ON THE DEFINITION OF QUANTUM HEISENBERG CATEGORY

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Abstract. We introduce a diagrammatic monoidal category $\mathcal{H}eis_k(z,t)$ which we call the *quantum Heisenberg category*; here, $k \in \mathbb{Z}$ is "central charge" and z and t are invertible parameters. Special cases were known before: for central charge $k = -1$ and parameters $z = q - q^{-1}$ and $t = -z^{-1}$ our quantum Heisenberg category may be obtained from the deformed version of Khovanov's Heisenberg category introduced by Licata and the second author by inverting its polynomial generator, while $\mathcal{H}eis_0(z,t)$ is the affinization of the HOMFLY-PT skein category. We also prove a basis theorem for the morphism spaces in $\mathcal{H}eis_k(z,t)$.

1. Introduction

Fix a commutative ground ring $\Bbbk$ and parameters $z, t \in \Bbbk^\times$. This paper introduces a family of pivotal monoidal categories $\mathcal{H}eis_k(z,t)$, one for each *central charge* $k \in \mathbb{Z}$. We refer to these categories as *quantum Heisenberg categories*. The terminology is due to a connection to Khovanov's Heisenberg category from [K]: our category for central charge $k = -1$ is a two parameter deformation of the category from *loc. cit.*, and is closely related to the one parameter deformation introduced already by Licata and the second author in [LS]. The category $\mathcal{H}eis_0(z,t)$ has also already appeared in the literature: it is the *affine HOMFLY-PT skein category* from [B2, §4]. For more general central charges, our categories are new. They were discovered by mimicking the approach of [B1], where the first author reformulated the definition of the degenerate Heisenberg categories introduced in [MS].

In fact, we will give three different monoidal presentations of $\mathcal{H}eis_k(z,t)$. They all start from the affine Hecke algebra AH_n associated to the symmetric group $\mathfrak{S}_n$. It is convenient to assemble these algebras for all $n \geq 0$ into a single monoidal category $\mathcal{AH}(z)$. By definition, this is the strict $\Bbbk$-linear monoidal category generated by one object $\uparrow$ and two morphisms $x : \uparrow \to \uparrow$ and $\tau : \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow$, subject to the relations

$$\tau \circ (1_\uparrow \otimes x) \circ \tau = x \otimes 1_\uparrow, \tag{1.1}$$

$$\tau \circ \tau = z\tau + 1_{\uparrow\otimes\uparrow}, \tag{1.2}$$

$$(\tau \otimes 1_\uparrow) \circ (1_\uparrow \otimes \tau) \circ (\tau \otimes 1_\uparrow) = (1_\uparrow \otimes \tau) \circ (\tau \otimes 1_\uparrow) \circ (1_\uparrow \otimes \tau). \tag{1.3}$$

The second relation here implies that τ is invertible. We also require that x is invertible, i.e., there is another generator x^{-1} such that

$$x \circ x^{-1} = x^{-1} \circ x = 1_\uparrow. \tag{1.4}$$

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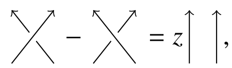
Adopting the usual string calculus for strict monoidal categories, we represent τ, τ^{-1}, x, and more generally $x^{\circ a}$ for any $a \in \mathbb{Z}$, by the diagrams


(1.5)

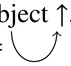
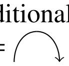
Then the relations (1.1)–(1.3) are equivalent to the following diagrammatic relations:


(1.6)


(1.7)

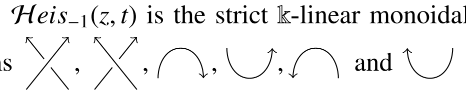

(1.8)

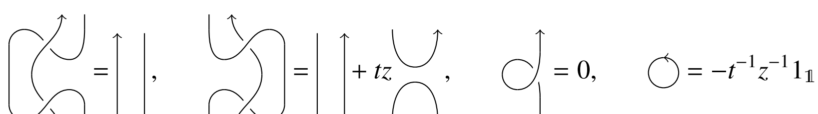
The affine Hecke algebra AH_n itself may be identified with $\mathrm{End}_{\mathcal{AH}(z)}(\uparrow^{\otimes n})$, with its standard generators x_i and τ_j coming from dot on the ith string and the positive crossing of the jth and $(j+1)$th strings, respectively; our convention for this numbers strings $1, \dots, n$ from right to left. It is often convenient to assume (passing to a quadratic extension if necessary) that $\Bbbk$ contains a root q of the quadratic equation $x^2 - zx - 1 = 0$, so that $z = q - q^{-1}$. The quadratic relation in AH_n may then be written as $(\tau_j - q)(\tau_j + q^{-1}) = 0$. Such a choice of parameter q is not needed in sections 2–4, but is essential for the applications in sections 5–10.

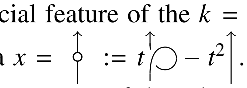
To obtain the quantum Heisenberg category $\mathcal{H}eis_k(z,t)$ from $\mathcal{AH}(z)$, we adjoin a right dual $\downarrow$ to the object $\uparrow$, i.e., we add an additional generating object $\downarrow$ and additional generating morphisms $c =$  $: \mathbb{1} \rightarrow \downarrow \otimes \uparrow$ and $d =$  $: \uparrow \otimes \downarrow \rightarrow \mathbb{1}$ subject to the relations

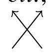

(1.9)

Then we add several more generating morphisms subject to relations which ensure that the resulting monoidal category is strictly pivotal, and moreover that there is a distinguished isomorphism $\uparrow \otimes \downarrow \cong \downarrow \otimes \uparrow \ \oplus \mathbb{1}^{\oplus k}$ if $k \geq 0$ or $\uparrow \otimes \downarrow \ \oplus \mathbb{1}^{\oplus(-k)} \cong \downarrow \otimes \uparrow$ if $k \leq 0$. There are various equivalent ways to accomplish this in practice; see sections 2–4. In these sections, we establish the equivalence of the three approaches, and record many other useful relations which follow from the defining ones, including the property already mentioned that $\mathcal{H}eis_k(z,t)$ admits a strictly pivotal structure.

In this paragraph, we explain the approach from section 4 in the special case $k = -1$. According to Definition 4.1 and (4.14), $\mathcal{H}eis_{-1}(z,t)$ is the strict $\Bbbk$-linear monoidal category generated by objects $\uparrow, \downarrow$ and morphisms  subject to (1.7)–(1.9), the relations



and one more relation, which is equivalent to (1.4). We have *not* included the generating morphism x since, due to a special feature of the $k = -1$ case, it can be recovered from the other generators via the formula . The relations in Definition 4.1 which involve x such as (1.6) are consequences of the other relations with one exception: we must still impose that x is invertible, that is, the relation (1.4).

The deformed Heisenberg category $\mathcal{H}(q^2)$ introduced in [LS] is (the additive envelope of) the strict $\mathbb{k}$-linear monoidal category defined by the same presentation as in the previous paragraph, with the parameters satisfying $tz = -1$, but *without* the relation (1.4). This follows easily on comparing our presentation with the one in *loc. cit.*, using also the fact that our category is strictly pivotal. The Hecke algebra generator $T =$  from [LS, Definition 2.1] is related to our τ by $T = q\tau$ (so that the quadratic relation becomes $(T_j - q^2)(T_j + 1) = 0$). The polynomial generator X appearing just before [LS, Lemma 3.8] is our $-x$. In fact, the category $\mathcal{H}(q^2)$ may be identified with the monoidal subcategory of our category $\mathcal{H}eis_{-1}(z, -z^{-1})$ consisting of all objects and all morphisms which do not involve negative powers of x.

Motivation for the definition of $\mathcal{H}eis_k(z,t)$ comes from the fact that it acts on various other well-known categories appearing in representation theory. If $k = 0$ and $t = q^n$ then $\mathcal{H}eis_k(z,t)$ acts on representations of $U_q(\mathfrak{gl}_n)$, with the generating objects $\uparrow$ and $\downarrow$ acting by tensoring with the natural $U_q(\mathfrak{gl}_n)$-module and its dual, respectively; see section 5. This action is an extension of the monoidal functor from the HOMFLY-PT skein category to the category of finite-dimensional $U_q(\mathfrak{gl}_n)$-modules constructed originally by Turaev [T]. If $k \neq 0$ then $\mathcal{H}eis_k(z,t)$ acts on representations of the cyclotomic Hecke algebras of level $|k|$ from [AK], with $\uparrow$ and $\downarrow$ acting by induction and restriction functors if $k < 0$, or vice versa if $k > 0$; see section 6. When $k = -1$, this specializes to the action of the deformed Heisenberg category on modules over the usual (finite) Hecke algebras associated to the symmetric groups constructed already in [LS]. The action of $\mathcal{H}eis_{-l}(z,t)$ on representations of cyclotomic Hecke algebras extends to an action on category $\mathcal{O}$ over the rational Cherednik algebras of type $\mathfrak{S}_n \wr \mathbb{Z}/l$ for all $n \geq 0$, with $\uparrow$ and $\downarrow$ acting by certain Bezrukavnikov-Etingof induction and restriction functors from [BE]; see section 7.

We also prove a basis theorem for the morphism spaces in $\mathcal{H}eis_k(z,t)$; see section 10 for the precise statement. In particular, our basis theorem implies that the *center* $\mathrm{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1})$ of the quantum Heisenberg category is the tensor product $\mathrm{Sym} \otimes \mathrm{Sym}$ of *two* copies of the algebra of symmetric functions. In the degenerate case studied in [B2], the basis theorem was proved by treating the cases $k = 0$ and $k \neq 0$ separately, appealing to results from [BCNR] and [MS]; the proofs in *loc. cit.* ultimately exploited analogs of the categorical actions mentioned above, on representations of degenerate cyclotomic Hecke algebras and representations of $\mathfrak{gl}_n(\mathbb{C})$, respectively. In the quantum case, it is still possible to prove the basis theorem when $k = 0$ by such an argument, but for non-zero k the approach from [MS] seems to be unmanageable due to the larger center. Instead, we prove the basis theorem here by following the approach developed in the degenerate case in [BSW1, Theorem 6.4] (and earlier, in the context of Kac-Moody 2-categories, in [W2]). It depends crucially on the existence of an action of $\mathcal{H}eis_k(z,t)$ on a "sufficiently large" module category, which is obtained by choosing $l \gg 0$ then taking the tensor product of actions of $\mathcal{H}eis_{-l}(z,t)$ and $\mathcal{H}eis_{k+l}(z,1)$ on representations of suitably generic cyclotomic Hecke algebras of levels l and $k+l$, respectively.

The construction of this categorical tensor product involves a remarkable monoidal functor from $\mathcal{H}eis_k(z,t)$ to a certain localization of the symmetric product $\mathcal{H}eis_l(z,u) \odot \mathcal{H}eis_m(z,v)$ for $k = l + m$ and $t = uv$. This functor is defined in section 8 and is the quantum analog of the categorical comultiplication from [BSW1, Theorem 5.3]. The particular tensor products exploited to prove the basis theorem are generic examples of *generalized cyclotomic quotients* of $\mathcal{H}eis_k(z,t)$; see section 9 for the general definition. In fact, these $\mathbb{k}$-linear categories first appeared in [W1, Proposition 5.6], but in a rather different form; the precise relationship between the categories of *loc. cit.* and the ones here will be explained in [BSW2].

We have stopped short of proving any results about the *decategorification* of $\mathcal{H}eis_k(z,t)$ here, but let us make some remarks about this. There are two complementary points of view:

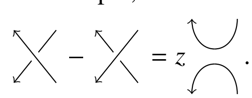
- One can consider the *Grothendieck ring* $K_0(\mathrm{Kar}(\mathcal{H}eis_k(z,t)))$ of the additive Karoubi envelope of $\mathcal{H}eis_k(z,t)$. For generic z (i.e., when q is not a root of unity), we expect that this is isomorphic to a $\mathbb{Z}$-form for a central reduction of the universal enveloping algebra of the infinite-dimensional Heisenberg Lie algebra, just as was established in the degenerate case in [BSW1, Theorem 1.1]. However, there is a significant obstruction to proving this result in the quantum case: we do not know how to show that the split Grothendieck group $K_0(AH_n)$ of the affine Hecke algebra is isomorphic to that of the finite Hecke algebra.

