate in the sense of Definition 5.8. This includes all of the cases with symmetric Cartan matrix and geometric parameters thanks to Theorem 5.10. We assume throughout the section that $\Bbbk_0$ is a field, necessarily of characteristic $\neq 2$ if $a_{i,\tau i} \neq 0$ for any $i \in I$, and that the 2-iquantum group $\mathfrak{U}^\imath$ is non-degenerate.

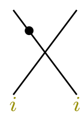
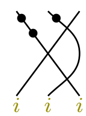
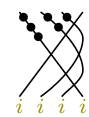
6.1. **Idempotents and divided/idivided powers.** We begin by constructing a Grothendieck ring from $\mathfrak{U}^\imath$ which, like $\dot{\mathrm{U}}^\imath_\mathbb{Z}$, is a locally unital $\mathbb{Z}[q, q^{-1}]$-algebra. Starting from $\mathfrak{U}^\imath$, we pass first to the q-envelope $\mathfrak{U}^\imath_q$ defined at the start of Subsection 4.1. Then we pass from there to the underlying $\Bbbk$-linear 2-category consisting of the same objects and 1-morphisms as $\mathfrak{U}^\imath_q$ but taking only the 2-morphisms that are homogeneous of degree 0. Finally we take the additive Karoubi envelope (=

idempotent completion of additive envelope) of each of the morphism categories of that to obtain a $\Bbbk$-linear 2-category denoted $\mathrm{Kar}(\mathfrak{U}_q^\imath)$. Then we define

$$K_0\left(\mathrm{Kar}(\mathfrak{U}_q^\imath)\right) := \bigoplus_{\kappa,\lambda\in X^\imath} K_0\left(\mathbb{1}_\kappa\,\mathrm{Kar}(\mathfrak{U}_q^\imath)\mathbb{1}_\lambda\right) \tag{6.1}$$

with $K_0\left(\mathbb{1}_\kappa\,\mathrm{Kar}(\mathfrak{U}_q^\imath)\mathbb{1}_\lambda\right)$ being the usual split Grothendieck group of $\mathbf{Hom}_{\mathrm{Kar}(\mathfrak{U}_q^\imath)}(\lambda,\kappa)$. We make $K_0\left(\mathrm{Kar}(\mathfrak{U}_q^\imath)\right)$ into a $\mathbb{Z}[q,q^{-1}]$-algebra with multiplication induced by horizontal composition, and the action of q defined by $q[B_{\boldsymbol{i}}\mathbb{1}_\lambda] := [qB_{\boldsymbol{i}}\mathbb{1}_\lambda]$ for $\boldsymbol{i}\in\langle I\rangle$. It is a locally unital with distinguished idempotents given by the isomorphism classes $1_\lambda := [\mathbb{1}_\lambda]$ for $\lambda\in X^\imath$.

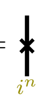
Fixing $i\in I$ and $\lambda\in X^\imath$ for a while, let $b_i1_\lambda = 1_{\lambda-\alpha_i}b_i \in K_0\left(\mathrm{Kar}(\mathfrak{U}_q^\imath)\right)$ be the 2-isomorphism class of the 1-morphism $B_i\mathbb{1}_\lambda$. There are also *divided powers* $B_i^{(n)}1_\lambda = 1_{\lambda-n\alpha_i}B_i^{(n)}$ if $i\neq\tau i$ or *idivided powers* $B_i^{(n)}1_\lambda = 1_{\lambda-n\alpha_i}B_i^{(n)}$ if $i=\tau i$. The distinction according to whether $i\neq\tau i$ or $i=\tau i$ is important—in the former case the defining relations imply that there is a homomorphism from the usual nil-Hecke algebra NH_n (graded so that the polynomial generators are all of degree $2d_i$) to $\mathrm{End}_{\mathfrak{U}^\imath}(B_i^n\mathbb{1}_\lambda)$ whereas in the latter case one needs to work with the nil-Brauer category from (3.46). In either case, we let $1_{i^{(n)}}\in\mathrm{End}_{\mathfrak{U}^\imath}(B_i^n\mathbb{1}_\lambda)$ be the 2-endomorphism defined by following the pattern:

$1_\varnothing := \mathrm{id}_{\mathbb{1}_\lambda}$, $\quad 1_i :=$, $\quad 1_{i^{(2)}} :=$ , $\quad 1_{i^{(3)}} :=$ , $\quad 1_{i^{(4)}} :=$ , $\quad \cdots$

We will use the "thick calculus" notation developed in [BWW23, Sec. 4] to denote this in general:

$1_{i^{(n)}} =$  (6.2)

The thick string labelled i^n here is a shorthand for n parallel thin strings each of color i. If $i\neq\tau i$, it is well known that $1_{i^{(n)}}$ is the image of an idempotent in NH_n. When $i=\tau i$, it is the image of an idempotent in the nil-Brauer category; see [BWW23, Cor. 4.24]. The essential property which implies that this is an idempotent is that

 $=$  . (6.3)

Applying the symmetry $\Xi^\imath$ from (3.68), we obtain another idempotent

${}'1_{i^{(n)}} := \Xi^\imath\left(1_{i^{(n)}}\right) =$  . (6.4)

The idempotents $1_{i^{(n)}}$ and ${}'1_{i^{(n)}}$ are conjugate: we have that $1_{i^{(n)}} = u_{i^{(n)}}\circ v_{i^{(n)}}$ and ${}'1_{i^{(n)}} = v_{i^{(n)}}\circ u_{i^{(n)}}$ where $u_{i^{(n)}}$ consists of the dots at the top and $v_{i^{(n)}}$ consists of the crossings at the bottom of (6.2). We need both since $\Xi^\imath$ will be used in some subsequent constructions. Let

$$B_i^{(n)}\mathbb{1}_\lambda := \left(q_i^{-\binom{n}{2}}B_i^n\mathbb{1}_\lambda, 1_{i^{(n)}}\right) \cong \left(q_i^{\binom{n}{2}}B_i^n\mathbb{1}_\lambda, {}'1_{i^{(n)}}\right). \tag{6.5}$$

With this definition, the notation (5.4) now makes sense. If $\boldsymbol{i} = i_1^{(n_1)}\cdots i_l^{(n_l)}\in\langle\!\langle I\rangle\!\rangle$ then

$$B_{\boldsymbol{i}}\mathbb{1}_\lambda = (q^{-\deg(\boldsymbol{i})}B_{i_1}^{n_1}\cdots B_{i_l}^{n_l}\mathbb{1}_\lambda, 1_{\boldsymbol{i}}) \cong (q^{\deg(\boldsymbol{i})}B_{i_1}^{n_1}\cdots B_{i_l}^{n_l}\mathbb{1}_\lambda, {}'1_{\boldsymbol{i}}) \tag{6.6}$$

where

$$1_{\boldsymbol{i}} := 1_{i_1^{(n_1)}}\cdots 1_{i_l^{(n_l)}}, \qquad {}'1_{\boldsymbol{i}} := {}'1_{i_1^{(n_1)}}\cdots {}'1_{i_l^{(n_l)}}, \qquad \deg(\boldsymbol{i}) := d_{i_1}\binom{n_1}{2}+\cdots+d_{i_l}\binom{n_l}{2}. \tag{6.7}$$

Lemma 6.1. *The elements $[B_i^{(n)}\mathbb{1}_\lambda]$ of $K_0\left(\mathrm{Kar}(\mathfrak{U}^\imath_q)\right)$ satisfy*

$$[B_i^{(n)}\mathbb{1}_\lambda] = \begin{cases} \dfrac{1}{[n]^!_{q_i}}\,[B_i^n\mathbb{1}_\lambda] & \text{if } i \neq \tau i \\ \dfrac{1}{[n]^!_{q_i}}\left[\displaystyle\prod_{\substack{m=0\\ m\equiv h_i(\hat{\lambda})\,(\mathrm{mod}\ 2)}}^{n-1}\left(B_i^2 - [m]^2_{q_i}\right)\mathbb{1}_\lambda\right] & \text{if } i = \tau i \text{ and } n \text{ is even} \\ \dfrac{1}{[n]^!_{q_i}}\left[B_i\displaystyle\prod_{\substack{m=1\\ m\equiv h_i(\hat{\lambda})\,(\mathrm{mod}\ 2)}}^{n-1}\left(B_i^2 - [m]^2_{q_i}\right)\mathbb{1}_\lambda\right] & \text{if } i = \tau i \text{ and } n \text{ is odd.}\end{cases} \tag{6.8}$$

This is the same as the identity satisfied by the elements $b_i^{(n)}1_\lambda$ of $\dot{\mathrm{U}}^\imath_{\mathbb{Z}}$ from (3.6).

Proof. When $i \neq \tau i$, this follows in the same way as the analogous statement for 2-quantum groups; e.g., see [Rou08, Lem. 4.1] (which ignores the grading) or [KL10, (3.54)–(3.55)]. Ultimately, it depends on a well-known result about primitive idempotents in the nilHecke algebra NH_n. When $i = \tau i$, the result follows from [BWW23, Th. 4.21], which shows that $B_i B_i^{(n)}\mathbb{1}_\lambda$ is isomorphic to a direct sum of graded shifts of $B_i^{(n+1)}\mathbb{1}_\lambda$ and $B_i^{(n-1)}\mathbb{1}_\lambda$ in a way that agrees with the recursive formula for $b_i b_i^{(n)}1_\lambda$ from [BW18c]. Although [BWW23] works over an algebraically closed field, the decomposition of $\mathrm{id}_{B_i^n\mathbb{1}_\lambda}$ into mutually orthogonal idempotents and the conjugacy of these idempotents established there is valid in our current setup. $\square$

6.2. **Classification of indecomposables and standard modules.** We next explain how to classify indecomposable objects in $\mathrm{Kar}(\mathfrak{U}^\imath_q)$. For this, we are going to switch to using the language of modules. For $\lambda \in X^\imath$, we let

$$\mathfrak{U}^\imath\mathbb{1}_\lambda := \coprod_{\kappa\in X^\imath}\mathbb{1}_\kappa\mathfrak{U}^\imath\mathbb{1}_\lambda \tag{6.9}$$

with q-envelope $\mathfrak{U}^\imath_q\mathbb{1}_\lambda$. Recall that ${}'\mathfrak{U}^\imath$ is the 2-iquantum group defined using the parameters ${}'Q_{i,j}(x,y) = r_{i,j}r_{j,i}Q_{i,j}(x,y)$. Let

$$\mathrm{H} = \mathrm{H}^\lambda := \bigoplus_{\boldsymbol{i},\boldsymbol{j}\in\langle I\rangle}\mathrm{Hom}_{{}'\mathfrak{U}^\imath}(B_{\boldsymbol{j}}\mathbb{1}_\lambda, B_{\boldsymbol{i}}\mathbb{1}_\lambda), \tag{6.10}$$

which is the path algebra of the graded category ${}'\mathfrak{U}^\imath\mathbb{1}_\lambda$. We keep λ fixed for the remainder of the subsection, so will drop the superscript, denoting H^λ simply by H. Most of our subsequent notation related to H also depends implicitly on λ (one could add back the superscript λ whenever necessary to avoid ambiguity).

For $\boldsymbol{i} = i_1\cdots i_l \in \langle I\rangle$, the idempotent $1_{\boldsymbol{i}} \in \mathrm{H}$ is the identity endomorphism of $B_{\boldsymbol{i}}\mathbb{1}_\lambda$, and H is a locally unital algebra with these as its distinguished idempotents, that is, we have that

$$\mathrm{H} = \bigoplus_{\boldsymbol{i},\boldsymbol{j}\in\langle I\rangle} 1_{\boldsymbol{i}}\mathrm{H}1_{\boldsymbol{j}}.$$

An important point is that each of the spaces $1_{\boldsymbol{i}}\mathrm{H}1_{\boldsymbol{j}}$ is locally finite-dimensional and bounded below as a vector space over $\Bbbk_0$, that is, its graded components are all finite-dimensional and they are zero in sufficiently negative degrees. This follows from Lemma 5.9. Thus, H is a graded algebra which is locally finite-dimensional and bounded below.

Let H-gmod be the category of graded left H-modules $V = \bigoplus_{\boldsymbol{i}\in\langle I\rangle} 1_{\boldsymbol{i}}V$. As a matter of notation, we denote the morphism space between two graded left H-modules in the category H-gmod by $\mathrm{Hom}_{\mathrm{H}}(U,V)_0$, reserving $\mathrm{Hom}_{\mathrm{H}}(U,V)$ for the graded hom $\bigoplus_{n\in\mathbb{Z}}\mathrm{Hom}_{\mathrm{H}}(U,V)_n$ where

$\mathrm{Hom}_{\mathrm{H}}(U,V)_n = \mathrm{Hom}_{\mathrm{H}}(U,q^{-n}V)_0$ consists of the H-module homomorphisms which map U_i into V_{i+n} for all $i \in \mathbb{Z}$. There are also the categories H-gproj and H-ginj of finitely generated graded projective left H-modules and finitely cogenerated graded injective left H-modules, respectively.

In the definition of H, we used the 2-iquantum group $'\mathfrak{U}^\imath$ rather than the usual $\mathfrak{U}^\imath$. The reason for this peculiarity is so that we can use the isomorphism of graded 2-categories $\Psi^\imath : \left(\mathfrak{U}^\imath\right)^{\mathrm{op}} \xrightarrow{\sim} {}'\mathfrak{U}^\imath$ from (3.68). Recalling (4.1), it induces an isomorphism $\mathrm{Kar}(\mathfrak{U}^\imath_q \mathbb{1}_\lambda) \xrightarrow{\sim} \mathrm{Kar}\left({}'\mathfrak{U}^\imath_{q^{-1}}\mathbb{1}_\lambda\right)^{\mathrm{op}}$. But also Yoneda gives an equivalence Contra : $\mathrm{Kar}\left({}'\mathfrak{U}^\imath_{q^{-1}}\mathbb{1}_\lambda\right)^{\mathrm{op}} \to$ H-gproj. Composing this with the isomorphism induced by $\Psi^\imath$, we obtain an equivalence we denote simply by

$$\mathrm{Cov} : \mathrm{Kar}\left(\mathfrak{U}^\imath_q \mathbb{1}_\lambda\right) \to \text{H-gproj} \tag{6.11}$$

such that $\mathrm{Cov} \circ q = q \circ \mathrm{Cov}$. Using (6.6) and (6.7), for $\boldsymbol{i} \in \langle\!\langle I \rangle\!\rangle$, we have that

$$P(\boldsymbol{i}) := q^{\deg(\boldsymbol{i})}\mathrm{H}\, 1_{\boldsymbol{i}} \cong q^{-\deg(\boldsymbol{i})}\mathrm{H}\, {}'1_{\boldsymbol{i}} = \mathrm{Cov}(B_{\boldsymbol{i}}\mathbb{1}_\lambda). \tag{6.12}$$

There has been an application of $\Psi^\imath$ here to switch $1_{\boldsymbol{i}}$ and ${}'1_{\boldsymbol{i}}$.

For a graded right H-module, let $V^\circledast$ be $\bigoplus_{\boldsymbol{i}\in\langle I\rangle}(V1_{\boldsymbol{i}})^*$ (linear dual over the field $\Bbbk_0$), which is a graded left H-module. If V is locally finite-dimensional and bounded below (resp., finitely generated and projective) then $V^\circledast$ is locally finite-dimensional and bounded above (resp., finitely cogenerated and injective). This duality functor is needed to define the *Nakayama functor*

$$\mathrm{Nak} := \mathrm{Hom}_{\mathrm{H}}(-,\mathrm{H})^\circledast : \text{H-gproj} \to \text{H-ginj}, \tag{6.13}$$

which is an equivalence of categories commuting with the grading shift functor. For $\boldsymbol{i} \in \langle\!\langle I \rangle\!\rangle$, it takes $P(\boldsymbol{i})$ to

$$I(\boldsymbol{i}) := q^{\deg(\boldsymbol{i})}(1_{\boldsymbol{i}}H)^\circledast \tag{6.14}$$

Now we are going to study H-gmod by appealing the general theory of algebras with *graded triangular bases* from [Bru25], taking the ground field to be $\Bbbk_0$. We need to equip H with a graded triangular basis consisting of the products xhy for $x \in X(\boldsymbol{i},\boldsymbol{j}), h \in H(\boldsymbol{j},\boldsymbol{k})$ and $y \in Y(\boldsymbol{k},\boldsymbol{\ell})$ and $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k},\boldsymbol{\ell} \in \langle I \rangle$. The sets $X(\boldsymbol{i},\boldsymbol{j}), H(\boldsymbol{j},\boldsymbol{k})$ and $Y(\boldsymbol{k},\boldsymbol{\ell})$ are defined next paragraph. Referring to [Bru25, Def. 1.1] for the other language being used, all of the distinguished idempotents $1_{\boldsymbol{i}}$ ($\boldsymbol{i} \in \langle I \rangle$) are special. The weight poset required in the definition of graded triangular basis is the poset Λ from (5.23) ordered by the partial order $\leq$ from (5.54). The required function $\langle I \rangle \to \Lambda$ is the function $\boldsymbol{i} \mapsto |\boldsymbol{i}|$ defined by (5.55). The fibers $\langle I \rangle_\alpha := \left\{\boldsymbol{i} \in \langle I \rangle \;\middle|\; |\boldsymbol{i}| = \alpha\right\}$ of this function are finite, so it makes sense to define

$$e_\alpha := \sum_{\boldsymbol{i}\in\langle I\rangle_\alpha} 1_{\boldsymbol{i}} \tag{6.15}$$

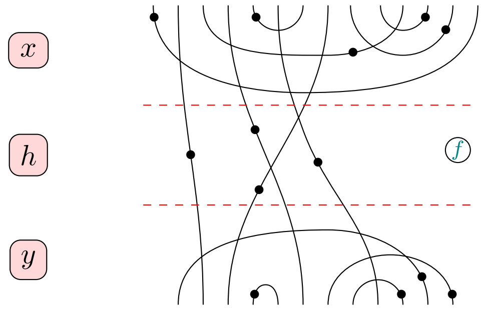
for $\alpha \in \Lambda$. We refer to these as *weight idempotents*. Any left H-module V decomposes as $V = \bigoplus_{\alpha\in\Lambda} e_\alpha V$. We refer to the subspace $e_\alpha V$ as the *α-weight space* of V.

Here, we define $X(\boldsymbol{i},\boldsymbol{j}), H(\boldsymbol{j},\boldsymbol{k})$ and $Y(\boldsymbol{k},\boldsymbol{\ell})$. For $\boldsymbol{i},\boldsymbol{j} \in \langle I \rangle$, we choose the set $\mathrm{sh}(\boldsymbol{i}\times\boldsymbol{j})$ of representatives for isotopy classes of reduced $\boldsymbol{i}\times\boldsymbol{j}$ shapes so that all cups in such a shape are in the top third of the diagram, all crossings of propagating strings are in the middle third of the diagram, and all caps are in the bottom third. We also fix a choice of basis for the algebra $\mathrm{R} := \mathrm{End}_{'\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$ as a vector space over $\Bbbk_0$. Then:

- The set $X(\boldsymbol{i},\boldsymbol{j})$ consists of all $D \in \mathrm{sh}(\boldsymbol{i},\boldsymbol{j})$ which only involve cups (no caps, no crossings of propagating strings or bubbles) with some number of dots added at the distinguished points on each cup.
- The set $H(\boldsymbol{j},\boldsymbol{k})$ consists of all $D \in \mathrm{sh}(\boldsymbol{j},\boldsymbol{k})$ which only involve propagating strings (no cups or caps) with some number of dots added at the distinguished points on these strings and a bubble labelled by one of the basis vectors for R on the right hand boundary.

- The set $Y(\boldsymbol{k},\boldsymbol{\ell})$ consists of all $D \in \mathrm{sh}(\boldsymbol{k},\boldsymbol{\ell})$ which only involve caps (no cups, no crossings of propagating strings or bubbles) with some number of dots added at the distinguished points on each cap.

The assumption of non-degeneracy implies that the monomials xhy indexed by these sets do indeed give a basis for H as a vector space over $\Bbbk_0$. The following picture illustrates a typical element of the resulting graded triangular basis, for strings colored by $\boldsymbol{i}$ at the top boundary and by $\boldsymbol{\ell}$ at the bottom boundary, some non-negative multicities labelling the dots, and a basis vector f of R:



The next step in the development from [Bru25] is to introduce the *Cartan algebras*, which are defined to be the subquotients

$$\mathrm{H}_\alpha := \bar{e}_\alpha \mathrm{H}_{\geq\alpha} \bar{e}_\alpha \tag{6.16}$$

for $\alpha \in \Lambda$, where $\mathrm{H}_{\geq\alpha}$ is quotient of H by the 2-sided ideal generated by $\{e_\beta \,|\, \beta \in \Lambda \text{ with } \beta \not\geq \alpha\}$, and $\bar{x}$ denotes the canonical image of an element $x \in \mathrm{H}$ in $\mathrm{H}_{\geq\alpha}$. Note this is a unital graded algebra, unlike H itself with is merely locally unital. We have that

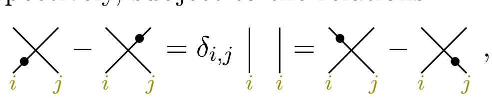
$$\mathrm{H}_\alpha = \bigoplus_{\boldsymbol{i},\boldsymbol{j} \in \langle I\rangle_\alpha} \bar{1}_{\boldsymbol{i}} \mathrm{H}_\alpha \bar{1}_{\boldsymbol{j}}$$

with $\bar{1}_{\boldsymbol{i}} \mathrm{H}_\alpha \bar{1}_{\boldsymbol{j}}$ having basis $\left\{\bar{h} \mid h \in H(\boldsymbol{i},\boldsymbol{j})\right\}$. In fact, as proved in the next lemma, the Cartan algebra H_α is a quiver Hecke algebra over the ground ring $\mathrm{R} = \mathrm{End}_{\prime\mathfrak{U}^\imath}(\mathbb{1}_\lambda)$. We remark that the rings R are isomorphic for different choices of λ, so that the dependency on λ in the following definition is mild. The definition is similar to Definition 5.3, but we are using the parameters $Q^\imath(i,j)$ with the modified signs from (3.21) and using unoriented rather than oriented strings in the string diagrams.

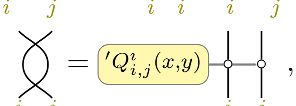
Definition 6.2. We define the *quiver Hecke category* $\mathbf{QH}^\imath$ to be the strict R-linear graded monoidal category with generating objects Θ_i ($i \in I$) and generating morphisms

(6.17)

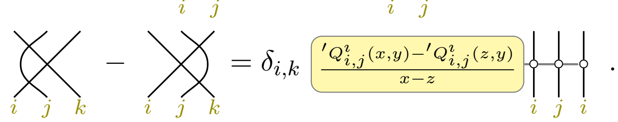
of degrees $2d_i$ and $-d_i a_{i,j}$, respectively, subject to the relations



(6.18)



(6.19)



(6.20)

Then, for $\alpha \in \Lambda$ of height l, we let

$$\mathrm{QH}^\imath_\alpha := \bigoplus_{\boldsymbol{i},\boldsymbol{j} \in \langle I\rangle_\alpha} \mathrm{Hom}_{\mathbf{QH}^\imath}(\Theta_{\boldsymbol{j}}, \Theta_{\boldsymbol{i}}), \tag{6.21}$$

notation as in (5.3).

Lemma 6.3. *We have that* $\mathrm{H}_\alpha \cong \mathrm{QH}^\imath_\alpha$ *as graded* R*-algebras.*

Proof. We remind that H was defined using ${}'\mathfrak{U}^\imath$ in (6.10) so that its parameters are ${}'Q_{i,j}(x,y) = r_{i,j}r_{j,i}Q_{i,j}(x,y)$, and ${}'Q^\imath_{i,j}(x,y) = (-1)^{\delta_{i,\tau j}}{}'Q_{i,j}(x,y)$. With this in mind, it is obvious on comparing the defining relations (6.18) to (6.20) with (3.33), (3.35) and (3.36) that there is a homomorphism $\mathrm{QH}^\imath_\alpha \to \mathrm{H}_\alpha$ taking diagrams to cosets of 2-morphisms represented by the same diagrams with λ labelling the rightmost 2-cell. It is an isomorphism because it maps a basis to a basis. □

We will identify $\mathrm{QH}^\imath_\alpha$ and H_α via the isomorphism from the proof of Lemma 6.3. The general theory developed by Khovanov and Lauda in [KL09] shows that $\mathbf{QH}^\imath$ categorifies the $\mathbb{Z}[q,q^{-1}]$-form $\mathbf{f}_\mathbb{Z}$ for the algebra $\mathbf{f}$ generated by the divided powers $\theta_i^{(n)} := \theta_i^n/[n]^!_{q_i}$. By this, we mean that $K_0\left(\mathrm{Kar}(\mathbf{QH}^\imath_q)\right) \cong \mathbf{f}_\mathbb{Z}$ as $\mathbb{Z}[q,q^{-1}]$-algebras. The canonical isomorphism

$$\mathtt{K} : \mathbf{f}_\mathbb{Z} \xrightarrow{\sim} K_0\left(\mathrm{Kar}(\mathbf{QH}^\imath_q)\right) \tag{6.22}$$

maps $\theta_i \mapsto [\Theta_i]$ for each $i \in I$. There are also divided powers

$$\Theta_i^{(n)} := \left(q_i^{-\binom{n}{2}}\Theta_i^{\otimes n}, 1_{i^{(n)}}\right) \tag{6.23}$$

where $1_{i^{(n)}} \in \mathrm{QH}^\imath_{n\alpha_i}$ is the idempotent defined by (6.2). Then we can define $\theta_{\boldsymbol{i}}$ and $\Theta_{\boldsymbol{i}}$ for $\boldsymbol{i} \in \langle\!\langle I\rangle\!\rangle$ as in (5.5), and have that $\mathtt{K}(\theta_{\boldsymbol{i}}) = [\Theta_{\boldsymbol{i}}]$.

There is one more essential piece of data: for each $\alpha \in \Lambda$, we need to be given a set $\mathbf{B}_\alpha$ and irreducible graded left H_α-modules $L_\alpha(\varpi)$ for $\varpi \in \mathbf{B}_\alpha$ such that

$$\left\{L_\alpha(\varpi)\;\middle|\;\varpi \in \mathbf{B}_\alpha\right\} \tag{6.24}$$

is a full set of irreducible graded left H_α-modules up to grading shift. As at the end of [KL09, Sec. 3.2], we assume the grading shift on each $L_\alpha(\varpi)$ is chosen so that it is graded self-dual. We will not concern ourselves with the much-studied problem of determining an *explicit parameterization* of irreducibles here—all that is important for us is that the *set* of isomorphism classes of the modules $L_\alpha(\varpi)$ ($\varpi \in \mathbf{B}_\alpha$) does not depend on any choices. Then we let $P_\alpha(\varpi)$ be a projective cover and $I_\alpha(\varpi)$ be an injective hull of $L_\alpha(\varpi)$ in H_α-gmod.

Similar to (6.11), there is an equivalence

$$\mathrm{Cov} : \mathrm{Kar}(\mathbf{QH}^\imath_q) \to \bigoplus_{\alpha\in\Lambda} \mathrm{H}_\alpha\text{-gproj} \tag{6.25}$$

defined using the contravariant Yoneda equivalence together with the anti-automorphism that reflects string diagrams in a horizontal axis. It maps $\Theta_{\boldsymbol{i}}$ ($\boldsymbol{i} \in \langle\!\langle I\rangle\!\rangle_\alpha$) to a finitely generated graded projective module that is isomorphic to

$$P_\alpha(\boldsymbol{i}) := q^{\deg(\boldsymbol{i})}\mathrm{H}_\alpha\, 1_{\boldsymbol{i}}. \tag{6.26}$$

For $\varpi \in \mathbf{B}_\alpha$, let Θ_ϖ be an indecomposable object of $\mathrm{Kar}(\mathrm{QH}^\imath_q)$ such that

$$\mathrm{Cov}(\Theta_\varpi) \cong P_\alpha(\varpi). \tag{6.27}$$

Similar to (6.13), there is an equivalence $\mathrm{Nak} : \mathrm{H}_\alpha\text{-gproj} \to \mathrm{H}_\alpha\text{-ginj}$, hence, $\mathrm{Kar}(\mathbf{QH}^\imath_q)$ is also equivalent to $\bigoplus_{\alpha\in\Lambda}\mathrm{H}_\alpha$-ginj. For $\boldsymbol{i} \in \langle\!\langle I\rangle\!\rangle_\alpha$, we let

$$I_\alpha(\boldsymbol{i}) := q^{\deg(\boldsymbol{i})}(1_{\boldsymbol{i}}\, \mathrm{H}_\alpha)^\circledast \cong \mathrm{Nak}(P_\alpha(\boldsymbol{i})). \tag{6.28}$$

With these preliminaries about Cartan algebras out of the way, we can return to the study of H-gmod. For $\alpha \in \Lambda$, the functor $\jmath^\alpha : \mathrm{H}_{\geq\alpha}\text{-gmod} \to \mathrm{H}_\alpha\text{-gmod}$ defined by truncation with respect to the idempotent $\bar{e}_\alpha$ has a left adjoint $\jmath^\alpha_!$ and a right adjoint $\jmath^\alpha_*$ defined by

$$\jmath^\alpha_! := \mathrm{H}_{\geq\alpha}\bar{e}_\alpha \otimes_{\mathrm{H}_\alpha} -, \qquad\qquad \jmath^\alpha_* := \bigoplus_{\beta\in\Lambda} \mathrm{Hom}_{\mathrm{H}_\alpha}(\bar{e}_\alpha \mathrm{H}_{\geq\alpha}\bar{e}_\beta, -). \tag{6.29}$$

Composing with inflation from $\mathrm{H}_{\geq\alpha}$ to H, we can view them as functors $\jmath_!^\alpha : \mathrm{H}_\alpha\text{-gmod} \to \mathrm{H}\text{-gmod}$ and $\jmath_*^\alpha : \mathrm{H}_\alpha\text{-gmod} \to \mathrm{H}\text{-gmod}$, which we call *standardization* and *costandardization*, respectively. An important point is that both of these functors are exact. This follows from the next lemma, which shows that $\mathrm{H}_{\geq\alpha}\bar{e}_\alpha$ is a projective right H_α-module and $\bar{e}_\alpha\mathrm{H}_{\geq\alpha}$ is a projective left H_α-module.

Lemma 6.4. *For $\boldsymbol{i} \in \langle I\rangle$, we have that*

$$\bar{1}_{\boldsymbol{i}}\mathrm{H}_{\geq\alpha}\bar{e}_\alpha = \bigoplus_{\substack{\boldsymbol{j}\in\langle I\rangle_\alpha \\ x\in X(\boldsymbol{i},\boldsymbol{j})}} \bar{x}\mathrm{H}_\alpha \qquad \textit{and} \qquad \bar{e}_\alpha\mathrm{H}_{\geq\alpha}\bar{1}_{\boldsymbol{i}} = \bigoplus_{\substack{\boldsymbol{j}\in\langle I\rangle_\alpha \\ y\in Y(\boldsymbol{j},\boldsymbol{i})}} \mathrm{H}_\alpha\bar{y}$$

with $\bar{x}\mathrm{H}_\alpha \cong q^{\deg(x)}\bar{1}_{\boldsymbol{j}}\mathrm{H}_\alpha$ as a graded right H_α-module, and $\mathrm{H}_\alpha\bar{y} \cong q^{\deg(y)}\mathrm{H}_\alpha\bar{1}_{\boldsymbol{j}}$ as a graded left H_α-module, respectively.