- Alternatively, one can pass to the *trace* (or zeroth Hochschild homology). In [CLLSS], this was computed already for the category $\mathcal{H}(q^2)$ of [LS], revealing an interesting connection to the elliptic Hall algebra. Using the basis theorem proved here, we expect it should be possible to extend the calculations made in *loc. cit.* to give a description of the trace of the full category $\mathcal{H}eis_k(z,t)$ for all $k \in \mathbb{Z}$.

In the main body of the article, proofs of all lemmas involving purely diagrammatic manipulations have been omitted. However, we have attempted to give enough details for the reader familiar with the analogous calculations in the degenerate case from [B2, §2] and [BSW1, §5] to be able to reconstruct the proofs. The first two authors are currently preparing a sequel [BS] in which we incorporate a (symmetric) Frobenius algebra into the definition of $\mathcal{H}eis_k(z,t)$, in a similar way to the Frobenius Heisenberg categories defined in the degenerate case in [Sa]. We will include full proofs of all of the diagrammatic lemmas in the more general Frobenius setting in this sequel.

2. First approach

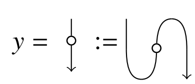
Before formulating our first definition of $\mathcal{H}eis_k(z,t)$, let us make some general remarks. We refer to the relation (1.7) as the *upward skein relation*. Rotating it through ±90° or 180°, one obtains three more skein relations; for example, here is the *leftward skein relation*


(2.1)

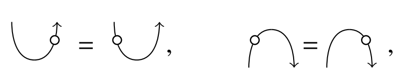
At present, this has no meaning since we have not defined the leftward cups, caps or crossings which it involves! However, already in the monoidal category obtained from $\mathcal{AH}(z)$ by adjoining a right dual ↓ to ↑ as explained in the introduction, we can introduce the rightward crossings:

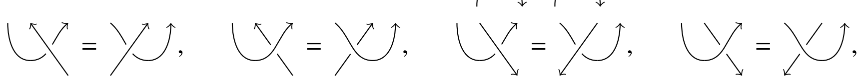

(2.2)

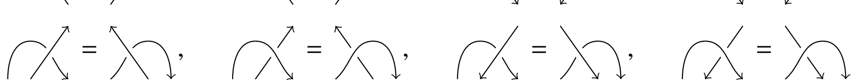
and then we see that the rightward skein relation holds from (1.7). Rotating the two rightward crossings once more by a similar procedure, we obtain positive and negative downward crossings satisfying the downward skein relation. We also define the downward dot:


(2.3)

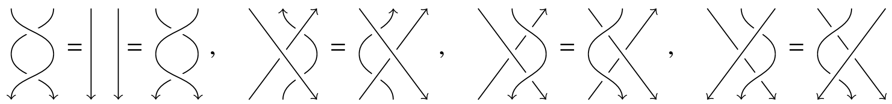
It is immediate from these definitions and (1.9) that dots and crossings slide past rightward cups and caps:


(2.4)


(2.5)


(2.6)

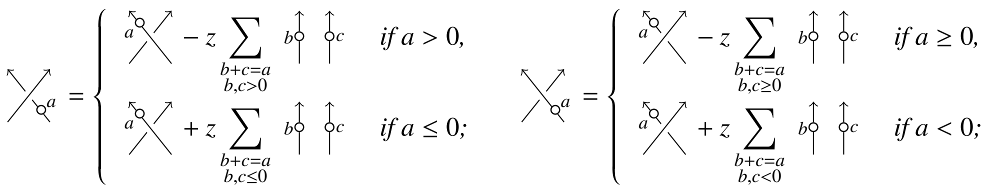
Also, the following relations are easily deduced by attaching rightward cups and caps to the relations in (1.8), then rotating the pictures using the definitions of the rightward/downward crossings:



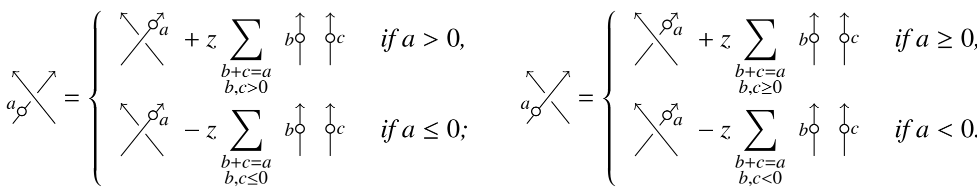
(2.7)

The following lemma will be used repeatedly (often without reference). There are analogous dot slide relations for the rightward and downward crossings (obtained by rotation).

Lemma 2.1. *The following relations hold for $a \in \mathbb{Z}$:*

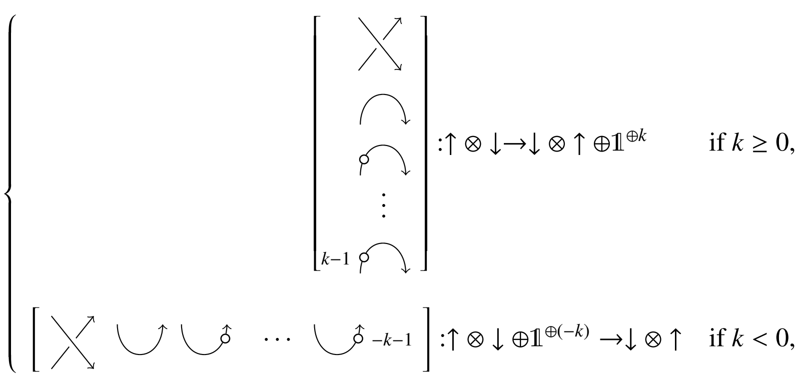


(2.8)



(2.9)

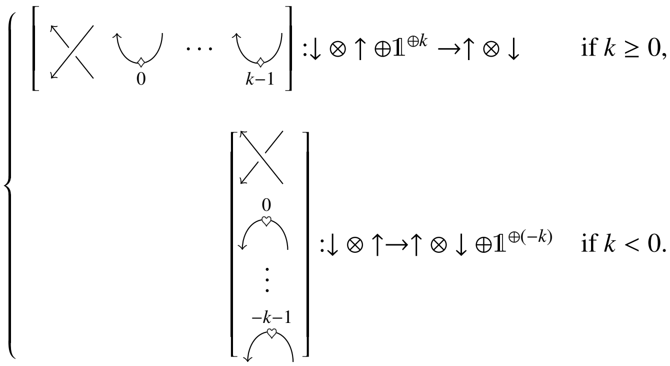
Now we can explain the first way to complete the definition of the quantum Heisenberg category following the scheme outlined in the introduction. The idea is to invert the morphism



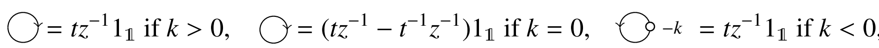
(2.10)

in $\mathrm{Add}(\mathcal{H}eis_k(z,t))$ (where Add denotes the additive envelope).

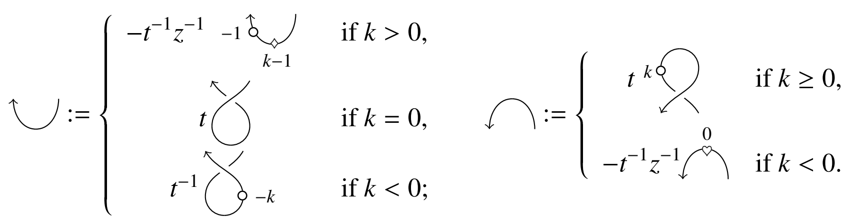
Definition 2.2. The *quantum Heisenberg category* $\mathcal{H}eis_k(z,t)$ is the strict $\mathbb{k}$-linear monoidal category obtained from $\mathcal{AH}(z)$ by adjoining a right dual $\downarrow$ to $\uparrow$ as explained in the introduction, together with the matrix entries of the following morphism which we declare to be a two-sided inverse to the morphism (2.10):


(2.11)

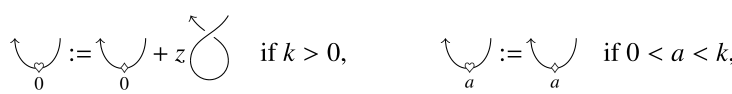
We impose one more essential relation:

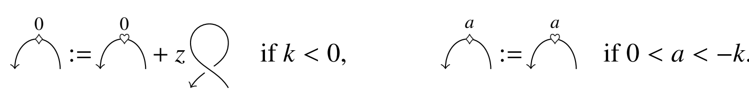

(2.12)

where the leftward cups and caps are defined by the formulas:

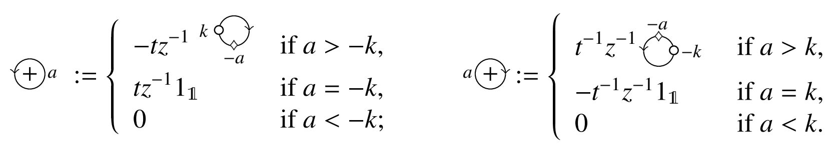

(2.13)

To complete the definition, we introduce a few more shorthands for morphisms. We have already introduced one of the two leftward crossings; define the other one so that the leftward skein relation (2.1) holds. Also set

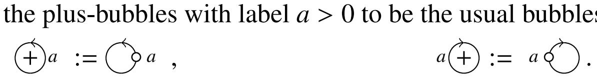

(2.14)


(2.15)

Next, introduce the following *plus-bubbles* assuming $a \leq 0$:


(2.16)

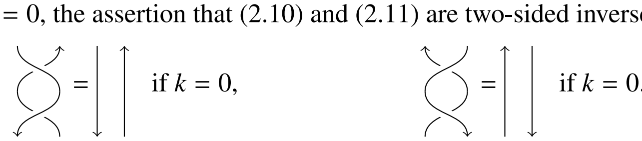
Finally, define the plus-bubbles with label $a > 0$ to be the usual bubbles with a dots:


(2.17)

Then define *minus-bubbles* for all $a \in \mathbb{Z}$ by setting


(2.18)

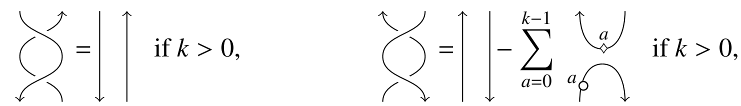
In the case $k = 0$, the assertion that (2.10) and (2.11) are two-sided inverses means that


(2.19)

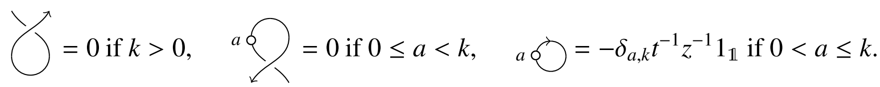
In fact, the defining relations for $\mathcal{H}eis_0(z,t)$ from Definition 2.2 are exactly the same as the ones for the affine HOMFLY-PT skein category $\mathcal{AOS}(z,t)$ from [B2, Theorem 1.1 and §4]. Thus,

$\mathcal{H}eis_0(z,t) = \mathcal{AOS}(z,t)$. Moreover, the leftward cup and cap introduced in Definition 2.2 are the same as the ones from [B2, (2.5)], and the plus- and minus-bubbles are the same as the ones from [B2, (4.13)–(4.14)]. In this case, most of the other relations that we need have already been proved in *loc. cit.*. However, the arguments there exploit a theorem of Turaev [T, Lemma I.3.3] to establish all of the relations that do not involve dots; the approach described below reproves all of these relations in a way that is indendent of Turaev's work.