Proof. This follows from [Bru25, (4.4)–(4.5)]. □

Now we introduce *standard* and *costandard modules*, denoted using Δ and ∇, respectively. Let

$$\mathbf{B} := \bigsqcup_{\alpha\in\Lambda} \mathbf{B}_\alpha, \tag{6.30}$$

assuming also that this set is disjoint from $\langle\!\langle I\rangle\!\rangle$. For $\alpha \in \Lambda$, $\varpi \in \mathbf{B}_\alpha$ and $\boldsymbol{i} \in \langle\!\langle I\rangle\!\rangle_\alpha$, we let

$$\Delta(\varpi) := \jmath_!^\alpha P_\alpha(\varpi), \qquad \nabla(\varpi) := \jmath_*^\alpha I_\alpha(\varpi), \tag{6.31}$$

$$\Delta(\boldsymbol{i}) := \jmath_!^\alpha P_\alpha(\boldsymbol{i}), \qquad \nabla(\boldsymbol{i}) := \jmath_*^\alpha I_\alpha(\boldsymbol{i}). \tag{6.32}$$

Since $P_\alpha(\boldsymbol{i})$ is a finite direct sum of grading shifts of the indecomposable projectives $P_\alpha(\varpi)(\varpi \in \mathbf{B}_\alpha)$, the standard module $\Delta(\boldsymbol{i})$ is a finite direct sum of grading shifts of the indecomposable standard modules $\Delta(\varpi)$ $(\varpi \in \mathbf{B}_\alpha)$. Similarly, the costandard module $\nabla(\boldsymbol{i})$ is a finite direct sum of grading shifts of the indecomposable costandard modules $\nabla(\varpi)$ $(\varpi \in \mathbf{B}_\alpha)$. Also, for $\varpi \in \mathbf{B}_\alpha$ again, there are the *proper standard* and *proper costandard modules*

$$\bar{\Delta}(\varpi) := \jmath_!^\alpha L_\alpha(\varpi), \qquad \bar{\nabla}(\varpi) := \jmath_*^\alpha L_\alpha(\varpi). \tag{6.33}$$

All of these are graded left H-modules. Fundamental to the entire theory is that

$$\mathrm{Ext}^i_{\mathrm{H}}(\Delta(\eta), \bar{\nabla}(\varpi)) \cong \mathrm{Ext}^i_{\mathrm{H}}(\bar{\Delta}(\eta), \nabla(\varpi)) \cong \begin{cases} \Bbbk & \text{if } i = 0 \text{ and } \eta = \varpi \\ \{0\} & \text{otherwise.} \end{cases} \tag{6.34}$$

See [Bru25, Cor. 4.6, Th. 7.5] for the proof of this. Also, $\Delta(\eta)$ has a $\bar{\Delta}$-flag in the sense of [Bru25, Def. 6.3] and $\nabla(\eta)$ has a $\bar{\nabla}$-flag in the sense of [Bru25, Def. 6.4], with

$$(\Delta(\eta) : \bar{\Delta}(\varpi))_q = \begin{cases} [P_\alpha(\eta) : L_\alpha(\varpi)]_q & \text{if } \eta, \varpi \in \mathbf{B}_\alpha \text{ for some } \alpha \in \Lambda \\ 0 & \text{otherwise,} \end{cases} \tag{6.35}$$

$$(\nabla(\eta) : \bar{\nabla}(\varpi))_q = \begin{cases} [I_\alpha(\eta) : L_\alpha(\varpi)]_q & \text{if } \eta, \varpi \in \mathbf{B}_\alpha \text{ for some } \alpha \in \Lambda \\ 0 & \text{otherwise.} \end{cases} \tag{6.36}$$

This is obvious from the definitions and exactness of the standardization/costandardization functors; see also [Bru25, (8.9), (8.15)].

Now everything is in place to be able to apply the general theory of graded triangular bases. It shows that the set $\mathbf{B}$ from (6.30) indexes a full set

$$\{L(\varpi) \mid \varpi \in \mathbf{B}\} \tag{6.37}$$

of irreducible graded left H-modules $L(\varpi)$ up to grading shift. By definition, $L(\varpi)$ is the irreducible head of $\bar{\Delta}(\varpi)$ and the irreducible socle of $\bar{\nabla}(\varpi)$. Assuming that $\varpi \in \mathbf{B}_\alpha$, the projective cover $P(\varpi)$ of $\bar{\Delta}(\varpi)$ in H-gmod has a Δ-flag in the sense of [Bru25, Def. 6.3] with top section $\Delta(\varpi)$ and finitely many other sections indexed by the weights $\beta \in \Lambda$ with $\beta < \alpha$. By the version of BGG reciprocity from [Bru25, Cor. 8.4], the β-section of this Δ-flag is isomorphic to $\bigoplus_{\eta \in \mathbf{B}_\beta} \Delta(\eta)^{\oplus (P(\varpi):\Delta(\eta))_q}$ with

$$(P(\varpi) : \Delta(\eta))_q := \dim_{q^{-1}} \mathrm{Hom}_{\mathrm{H}}(P(\varpi), \bar{\nabla}(\eta)) = [\bar{\nabla}(\eta) : L(\varpi)]_{q^{-1}} \in \mathbb{N}(\!(q)\!). \tag{6.38}$$

Also $[\bar{\nabla}(\eta) : L(\varpi)]_q$ is the usual graded composition multiplicity and $[\bar{\nabla}(\eta) : L(\varpi)]_{q^{-1}}$ is its image under the bar involution. There is a dual statement describing the injective hull $I(\varpi)$ of $\bar{\nabla}(\varpi)$, which will not be needed subsequently; see [Bru25, Cor. 8.9]. The following theorem is just a restatement of part of this discussion:

Theorem 6.5. *For each $\varpi \in \mathbf{B}$, let $B_\varpi \mathbb{1}_\lambda$ be the unique (up to isomorphism) indecomposable object of $\mathrm{Kar}(\mathfrak{U}_q^\imath \mathbb{1}_\lambda)$ such that $\mathrm{Cov}(B_\varpi \mathbb{1}_\lambda) \cong P(\varpi)$. Then the 1-morphisms $B_\varpi \mathbb{1}_\lambda$ $(\varpi \in \mathbf{B})$ give a full set of indecomposable objects in $\mathrm{Kar}(\mathfrak{U}_q^\imath \mathbb{1}_\lambda)$ up to grading shift.*

There are two more useful categories of H-modules, $\mathrm{H}\text{-gmod}_\Delta$ and $\mathrm{H}\text{-gmod}_\nabla$, which are the full subcategories of H-gmod consisting of modules with a Δ-flag in the sense of [Bru25, Def. 6.3] or a ∇-flag in the sense of [Bru25, Def. 6.4], respectively. These are exact categories (but not Abelian). If V is a module with a Δ-flag, it has a filtration with sections indexed by Λ, all but finitely many of which are zero, with β-section isomorphic to $\bigoplus_{\varpi \in \mathbf{B}_\beta} \Delta(\varpi)^{\oplus (V:\Delta(\varpi))_q}$ where

$$(V : \Delta(\varpi))_q = \dim_{q^{-1}} \mathrm{Hom}_{\mathrm{H}}(V, \bar{\nabla}(\varpi)) \in \mathbb{N}(\!(q)\!). \tag{6.39}$$

The Grothendieck group $K_0(\mathrm{H}\text{-gmod}_\Delta)$ is the free $\mathbb{Z}(\!(q)\!)$-module with basis $[\Delta(\varpi)]$ $(\varpi \in \mathbf{B})$ and, for V as in (6.39), we have that $[V] = \sum_{\varpi \in \mathbf{B}} (V : \Delta(\varpi))_q [\Delta(\varpi)]$. By the BGG reciprocity discussed before (6.38), finitely generated projectives have Δ-flags, and the inclusion functor H-gproj $\rightarrow$ $\mathrm{H}\text{-gmod}_\Delta$ induces an embedding

$$K_0\big(\mathrm{H}\text{-gproj}\big) \hookrightarrow K_0\big(\mathrm{H}\text{-gmod}_\Delta\big) \tag{6.40}$$

of $\mathbb{Z}[q, q^{-1}]$-modules. There are analogous dual statements about modules with ∇-flags. In particular, H-ginj embeds into $\mathrm{H}\text{-gmod}_\nabla$, and $K_0(\mathrm{H}\text{-gmod}_\nabla)$ is the free $\mathbb{Z}(\!(q^{-1})\!)$-module with basis $[\nabla(\varpi)]$ $(\varpi \in \mathbf{B})$. We refer to [Bru25] for further background.

Lemma 6.6. *For $\alpha \in \Lambda$, there is a partial order $\leq_\alpha$ on $\mathbf{B}_\alpha$ and an injective function $\boldsymbol{i}_\alpha : \mathbf{B}_\alpha \hookrightarrow \langle\!\langle I \rangle\!\rangle_\alpha$ such that*

$$P_\alpha(\boldsymbol{i}_\alpha(\varpi)) \cong P_\alpha(\varpi) \oplus \bigoplus_{\eta \in \mathbf{B}_\alpha \text{ with } \eta <_\alpha \varpi} P_\alpha(\eta)^{\oplus m_{\varpi,\eta}(q)}$$

for some $m_{\varpi,\eta}(q) \in \mathbb{N}[q, q^{-1}]$.

Proof. This is a well-known general property of quiver Hecke algebras. Its proof is explained in [KL09, Sec. 3.2] (the strategy of the proof can be seen already in [Gro98, Sec. 11]). ☐

Lemma 6.7. *For $\boldsymbol{i} \in \langle\!\langle I \rangle\!\rangle_\alpha$, $P(\boldsymbol{i})$ is isomorphic to the direct sum of the projective cover of $\Delta(\boldsymbol{i})$ and grading shifts of finitely many other indecomposable projectives $P(\eta)$ for $\eta \in \bigcup_{\beta < \alpha} \mathbf{B}_\beta$.*

Proof. Take any $\beta \in \Lambda$ and $\varpi \in \mathbf{B}_\beta$. Since $P(\boldsymbol{i}) = q^{\deg(\boldsymbol{i})} \mathrm{H} 1_{\boldsymbol{i}}$, we have that

$$\mathrm{Hom}_{\mathrm{H}}(P(\boldsymbol{i}), L(\varpi)) \cong q^{-\deg(\boldsymbol{i})} 1_{\boldsymbol{i}} L(\varpi).$$

This is $\{0\}$ unless the α-weight space of $L(\varpi)$ is non-zero. By the general theory of graded triangular bases, β is the lowest weight of $L(\varpi)$, so this implies that $\beta \leq \alpha$. Consequently, $P(\boldsymbol{i})$ is a direct sum of finitely many indecomposable projectives that are grading shifts of $P(\varpi)$ for $\varpi \in \bigcup_{\beta \leq \alpha} \mathbf{B}_\beta$. It remains to observe for $\varpi \in \mathbf{B}_\alpha$ that

$$\mathrm{Hom}_{\mathrm{H}}(\Delta(\boldsymbol{i}), L(\varpi)) = \mathrm{Hom}_{\mathrm{H}_{\geq \alpha}}(\Delta(\boldsymbol{i}), L(\varpi)) \cong \mathrm{Hom}_{\mathrm{H}_\alpha}(P_\alpha(\boldsymbol{i}), e_\alpha L(\varpi)) \cong q^{-\deg(\boldsymbol{i})} 1_{\boldsymbol{i}} L(\varpi)).$$

Here, we used adjunction plus the definition $P_\alpha(\boldsymbol{i}) = q^{\deg(\boldsymbol{i})} \mathrm{H}_\alpha 1_{\boldsymbol{i}}$. So

$$\mathrm{Hom}_{\mathrm{H}}(\Delta(\boldsymbol{i}), L(\varpi)) \cong \mathrm{Hom}_{\mathrm{H}}(P(\boldsymbol{i}), L(\varpi)).$$

We deduce for $\varpi \in \mathbf{B}_\alpha$ that the graded multiplicity of the summand $P(\varpi)$ of $P(\boldsymbol{i})$ is the same as the graded multiplicity of $P(\varpi)$ in the projective cover of $\Delta(\boldsymbol{i})$. The lemma follows. $\square$

Theorem 6.8. *There is a partial order $\leq$ on $\mathbf{B}$ and an injective function $\boldsymbol{i} : \mathbf{B} \hookrightarrow \langle\!\langle I \rangle\!\rangle$ such that the following hold:*

(1) $\boldsymbol{i}$ maps $\mathbf{B}_\alpha$ into $\langle\!\langle I \rangle\!\rangle_\alpha$ for each $\alpha \in \Lambda$;
(2) if $\eta \in \mathbf{B}_\alpha$, $\varpi \in \mathbf{B}_\beta$ and $\eta \leq \varpi$ then $\alpha \leq \beta$;
(3) for any $\varpi \in \mathbf{B}$,

$$P(\boldsymbol{i}(\varpi)) \cong P(\varpi) \oplus \bigoplus_{\eta \in \mathbf{B} \text{ with } \eta < \varpi} P(\eta)^{\oplus m_{\varpi,\eta}(q)}$$

for some $m_{\varpi,\eta}(q) \in \mathbb{N}[q, q^{-1}]$.

Proof. We define the required partial order $\leq$ on $\mathbf{B}$ by declaring for $\eta \in \mathbf{B}_\beta$ and $\varpi \in \mathbf{B}_\alpha$ that $\eta \leq \varpi$ if *either* $\beta < \alpha$ in the partial order on Λ, *or* $\beta = \alpha$ and $\eta \leq_\alpha \varpi$ in the partial order from Lemma 6.6. Of course this has the property (1). We define the function $\boldsymbol{i} : \mathbf{B} \hookrightarrow \langle\!\langle I \rangle\!\rangle$ so that $\boldsymbol{i}|_{\mathbf{B}_\alpha}$ is the function $\boldsymbol{i}_\alpha$ from Lemma 6.6. Of course this has the property (2). It remains to prove the property (3). Suppose that $\varpi \in \mathbf{B}_\alpha$. Applying the exact standardization functor $\jmath_!^\alpha$ to the decomposition in Lemma 6.6, we get that

$$\Delta(\boldsymbol{i}(\varpi)) \cong \Delta(\varpi) \oplus \bigoplus_{\eta \in \mathbf{B}_\alpha \text{ with } \eta <_\alpha \varpi} \Delta(\eta)^{\oplus m_{\varpi,\eta}(q)}.$$

By Lemma 6.7, $P(\boldsymbol{i}(\varpi))$ is the direct sum of the projective cover of this module, which is

$$P(\varpi) \oplus \bigoplus_{\eta \in \mathbf{B}_\alpha \text{ with } \eta <_\alpha \varpi} P(\eta)^{\oplus m_{\varpi,\eta}(q)},$$

and grading shifts of finitely many other indecomposable projectives $P(\eta)$ for $\eta \in \mathbf{B}_\beta$ with $\beta < \alpha$. This proves the theorem. $\square$

Corollary 6.9. *The classes $[B_{\boldsymbol{i}} \mathbb{1}_\lambda]$ for $\boldsymbol{i} \in \langle\!\langle I \rangle\!\rangle$ span $K_0\left(\mathrm{Kar}(\mathfrak{U}_q^\imath \mathbb{1}_\lambda)\right)$ as a $\mathbb{Z}[q, q^{-1}]$-module.*

Proof. Using the equivalence $\mathrm{Cov} : \mathrm{Kar}(\mathfrak{U}_q^\imath \mathbb{1}_\lambda) \to \mathrm{H}\text{-gproj}$, the uni-triangularity property from Theorem 6.8(3) implies that each $[B_\varpi]$ ($\varpi \in \mathbf{B}$) can be written as a $\mathbb{Z}[q, q^{-1}]$-linear combination of $[B_{\boldsymbol{i}}]$ for $\boldsymbol{i} \in \langle\!\langle I \rangle\!\rangle$. $\square$

For the final result in this subsection, which generalizes [BWW23, Th. 5.13], we use a well-known adjoint pair $(I_{\alpha_i,\alpha}, R_{\alpha_i,\alpha})$ of functors

$$I_{\alpha_i,\alpha} : \mathrm{H}_\alpha\text{-gmod} \to \mathrm{H}_{\alpha_i+\alpha}\text{-gmod}, \qquad R_{\alpha_i,\alpha} : \mathrm{H}_{\alpha_i+\alpha}\text{-gmod} \to \mathrm{H}_\alpha\text{-gmod}. \tag{6.41}$$

To define them, let $\bar{e}_{\alpha_i,\alpha}$ be the idempotent in $\mathrm{H}_{\alpha_i+\alpha}$ that is the image of $e_{\alpha_i,\alpha} := 1_i \otimes e_\alpha$. There is an algebra homomorphism $\iota_{\alpha_i,\alpha} : \mathrm{H}_\alpha \hookrightarrow \mathrm{H}_{\alpha_i+\alpha}$ defined in terms of string diagrams by tensoring on the left with a vertical string of color i. This maps $\bar{e}_\alpha$ to $\bar{e}_{\alpha_i,\alpha}$. Using it, we view H_α as a unital subalgebra of $\bar{e}_{\alpha_i,\alpha}\mathrm{H}_{\alpha_i+\alpha}\bar{e}_{\alpha_i,\alpha}$. Then $\mathrm{H}_{\alpha_i+\alpha}\bar{e}_{\alpha_i,\alpha}$ is a graded $(\mathrm{H}_{\alpha_i+\alpha}, \mathrm{H}_\alpha)$-bimodule and we can define

$$I_{\alpha_i,\alpha} := \mathrm{H}_{\alpha_i+\alpha}\bar{e}_{\alpha_i,\alpha} \otimes_{\mathrm{H}_\alpha} -, \quad R_{\alpha_i,\alpha} := \bar{e}_{\alpha_i,\alpha}\mathrm{H}_{\alpha+\alpha_i} \otimes_{\mathrm{H}_{\alpha+\alpha_i}} - \cong \mathrm{Hom}_{\mathrm{H}_{\alpha+\alpha_i}}(\mathrm{H}_{\alpha_i+\alpha}\bar{e}_{\alpha_i,\alpha}, -). \tag{6.42}$$

It is clear that $(I_{\alpha_i,\alpha}, R_{\alpha_i,\alpha})$ is an adjoint pair (with adjunction that is homogeneous of degree zero). Both functors are additive graded functors which take finitely generated projectives to finitely generated projectives, so they induce $\mathbb{Z}[q, q^{-1}]$-linear maps between the split Grothendieck groups $K_0(\mathrm{H}_\alpha\text{-gproj})$ and $K_0(\mathrm{H}_{\alpha+\alpha_i}\text{-gproj})$. The following lemma is well known; e.g., see [KK12].

Lemma 6.10. *Under the identification of split Grothendieck groups of H_α-gproj and $\mathrm{H}_{\alpha+\alpha_i}$-gproj with the $\mathbb{Z}[q, q^{-1}]$-forms for $\mathbf{f}_\alpha$ and $\mathbf{f}_{\alpha+\alpha_i}$, the map induced by $I_{\alpha_i,\alpha}$ corresponds to left multiplication by θ_i, and the map induced by $R_{\alpha_i,\alpha}$ corresponds to the map ${}_i\widetilde{R}$ from (5.46).*

Acting on the left with B_i defines an endofunctor of $\mathfrak{U}^\imath\mathbb{1}_\lambda$. It induces a graded functor we also denote by

$$B_i : \text{H-gmod} \to \text{H-gmod}. \tag{6.43}$$

This is defined explicitly by tensoring with the (H,H)-bimodule $\bigoplus_{\boldsymbol{j}\in\langle I\rangle} \mathrm{H}1_{i\boldsymbol{j}}$ viewed as a left H-module in the natural way and as a right H-module with action defined by pulling-back the natural right action along the homomorphism $\mathrm{H} \to \mathrm{H}$ which maps a string diagram to the string diagram obtained by adding a vertical string labelled i on the left boundary.

Lemma 6.11. *The functor* B_i : H-gmod $\to$ H-gmod *is isomorphic to the functor defined by tensoring with the graded* (H,H)*-bimodule*

$$\bigoplus_{\beta\in\Lambda}\bigoplus_{\boldsymbol{i}\in\langle I\rangle_\beta} q_i^{(\lambda-\beta)_i-\varsigma_i-1} 1_{(\tau i)\boldsymbol{i}}\mathrm{H}$$

with the natural right action of H *and the left action defined by pull-back along the homomorphism which adds a vertical string labelled* τi *on the left boundary.*

Proof. This is similar to [BV22, Lem. 2.5]. The idea is to define mutually inverse bimodule homomorphisms by attaching a suitable cup or cap to the string diagrams. $\square$

The short exact sequence in the following theorem is important because it categorifies the equality

$$b_i\widetilde{\jmath}^{-1}(x) = \widetilde{\jmath}^{-1}(\theta_i x) + q_i^{(\lambda-\alpha)_i-\varsigma_i-1}\widetilde{\jmath}^{-1}\big({}_{\tau i}\widetilde{R}(x)\big) \tag{6.44}$$

for $x \in \mathbf{f}_\alpha$ from (5.48), where $\widetilde{\jmath}^{-1}$ is the inverse of the isomorphism $\widetilde{\jmath} : \mathbf{f} \to \dot{\mathrm{U}}^\imath 1_\lambda$. This follows using also Lemma 6.10.

Theorem 6.12. *Take* $\alpha \in \Lambda$ *and* $i \in I$. *There is a short exact sequence of functors*

$$0 \longrightarrow q_i^{(\lambda-\alpha)_i-\varsigma_i-1} \jmath_!^{\alpha-\alpha_{\tau i}} \circ R_{\alpha_{\tau i},\alpha-\alpha_{\tau i}} \longrightarrow B_i \circ \jmath_!^{\alpha} \longrightarrow \jmath_!^{\alpha+\alpha_i} \circ I_{\alpha_i,\alpha} \longrightarrow 0$$

from H_α-gmod *to* H-gmod. *The first term should be interpreted as the zero functor if* $\alpha - \alpha_{\tau i} \notin \Lambda$.

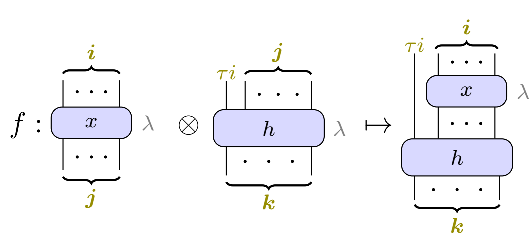
Proof. All three functors appearing in the short exact sequence are isomorphic to the functors defined by tensoring with certain graded $(\mathrm{H},\mathrm{H}_\alpha)$-bimodules, as follows:

- $\jmath_!^{\alpha-\alpha_{\tau i}} \circ R_{\alpha_{\tau i},\alpha-\alpha_{\tau i}}$ is tensoring with $q_i^{(\lambda-\alpha)_i-\varsigma_i-1}\mathrm{H}_{\geq(\alpha-\alpha_{\tau i})}\bar{e}_{\alpha-\alpha_{\tau i}} \otimes_{\mathrm{H}_{\alpha-\alpha_{\tau i}}} \bar{e}_{\alpha_{\tau i},\alpha-\alpha_{\tau i}}\mathrm{H}_\alpha$;
- $B_i \circ \jmath_!^{\alpha}$ is tensoring with $\bigoplus_{\beta\in\Lambda}\bigoplus_{\boldsymbol{i}\in\langle I\rangle_\beta} q_i^{(\lambda-\beta)_i-\varsigma_i-1}\bar{e}_{(\tau i)\boldsymbol{i}}\mathrm{H}_{\geq\alpha}\bar{e}_\alpha$ (see Lemma 6.11);
- $\jmath_!^{\alpha+\alpha_i} \circ I_{\alpha_i,\alpha}$ is tensoring with $\mathrm{H}_{\geq(\alpha+\alpha_i)}\bar{e}_{\alpha+\alpha_i} \otimes_{\mathrm{H}_{\alpha+\alpha_i}} \mathrm{H}_{\alpha+\alpha_i}\bar{e}_{\alpha_i,\alpha}$.

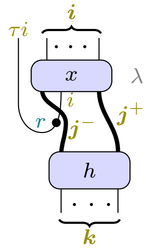
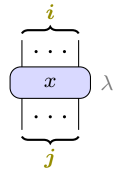
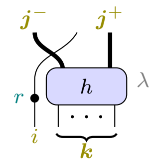
We claim that there is a short exact sequence of graded bimodules and degree-preserving bimodule homomorphisms:

$$\begin{aligned}0 \longrightarrow q_i^{(\lambda-\alpha)_i-\varsigma_i-1}\mathrm{H}_{\geq(\alpha-\alpha_{\tau i})}\bar{e}_{\alpha-\alpha_{\tau i}} \otimes_{\mathrm{H}_{\alpha-\alpha_{\tau i}}} \bar{e}_{\alpha_{\tau i},\alpha-\alpha_{\tau i}}\mathrm{H}_\alpha \xrightarrow{f}&\\ \bigoplus_{\beta\in\Lambda}\bigoplus_{\boldsymbol{i}\in\langle I\rangle_\beta} q_i^{(\lambda-\beta)_i-\varsigma_i-1}\bar{1}_{(\tau i)\boldsymbol{i}}\mathrm{H}_{\geq\alpha}\bar{e}_\alpha \xrightarrow{g} \mathrm{H}_{\geq(\alpha+\alpha_i)}\bar{e}_{\alpha+\alpha_i} \otimes_{\mathrm{H}_{\alpha+\alpha_i}} \mathrm{H}_{\alpha+\alpha_i}\bar{e}_{\alpha_i,\alpha} \longrightarrow 0.&\end{aligned}$$

The graded bimodule homomorphism f is defined on basis vectors so that



for $\beta \geq \alpha, \boldsymbol{i} \in \langle I\rangle_\beta, \boldsymbol{j} \in \langle I\rangle_{\alpha-\alpha_{\tau i}}, \boldsymbol{k} \in \langle I\rangle_\alpha$ and $x \in X(\boldsymbol{i},\boldsymbol{j}), h \in H((\tau i)\boldsymbol{j},\boldsymbol{k})$. The graded bimodule homomorphism g is defined on basis vectors so that the vectors in the image of f are in the kernel of g and

$g:$  $\mapsto$  $\otimes$ 

for $\beta \geq \alpha + \alpha_i$, $\boldsymbol{i} \in \langle I\rangle_\beta, \boldsymbol{j} = \boldsymbol{j}^- i \boldsymbol{j}^+ \in \langle I\rangle_{\alpha+\alpha_i}, \boldsymbol{k} \in \langle I\rangle_\alpha$ and $r \geq 0$. The proof that these are well-defined bimodule homomorphisms and that the sequence is exact is essentially the same as the prooof of [BWW23, Th. 5.13] so we omit any further details here. To deduce the theorem from the claim, it remains to observe by Lemma 6.4 that the bimodule in the third position in the short exact sequence is projective as a right H_α-module. Hence, this is a split short exact sequence of right H_α-modules, so it remains exact on applying $- \otimes_{\mathrm{H}_\alpha} V$ for any graded left H_α-module V. Thus, it defines an exact sequence of functors. □

6.3. **Categorification of the iSerre relation.** We continue to assume that $\mathfrak{U}^\imath$ is non-degenerate and $\Bbbk_0$ is a field. The two theorems in this subsection show that the analog of (3.7) holds in $K_0\left(\mathrm{Kar}(\mathfrak{U}^\imath_q)\right)$.

First, we treat the that case $i \neq \tau j$. Explicit formulae for the differential d and the splitting s in the following theorem can be found in the proof.

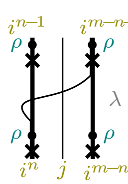
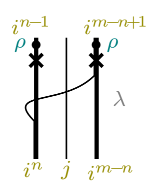
Theorem 6.13. *Suppose that $i \neq j$ and $i \neq \tau j$, and let $m := 1 - a_{i,j}$. For any $\lambda \in X^\imath$, there is a split exact complex*

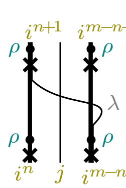
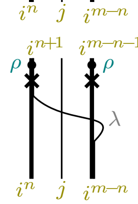
$$\begin{tikzcd} 0 \arrow[r, "d_{m+1}"] & B_i^{(m)}B_j 1_\lambda \arrow[l, dotted, bend left, "s_m"] \arrow[r, "d_m"] & \cdots \arrow[l, dotted, bend left, "s_{m-1}"] \arrow[r, "d_2"] & B_iB_jB_i^{(m-1)}1_\lambda \arrow[l, dotted, bend left, "s_1"] \arrow[r, "d_1"] & B_jB_i^{(m)}1_\lambda \arrow[l, dotted, bend left, "s_0"] \arrow[r, "d_0"] & 0 \arrow[l, dotted, bend left, "s_{-1}"] \end{tikzcd} \tag{6.45}$$

in $\mathbf{Hom}_{\mathrm{Kar}(\mathfrak{U}^\imath_q)}(\lambda, \lambda - m\alpha_i - \alpha_j)$.

Proof. When $i \neq \tau i$, this is proved in exactly the same way as the analogous result for 2-quantum groups in [Rou08, Prop. 4.2] (generalizing [KL11, Cor. 7]).

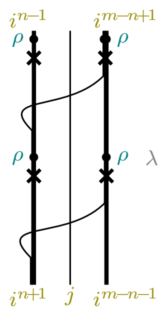
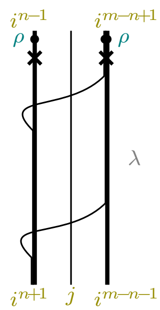
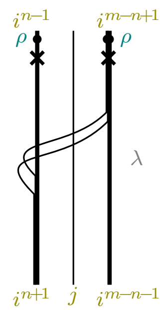
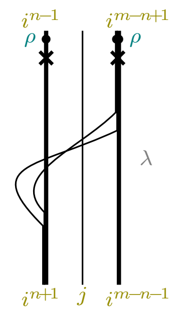
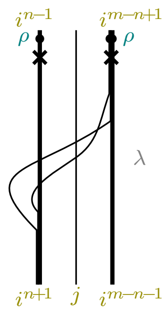
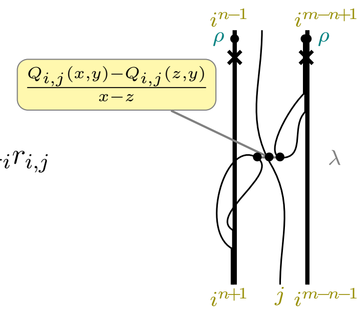
Now assume that $i = \tau i$. The relation looks like the usual Serre relation, but it is not the same since the idivided power is more complicated. However, the same proof strategy also works in the case $i = \tau i$. This is quite subtle since the nil-Brauer category is fussier than the nil-Hecke algebras, so we go through the details. We define 2-morphisms in $\mathrm{Kar}(\mathfrak{U}^\imath_q)$ by

$d_n :=$  $\overset{(6.3)}{=}$  $: B_i^{(n)}B_jB_i^{(m-n)}1_\lambda \Rightarrow B_i^{(n-1)}B_jB_i^{(m-n+1)}1_\lambda,$

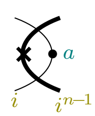
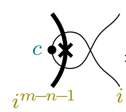
$s_n := (-1)^n t_{i,j}^{-1}$  $\overset{(6.3)}{=} (-1)^n t_{i,j}^{-1}$  $: B_i^{(n)}B_jB_i^{(m-n)}1_\lambda \Rightarrow B_i^{(n+1)}B_jB_i^{(m-n-1)}1_\lambda,$

interpreting $B_i^{(n)}B_jB_i^{(m-n)}1_\lambda$ (hence, 2-morphisms with this as their domain or range) as 0 if $n < 0$ or $n > m$.

The following calculation checks that d defines a differential making (6.45) into a complex:

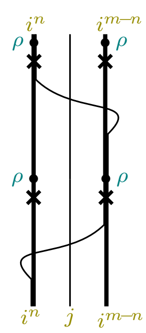
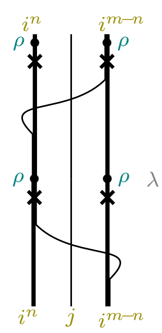
$$d_n \circ d_{n+1} =$$

$$\overset{(6.3)}{=}$$

$$=$$

$$\overset{(3.36)}{=}$$

$$\overset{(3.36)}{=}$$

$$-\,\delta_{i,\tau i} r_{i,j}$$

$$= 0.$$

The final equality needs some explanation. The first term is zero because it involves the square of a crossing of strings of color i, which is 0 by (3.35). Now assume that $i = \tau i$. Then the second term is zero because $\frac{Q_{i,j}(x,y)-Q_{i,j}(z,y)}{x-z}$ is a linear combination of monomials of the form $x^a y^b z^c$ with $a+c \leq m-2$, so either $a < n-1$ or $c \leq m-n-1$. The term arising from the monomial $x^a y^b z^c$ is zero because

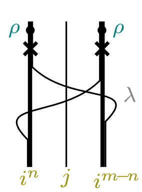
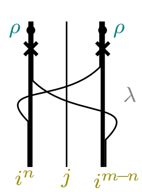
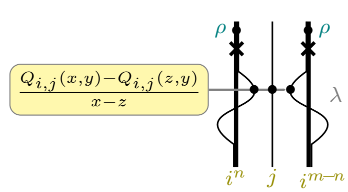

$$= 0 \text{ if } a < n-1,$$

$$= 0 \quad \text{if } c \leq m-n-1.$$

These identities are [BWW23, Lem. 4.2 and Cor. 4.3].

Finally, we prove that s is a splitting. We need to show that

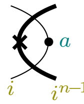
$$s_{n-1} \circ d_n + d_{n+1} \circ s_n = (-1)^{n-1} t_{i,j}^{-1} \Bigg($$

$$-$$

$$\Bigg) = \mathrm{id}_{B_i^{(n)} B_j B_i^{(m-n)} \mathbb{1}_\lambda}.$$

To see this, we rewrite the expression inside the parentheses: it equals

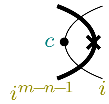

$$-$$

$$\overset{(3.36)}{=}$$

$$.$$

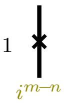
Like in the previous paragraph, $\frac{Q_{i,j}(x,y)-Q_{i,j}(z,y)}{x-z}$ is a linear combination of monomials of the form $x^a y^b z^c$ with $a+c \leq m-2$, and the monomials with $a+c = m-2$ come from the expansion

$t_{i,j}(x^{m-1}-z^{m-1})/(x-z)=\sum_{a+c=m-2} t_{i,j}x^a y^c$. Now we use the identities



$=\delta_{a,n-1}(-1)^{n-1}$


if $a \le n-1$,


$=\delta_{c,m-n-1}$


if $c \le m-n-1$

from [BWW23, Lem. 4.2] (this treats the new case when $i=\tau i$ but these identities are also well known when $i \ne \tau i$). We deduce that the terms coming from all of the monomials except for $t_{i,j}x^{n-1}z^{m-n-1}$ evaluate to 0, and the remaining term contributes $(-1)^{n-1}t_{i,j}\,\mathrm{id}_{B_i^{(n)}B_jB_i^{(m-n)}\mathbb{1}_\lambda}$, as required to finish the argument. □

Corollary 6.14. *If $i \ne j$, $i \ne \tau j$ and $m := 1-a_{i,j}$, we have that*

$$\bigoplus_{\substack{n=0\\ n\text{ even}}}^{m} B_i^{(n)}B_jB_i^{(m-n)}\mathbb{1}_\lambda \cong \bigoplus_{\substack{n=0\\ n\text{ odd}}}^{m} B_i^{(n)}B_jB_i^{(m-n)}\mathbb{1}_\lambda.$$

For the categorical iSerre relation in the case $i \ne j = \tau j$, we will work in terms of modules over the path algebra $\mathrm{H}=\mathrm{H}^\lambda$ from (6.10), adopting all of the module-theoretic notation from the previous subsection.