When $k > 0$, the assertion that the morphisms (2.10) and (2.11) are two-sided inverses implies the following relations:

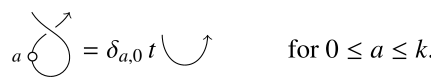


(2.20)



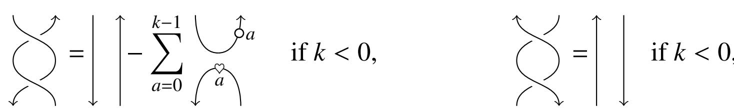
(2.21)

To derive these relations, we multiplied the matrices (2.10) and (2.11) in both orders, then equated the result with the appropriate identity matrix. The following useful relation is an easy exercise at this point; one needs to use (2.8), (2.12), (2.13) and (2.21):

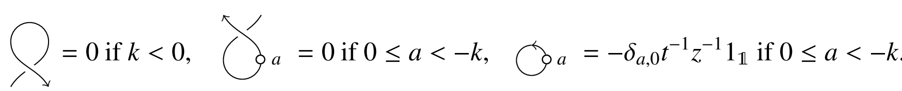


(2.22)

Finally, when $k < 0$, we will need the following relations which are deduced from (2.10) and (2.11) by the same argument as explained in the previous paragraph:

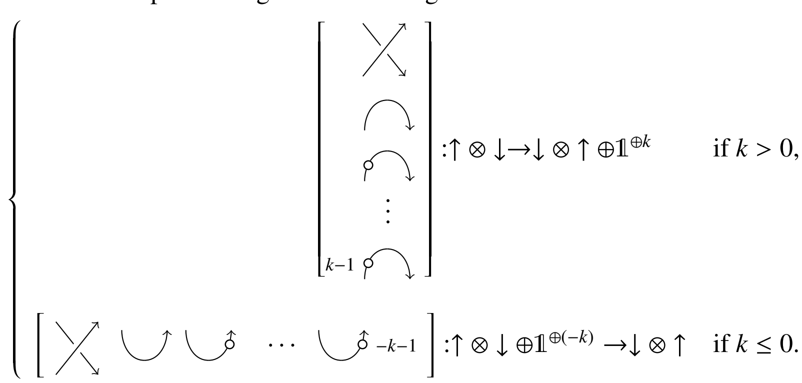


(2.23)



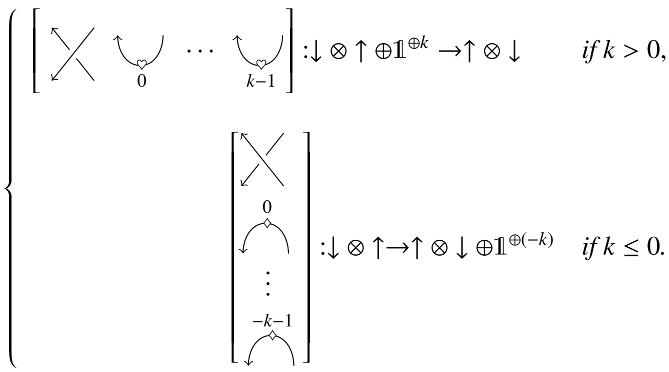
(2.24)

Now we are going to consider the counterpart of the morphism (2.10) defined using the negative instead of positive rightward crossing:



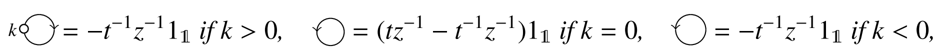
(2.25)

Lemma 2.3. *The morphism (2.25) is invertible with two-sided inverse*

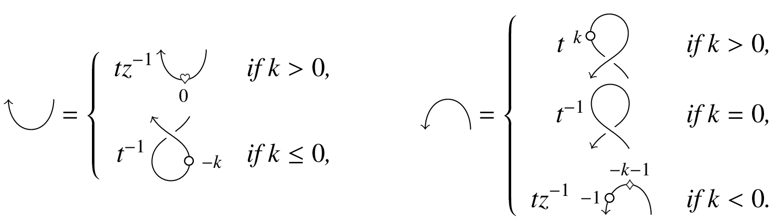


(2.26)

Moreover, we have that



(2.27)

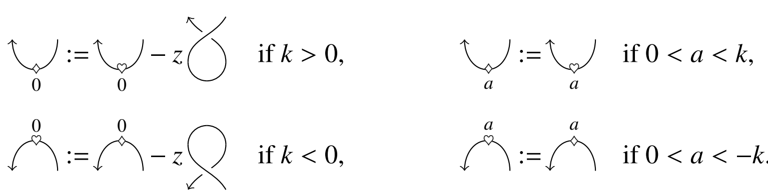


(2.28)

3. Second approach

Our second presentation for $\mathcal{H}eis_k(z,t)$ is very similar to the first presentation, but we invert the morphism (2.25) instead of (2.10).

Definition 3.1. The *quantum Heisenberg category* $\mathcal{H}eis_k(z,t)$ is the strict $\mathbb{k}$-linear monoidal category obtained from $\mathcal{AH}(z)$ by adjoining a right dual $\downarrow$ to $\uparrow$ as explained in the introduction, together with the matrix entries of the morphism (2.26), which we declare to be a two-sided inverse to (2.25). In addition, we impose the relation (2.27) for the leftward cups and caps which are defined in this approach from (2.28). Define the other leftward crossing, i.e., the one which does not appear in (2.26), so the leftward skein relation (2.1) holds. Also set



(3.1)

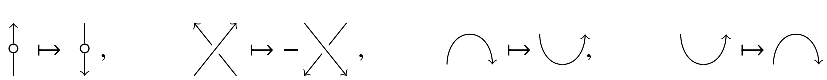
(3.2)

Finally define the plus- and minus-bubbles from (2.16)–(2.18) as before.

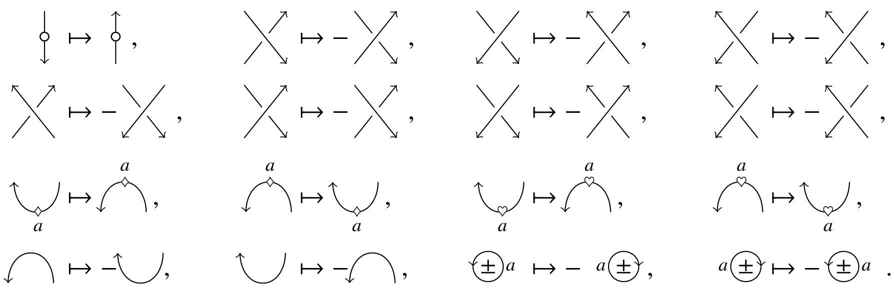
Theorem 3.2. *Definitions 2.2 and 3.1 give two different presentations for the same monoidal category, with all of the named morphisms introduced in the two definitions being the same. Moreover, there is a unique isomorphism of* $\mathbb{k}$*-linear monoidal categories*

$$\Omega_k : \mathcal{H}eis_k(z,t) \to \mathcal{H}eis_{-k}(z,t^{-1})^{\mathrm{op}} \tag{3.3}$$

sending



The effect of Ω_k on the other morphisms is as follows:



Proof. To avoid confusion, denote the category $\mathcal{H}eis_k(z,t)$ from Definition 2.2 by $\mathcal{H}eis_k^{\mathrm{old}}(z,t)$ and the one from Definition 3.1 by $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$. The relations and other definitions for the category $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$ in Definition 3.1 and the ones for $\mathcal{H}eis_{-k}^{\mathrm{old}}(z,t^{-1})$ from Definition 2.2 are related by reflecting all diagrams in a horizontal plane and multiplying by $(-1)^{x+y}$, where x is the number of crossings and y is the number of leftward cups and caps (including leftward cups and caps in plus- and minus-bubbles but not ones labelled by $\diamond$ or $\heartsuit$). It follows that there are mutually inverse isomorphisms

$$\mathcal{H}eis_{-k}^{\mathrm{old}}(z,t^{-1}) \underset{\Omega_+}{\overset{\Omega_-}{\rightleftarrows}} \mathcal{H}eis_k^{\mathrm{new}}(z,t)^{\mathrm{op}}$$

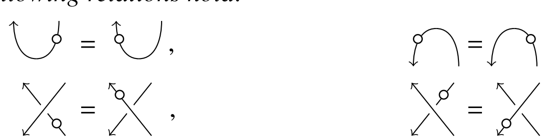
both defined in the same way as the functor Ω_k in the statement of the theorem. Now we apply Lemma 2.3 and Definition 3.1 to construct a strict $\Bbbk$-linear monoidal functor

$$\Theta_k : \mathcal{H}eis_k^{\mathrm{new}}(z,t) \to \mathcal{H}eis_k^{\mathrm{old}}(z,t)$$

which is the identity on diagrams. This functor is an isomorphism because it has a two-sided inverse, namely, $\Omega_+ \circ \Theta_{-k} \circ \Omega_-$. Thus, using Θ_k, we may identify $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$ and $\mathcal{H}eis_k^{\mathrm{old}}(z,t)$. Finally, $\Omega_k := \Omega_+$ gives the required symmetry. □

In the remainder of the section, we record some further consequences of the defining relations, thereby showing that $\mathcal{H}eis_k(z,t)$ is strictly pivotal. The first lemma explains how dots slide past leftward cups, caps and crossings. Its generalization to dots with arbitrary multiplicities $n \in \mathbb{Z}$ may also be deduced using induction and the leftward skein relation like in Lemma 2.1.

Lemma 3.3. *The following relations hold:*



(3.4)

(3.5)

Let Sym be the algebra of symmetric functions over $\Bbbk$. This is an infinite rank polynomial algebra with two algebraically independent sets of generators, namely, the *elementary symmetric functions* $e_1, e_2, \dots$ and the *complete symmetric functions* $h_1, h_2, \dots$. Adopting the convention that $e_n = h_n := \delta_{n,0}$ for $n \leq 0$, the elementary and complete symmetric functions are related by the following well-known identity [M, (I.2.6)]:

$$\sum_{r+s=n} (-1)^s e_r h_s = \delta_{n,0}. \tag{3.6}$$

The following lemma, which we may refer to as the *infinite Grassmannian relation* (following Lauda), shows that there is a well-defined homomorphism

$$\beta : \mathrm{Sym} \otimes \mathrm{Sym} \to \mathrm{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1}) \tag{3.7}$$

such that

$$h_n \otimes 1 \mapsto (-1)^{n-1} tz \ {}_{n+k}(+), \qquad 1 \otimes h_n \mapsto (-1)^n t^{-1} z \ {}_{-n}(-), \tag{3.8}$$

$$e_n \otimes 1 \mapsto t^{-1} z \, \overset{\circlearrowleft}{\oplus}_{n-k}\,, \qquad\qquad 1 \otimes e_n \mapsto -tz \, \overset{\circlearrowleft}{\ominus}_{-n}\,. \tag{3.9}$$

We will prove in Corollary 10.2 that β is actually an *isomorphism*.

Lemma 3.4. *For any $a \in \mathbb{Z}$, we have that*

$$\sum_{\substack{b,c\in\mathbb{Z}\\ b+c=a}} \overset{\circlearrowleft}{\oplus}_b \; {}_c\overset{\circlearrowright}{\oplus} = \sum_{\substack{b,c\in\mathbb{Z}\\ b+c=a}} \overset{\circlearrowleft}{\ominus}_b \; {}_c\overset{\circlearrowright}{\ominus} = -\delta_{a,0}\, z^{-2} 1_{\mathbb{1}}. \tag{3.10}$$

Moreover:

$$\overset{\circlearrowleft}{\oplus}_a = \delta_{a,-k} t z^{-1} 1_{\mathbb{1}} \quad \text{if } a \leq -k, \qquad {}_a\overset{\circlearrowright}{\oplus} = -\delta_{a,k} t^{-1} z^{-1} 1_{\mathbb{1}} \quad \text{if } a \leq k, \tag{3.11}$$

$${}_a\overset{\circlearrowright}{\ominus} = \delta_{a,0} t z^{-1} 1_{\mathbb{1}} \quad \text{if } a \geq 0, \qquad \overset{\circlearrowleft}{\ominus}_a = -\delta_{a,0} t^{-1} z^{-1} 1_{\mathbb{1}} \quad \text{if } a \geq 0. \tag{3.12}$$

Corollary 3.5. *For an indeterminate w, we have that*

$$\overset{\circlearrowleft}{\oplus}(w)\; \overset{\circlearrowright}{\oplus}(w) = \overset{\circlearrowleft}{\ominus}(w)\; \overset{\circlearrowright}{\ominus}(w) = -z^{-2}, \tag{3.13}$$

where

$$\overset{\circlearrowleft}{\oplus}(w) := \sum_{n\in\mathbb{Z}} \overset{\circlearrowleft}{\oplus}_n \; w^{-n} \in t z^{-1} w^k 1_{\mathbb{1}} + w^{k-1} \operatorname{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1})[\![w^{-1}]\!], \tag{3.14}$$

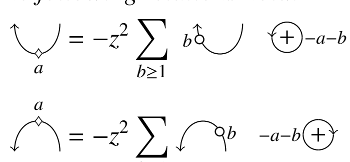
$$\overset{\circlearrowright}{\oplus}(w) := \sum_{n\in\mathbb{Z}} {}_n\overset{\circlearrowright}{\oplus} \; w^{-n} \in -t^{-1} z^{-1} w^{-k} 1_{\mathbb{1}} + w^{-k-1} \operatorname{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1})[\![w^{-1}]\!], \tag{3.15}$$

$$\overset{\circlearrowleft}{\ominus}(w) := \sum_{n\in\mathbb{Z}} \overset{\circlearrowleft}{\ominus}_n \; w^{-n} \in -t^{-1} z^{-1} 1_{\mathbb{1}} + w \operatorname{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1})[\![w]\!], \tag{3.16}$$

$$\overset{\circlearrowright}{\ominus}(w) := \sum_{n\in\mathbb{Z}} {}_n\overset{\circlearrowright}{\ominus} \; w^{-n} \in t z^{-1} 1_{\mathbb{1}} + w \operatorname{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1})[\![w]\!]. \tag{3.17}$$

Using the next relations plus (2.14) and (3.2), the leftward cups and caps decorated by $\diamond$ or $\heartsuit$ can be eliminated from any diagram.