Lemma 6.15. *Let $\mathrm{H}=\mathrm{H}^\lambda$ for $\lambda \in X^\imath$. Suppose that $i \ne j$ and $i=\tau j$. For $m \ge 1$ and $0 \le n \le m$, there is a short exact sequence of graded* H*-modules*

$$0 \longrightarrow P(i^{(m-1)})^{\oplus f^\lambda_{n,m;i}(q)} \longrightarrow P(i^{(n)}ji^{(m-n)}) \longrightarrow \Delta(i^{(n)}ji^{(m-n)}) \longrightarrow 0$$

where $f^\lambda_{n,m;i}(q) \in \mathbb{N}(\!(q)\!)$ is defined by (5.58).

Proof. There is just one smaller weight that $m\alpha_i+\alpha_j$ in the poset Λ, namely, $(m-1)\alpha_i$. Since $\mathrm{H}_{(m-1)\alpha_i}$ is a nil-Hecke algebra, the set $\mathbf{B}_{(m-1)\alpha_i}$ contains just one element which we denote by ϖ, and $\Delta(\varpi)=\Delta(i^{(m-1)})$. Also let $\boldsymbol{i} := i^{(n)}ji^{(m-n)}$ for short. By Lemma 6.7, $P(\boldsymbol{i})$ is isomorphic to the projective cover of $\Delta(\boldsymbol{i})$ plus a finite direct sum of grading shifts of $P(\varpi)$. It follows that $P(\boldsymbol{i})$ has a two-step Δ-flag with top section isomorphic to $\Delta(\boldsymbol{i})$ and bottom section isomorphic to $\Delta(i^{(m-1)})^{\oplus g(q)}$ for some $g(q) \in \mathbb{N}(\!(q)\!)$. Since $\Delta(i^{(m-1)}) \cong P(i^{(m-1)})$ by Lemma 6.7 again, it just remains to prove that $g(q)=f^\lambda_{n,m;i}(q^{-1})$. Using (6.39) for the first equality and $P(i^nji^{m-n}) \cong P(\boldsymbol{i})^{\oplus [n]^!_{q_i}[m-n]^!_{q_i}}$ for the second, we have that

$$g(q)=\dim_{q^{-1}}\mathrm{Hom}_{\mathrm{H}}(P(\boldsymbol{i}),\bar\nabla(\varpi))=\frac{1}{[n]^!_{q_i}[m-n]^!_{q_i}}\dim_{q^{-1}} 1_{i^nji^{m-n}}\bar\nabla(\varpi).$$

By Lemma 6.4, $\bar\nabla(\varpi)$ has an explicit basis consisting of the functions $\varphi_{y,v}$ for $y \in Y(i^{m-1}, i^nji^{m-n})$ and v running over a basis for $L_{(m-1)\alpha_i}(\varpi)$, this being the function that maps $\bar y \mapsto v$ and all other basis vectors of $\bar e_{(m-1)\alpha_i}H_{\ge(m-1)\alpha_i}$ to 0. We have that $\deg(\varphi_{y,v})=\deg(v)-\deg(y)$, the graded dimension of $L(m-1)\alpha_i(\varpi)$ is $[m-1]^!_{q_i}$, and the set $Y(i^{m-1}, i^nji^{m-n})$ may be chosen so that its elements are the mirror images of the diagrams in (5.59). It follows that $g(q)=[m-1]^!_{q_i}\times\sum_D q^{\deg(D)}$ summing over these diagrams, that is, $g(q)=[m-1]^!_{q_i}f(q)/[n]^!_{q_i}[m-n]^!_{q_i}$ where $f(q)$ was computed in (5.60). This equals $f^\lambda_{n,m;i}(q)$. □

Now we establish the categorical iSerre relation in the case $i=\tau j$. Unlike Theorem 6.13, we do not know formulae for the differential or an explicit splitting.

Theorem 6.16. *Suppose that $i \ne j$ and $i=\tau j$. Fix $\lambda \in X^\imath$ and let $m := 1-a_{i,j}$. Let $f^\lambda_{n,m;i}(q) \in \mathbb{N}(\!(q)\!)$ be as in* (5.58). *There is a split exact complex*

$$0 \longrightarrow B_i^{(m-1)}\mathbb{1}_\lambda^{\oplus f^\lambda_{m,m;i}(q)} \longrightarrow B_i^{(m)}B_j\mathbb{1}_\lambda \oplus B_i^{(m-1)}\mathbb{1}_\lambda^{\oplus f^\lambda_{m-1,m;i}(q)} \longrightarrow$$

$$\cdots \longrightarrow B_i^{(n)} B_j B_i^{(m-n)} \mathbb{1}_\lambda \oplus B_i^{(m-1)} \mathbb{1}_\lambda^{\oplus f^\lambda_{n-1,m;i}(q)} \longrightarrow \cdots$$
$$\longrightarrow B_i B_j B_i^{(m-1)} \mathbb{1}_\lambda \oplus B_i^{(m-1)} \mathbb{1}_\lambda^{\oplus f^\lambda_{0,m;i}(q)} \longrightarrow B_j B_i^{(m)} \mathbb{1}_\lambda \longrightarrow 0$$

in $\mathbf{Hom}_{\mathrm{Kar}(\mathfrak{U}^\imath_q)}(\lambda, \lambda - m\alpha_i - \alpha_j)$*.*

Proof. Let $\alpha := m\alpha_i + \alpha_j \in \Lambda$. Applying the standardization functor $\jmath^\alpha_!$ to the split exact complex in H_α-gmod from [KL11, Cor. 7] and [Rou08, Prop. 4.2] (which is proved in the same way as the $i \neq \tau i$ case of Theorem 6.13)) gives a split exact sequence which is the bottom row of the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & P(i^{(m-1)})^{\oplus f^\lambda_{m,m;i}(q)} \arrow[r] \arrow[d] & \cdots \arrow[r] & P(i^{(m-1)})^{\oplus f^\lambda_{1,m;i}(q)} \arrow[r] \arrow[d] & P(i^{(m-1)})^{\oplus f^\lambda_{0,m;i}(q)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(i^{(m)}j) \arrow[r] \arrow[d] & \cdots \arrow[r] & P(iji^{(m-1)}) \arrow[r] \arrow[d] & P(ji^{(m)}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Delta(i^{(m)}j) \arrow[r] \arrow[d] & \cdots \arrow[r] & \Delta(iji^{(m-1)}) \arrow[r] \arrow[d] & \Delta(ji^{(m)}) \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 &
\end{tikzcd}$$

The columns are the short exact sequences arising from Lemma 6.15. The maps along the bottom row lift to give horizontal maps (not necessarily differentials) along the middle and top rows making the diagram commute. While this is not a double complex, we can still define a total complex for it of the form

$$0 \longrightarrow P(i^{(m-1)})^{\oplus f^\lambda_{m,m;i}(q)} \longrightarrow P(i^{(m)}j) \oplus P(i^{(m-1)})^{\oplus f^\lambda_{m-1,m;i}(q)} \longrightarrow$$
$$\cdots \longrightarrow P(i^{(n)}ji^{(m-n)}) \oplus P(i^{(m-1)})^{\oplus f^\lambda_{n-1,m;i}(q)} \longrightarrow \cdots$$
$$\longrightarrow P(iji^{(m-1)}) \oplus P(i^{(m-1)})^{\oplus f^\lambda_{0,m;i}(q)} \longrightarrow P(ji^{(m)}) \longrightarrow 0$$

by adding maps $P(i^{(n)}ji^{(m-n)}) \to P(i^{(m-1)})^{\oplus f^\lambda_{n-2,m;i}(q)}$ in the diagram above to make the total differential square to 0; these maps can be obtained by noting that the composition of two horizontal maps must lie in the image of the vertical map, since the bottom row is a complex and the columns are exact. The obvious projection to the bottom row is an isomorphism, so the total complex is exact. Since all of the modules involved are finitely generated and projective, the complex is again split exact. The complex in the statement of the theorem is obtained from this using the equivalence Cov. $\square$

Corollary 6.17. *For* $i \neq j$*,* $i = \tau j$ *and* $m := 1 - a_{i,j}$*, we have that*

$$\bigoplus_{\substack{n=0\\ n \text{ even}}}^{m} B_i^{(n)} B_j B_i^{(m-n)} \mathbb{1}_\lambda \oplus \bigoplus_{\substack{n=0\\ n \text{ odd}}}^{m} B_i^{(m-1)} \mathbb{1}_\lambda^{\oplus f^\lambda_{n,m;i}(q)} \cong \bigoplus_{\substack{n=0\\ n \text{ odd}}}^{m} B_i^{(n)} B_j B_i^{(m-n)} \mathbb{1}_\lambda \oplus \bigoplus_{\substack{n=0\\ n \text{ even}}}^{m} B_i^{(m-1)} \mathbb{1}_\lambda^{\oplus f^\lambda_{n,m;i}(q)}.$$

6.4. **Main theorem.** Now everything is in place in order to be able to state and prove the main theorem. Recall the isomorphism $\mathtt{K}$ from (6.22) and the equivalences Cov from (6.11) and (6.25). Setting $\theta_\varpi := \mathtt{K}^{-1}\big([\Theta_\varpi]\big)$, we obtain a basis $\big\{\theta_\varpi \bigm| \varpi \in \mathbf{B}\big\}$ for $\mathbf{f}_\mathbb{Z}$ as a free $\mathbb{Z}[q,q^{-1}]$-module which we call the *orthodox basis*[6]. Let $\dot{\mathrm{U}}^\imath_{\mathbb{Z}(\!(q)\!)} := \dot{\mathrm{U}}^\imath_\mathbb{Z} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}(\!(q)\!)$ and $\dot{\mathrm{U}}^\imath_{\mathbb{Q}(\!(q)\!)} := \dot{\mathrm{U}}^\imath_{\mathbb{Z}(\!(q)\!)} \otimes_\mathbb{Z} \mathbb{Q}$. There is a

[6]This terminology was introduced in [Web15].

natural inclusion $\dot{\mathrm{U}}^\imath_{\mathbb{Z}} \hookrightarrow \dot{\mathrm{U}}^\imath_{\mathbb{Z}((q))}$, and both $\dot{\mathrm{U}}^\imath_{\mathbb{Z}((q))}$ and $\dot{\mathrm{U}}^\imath$ embed into $\dot{\mathrm{U}}^\imath_{\mathbb{Q}((q))}$. Also let $\jmath_\lambda, \widetilde{\jmath}_\lambda : \dot{\mathrm{U}}^\imath 1_\lambda \xrightarrow{\sim} \mathbf{f}$ be the $\mathbb{Q}(q)$-linear isomorphisms from Theorem 5.12 and (5.48); we have added the subscript $\lambda \in X^\imath$ to avoid any confusion as λ varies.

Theorem 6.18. *Assume that $\Bbbk_0$ is a field and the 2-iquantum group $\mathfrak{U}^\imath$ is non-degenerate in the sense of Definition 5.8.*

(1) There is a unique isomorphism of locally unital $\mathbb{Z}[q,q^{-1}]$-algebras

$$\mathtt{I} : \dot{\mathrm{U}}^\imath_{\mathbb{Z}} \xrightarrow{\sim} K_0\left(\mathrm{Kar}(\mathfrak{U}^\imath_q)\right)$$

mapping $b_i 1_\lambda \mapsto [B_i \mathbb{1}_\lambda]$ for each $i \in I$ and $\lambda \in X^\imath$. It maps $b_{\boldsymbol{i}} 1_\lambda \mapsto B_{\boldsymbol{i}} \mathbb{1}_\lambda$ for any $\boldsymbol{i} \in \langle\!\langle I \rangle\!\rangle$. Setting $b_\varpi 1_\lambda := \mathtt{I}^{-1}\big([B_\varpi \mathbb{1}_\lambda]\big)$, we obtain a basis $\left\{ b_\varpi 1_\lambda \,\middle|\, \varpi \in \mathbf{B}, \lambda \in X^\imath \right\}$ for $\dot{\mathrm{U}}^\imath_{\mathbb{Z}}$ as a free $\mathbb{Z}[q,q^{-1}]$-module, which we call the iorthodox basis.

(2) For $\lambda \in X^\imath$ and $\mathrm{H} = \mathrm{H}^\lambda$ as in (6.10), there is a unique isomorphism of $\mathbb{Z}((q))$-modules

$$\widetilde{\mathtt{J}} : \dot{\mathrm{U}}^\imath_{\mathbb{Z}((q))} 1_\lambda \xrightarrow{\sim} K_0\left(\mathrm{H}\text{-gmod}_\Delta\right)$$

such that the following diagram commutes:

$$\begin{tikzcd}[column sep=large]
\substack{\dot{\mathrm{U}}^\imath_{\mathbb{Z}} 1_\lambda \\ b_{\boldsymbol{i}} 1_\lambda \quad b_\varpi 1_\lambda} \arrow[r, "\sim", "\mathtt{I}"'] \arrow[d, hook] & \substack{K_0\left(\mathrm{Kar}(\mathfrak{U}^\imath_q \mathbb{1}_\lambda)\right) \\ [B_{\boldsymbol{i}} \mathbb{1}_\lambda] \quad [B_\varpi \mathbb{1}_\lambda]} \arrow[r, "\sim", "{[\mathrm{Cov}]}"'] & \substack{K_0\left(\mathrm{H}^\lambda\text{-gproj}\right) \\ [P(\boldsymbol{i})] \quad [P(\varpi)]} \arrow[d, hook] \\
\substack{b_{\boldsymbol{i}} 1_\lambda \quad b_\varpi 1_\lambda \\ \dot{\mathrm{U}}^\imath_{\mathbb{Z}((q))} 1_\lambda \\ \delta_{\boldsymbol{i}} 1_\lambda \quad \delta_\varpi 1_\lambda} \arrow[rr, "\sim", "\widetilde{\mathtt{J}}"'] & & \substack{[P(\boldsymbol{i})] \quad [P(\varpi)] \\ K_0\left(\mathrm{H}^\lambda\text{-gmod}_\Delta\right) \\ [\Delta(\boldsymbol{i})] \quad [\Delta(\varpi)]} \\
\substack{\theta_{\boldsymbol{i}} \quad \theta_\varpi \\ \mathbf{f}_{\mathbb{Z}}} \arrow[r, "\sim", "\mathtt{K}"'] \arrow[u, hook, "\widetilde{\jmath}_\lambda^{-1}"] & \substack{[\Theta_{\boldsymbol{i}}] \quad [\Theta_\varpi] \\ K_0\left(\mathrm{Kar}(\mathbf{QH}^\imath_q)\right)} \arrow[r, "\sim", "{[\mathrm{Cov}]}"'] & \substack{[P_\alpha(\boldsymbol{i})] \quad [P_\alpha(\varpi)] \\ \bigoplus_{\alpha \in \Lambda} K_0\left(\mathrm{QH}^\imath_\alpha\text{-gproj}\right)} \arrow[u, hook, "{[\jmath^\alpha_!]}"']
\end{tikzcd} \tag{6.46}$$

For $\boldsymbol{i} \in \langle\!\langle I \rangle\!\rangle$, the element $\widetilde{\mathtt{J}}^{-1}\big([\Delta(\boldsymbol{i})]\big) \in \dot{\mathrm{U}}^\imath_{\mathbb{Z}((q))}$ is equal to $\delta_{\boldsymbol{i}} = \widetilde{\jmath}_\lambda^{-1}(\theta_{\boldsymbol{i}})$ defined in (5.50). Setting $\delta_\varpi 1_\lambda := \widetilde{\mathtt{J}}^{-1}\big([\Delta_\varpi \mathbb{1}_\lambda]\big) = \widetilde{\jmath}_\lambda^{-1}(\theta_\varpi)$, we obtain a basis $\left\{ \delta_\varpi 1_\lambda \,\middle|\, \varpi \in \mathbf{B}, \lambda \in X^\imath \right\}$ for $\dot{\mathrm{U}}^\imath_{\mathbb{Z}((q))}$ as a free $\mathbb{Z}((q))$-module, which we call the standardized orthodox basis.

Proof. (1) This is essentially the same as the proof of the analogous result for 2-quantum groups in [KL10]. First, we construct $\mathtt{I}$. Morphism spaces in $\mathbf{Hom}_{\mathfrak{U}^\imath_q}(\lambda, \kappa)$ are finite-dimensional as vector spaces over $\Bbbk_0$. This follows from Lemma 5.9. So this category has the Krull-Schmidt property. It follows that $K_0\left(\mathrm{Kar}(\mathfrak{U}^\imath_q)\right)$ is free as a $\mathbb{Z}[q,q^{-1}]$-module with basis given by isomorphism classes of indecomposable objects, up to grading shift. This means that we can identify $K_0\left(\mathrm{Kar}(\mathfrak{U}^\imath_q)\right)$ with a $\mathbb{Z}[q,q^{-1}]$-subalgebra of $K_0\left(\mathrm{Kar}(\mathfrak{U}^\imath_q)\right) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$. Also, by its definition, $\dot{\mathrm{U}}^\imath_{\mathbb{Z}}$ is a $\mathbb{Z}[q,q^{-1}]$-subalgebra of $\dot{\mathrm{U}}^\imath$. Now we define a $\mathbb{Q}(q)$-algebra homomorphism

$$\mathtt{I}_{\mathbb{Q}} : \dot{\mathrm{U}}^\imath \to K_0\left(\mathrm{Kar}(\mathfrak{U}^\imath_q)\right) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$$

on generators by mapping $1_\lambda \mapsto 1_\lambda$ and $b_i 1_\lambda \mapsto b_i 1_\lambda$ for $i \in I$ and $\lambda \in X^\imath$. Lemma 6.1 implies that $b_i^{(n)} 1_\lambda \mapsto b_i^{(n)} 1_\lambda$. Corollaries 6.14 and 6.17 show that the defining relations of $\dot{\mathrm{U}}^\imath$ from (3.7) hold in $K_0\left(\mathrm{Kar}(\mathfrak{U}^\imath_q)\right) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$ (to see this in the $i = \tau j$ case also needs Lemma 5.20). Hence, $\mathtt{I}_{\mathbb{Q}}$ is well defined. Then we define $\mathtt{I} : \dot{\mathrm{U}}^\imath_{\mathbb{Z}} \to K_0\left(\mathrm{Kar}(\mathfrak{U}^\imath_q)\right)$ to be the restriction of $\mathtt{I}_{\mathbb{Q}}$. This produces the desired $\mathbb{Z}[q,q^{-1}]$-algebra homomorphism. The homomorphism $\mathtt{I}$ is surjective thanks to Corollary 6.9. Finally we show that it is injective: by Theorem 5.17, if $y \in \ker \mathtt{I}$ then $\langle x, y \rangle^\imath = 0$ for all $x \in \dot{\mathrm{U}}^\imath_{\mathbb{Z}}$, hence, $y = 0$ since the form $\langle \cdot, \cdot \rangle^\imath$ is non-degenerate.

(2) It is always the case that $K_0(\text{H-gmod}_\Delta) \cong H_0(\text{H-gproj}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}(\!(q)\!)$ for any algebra H with a graded triangular basis. The existence of a unique isomorphism $\widetilde{\mathtt{J}}$ making the top square commute follows immediately. The bottom square in the diagram commutes by Theorem 6.12, as noted already in (6.44). □

Remark 6.19. The orthodox basis for $\mathbf{f}_{\mathbb{Z}}$ is closely related (but not necessarily equal) to the *canonical basis* of $\mathbf{f}_{\mathbb{Z}}$, which we denote by $\left\{c_\varpi \mid \varpi \in \mathbf{B}\right\}$. We are using the same labelling set $\mathbf{B}$ both for the orthodox basis and for the canonical basis. This is justified since both bases are *dual perfect bases* whose *associated crystal* is Kashiwara's crystal $\mathbf{B}(\infty)$, and via this theory all labelling sets are canonically isomorphic. By [VV11, Rou12], for symmetric Cartan matrices with geometric parameters as in Example 2.3, and assuming $\mathrm{char}(\Bbbk_0) = 0$, the orthodox basis and the canonical basis coincide: we have that $[\Theta_\varpi] = c_\varpi$ for all $\varpi \in \mathbf{B}$. There is also another basis for $\dot{\mathrm{U}}^\imath_{\mathbb{Z}(\!(q)\!)}$,

Remark 6.20. The iorthodox basis $\left\{b_\varpi 1_\lambda \mid \varpi \in \mathbf{B}, \lambda \in X^\imath\right\}$ defined by Theorem 6.18(1) should be closely related (but not necessarily equal) to the *icanonical basis* for $\dot{\mathrm{U}}^\imath_{\mathbb{Z}}$ constructed in [BW18a]. Both bases are related to the standardized orthodox basis from Theorem 6.18(2) in a uni-triangular way, so they are naturally indexed by the same set $\mathbf{B}$. There is also the *standardized canonical basis* $\left\{\widetilde{\jmath}_\lambda^{-1}(c_\varpi) \mid \varpi \in \mathbf{B}, \lambda \in X^\imath\right\}$, with similarly uni-triangular transition matrices to the iorthodox and icanonical bases. Entries in the transition matrix from the iorthodox to the standardized orthodox basis lie in $\mathbb{N}(\!(q)\!)$, and off-diagonal entries in the transition matrix from the icanonical basis to the standardized canonical basis lie in $q\mathbb{Z}[\![q]\!]$.

Remark 6.21. In this section, we have generally preferred to work with finitely generated projectives and standard modules. But it is just as natural to work in terms of finitely cogenerated injectives and costandard modules. These fit into a similar commutative diagram to (6.46):

$$\begin{tikzcd}[column sep=large]
\underset{b_{\boldsymbol{i}}1_\lambda \quad b_\varpi 1_\lambda}{\dot{\mathrm{U}}^\imath_{\mathbb{Z}}1_\lambda} \arrow[r, "\sim", "\mathtt{I}"'] \arrow[d, hook] & \underset{[B_{\boldsymbol{i}}\mathbb{1}_\lambda] \quad [B_\varpi \mathbb{1}_\lambda]}{K_0\left(\mathrm{Kar}(\mathfrak{U}^\imath_q \mathbb{1}_\lambda)\right)} \arrow[r, "\sim", "{[\mathrm{Nak}\circ\mathrm{Cov}]}"'] & \underset{[I(\boldsymbol{i})] \quad [I(\varpi)]}{K_0\left(\mathrm{H}^\lambda\text{-ginj}\right)} \arrow[d, hook] \\
\overset{b_{\boldsymbol{i}}1_\lambda \quad b_\varpi 1_\lambda}{\underset{\varrho_{\boldsymbol{i}}1_\lambda \quad \varrho_\varpi 1_\lambda}{\dot{\mathrm{U}}^\imath_{\mathbb{Z}(\!(q^{-1})\!)}1_\lambda}} \arrow[rr, "\sim", "\mathtt{J}"'] & & \overset{[I(\boldsymbol{i})] \quad [I(\varpi)]}{\underset{[\nabla(\boldsymbol{i})] \quad [\nabla(\varpi)]}{K_0\left(\mathrm{H}^\lambda\text{-gmod}_\nabla\right)}} \\
\overset{\theta_{\boldsymbol{i}} \quad \theta_\varpi}{\mathbf{f}_{\mathbb{Z}}} \arrow[u, hook, "\jmath_\lambda^{-1}"'] \arrow[r, "\sim", "\mathtt{K}"'] & \overset{[\Theta_{\boldsymbol{i}}] \quad [\Theta_\varpi]}{K_0\left(\mathrm{Kar}(\mathbf{QH}^\imath_q)\right)} \arrow[r, "\sim", "{[\mathrm{Nak}\circ\mathrm{Cov}]}"'] & \overset{[I_\alpha(\boldsymbol{i})] \quad [I_\alpha(\varpi)]}{\bigoplus_{\alpha\in\Lambda} K_0\left(\mathrm{QH}^\imath_\alpha\text{-ginj}\right)} \arrow[u, hook, "{[\jmath^\alpha_*]}"']
\end{tikzcd} \tag{6.47}$$

The proof is similar to the above and will be omitted. We call the basis ϱ_ϖ ($\varpi \in \mathbf{B}$) for $\dot{\mathrm{U}}^\imath_{\mathbb{Z}(\!(q^{-1})\!)}$ as a free $\mathbb{Z}(\!(q^{-1})\!)$-module obtained by applying $\jmath_\lambda^{-1}$ to θ_ϖ ($\varpi \in \mathbf{B}$) the *costandardized orthodox basis*; we have simply that $\varrho_\varpi = \psi^\imath(\delta_\varpi)$. Applying $\jmath_\lambda^{-1}$ instead to the canonical basis c_ϖ ($\varpi \in \mathbf{B}$) for $\mathbf{f}$ produces the *costandardized canonical basis*. We mention this in order to point out that the *fused canonical basis* for modified forms of quantum groups defined in [Wan25] is the costandardized canonical basis in the special case of iquantum groups of diagonal type.

Remark 6.22. The sesquilinear form $\langle\cdot,\cdot\rangle^\imath$ from (5.45) also has a natural interpretation in the categorification framework. For finitely generated graded projective H-modules V and W corresponding to $v, w \in \dot{\mathrm{U}}^\imath_{\mathbb{Z}}$ under the isomorphism $\mathrm{Cov}\circ\mathtt{I}$, we have that

$$\langle v, w\rangle^\imath = \dim_{q^{-1}} \mathrm{Hom}_{\mathrm{H}}(W, V) \,\big/\, \dim_{q^{-1}} \mathrm{R} \tag{6.48}$$

where $\mathrm{R} = \mathrm{End}_{\mathfrak{U}^\imath}(\lambda)$. To prove this, we may assume by Theorem 6.8 that $V = P(\boldsymbol{i})$ and $W = P(\boldsymbol{j})$ for $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle$, and then it follows by Theorem 5.17. Also, for H-modules V and W with a Δ-flag and a ∇-flag, respectively, corresponding to $v \in \dot{\mathrm{U}}^\imath_{\mathbb{Z}((q))}$ and $w \in \dot{\mathrm{U}}^\imath_{\mathbb{Z}((q^{-1}))}$ under the isomorphisms $\widetilde{\mathtt{J}}$ and $\mathtt{J}$, we have that

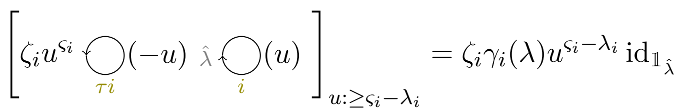
$$\langle v, w \rangle^\imath = \dim_q \mathrm{Hom}_{\mathrm{H}}(V, W) \big/ \dim_q \mathrm{R}. \tag{6.49}$$

To prove this, one can reduce to the case that $V = P(\boldsymbol{i})$ and $W = I(\boldsymbol{j})$, and in this case the formula can be deduced from (6.48) using some duality.

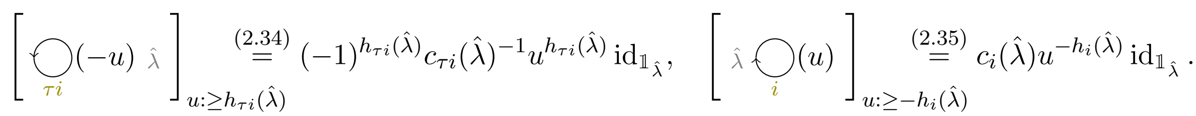
Appendix A. Checking relations

This appendix provides detailed relation checks needed in the proof of Theorem 4.13.

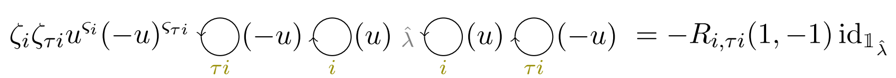
Relation (3.27): In view of (4.35), we need to check that



This follows using $\lambda_i = h_i(\hat{\lambda}) - h_{\tau i}(\hat{\lambda})$, the definition (3.19), and the identities

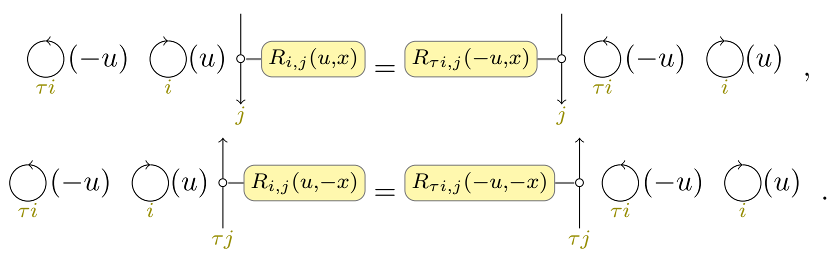


Relation (3.28): By (4.35), we need to show that

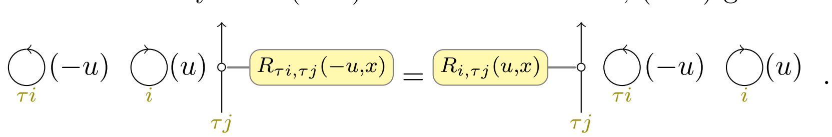


This follows from (2.32) and (3.18).

Relation (3.29): Using (4.35), this reduces to the checking the following two identities:



The first of these follows directly from (2.37). For the second one, (2.37) gives instead that



Now we use the definitions (2.36) and (3.13), which show that $R_{\tau i, \tau j}(-u, x) = r R_{i,j}(u, -x)$ and $R_{i,\tau j}(u, x) = r R_{\tau i, j}(-u, -x)$. for the same non-zero scalar $r = r_{\tau i,\tau j} r_{j,i} r_{i,\tau j} r_{j,\tau i}$.

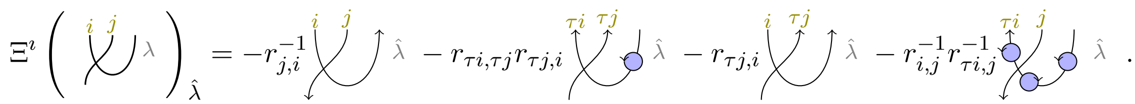
Relation (3.30): This follows from (2.13) and (2.19) using also Lemma 4.4.

Relations (3.31): These follow from (2.20)—this is the reason for the sign in (4.32).

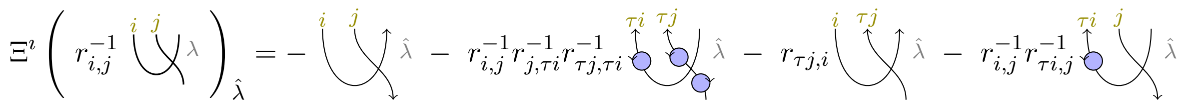
Relations (3.32): It suffices to prove the first relation involving the cup. The other one then follows by rotation, i.e. we attach caps to the top left and top right strings and use the zig-zag identity.

If $i = j = \tau i$ the relation is checked in the proof of [BWW24, Th. 4.2] (it is the relation referred to there as "the second relation from (2.4)").

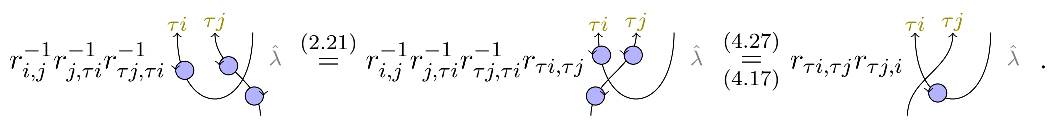
Suppose next that $i \neq j$ and $i \neq \tau j$. We have that



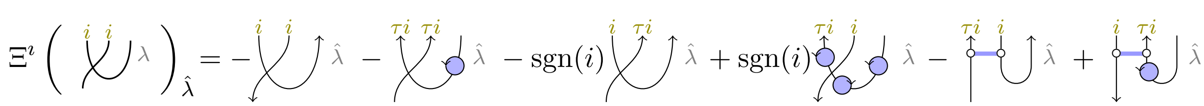
We need to show that this equals



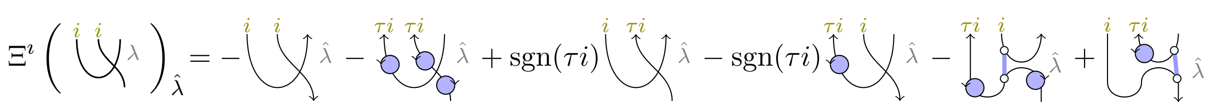
That the first and third terms are equal follows directly from (2.21), and the fourth terms are equal by (2.21) plus an application of Lemma 4.4. To see that the second terms are equal, we use Lemma 4.8:



Next suppose that $i = j \neq \tau i$. We must show that



equals

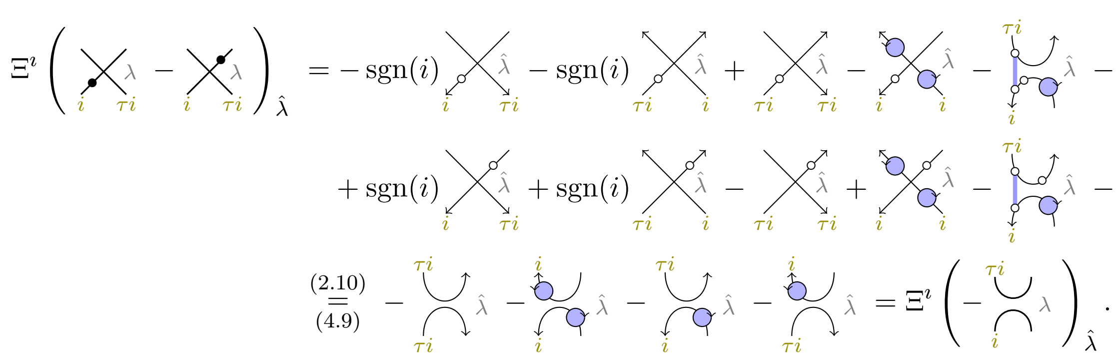


The last two terms are easily seen to be equal using the zig-zag identities and Lemma 4.4. The equality of the first four terms follows like in the previous paragraph, using also that $\operatorname{sgn}(\tau i) = -\operatorname{sgn}(i)$.