Lemma 3.6. *The following relations hold:*

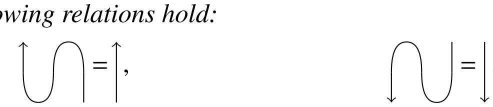


if $0 \leq a < k$, (3.18)

if $0 \leq a < -k$. (3.19)

The next lemma shows that $\downarrow$ is left dual to $\uparrow$ (as well as being right dual by the original construction). Thus, the monoidal category $\mathcal{H}eis_k(z,t)$ is rigid.

Lemma 3.7. *The following relations hold:*



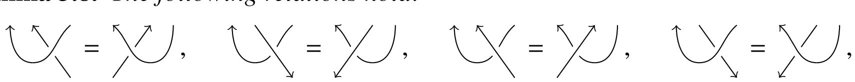
(3.20)

The final lemma together with (3.4) implies that $\mathcal{H}eis_k(z,t)$ is strictly pivotal, with duality functor

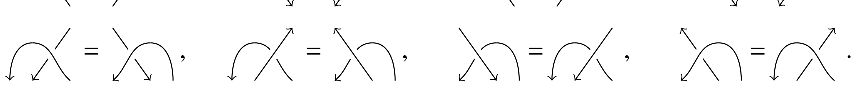
$$* : \mathcal{H}eis_k(z,t) \xrightarrow{\sim} (\mathcal{H}eis_k(z,t)^{\mathrm{op}})^{\mathrm{rev}} \tag{3.21}$$

defined on morphisms by rotating diagrams through 180°.

Lemma 3.8. *The following relations hold:*



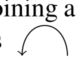
(3.22)

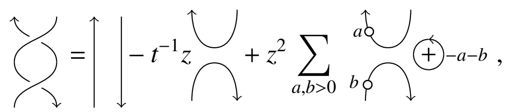


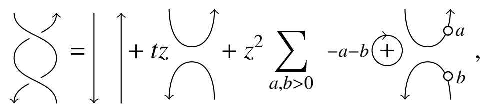
(3.23)

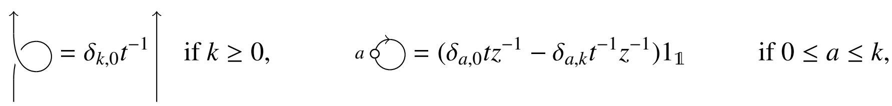
4. Third approach

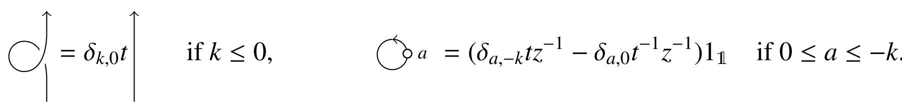
Now we have enough relations in hand to formulate our third presentation for $\mathcal{H}eis_k(z,t)$. This presentation does not involve any leftward cups or caps decorated by $\diamond$ or $\heartsuit$; Lemma 3.6 showed already that these are redundant as generators.

Definition 4.1. The *quantum Heisenberg category* $\mathcal{H}eis_k(z,t)$ is the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{AH}(z)$ by adjoining a right dual $\downarrow$ to $\uparrow$ as explained in the introduction, plus two more generating morphisms  and  subject to the following additional relations:

 (4.1)

 (4.2)

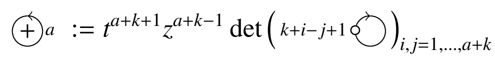
 (4.3)

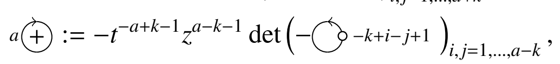
 (4.4)

Here, we have used the leftward crossings which are defined in this approach by

 (4.5)

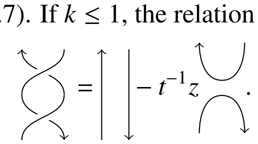
and the plus-bubbles which are defined for $a \leq 0$ by

 (4.6)

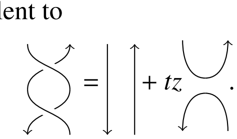
 (4.7)

interpreting the determinant of an $n \times n$ matrix as $\delta_{n,0}$ in case $n \leq 0$. Finally, define the plus-bubbles with label $a > 0$ to be the usual bubbles with a dots as in (2.17), then define the minus-bubbles for all $a \in \mathbb{Z}$ so that (2.18) holds.

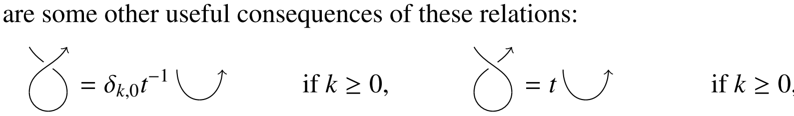
Before proving the equivalence of this definition with the earlier ones, we make some remarks about the relations (4.1)–(4.7). If $k \leq 1$, the relation (4.1) is equivalent to

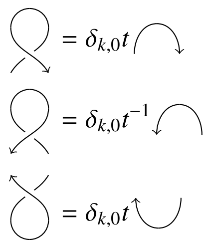
 (4.8)

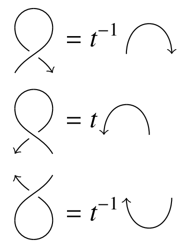
This follows immediately from the definition of the plus-bubbles from (4.6). Similarly, when $k \geq -1$, the relation (4.2) is equivalent to

 (4.9)

Here are some other useful consequences of these relations:

 (4.10)



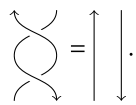


if $k \leq 0$, if $k \leq 0$, (4.11)

if $k \geq 0$, if $k \geq 0$, (4.12)

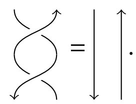
if $k \leq 0$, if $k \leq 0$. (4.13)

These follow from (4.3)–(4.4) on expanding the definitions of the sideways crossings. Then, using (4.13) and the leftward skein relation to convert the negative crossings in (4.8) to positive ones, relation (4.8) can be further simplified in case that $k < 0$: it is equivalent to



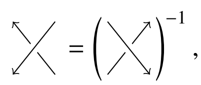
(4.14)

Similarly, (4.9) is equivalent to the following when $k > 0$:



(4.15)

Finally, when $k = 0$, the relations (4.8)–(4.9) together are equivalent to the single assertion



(4.16)

i.e., both of the relations from (2.19).

Theorem 4.2. *The category $\mathcal{H}eis_k(z,t)$ defined by Definition 4.1 is the same as the one from Definitions 2.2 and 3.1, with all morphisms introduced in the third definition being the same as the ones from before.*

Proof. To avoid confusion in the proof, we denote the category from the equivalent Definitions 2.2 and 3.1 by $\mathcal{H}eis_k^{\mathrm{old}}(z,t)$, and the one from Definition 4.1 by $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$. From the evident symmetry in the relations (4.1)–(4.7), it follows that there is an isomorphism

$$\Omega_k : \mathcal{H}eis_k^{\mathrm{new}}(z,t) \to \mathcal{H}eis_{-k}^{\mathrm{new}}(z,t^{-1})^{\mathrm{op}}$$

which reflects diagrams in a horizontal plane and multiplies by $(-1)^{x+y}$ where x is the number of crossings and y is the number of leftward cups and caps. Combining this with (3.3), we are reduced to proving the theorem under the assumption that $k \leq 0$.

We first check that all of the defining relations (4.1)–(4.7) of $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$ are satisfied in $\mathcal{H}eis_k^{\mathrm{old}}(z,t)$, so that there is a strict $\Bbbk$-linear monoidal functor

$$\Theta : \mathcal{H}eis_k^{\mathrm{new}}(z,t) \to \mathcal{H}eis_k^{\mathrm{old}}(z,t)$$

which is the identity on diagrams. For this, note to start with that (4.5) holds in $\mathcal{H}eis_k^{\mathrm{old}}(z,t)$ as we have shown that the latter category is strictly pivotal. The relation (4.6) is almost trivial when $k \leq 0$ and holds thanks to (3.11). For (4.7), the identity holds if $a - k \leq 0$ due again to (3.11), so assume that $a - k > 0$. Then the desired identity is the image under the homomorphism β from (3.7) of the identity

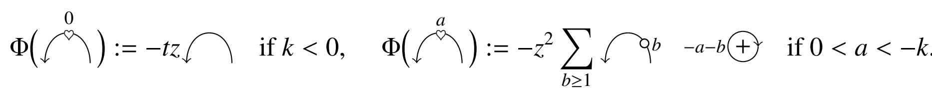
$$(-1)^{a-k-1}t^{-1}z^{-1}h_{a-k} \otimes 1 = -z^{a-k-1}t^{-a+k-1}\det\left(-tz^{-1}e_{i-j+1} \otimes 1\right)_{i,j=1,\dots,a-k}$$

in $\mathrm{Sym} \otimes \mathrm{Sym}$. This follows from the well-known identity $h_n = \det\left(e_{i-j+1}\right)_{i,j=1,\dots,n}$; see [M, Exercise I.2.8]. It remains to check the relations (4.1)–(4.4). For (4.1)–(4.2) when $k = 0$, we just need to check the equivalent form (4.16), which follows by (2.19). For (4.1) when $k < 0$, we check the equivalent form (4.14), which holds due to the second relation from (2.23). For (4.2) when $k < 0$, we use the first relation from (2.23), expanding the leftward caps decorated by ♡ using (2.13) when $a = 0$ or (2.15) and (3.19) when $a > 0$. Finally, the relations (4.3)–(4.4) follow easily from (2.24), (2.12)–(2.13) and (2.27)–(2.28).

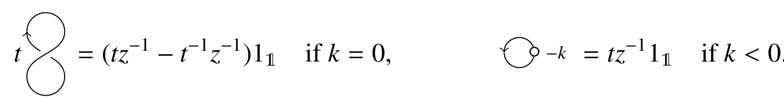
Now we want to show that Θ is an isomorphism. We do this by using the presentation from Definition 2.2 to construct a two-sided inverse

$$\Phi : \mathcal{H}eis_k^{\mathrm{old}}(z,t) \to \mathcal{H}eis_k^{\mathrm{new}}(z,t),$$

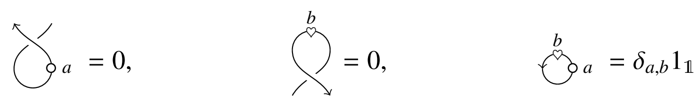
still assuming that $k \leq 0$. We define Φ on morphisms by declaring that it takes the rightward cup, the rightward cap, and all dots and crossings (with any orientation) to the corresponding morphisms in $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$, and also



To see that Φ is well defined, we must verify the relations from Definition 2.2. For (2.12), we must check the following in $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$:



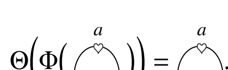
These follow from (4.4) and (4.12). Then the main work is to show that the images under Φ of the morphisms (2.10) and (2.11) are two-sided inverses in $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$. When $k = 0$, this is immediate from (4.16), so suppose that $k < 0$. The images under Φ of the two equations in (2.23) are precisely the known relations (4.2) and (4.14). We are left with checking that the images under Φ of the relations



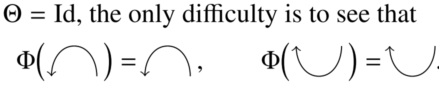
hold in $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$ for $0 \leq a, b < -k$. The first of these when $a = 0$ follows by (4.13). To see it for $0 < a < -k$, we first apply the leftward skein relation, then slide the dots past the crossing using the leftward analog of (2.9) which may be deduced from the definition (4.5), and finally appeal to (4.4). The second and third relations follow from (4.11) and (4.4) in the case that $b = 0$. To prove them when $0 < b < -k$, we must show that

in $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$. For the first identity, it is zero if $b \geq -k$ as the plus-bubble vanishes by (1.3). To see it for $0 < b < -k$, use the skein relation, commute the dots past the crossing, then appeal to (4.4) and (4.11). For the second identity, define a homomorphism $\gamma : \mathrm{Sym} \to \mathrm{End}_{\mathcal{H}eis_k^{\mathrm{new}}(z,t)}(\mathbb{1})$ by sending $e_n \mapsto t^{-1}z$ ○ $n-k$ for $n \geq 0$. Using $h_n = \det\left(e_{i-j+1}\right)_{i,j=1,\dots,n}$ and (4.7), it follows that γ sends $h_n \mapsto (-1)^{n-1}tz$ $n+k$ ⊕ for $n \leq -k$. Then the identity we are trying to prove follows by applying γ to the identity $\sum_{c\geq 1}(-1)^{-k-b-c}e_{k+a+c}h_{-k-b-c} = \delta_{a,b}$, which is (3.6).