The final case $i = \tau j \neq \tau i$ follows in a similar way to the case $i = j \neq \tau i$ just treated.

Relation (3.33): In the case $i = j = \tau i$, this is checked in the proof of [BWW24, Th. 4.2] (it is the relation referred to there as "the first relation from (2.5)").

Suppose that $i = \tau j \neq \tau i$. We prove the first equality in (3.33). We have that

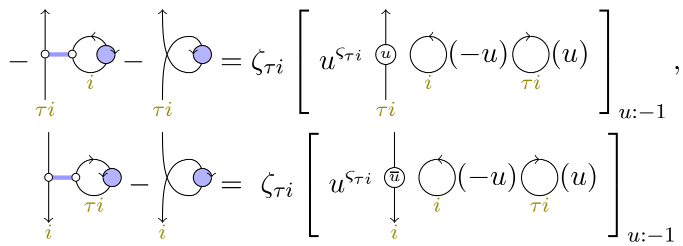


The second equality in (3.33) follows by similar considerations.

The case $i = j \neq \tau i$ follows by similar considerations.

The case that $i \neq j$ and $i \neq \tau j$ is even a bit easier.

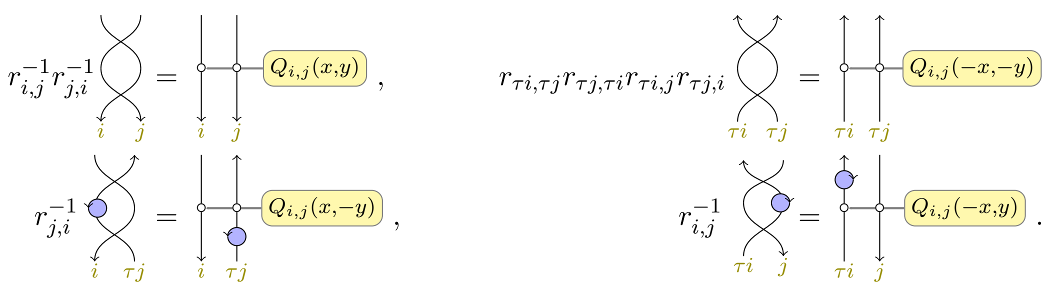
Relation (3.34): If $i = \tau i$, this is dealt with in the proof of [BWW24, Th. 4.2] (where it is "the first relation from (2.4)"). Now assume that $i \neq \tau i$. Applying $\Xi^\imath$ to (3.34) using (4.35) reduces the proof to checking the following two identities:



The first identity is (4.29) with i replaced by τi. The second identity follows from the first one on applying $\bar{\Omega}$ as explained in Remark 4.2.

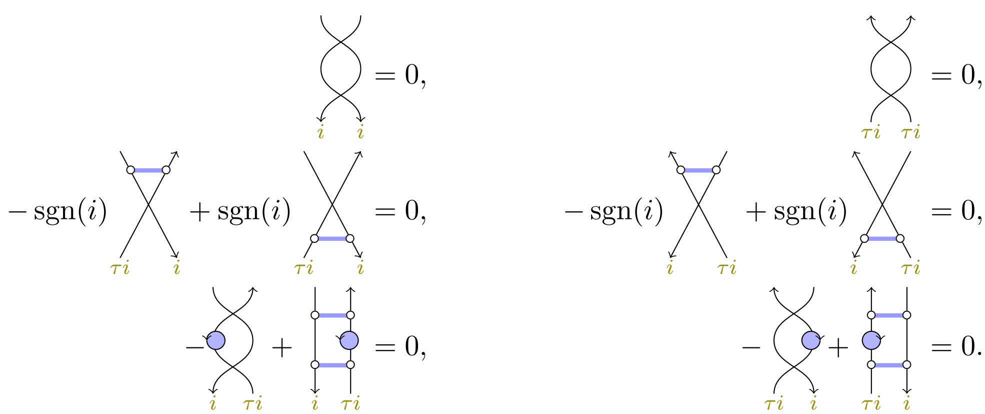
Relation (3.35): If $i = j = \tau i$, this is checked in the proof of [BWW24, Th. 4.2] (it is "the first relation from (2.2)").

If $i \neq j$ and $i \neq \tau j$, the relation to be checked expands to the following four relations (we have also used Lemma 4.4 to redistribute some internal bubbles):

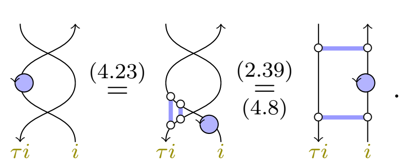


The first of these follows by (2.42), and the second by (2.11) and (3.13). The third and the fourth relations follow using (4.20) and (2.39).

Next suppose that $i = j \neq \tau i$. Using that $Q_{i,j}(x, y) = 0$, the relation to be checked expands to six relations:



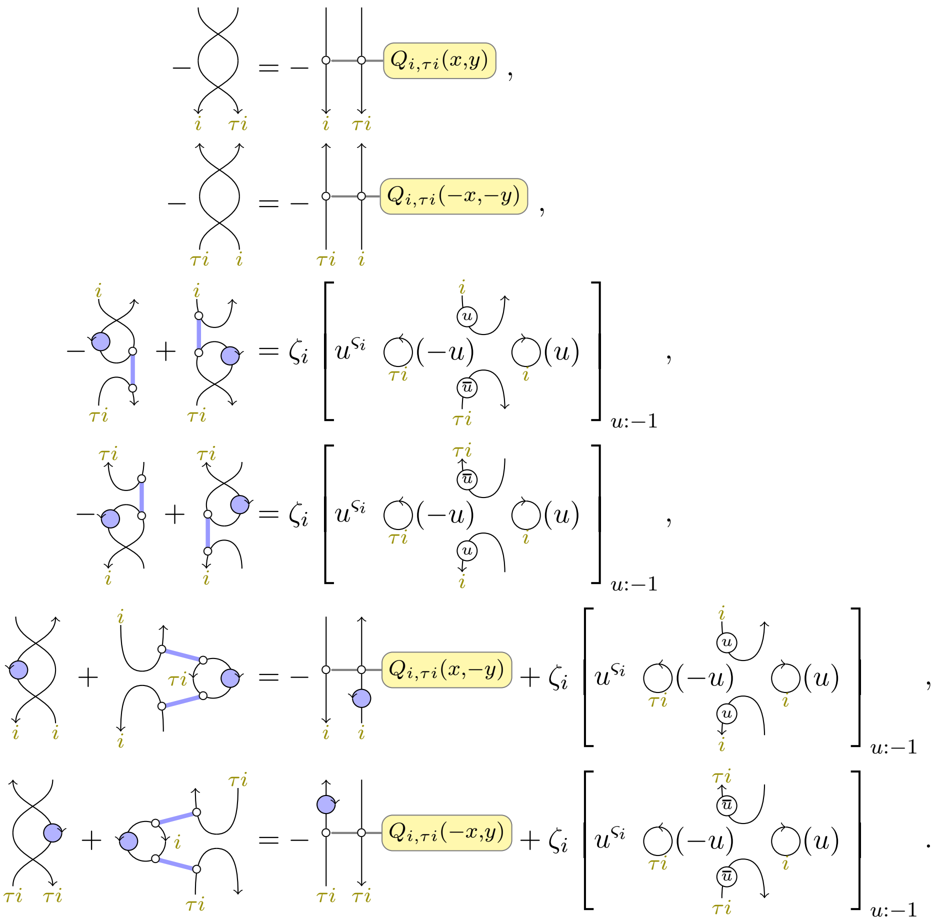
The first four of these follow easily from (2.10), (2.11) and (2.42). The fifth one follows because

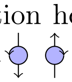


(A.1)

Then the final one follows from the fifth one by applying Σ.

Finally suppose that $i = \tau j \neq \tau i$. Again, there are six relations to check (after redistributing some internal bubbles):



The first follows from (2.42), the second from (2.11) and (3.13), the third from Lemma 4.11, the fourth follows from the third (with i replaced by τi) on applying Σ as in Remark 4.2, and the fifth relation holds by Lemma 4.12. The final relation follows from the fifth one on composing on top with , cancelling internal bubbles with Lemmas 4.4 and 4.8, then replacing i by τi and applying $\bar{\Omega}$.

Relation (3.36): Let (i, j, k) be the labels of the strings at the bottom of the braids. We noted in Remark 3.5 that the relation (3.36) is invariant under partial rotations. In view of the relations already proved, we can exploit this to reduce to proving the braid relation for just one triple (i, j, k) from each of the rotation classes

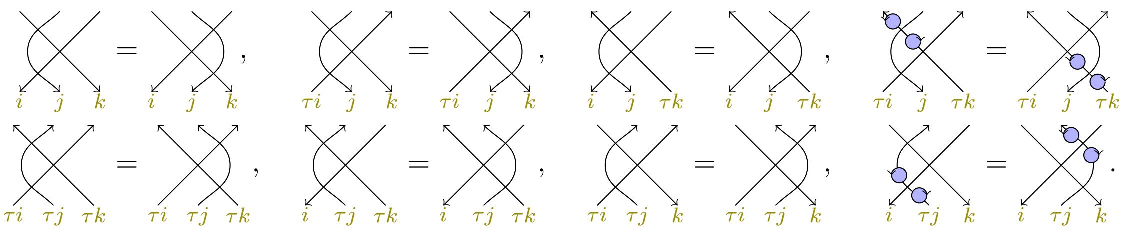
$$\{(i, j, k), (j, k, \tau i), (k, \tau i, \tau j), (\tau i, \tau j, \tau k), (\tau j, \tau k, i), (\tau k, i, j)\}.$$

Representatives for these classes are as follows:

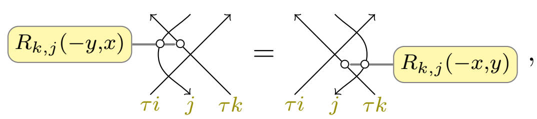
- (i, j, k) for $i, j, k \in I$ with $i \neq j, i \neq \tau j, j \neq k, j \neq \tau k, i \neq k$ and $i \neq \tau k$.
- $(i, j, \tau i)$ for $i, j \in I$ with $i \neq \tau i$, $i \neq j$ and $i \neq \tau j$.
- (i, j, i) for $i, j \in I$ with $i \neq \tau i$, $i \neq j$ and $i \neq \tau j$.
- (i, j, i) for $i, j \in I$ with $i = \tau i$ and $i \neq j$.
- (i, i, i) for $i \in I$ with $i = \tau i$.
- (i, i, i) for $i \in I$ with $i \neq \tau i$.
- $(i, \tau i, i)$ for $i \in I$ with $i \neq \tau i$.

We will check these one by one.

Case (i,j,k): This is the easiest case. Applying $\Xi^\imath$ to the relation (3.36) for i,j,k with $i \neq j, i \neq \tau j, j \neq k, j \neq \tau k, i \neq k, i \neq \tau k$ (so that the right hand side is 0), also cancelling the constants which are the same on both sides and some pairs of oppositely oriented internal bubbles, this reduces to checking the following eight equalities:

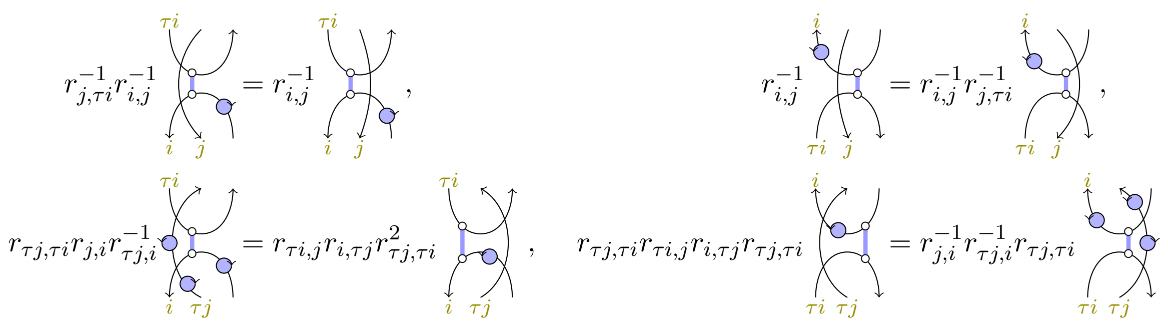


The bottom four equalities follow from the top four on applying $\bar{\Omega}$; for the last one, this also uses Corollary 4.9. Thus, we are reduced to checking the top four equalities. The first three follow directly from (2.12) and its rotations. For the last one, by (2.10) and (4.24), the identity to be checked is equivalent to



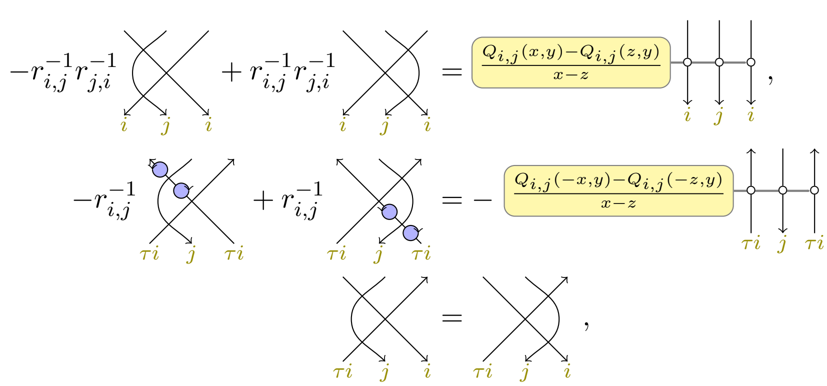
which follows by (2.10) and (2.40).

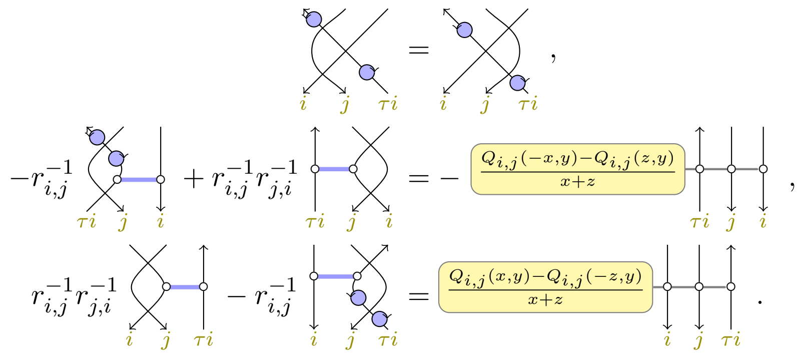
Case $(i,j,\tau i)$ *for* $i \neq \tau i$: This is quite similar to Case (i,j,k) taking $k = \tau i$. We obtain the same eight equalities appearing as before, plus there are four additional equalities arising from the additional terms in (4.38) compared to (4.36). The first eight equalities follow by the same argument as in the previous case. After cancelling some inverse pairs of internal bubbles, the remaining four are as follows:



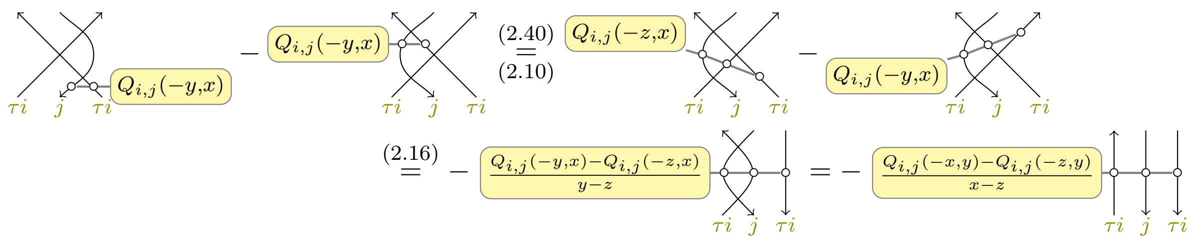
The first two follow using (2.20) to (2.22) and (4.8). For the last two, one has to use Corollary 4.9 first to replace the terms with three internal bubbles with terms with just one, then they follow using (2.20) to (2.22) and (4.8) once again.

Case (i,j,i) *for* $i \neq \tau i$: This expands to 12 relations. The six in which the j string is downward are

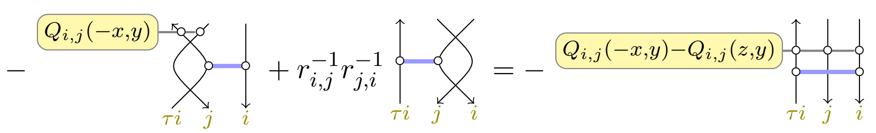




The first of these is (2.42). To prove the second, we use (4.28) to see that the left hand side equals

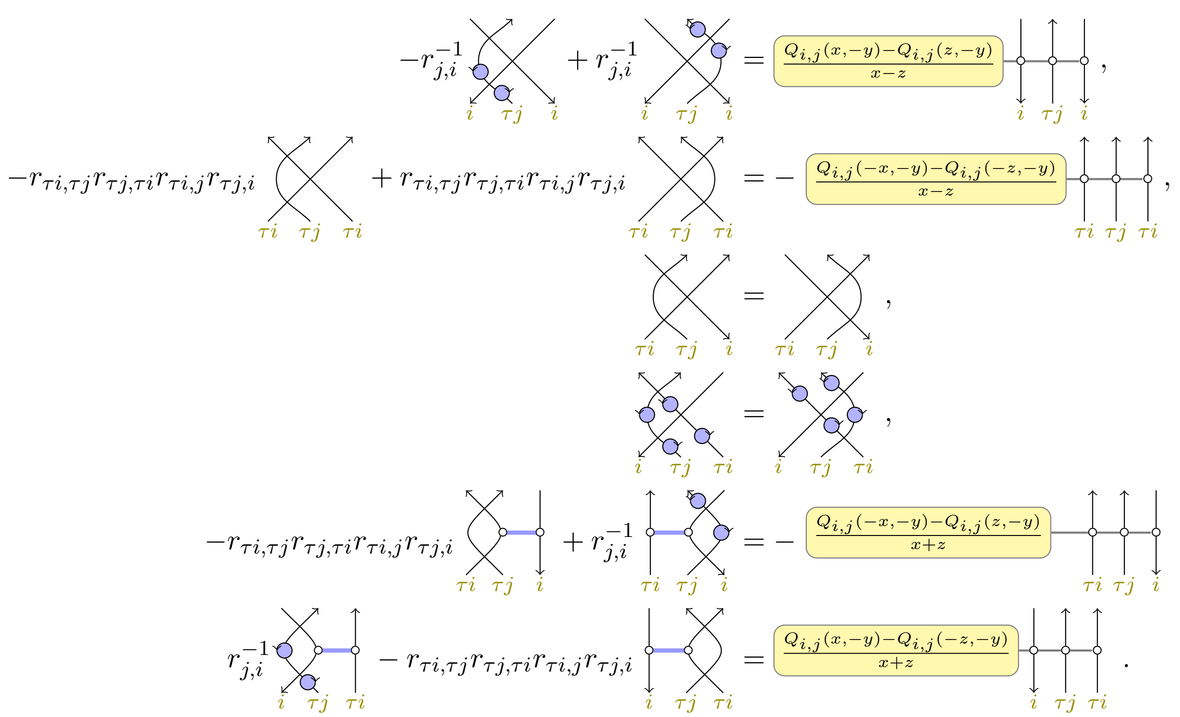


The third and the fourth equalities follow from the rotation of (2.11). By (2.42) and (4.28), the fifth identity to be proved is equivalent to



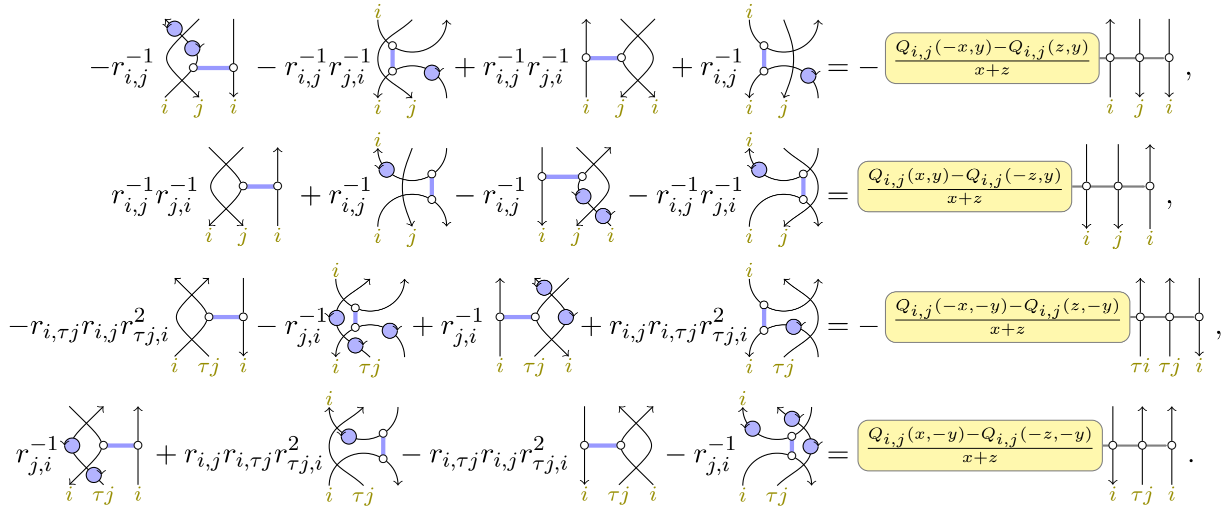
which follows using (2.39) and (2.42) to simplify the left hand side. The proof of the sixth equation is similar to the proof of the fifth one.

The other six cases with upward middle string are

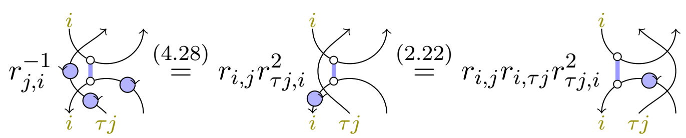


These may be verified in a similar way to the first six equalities, using also Corollary 4.9 to cancel internal bubbles in the fourth one. Alternatively, they can be deduced from identities obtained from the first six by applying $\bar{\Omega}$.

Case (i,j,i) *for* $i=\tau i$: This expands to 12 equations, eight of which (the first four of the downward six and the first four of the upward six) are the same as in the previous case, with the same proofs. However, the remaining four identities are more complicated compared to the previous case since a couple more terms arise from the extra two terms in (4.39) compared to (4.37). So we now need to show that



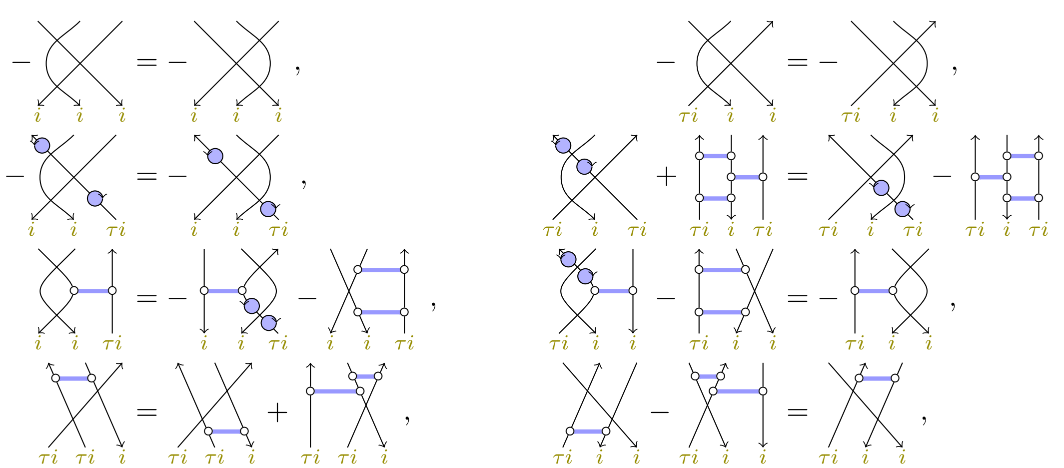
In fact, the extra terms with cups and caps on the left hand sides of these equations cancel, leaving the same four identities that were already checked in the previous case. For example, in the third equality here, we have that



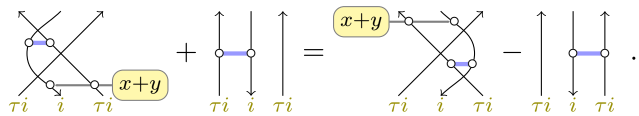
The other three cases are similar.

Case (i,i,i) *for* $i=\tau i$: This difficult case is checked in the proof of [BWW24, Th. 4.2] (where it is "the second relation from (2.2)").

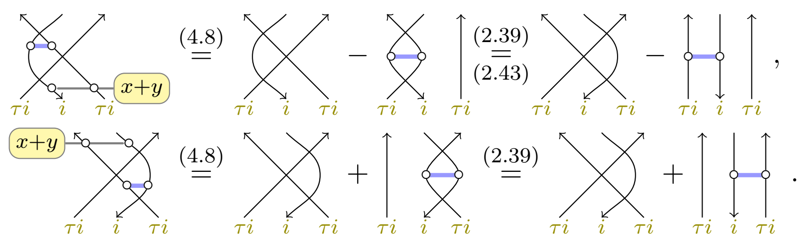
Case (i,i,i) *for* $i\neq\tau i$: There are 20 relations to be checked. We just list the first 10, with the rest essentially being obtained by applying $\bar{\Omega}$.



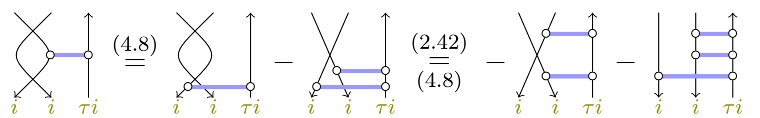
The first three of these follow from (2.43) plus some rotation. To establish the fourth equation, we apply (4.26) to remove the internal bubbles, also cancelling a couple of teleporters on both sides, to reduce to proving the identity



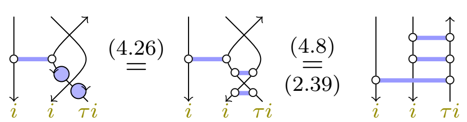
This is true because



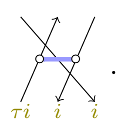
For the fifth relation, we expand the left hand side:



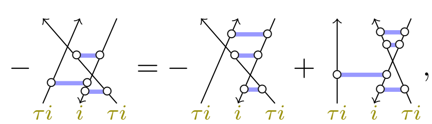
This is what we want because



The proof of the sixth relation is similar. To prove the seventh equation, a couple of applications of (4.8) shows that both sides equal

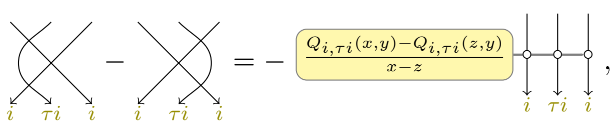


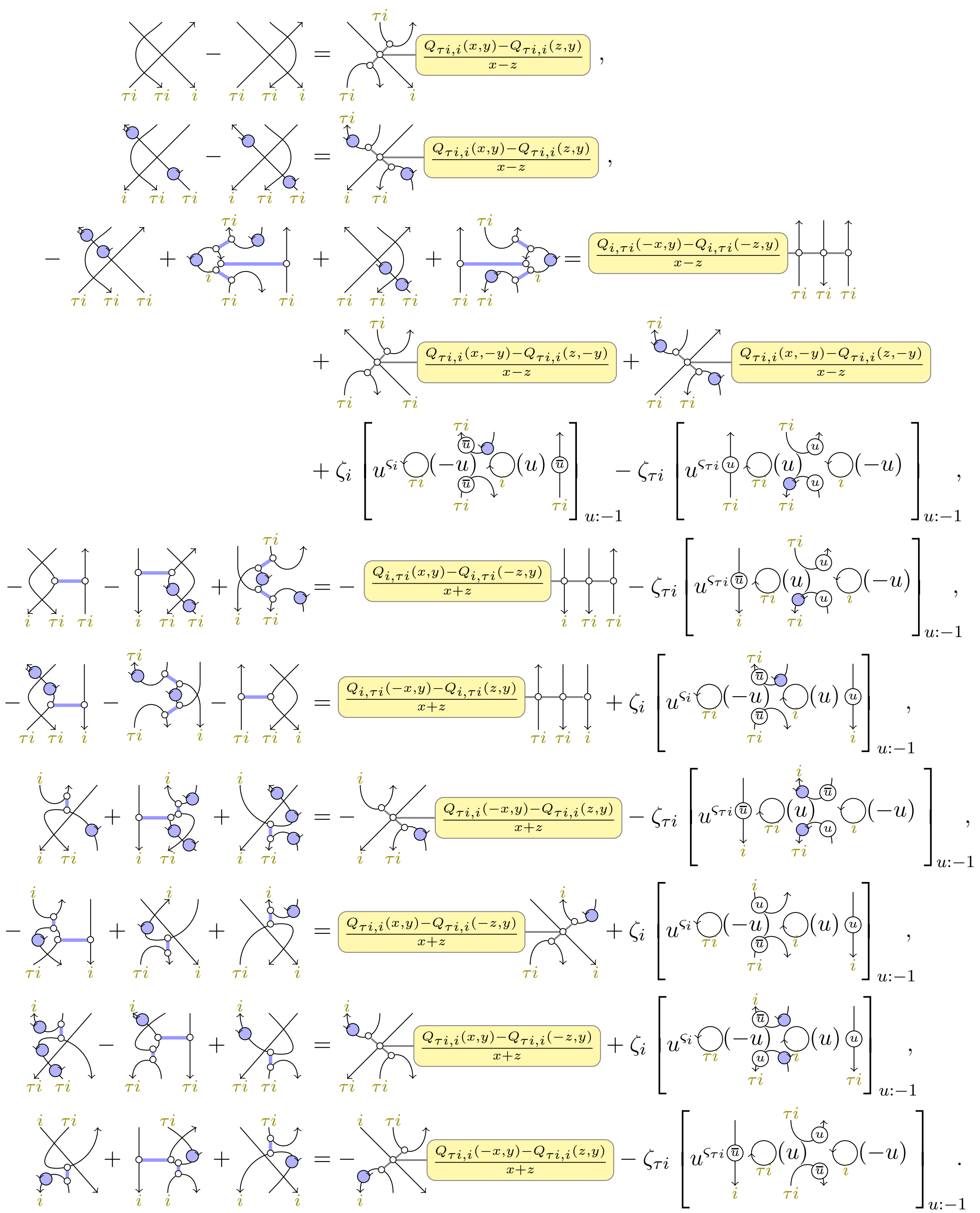
The eighth equation follows in a similar way. Finally, for the ninth equation, we first apply (4.26) to see that it is equivalent to



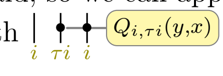
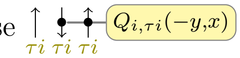
and this last equation follows by applying (4.8) to the top teleporter in the first term on the right hand side. The tenth equation is proved similarly.

Case $(i, \tau i, i)$ *for* $i \neq \tau i$: This is the ibraid relation. A more complicated expansion leads to 20 relations to be checked, which we split into two sets of 10. The second set is essentially the image of the first set under Ω up to redistributing some internal bubbles, and will be skipped. The first 10 equations to be checked are





We proceed to check these 10 relations, starting with the fourth one which we found was beyond our computational ability to check directly. Instead, we will prove it indirectly using Theorem 3.16 and the invertibility of the morphisms (4.2) (this is why we included those).

At this point we have checked all of the defining relations except for ibraid, so we can appeal to Theorem 3.16 to deduce that the image of the ibraid relation composed with  holds in $\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta)$. In particular, the fourth equation follows because 

is invertible in $\widehat{\underline{\mathfrak{U}}}(\varsigma,\zeta)$ as we have inverted all of the 2-morphisms (4.2). It is not hard to verify the second, fifth, seventh, eighth and tenth relations directly, but there is no need to do that since they follow using the same trick as for the fourth one.

It remains to prove the first, third, sixth and ninth relations. The first one is (2.43). The third follows from (2.43) transformed by a partial rotation. The sixth relation follows from the fifth one by adding a clockwise internal bubble to the top right string and a counterclockwise internal bubble to the bottom right string, cancelling internal bubbles with Lemma 4.4, then applying Σ. The ninth relation follows from the fifth one by attaching a rightward cap to the top right string and a leftward cup to the bottom left string.

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