To complete the proof, we must show that Θ and Φ are indeed two-sided inverses. To check that $\Theta \circ \Phi = \mathrm{Id}$, the only difficulty is to see that



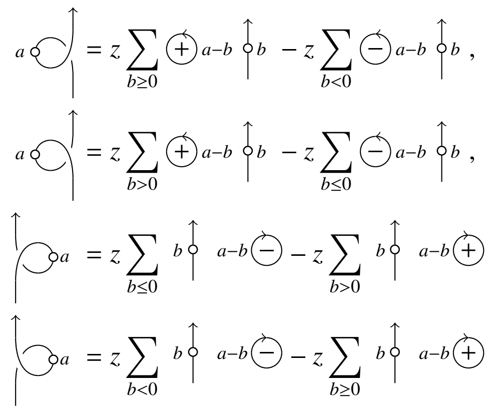
When $a = 0$, this is immediate from (2.13), while if $0 < a < -k$ it follows from (2.15) and (3.19). To check that $\Phi \circ \Theta = \mathrm{Id}$, the only difficulty is to see that



These follow from (2.13) and (4.12)–(4.13). □

To conclude the section, we formulate three more important sets of relations. The first of these explains how to expand *curls*. It is quite surprising that we have never needed to simplify left curls when $k > 0$ (or right curls when $k < 0$) before this point.

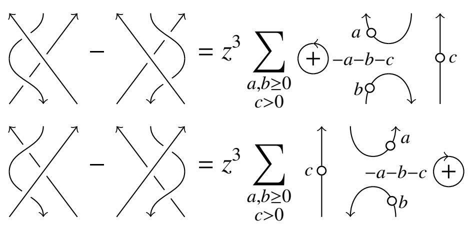
Lemma 4.3. *The following relations hold for any $a \in \mathbb{Z}$:*



(4.17) (4.18) (4.19) (4.20)

The following lemma gives a braid relation for *alternating crossings*. All other variations on the braid relation can be deduced from this plus the original braid relation from (1.8), by arguments similar to the proof of the braid relations in (2.7).

Lemma 4.4. *The following relation holds:*

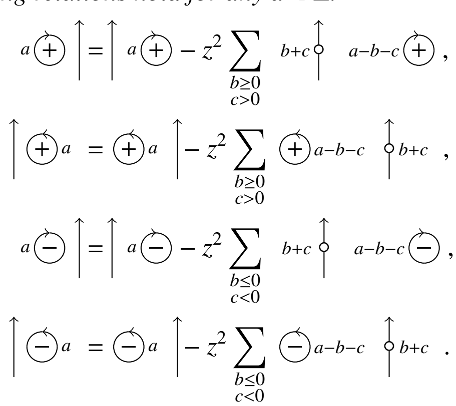


if $k \geq 0$, (4.21)

if $k \leq 0$. (4.22)

Finally we have the *bubble slides*:

Lemma 4.5. *The following relations hold for any $a \in \mathbb{Z}$:*



(4.23) (4.24) (4.25) (4.26)

5. Action on representations of quantum GL_n

In this section, we construct an action of $\mathcal{H}eis_0(z,t)$ on the category of modules over $U_q(\mathfrak{gl}_n)$ and use this action to produce a family of generators for the center of $U_q(\mathfrak{gl}_n)$. These central elements were introduced originally by Bracken, Gould and Zhang [BGZ]. We also determine their images under the Harish-Chandra homomorphism, giving a new approach to some results of Li [Li]. We work in the generic case over the field $\Bbbk := \mathbb{Q}(q)$, setting $z := q - q^{-1}$ and $t := q^n$.

In fact, the formulae which we derive are defined over $\mathbb{Z}[q, q^{-1}]$, hence, they make sense over any ground ring for any invertible q (including roots of unity).

For the precise definition of $U_q(\mathfrak{gl}_n)$, we follow the conventions of [B2, §3], denoting its standard generators by $\left\{e_i, f_i, d_j^{\pm 1} \;\middle|\; i = 1, \dots, n-1, j = 1, \dots, n\right\}$. The usual diagonal generator k_i of the subalgebra $U_q(\mathfrak{sl}_n)$ is $d_i d_{i+1}^{-1}$. The subalgebras of $U_q(\mathfrak{gl}_n)$ generated by the e_i, f_i and $d_j^{\pm}$ are $U_q(\mathfrak{gl}_n)^+$, $U_q(\mathfrak{gl}_n)^-$ and $U_q(\mathfrak{gl}_n)^0$, respectively. We also have the Borel subalgebras $U_q(\mathfrak{gl}_n)^\sharp := U_q(\mathfrak{gl}_n)^0 U_q(\mathfrak{gl}_n)^+$ and $U_q(\mathfrak{gl}_n)^\flat := U_q(\mathfrak{gl}_n)^0 U_q(\mathfrak{gl}_n)^-$. We will often cite Lusztig's book [Lu], noting that our q and k_i are Lusztig's v^{-1} and K_i^{-1}.

The natural module V^+ and dual natural module V^- are the left $U_q(\mathfrak{gl}_n)$-modules with bases $\left\{v_i^+ \;\middle|\; 1 \leq i \leq n\right\}$ and $\left\{v_i^- \;\middle|\; 1 \leq i \leq n\right\}$, respectively, on which the generators act by

$$f_i v_j^+ = \delta_{i,j} v_{i+1}^+, \qquad e_i v_j^+ = \delta_{i+1,j} v_i^+, \qquad d_i v_j^+ = q^{\delta_{i,j}} v_j^+, \tag{5.1}$$

$$f_i v_j^- = \delta_{i+1,j} v_i^-, \qquad e_i v_j^- = \delta_{i,j} v_{i+1}^-, \qquad d_i v_j^- = q^{-\delta_{i,j}} v_j^-. \tag{5.2}$$

We denote the weight of v_i^+ by ε_i; then v_i^- is of weight $-\varepsilon_i$. Let $\Lambda := \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i$ be the *weight lattice* with inner product $(\cdot,\cdot)$ defined so that $\varepsilon_1, \dots, \varepsilon_n$ are orthonormal. The *positive roots* are $\{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$. By a *weight module* we mean a $U_q(\mathfrak{gl}_n)$-module V that is the sum of its weight spaces $V_\lambda := \left\{v \in V \;\middle|\; d_i v = q^{(\lambda,\varepsilon_i)} v\right\}$ for all $\lambda \in \Lambda$. The *Weyl group* is the symmetric group $\mathfrak{S}_n$. It acts in obvious ways on Λ and on $U_q(\mathfrak{gl}_n)^0 = \Bbbk[d_1^{\pm 1}, \dots, d_n^{\pm 1}]$, permuting the generators. Denote the longest element of $\mathfrak{S}_n$ by w_0.

We work with the Hopf algebra structure on $U_q(\mathfrak{gl}_n)$ whose comultiplication Δ satisfies

$$\Delta(e_i) = d_i^{-1} d_{i+1} \otimes e_i + e_i \otimes 1, \qquad \Delta(f_i) = 1 \otimes f_i + f_i \otimes d_i d_{i+1}^{-1}, \qquad \Delta(d_j) = d_j \otimes d_j. \tag{5.3}$$

We also need various (anti)automorphisms. First, we have the *bar involution*, which is the antilinear automorphism $- : U_q \to U_q$ defined from $\overline{e_i} := e_i, \overline{f_i} := f_i$ and $\overline{d_i} := d_i^{-1}$. Then there are linear antiautomorphisms T and G defined from

$$T(e_i) := f_i, \qquad T(f_i) := e_i, \qquad T(d_i) := d_i, \tag{5.4}$$

$$G(e_i) := e_{n-i}, \qquad G(f_i) := f_{n-i}, \qquad G(d_i) := d_{n+1-i}. \tag{5.5}$$

The maps $-, T$ and G commute with each other. Finally, we have Lusztig's braid group action, under which the ith generator of the braid group acts by the automorphism $T_i : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n)$ (which is $T''_{i,-}$ from [Lu, §37.1.3]) defined for $|j - i| > 1$ and $k \neq i, i+1$ by

$$\begin{aligned} &T_i(e_i) = -f_i d_i d_{i+1}^{-1}, && T_i(e_{i\pm1}) = e_i e_{i\pm1} - q^{-1} e_{i\pm1} e_i, && T_i(e_j) = e_j,\\ &T_i(f_i) = -d_i^{-1} d_{i+1} e_i, && T_i(f_{i\pm1}) = f_{i\pm1} f_i - q f_i f_{i\pm1}, && T_i(f_j) = f_j,\\ &T_i(d_i) = d_{i+1}, && T_i(d_{i+1}) = d_i, && T_i(d_k) = d_k. \end{aligned}$$

A key role is played by the R-matrix. We recall its definition following the approach from [Lu, §32.1]. Let Θ be the *quasi-R-matrix* from [Lu, §4.1]. This is an infinite sum of components $\Theta_\alpha \in U_q(\mathfrak{gl}_n)^-_{-\alpha} \otimes U_q(\mathfrak{gl}_n)^+_\alpha$ as α runs over the positive root lattice $\bigoplus_{i=1}^{n-1} \mathbb{N}(\varepsilon_i - \varepsilon_{i+1})$. Let $P : V \otimes W \to W \otimes V$ be the tensor flip. Assuming in addition that V and W are weight modules, let $\Pi : V \otimes W \to V \otimes W$ be the diagonal map defined from

$$\Pi(v \otimes w) := q^{(\lambda,\mu)} v \otimes w$$

for v of weight λ and w of weight μ. Then, for finite-dimensional weight modules V and W, the *R-matrix*

$$R_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V \tag{5.6}$$

is the $U_q(\mathfrak{gl}_n)$-module isomorphism defined by the composition $\Theta \circ P \circ \Pi$, which makes sense since all but finitely many of the components Θ_α act as zero. For finite-dimensional weight modules U, V and W, we have the *hexagon property*:

$$R_{U,W} \otimes \mathrm{id}_V \circ \mathrm{id}_U \otimes R_{V,W} = R_{U\otimes V,W}, \qquad \mathrm{id}_V \otimes R_{U,W} \circ R_{U,V} \otimes \mathrm{id}_W = R_{U,V\otimes W}. \tag{5.7}$$

In Lusztig's treatment, this is proved in [Lu, Proposition 32.2.2] (our $R_{V,W}$ is Lusztig's ${}_f\mathcal{R}_{W,V}$ taking the function f from [Lu, §31.1.3] to be $f(\lambda,\mu) := -(\lambda,\mu)$).

In fact, to define the isomorphism $R_{V,W}$, one only needs *one* of the modules V or W to be a finite-dimensional weight module; the other can be an arbitrary $U_q(\mathfrak{gl}_n)$-module. To see this, one just needs to observe that Π extends to a linear map $V\otimes W \to V\otimes W$ when just one of V or W is a weight module on setting

$$\Pi(v\otimes w) := \begin{cases} (d_\lambda\otimes 1)(v\otimes w) & \text{if } w \text{ is a weight vector of weight } \lambda,\\ (1\otimes d_\lambda)(v\otimes w) & \text{if } v \text{ is a weight vector of weight } \lambda,\end{cases}$$

where $d_\lambda := d_1^{(\lambda,\varepsilon_1)}\cdots d_n^{(\lambda,\varepsilon_n)}$. Then the same formula $R_{V,W} := \Theta\circ P\circ\Pi$ makes sense when only one of V or W is a finite-dimensional weight module. It still gives an isomorphism of $U_q(\mathfrak{gl}_n)$-modules; the proof of this reduces to the case that both V and W are finite-dimensional weight modules using the fact that the intersection of the annihilators of all finite-dimensional weight modules is zero. Similarly, the hexagon property (5.7) remains true if only two of U, V and W are finite-dimensional weight modules.

The goal now is to derive explicit formulae for $R_{V^\pm,M}$ and $R_{M,V^\pm}$ for any module M. Similar formulae were established already in [BGZ, §III] following the older conventions of Drinfeld and Jimbo. They involve the *higher root elements* defined as follows. Let

$$e_{i,i} = f_{i,i} := z^{-1}, \qquad e_{i,i+1} := e_i, \qquad f_{i,i+1} := f_i. \tag{5.8}$$

Then when $j-i>1$ we recursively define

$$e_{i,j} := e_{i,r}e_{r,j} - q^{-1}e_{r,j}e_{i,r}, \qquad f_{i,j} := f_{r,j}f_{i,r} - q^{-1}f_{i,r}f_{r,j}, \tag{5.9}$$

where r is any index chosen so that $i<r<j$. It is an induction exercise to see that these elements are well-defined independent of the choice of r; see the proof of the following lemma for a more conceptual explanation of this. Alternatively, $e_{i,j}$ and $f_{i,j}$ can be defined using the braid group action: we have that

$$e_{i,j} = T_{j-1}\cdots T_{i+1}(e_i), \qquad f_{i,j} = \overline{T_{j-1}\cdots T_{i+1}(f_i)}.$$

Note that

$$T(e_{i,j}) = f_{i,j}, \qquad T(f_{i,j}) = e_{i,j}, \tag{5.10}$$

$$G(e_{i,j}) = e_{n+1-j,n+1-i}, \qquad G(f_{i,j}) = f_{n+1-j,n+1-i}. \tag{5.11}$$

However, the bar involution does not fix $e_{i,j}$ or $f_{i,j}$ (except when $j=i+1$).

Lemma 5.1. *For any $i<j$, the $(\varepsilon_i-\varepsilon_j)$-component $\Theta_{i,j}$ of the quasi-R-matrix Θ satisfies*

$$\Theta_{i,j} = \sum_{\substack{r\geq 1\\ i=i_0<i_1<\cdots<i_r=j}} z^r f_{i_{r-1},i_r}\cdots f_{i_0,i_1}\otimes\overline{e_{i_{r-1},i_r}\cdots e_{i_0,i_1}} = \sum_{\substack{r\geq 1\\ i=i_0<i_1<\cdots<i_r=j}} z^r\,\overline{f_{i_0,i_1}\cdots f_{i_{r-1},i_r}}\otimes e_{i_0,i_1}\cdots e_{i_{r-1},i_r}.$$

Proof. It suffices to derive the first expression. Then the second follows using (5.10) and the identitiy $(T\otimes T)(\Theta_\alpha) = P(\Theta_\alpha)$, which may easily be deduced from the characterization in [Lu, Theorem 4.1.2(a)]. To prove the first expression, we appeal to further results of Lusztig from [Lu]. Let $\mathbf{f}$ be Lusztig's "half" quantum group with its standard generators $\theta_1,\dots,\theta_{n-1}$; see also [BKM, §2.1] which follows the same conventions as here. There are two isomorphisms

$$(-)^+ : \mathbf{f}\xrightarrow{\sim} U_q(\mathfrak{gl}_n)^+,\quad \theta_i^+ := e_i, \qquad (-)^- : \mathbf{f}\xrightarrow{\sim} U_q(\mathfrak{gl}_n)^-,\quad \theta_i^- := f_i.$$

Consider the convex ordering on the positive roots defined so that $\varepsilon_i-\varepsilon_j<\varepsilon_p-\varepsilon_q$ if either $i<p$ or ($i=p$ and $j<q$); this is the "standard order" as in [BKM, Example A.1]. Let $\theta_{i,j}$ be Lusztig's higher root element associated to this ordering, which was denoted $r_{\varepsilon_i-\varepsilon_j}$ in [BKM, §2.4]. Noting that $(\varepsilon_m-\varepsilon_j,\varepsilon_i-\varepsilon_m)$ is a minimal pair for $\varepsilon_i-\varepsilon_j$, [BKM, Theorem 4.2] implies that these satisfy the following recursion: $\theta_{i,i+1}=\theta_i$ and $\theta_{i,j}=\theta_{i,r}\theta_{r,j}-q\theta_{r,j}\theta_{i,r}$ for any $i<r<j$. Comparing with (5.9), it follows that $\theta^+_{i,j}=\overline{e_{i,j}}$ and $\theta^-_{i,j}=(-q)^{j-i-1}f_{i,j}$; in particular, these equalities justify the independence of r in (5.9). Then we appeal to [BKM, Theorem

2.7] (which was extracted from [Lu]) to see that $\left\{\theta_{i_{r-1},i_r}\cdots\theta_{i_0,i_1}\;\middle|\; r\geq 1, i=i_0<\cdots<i_r=j\right\}$ and $\left\{(1-q^2)^r\theta_{i_{r-1},i_r}\cdots\theta_{i_0,i_1}\;\middle|\; r\geq 1, i=i_0<\cdots<i_r=j\right\}$ are a pair of dual bases for $\mathbf{f}_{\varepsilon_i-\varepsilon_j}$ with respect to Lusztig's form. Finally the formula from [Lu, Theorem 4.1.2(b)] gives that

$$\Theta_{i,j}=\sum_{\substack{r\geq 1\\ i=i_0<\cdots<i_r=j}}(-q)^{i-j}(1-q^2)^r\theta^-_{i_{r-1},i_r}\cdots\theta^-_{i_0,i_1}\otimes\theta^+_{i_{r-1},i_r}\cdots\theta^+_{i_0,i_1}.$$

This simplifies to the desired formula. ◻

For $1\leq i,j\leq n$ let $e^+_{i,j}\in\operatorname{End}_{\Bbbk}(V^+)$ (resp. $e^-_{i,j}\in\operatorname{End}_{\Bbbk}(V^-)$) be the ij-matrix unit with respect to the basis $v_1^+,\dots,v_n^+$ (resp. $v_1^-,\dots,v_n^-$). Then for $i<j$ and $v^\pm\in V^\pm$ we have that

$$e_{i,j}v^+=e^+_{i,j}v^+,\quad f_{i,j}v^+=e^+_{j,i}v^+,\quad e_{i,j}v^-=(-q)^{i-j+1}e^-_{j,i}v^-,\quad f_{i,j}v^-=(-q)^{i-j+1}e^-_{i,j}v^-,\tag{5.12}$$

$$\overline{e_{i,j}}v^+=e^+_{i,j}v^+,\quad \overline{f_{i,j}}v^+=e^+_{j,i}v^+,\quad \overline{e_{i,j}}v^-=(-q)^{j-i-1}e^-_{j,i}v^-,\quad \overline{f_{i,j}}v^-=(-q)^{j-i-1}e^-_{i,j}v^-.\tag{5.13}$$

These follow easily by induction on $j-i$ using (5.1)–(5.2) and (5.9). Also let

$$x_{i,j}:=z^2\sum_{r=1}^{\min(i,j)}e_{r,i}d_rf_{r,j}d_j,\qquad\qquad y_{i,j}:=z^2\sum_{r=\max(i,j)}^{n}d_if_{i,r}d_re_{j,r}\tag{5.14}$$

for any $1\leq i,j\leq n$. Then for $m\geq 0$ we set

$$x^{(m)}_{i,j}:=\sum_{i=i_0,i_1,\dots,i_{m-1},i_m=j}x_{i_0,i_1}\cdots x_{i_{m-1},i_m},\qquad y^{(m)}_{i,j}:=\sum_{i=i_0,i_1,\dots,i_{m-1},i_m=j}y_{i_0,i_1}\cdots y_{i_{m-1},i_m}.\tag{5.15}$$

In particular, $x^{(0)}_{i,j}=y^{(0)}_{i,j}=\delta_{i,j}$. From (5.11), we get that

$$G\left(x^{(m)}_{i,j}\right)=y^{(m)}_{n+1-j,n+1-i},\qquad\qquad G\left(y^{(m)}_{i,j}\right)=x^{(m)}_{n+1-j,n+1-i}.\tag{5.16}$$

Lemma 5.2. *For any $U_q(\mathfrak{gl}_n)$-module M, the endomorphisms $R_{V^\pm,M}$ and $R_{M,V^\pm}$ and their inverses are given explicitly by the following operators:*

$$R_{V^+,M}=zP\circ\sum_{i\leq j}e^+_{i,j}\otimes f_{i,j}d_j,\qquad R^{-1}_{V^+,M}=-zP\otimes\sum_{i\leq j}\overline{d_if_{i,j}}\otimes e^+_{i,j},$$

$$R_{M,V^+}=zP\circ\sum_{i\leq j}e_{i,j}d_i\otimes e^+_{j,i},\qquad R^{-1}_{M,V^+}=-zP\circ\sum_{i\leq j}e^+_{j,i}\otimes\overline{d_je_{i,j}},$$

$$R_{V^-,M}=-zP\circ\sum_{i\leq j}(-q)^{i-j}e^-_{j,i}\otimes\overline{d_if_{i,j}},\qquad R^{-1}_{V^-,M}=zP\circ\sum_{i\leq j}(-q)^{i-j}f_{i,j}d_j\otimes e^-_{j,i},$$

$$R_{M,V^-}=-zP\circ\sum_{i\leq j}(-q)^{i-j}\overline{d_je_{i,j}}\otimes e^-_{i,j},\qquad R^{-1}_{M,V^-}=zP\circ\sum_{i\leq j}(-q)^{i-j}e^-_{i,j}\otimes e_{i,j}d_i.$$

Proof. These are all proved by similar calculations, so we just go through the argument for R_{M,V^-}. Take $v\otimes v_j^-\in M\otimes V^-$. By definition, $R_{M,V^-}(v\otimes v_j^-)=\Theta(v_j^-\otimes d_j^{-1}v)$. To compute the action of Θ, we observe by weight considerations that only its weight components $\Theta_{\varepsilon_i-\varepsilon_j}$ for $i\leq j$ are non-zero on $v_j^-\otimes d_j^{-1}v$. Moreover, in the first expression for $\Theta_{i,j}$ from Lemma 5.1, all of the monomials with $r>1$ act on v_j^- as zero. We deduce that

$$R_{M,V^-}(v\otimes v_j^-)=v_j^-\otimes d_j^{-1}v+z\sum_{i<j}f_{i,j}v_j^-\otimes\overline{e_{i,j}d_j}v.$$

Then we use (5.12) to replace $f_{i,j}$ with $(-q)^{i-j+1}e^-_{i,j}$, the relation $e_{i,j}d_j=qd_je_{i,j}$, and the definition $\overline{e_{j,j}}=-z^{-1}$ to get

$$R_{M,V^-}(v\otimes v_j^-)=-ze^-_{j,j}v_j^-\otimes\overline{d_je_{j,j}}v-z\sum_{i<j}(-q)^{i-j}e^-_{i,j}v_j^-\otimes\overline{e_{i,j}d_j}v.$$

Now observe that the expression on the right hand side of the formula we are trying to prove acts on $v\otimes v_j^-$ in the same way. ◻

Corollary 5.3. *For any $U_q(\mathfrak{gl}_n)$-module M and $m \in \mathbb{Z}$, we have that*

$$(R_{M,V^+} \circ R_{V^+,M})^m = \begin{cases} \displaystyle\sum_{i,j=1}^n e^+_{i,j} \otimes x^{(m)}_{i,j} & \text{if } m \geq 0, \\ \displaystyle\sum_{i,j=1}^n e^+_{i,j} \otimes \overline{y^{(-m)}_{i,j}} & \text{if } m \leq 0. \end{cases}$$

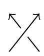
Proof. This follows from Lemma 5.2 and the definitions (5.14)–(5.15). □

Now we return to the Heisenberg category $\mathcal{H}eis_0(z,t)$ taking $t := q^n$. Let $\mathcal{OS}(z,t)$ be the *HOMFLY-PT skein category* as defined in the introduction of [B2], which is Turaev's Hecke category from [T]. By [B2, Theorem 1.1], $\mathcal{OS}(z,t)$ has a presentation by generators and relations which is very similar to the presentation of $\mathcal{H}eis_0(z,t)$ from Definition 2.2 but *without* the morphism x. Consequently, there is a strict $\Bbbk$-linear monoidal functor $\mathcal{OS}(z,t) \to \mathcal{H}eis_0(z,t)$. By [B2, Lemma 4.2], this functor is faithful, so we may use it to *identify* $\mathcal{OS}(z,t)$ with a subcategory of $\mathcal{H}eis_0(z,t)$. Thus, $\mathcal{OS}(z,t)$ is the monoidal subcategory of $\mathcal{H}eis_0(z,t)$ consisting of all objects and all morphisms which do not involve dots (i.e., x or y). In fact, as noted already after Definition 2.2, $\mathcal{H}eis_0(z,t)$ is the *affine* HOMFLY-PT skein category from [B2, §4].

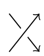
Let $U_q(\mathfrak{gl}_n)$-mod be the category of all left $U_q(\mathfrak{gl}_n)$-modules. By [B2, Lemma 3.1] (although the result is much older, e.g., it was exploited already in [T]), there is a monoidal functor

$$\Psi : \mathcal{OS}(z,t) \to U_q(\mathfrak{gl}_n)\text{-mod} \tag{5.17}$$

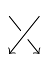
to the category of left $U_q(\mathfrak{gl}_n)$-modules. The functor Ψ sends the generating objects $\uparrow$ and $\downarrow$ to V^+ and V^-, respectively. It maps the various generating morphisms to the following $U_q(\mathfrak{gl}_n)$-module homomorphisms:



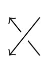
$$: v^+_i \otimes v^+_j \mapsto \begin{cases} v^+_j \otimes v^+_i & \text{if } i < j, \\ q v^+_j \otimes v^+_i & \text{if } i = j, \\ v^+_j \otimes v^+_i + z v^+_i \otimes v^+_j & \text{if } i > j; \end{cases} \tag{5.18}$$



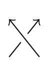
$$: v^+_i \otimes v^-_j \mapsto \begin{cases} v^-_j \otimes v^+_i & \text{if } i \neq j, \\ q^{-1} v^-_j \otimes v^+_i - z \displaystyle\sum_{r=1}^{i-1} (-q)^{-r} v^-_{j-r} \otimes v^+_{i-r} & \text{if } i = j; \end{cases} \tag{5.19}$$



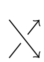
$$: v^-_i \otimes v^-_j = \begin{cases} v^-_j \otimes v^-_i & \text{if } i > j, \\ q v^-_j \otimes v^-_i & \text{if } i = j, \\ v^-_j \otimes v^-_i + z v^-_i \otimes v^-_j & \text{if } i < j; \end{cases} \tag{5.20}$$



$$: v^-_i \otimes v^+_j \mapsto \begin{cases} v^+_j \otimes v^-_i & \text{if } i \neq j, \\ q^{-1} v^+_j \otimes v^-_i - z \displaystyle\sum_{r=1}^{n-i} (-q)^{-r} v^+_{j+r} \otimes v^-_{i+r} & \text{if } i = j; \end{cases} \tag{5.21}$$



$$: v^+_i \otimes v^+_j \mapsto \begin{cases} v^+_j \otimes v^+_i & \text{if } i > j, \\ q^{-1} v^+_j \otimes v^+_i & \text{if } i = j, \\ v^+_j \otimes v^+_i - z v^+_i \otimes v^+_j & \text{if } i < j; \end{cases} \tag{5.22}$$



$$: v^+_i \otimes v^-_j \mapsto \begin{cases} v^-_j \otimes v^+_i & \text{if } i \neq j, \\ q v^-_j \otimes v^+_i + z \displaystyle\sum_{r=1}^{n-i} (-q)^{r} v^-_{j+r} \otimes v^+_{i+r} & \text{if } i = j; \end{cases} \tag{5.23}$$

$$: v^-_i \otimes v^-_j = \begin{cases} v^-_j \otimes v^-_i & \text{if } i < j, \\ q^{-1} v^-_j \otimes v^-_i & \text{if } i = j, \\ v^-_j \otimes v^-_i - z v^-_i \otimes v^-_j & \text{if } i > j; \end{cases} \tag{5.24}$$

$$: v_i^- \otimes v_j^+ \mapsto \begin{cases} v_j^+ \otimes v_i^- & \text{if } i \neq j, \\ q v_j^+ \otimes v_i^- + z \sum_{r=1}^{i-1} (-q)^r v_{j-r}^+ \otimes v_{i-r}^- & \text{if } i = j; \end{cases} \tag{5.25}$$

$$: 1 \mapsto \sum_{j=1}^{n} (-1)^j q^j v_j^- \otimes v_j^+, \qquad : 1 \mapsto \sum_{j=1}^{n} (-1)^j q^{n+1-j} v_j^+ \otimes v_j^-, \tag{5.26}$$

$$: v_i^+ \otimes v_j^- \mapsto (-1)^i q^{-i} \delta_{i,j}, \qquad : v_i^- \otimes v_j^+ \mapsto (-1)^i q^{i-n-1} \delta_{i,j}. \tag{5.27}$$

These formulae are recorded in many places in the literature going back to the original work [T], but one finds many different choices of normalization. For our choices, (5.18)–(5.21) and (5.22)–(5.25) follow from the formulae for the R-matrix and its inverse from Lemma 5.2, while the formulae (5.26)–(5.27) are explained in [B2, §3].

Theorem 5.4. *Assuming $t = q^n$ and $z = q - q^{-1}$, there is a monoidal functor*

$$\widehat{\Psi} : \mathcal{H}eis_0(z,t) \to \mathcal{E}nd\left(U_q(\mathfrak{gl}_n)\text{-mod}\right)$$

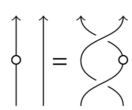
such that $\Psi = \mathrm{Ev} \circ \widehat{\Psi}\big|_{\mathcal{OS}(z,t)}$, where Ev *denotes evaluation on the trivial module. On objects, $\widehat{\Psi}$ takes X to the endofunctor $\Psi(X) \otimes -$, e.g., $\widehat{\Psi}(\uparrow) = V^+ \otimes -$ and $\widehat{\Psi}(\downarrow) = V^- \otimes -$. On morphisms, $\widehat{\Psi}$ sends $f \in \mathrm{Hom}_{\mathcal{OS}(z,t)}(X,Y)$ to the natural transformation $\Psi(f) \otimes 1 : \Psi(X) \otimes - \to \Psi(Y) \otimes -$. Finally, on the additional generating morphism x, it is defined by*

$$\widehat{\Psi}(x)_M := R_{M,V^+} \circ R_{V^+,M} : V^+ \otimes M \to V^+ \otimes M, \qquad v_j^+ \otimes m \mapsto \sum_{i=1}^{n} v_i^+ \otimes x_{i,j} m.$$

Proof. We just need to verify that the relations from Definition 2.2 are satisfied. All of the ones that do not involve x follow immediately since they are already satisfied by the morphisms in the image of the monoidal functor Ψ. Also $R_{V^+,M} \circ R_{M,V^+}$ is invertible since each of these R-matrices is invertible. It just remains to check the relation (1.6). In fact, this is a formal consequence of the hexagon property; see e.g. [V, Proposition 3.1.1]. The argument goes as follows. By (5.7), we have for any $U_q(\mathfrak{gl}_n)$-module M that

$$R_{V^+ \otimes M, V^+} \circ R_{V^+, V^+ \otimes M} = R_{V^+,V^+} \otimes \mathrm{id}_M \circ \mathrm{id}_{V^+} \otimes R_{M,V^+} \circ \mathrm{id}_{V^+} \otimes R_{V^+,M} \circ R_{V^+,V^+} \otimes \mathrm{id}_M.$$

This establishes that the image under $\widehat{\Psi}$ of the relation



is satisfied, from which (1.6) easily follows. □

Let $Z_q(\mathfrak{gl}_n)$ be the center of $U_q(\mathfrak{gl}_n)$. It is identified with the endomorphism algebra of the identity functor $\mathrm{Id}_{U_q(\mathfrak{gl}_n)\text{-mod}}$; indeed, evaluation on the identity element of the regular representation defines a canonical algebra isomorphism $\mathcal{E}nd\left(\mathrm{Id}_{U_q(\mathfrak{gl}_n)\text{-mod}}\right) \xrightarrow{\sim} Z_q(\mathfrak{gl}_n)$. Dotted bubbles are endomorphisms of the unit object of $\mathcal{H}eis_0(z,t)$. Applying the monoidal functor $\widehat{\Psi}$ from Theorem 5.4, we obtain natural transformations

$$\widehat{\Psi}\left(\;\; m \right) : \mathrm{Id}_{U_q(\mathfrak{gl}_n)\text{-mod}} \to \mathrm{Id}_{U_q(\mathfrak{gl}_n)\text{-mod}}, \tag{5.28}$$

hence, central elements $z_m \in Z(U_q(\mathfrak{gl}_n))$ for each $m \in \mathbb{Z}$. A calculation using (5.26)–(5.27) and Corollary 5.3 shows that

$$z_m = \begin{cases} \sum_{i=1}^{n} q^{2i-n-1} x_{i,i}^{(m)} & \text{if } m \geq 0, \\ \sum_{i=1}^{n} q^{2i-n-1} \overline{y_{i,i}^{(-m)}} & \text{if } m \leq 0. \end{cases} \tag{5.29}$$

We have trivially that $z_0 = [n]_q$. The goal in the remainder of the section is to compute explicit formulae for the images of all the others under the Harish-Chandra homomorphism.

Our argument uses the Harish-Chandra homomorphism in two different forms adapted to the positive and negative Borel subalgebras, respectively. To review the definitions, let $\rho_+ := -\varepsilon_2 - 2\varepsilon_3 - \cdots - (n-1)\varepsilon_n$ and $\rho_- := -(n-1)\varepsilon_1 - \cdots - 2\varepsilon_{n-2} - \varepsilon_{n-1}$, i.e., $\rho_- = w_0(\rho_+)$. For any $\lambda \in \Lambda$, we have the *shift automorphism*

$$S_\lambda : U_q(\mathfrak{gl}_n)^0 \to U_q(\mathfrak{gl}_n)^0, \qquad d_i \mapsto q^{(\lambda,\varepsilon_i)} d_i. \tag{5.30}$$

For example, $S_{-\rho_+}(d_i) = q^{i-1} d_i$ and $S_{-\rho_-}(d_i) = q^{n-i} d_i$. Let $U_q(\mathfrak{gl}_n)_0$ be the zero weight space of $U_q(\mathfrak{gl}_n)$, which is a subalgebra containing $U_q(\mathfrak{gl}_n)^0$. Let I_+ (resp. I_-) be the intersection of $U_q(\mathfrak{gl}_n)_0$ with the left ideal of $U_q(\mathfrak{gl}_n)$ generated by $e_1, \dots, e_{n-1}$ (resp. $f_1, \dots, f_{n-1}$). Equivalently, I_+ (resp. I_-) is the intersection of $U_q(\mathfrak{gl}_n)_0$ with the right ideal generated by $f_1, \dots, f_{n-1}$ (resp. $e_1, \dots, e_{n-1}$). It follows that $I_\pm$ is a two-sided ideal of $U_q(\mathfrak{gl}_n)_0$. Let $\mathrm{pr}_\pm : U_q(\mathfrak{gl}_n)_0 \to U_q(\mathfrak{gl}_n)^0$ be the algebra homomorphism defined by projection along the direct sum decomposition $U_q(\mathfrak{gl}_n)_0 = U_q(\mathfrak{gl}_n)^0 \oplus I_\pm$. The two versions of the *Harish-Chandra homomorphism* are

$$HC_\pm := S_{-\rho_\pm} \circ \mathrm{pr}_\pm : U_q(\mathfrak{gl}_n)_0 \to U_q(\mathfrak{gl}_n)^0. \tag{5.31}$$

The following is an extension of the well-known description of $Z_q(\mathfrak{sl}_n)$ from e.g. [J, 6.25].

Lemma 5.5 ([Li, Lemma 2.1])**.** *The restriction $HC := HC_+\big|_{Z_q(\mathfrak{gl}_n)}$ defines an algebra isomorphism between $Z_q(\mathfrak{gl}_n)$ and the algebra $\Bbbk[(d_1 \cdots d_n)^{-1}, d_1^2, \dots, d_n^2]^{\mathfrak{S}_n}$.*

The following facts are also well known, but we could not find a suitable reference.

Lemma 5.6. *Each braid group generator $T_i : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n)$ fixes $Z_q(\mathfrak{gl}_n)$ pointwise.*

Proof. Take $c \in Z_q(\mathfrak{gl}_n)$. Let V be an integrable highest weight module. Since V is irreducible, both c and $T_i(c)$ act on V as scalars. These scalars are equal because there is an automorphism $T_i : V \to V$ such that $T_i(cv) = T_i(c)T_i(v)$; see [Lu, §37.1.1]. This shows that $c - T_i(c)$ acts as zero on every integrable highest weight module. The intersection of the annihilators of all integrable highest weight modules is zero, so this proves that $c = T_i(c)$. □

Lemma 5.7. *The restriction $HC = HC_+\big|_{Z_q(\mathfrak{gl}_n)}$ is equal also to the restriction $HC_-\big|_{Z_q(\mathfrak{gl}_n)}$.*

Proof. Let T_{w_0} be the product of simple braid group generators T_i taken in some order corresponding to a reduced expression of w_0. This is an automorphism of $U_q(\mathfrak{gl}_n)$ which switches $U_q(\mathfrak{gl}_n)^\sharp$ and $U_q(\mathfrak{gl}_n)^\flat$, and it sends $d_i \mapsto d_{n+1-i}$. It follows that

$$HC_\mp \circ T_{w_0} = T_{w_0} \circ HC_\pm. \tag{5.32}$$

Clearly, T_{w_0} fixes $\Bbbk[(d_1 \cdots d_n)^{-1}, d_1^{\pm 2}, \dots, d_n^{\pm 2}]^{\mathfrak{S}_n}$ pointwise. It also fixes $Z_q(\mathfrak{gl}_n)$ pointwise by Lemma 5.6. Hence, $HC_-\big|_{Z_q(\mathfrak{gl}_n)} = HC_- \circ T_{w_0}\big|_{Z_q(\mathfrak{gl}_n)} = T_{w_0} \circ HC_+\big|_{Z_q(\mathfrak{gl}_n)} = HC_+\big|_{Z_q(\mathfrak{gl}_n)}$. □

Lemma 5.8. *The antiautomorphism G fixes $Z_q(\mathfrak{gl}_n)$ pointwise.*

Proof. We have that

$$HC_\mp \circ G = G \circ HC_\pm. \tag{5.33}$$

Combined with Lemma 5.7, it follows that $HC_+ \circ G\big|_{Z_q(\mathfrak{gl}_n)} = G \circ HC_+\big|_{Z_q(\mathfrak{gl}_n)}$. Also G clearly fixes $\Bbbk[(d_1 \cdots d_n)^{-1}, d_1^{\pm 2}, \dots, d_n^{\pm 2}]^{\mathfrak{S}_n}$ pointwise. Hence, $HC_+ \circ G\big|_{Z_q(\mathfrak{gl}_n)} = HC_+\big|_{Z_q(\mathfrak{gl}_n)}$, which implies the result since HC_+ is injective on $Z_q(\mathfrak{gl}_n)$. □

In particular, this shows that $G(z_m) = z_m$, hence, on applying G to the right hand side of (5.29) using (5.16), we obtain another formula for z_m:

$$z_m = \begin{cases} \sum_{i=1}^n q^{n+1-2i} y_{i,i}^{(m)} & \text{if } m \geq 0, \\ \sum_{i=1}^n q^{n+1-2i} \overline{x_{i,i}^{(-m)}} & \text{if } m \leq 0. \end{cases} \tag{5.34}$$

Comparing with (5.29), it follows that

$$z_{-m} = \overline{z_m} \tag{5.35}$$

for every $m \in \mathbb{Z}$. From now on, we only consider z_m for $m \geq 1$.

Finally, consider the *modified complete symmetric polynomials*

$$\widetilde{h}_m(x_1, \dots, x_n) := \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq n} \left(q^{-1}z\right)^{\#\{i_1,\dots,i_m\}-1} x_{i_1} \cdots x_{i_m}. \tag{5.36}$$

We will use these for all values of $n \geq 0$ (not just the n fixed above for $\mathfrak{gl}_n$). We have that $\widetilde{h}_m(x_1, \dots, x_n) = qz^{-1}$ if $m = 0$, and $\widetilde{h}_m(x_1, \dots, x_n) = 0$ if $m > 0$ but $n = 0$. These elements obviously satisfy the recurrence relation

$$\widetilde{h}_m(x_1, \dots, x_n) = \widetilde{h}_m(x_1, \dots, x_{n-1}) + q^{-1}z \sum_{r=1}^m \widetilde{h}_{m-r}(x_1, \dots, x_{n-1}) x_n^r \tag{5.37}$$

for $n > 0$.

Lemma 5.9. $\widetilde{h}_m(x_1, \dots, x_n) = \widetilde{h}_m(x_1, \dots, x_{n-1}) + \widetilde{h}_{m-1}(x_1, \dots, x_n)x_n - q^{-2}\widetilde{h}_{m-1}(x_1, \dots, x_{n-1})x_n.$

Proof. By (5.37) with m replaced by $m-1$, we have that

$$\begin{aligned} \widetilde{h}_{m-1}(x_1, \dots, x_n)x_n &= \widetilde{h}_{m-1}(x_1, \dots, x_{n-1})x_n + q^{-1}z \sum_{r=1}^{m-1} \widetilde{h}_{m-r-1}(x_1, \dots, x_{n-1}) x_n^{r+1} \\ &= \widetilde{h}_{m-1}(x_1, \dots, x_{n-1})x_n + q^{-1}z \sum_{r=2}^{m} \widetilde{h}_{m-r}(x_1, \dots, x_{n-1}) x_n^{r} \\ &= q^{-2}\widetilde{h}_{m-1}(x_1, \dots, x_{n-1})x_n + q^{-1}z \sum_{r=1}^{m} \widetilde{h}_{m-r}(x_1, \dots, x_{n-1}) x_n^{r}. \end{aligned}$$

Given this, it is easy to see that the right hand side of the identity we are trying to prove is equal to the right hand side of (5.37). □

Theorem 5.10. *For any $m \geq 1$ we have that $HC(z_m) = q^{n-1}\widetilde{h}_m\left(d_1^2, \dots, d_n^2\right)$.*

Proof. Noting that $q^{1-n}z_m = \sum_{i=1}^n q^{2i-2n} x_{i,i}^{(m)}$ according to (5.29), this follows from the following claim: *for any $m \geq 1$ and $i = 1, \dots, n$, we have that*

$$HC_+\left(x_{i,i}^{(m)}\right) = \widetilde{h}_m\left(d_1^2, \dots, d_i^2\right) - q^{-2}\widetilde{h}_m\left(d_1^2, \dots, d_{i-1}^2\right). \tag{5.38}$$

To prove (5.38), we proceed by induction on $m+n$. The result is easy to check when $n = 1$. Now assume that $n > 1$. The Harish-Chandra homomorphism HC_+ is compatible with the usual "top left corner" embedding of $U_q(\mathfrak{gl}_{n-1})$ into $U_q(\mathfrak{gl}_n)$. This follows because the restriction of ρ_+ for $\mathfrak{gl}_n$ is the weight ρ_+ for $\mathfrak{gl}_{n-1}$. Also the elements $x_{1,1}^{(m)}, \dots, x_{n-1,n-1}^{(m)}$ of $U_q(\mathfrak{gl}_{n-1})$ are the same as these elements in $U_q(\mathfrak{gl}_n)$. Thus we get (5.38) for each $i < n$ from the induction hypothesis. It remains to prove (5.38) when $i = n$. We have that

$$q^{1-n}HC_-(z_m) = \sum_{i=1}^n q^{2i-2n} \sum_{j_1,\dots,j_m} HC_-\left(z^{2m} e_{j_1,i} d_{j_1} f_{j_1,j_2} d_{j_2} \cdots e_{j_m,j_{m-1}} d_{j_m} f_{j_m,i} d_i\right).$$

By the definition of HC_-, the terms in this expansion are zero if either $j_1 < i$ or $j_m < i$. Thus, the sum simplifies to give

$$q^{1-n} HC_-(z_m) = \sum_{i=1}^{n} q^{2i-2n} HC_-\left(y_{i,i}^{(m-1)} d_i^2\right) = \sum_{i=1}^{n} HC_-\left(y_{i,i}^{(m-1)}\right) d_i^2.$$

Now we apply G, using Lemma 5.8, (5.33) and (5.11), to see that

$$q^{1-n} HC_+(z_m) = \sum_{i=1}^{n} HC_+\left(x_{i,i}^{(m-1)}\right) d_i^2.$$

Remembering (5.29), we have now proved that

$$\sum_{i=1}^{n} q^{2i-2n} HC_+\left(x_{i,i}^{(m)}\right) = \sum_{i=1}^{n} HC_+\left(x_{i,i}^{(m-1)}\right) d_i^2. \tag{5.39}$$

The same identity with n replaced by $(n-1)$ gives

$$\sum_{i=1}^{n-1} q^{2i-2(n-1)} HC_+\left(x_{i,i}^{(m)}\right) = \sum_{i=1}^{n-1} HC_+\left(x_{i,i}^{(m-1)}\right) d_i^2. \tag{5.40}$$

By the induction hypothesis, the left hand side of (5.40) is equal to $\widetilde{h}_m\left(d_1^2, \dots, d_{n-1}^2\right)$. Hence, (5.39) can be rewritten to obtain

$$\begin{aligned} HC_+\left(x_{n,n}^{(m)}\right) + q^{-2}\widetilde{h}_m\left(d_1^2, \dots, d_{n-1}^2\right) &= HC_+\left(x_{n,n}^{(m-1)}\right) d_n^2 + \widetilde{h}_m\left(d_1^2, \dots, d_{n-1}^2\right) \\ &= \widetilde{h}_m\left(d_1^2, \dots, d_{n-1}^2\right) + \widetilde{h}_{m-1}\left(d_1^2, \dots, d_n^2\right) d_n^2 - q^{-2}\widetilde{h}_{m-1}\left(d_1^2, \dots, d_{n-1}^2\right) d_n^2, \end{aligned}$$

where we have used the induction hypothesis again to establish the second equality. This is equal to $\widetilde{h}_m\left(d_1^2, \dots, d_n^2\right)$ thanks to Lemma 5.9. The conclusion follows. □

Corollary 5.11 ([Li, Theorem 4.1])**.** *$Z_q(\mathfrak{gl}_n)$ is generated by $z_1, \dots, z_n$ and $(d_1 \cdots d_n)^{-1}$.*

Proof. This follows from Lemma 5.5 and Theorem 5.10 since $\Bbbk[x_1, \dots, x_n]^{\mathfrak{S}_n}$ is generated by the modified complete symmetric functions $\widetilde{h}_1(x_1, \dots, x_n), \dots, \widetilde{h}_n(x_1, \dots, x_n)$. □

6. Action on modules over cyclotomic Hecke algebras

Throughout the section, we assume that we are given a polynomial

$$f(w) = f_0 w^l + f_1 w^{l-1} + \cdots + f_l \in \Bbbk[w] \tag{6.1}$$

of degree $l \geq 0$ such that $f_0 f_l = 1$. Recall from the introduction that the affine Hecke algebra AH_n with its standard generators $x_1, \dots, x_n, \tau_1, \dots, \tau_{n-1}$ is identified with the endomorphism algebra $\operatorname{End}_{\mathcal{AH}(z)}(\uparrow^{\otimes n})$ so that and x_i is the dot on the ith string and τ_j is the positive crossing of the jth and $(j+1)$th strings (numbering strings $1, \dots, n$ from right to left). The *cyclotomic Hecke algebra H_n^f* of level l associated to the polynomial $f(w)$ is the quotient of AH_n by the two-sided ideal generated by $f(x_1)$. We also include the possibility $n = 0$ with the convention that $H_0^f = \Bbbk$.

The basis theorem proved in [AK, Theorem 3.10] shows that the following gives a basis for H_n^f as a free $\Bbbk$-module:

$$\left\{x_1^{r_1} \cdots x_n^{r_n} \tau_g \;\middle|\; 0 \leq r_1, \dots, r_n < l, g \in \mathfrak{S}_n\right\}, \tag{6.2}$$

where τ_g denotes the element of the finite Hecke algebra defined from a reduced expression for the permutation g. By the basis theorem, the obvious homomorphism $H_n^f \to H_{n+1}^f$ sending the generators x_i and τ_j to the elements of H_{n+1}^f with the same names is *injective*. So we may identify H_n^f with a subalgebra of H_{n+1}^f. We denote the induction and restriction functors by

$$\operatorname{ind}_n^{n+1} := H_{n+1}^f \otimes_{H_n^f} - : H_n^f\text{-mod} \to H_{n+1}^f\text{-mod}, \tag{6.3}$$

$$\mathrm{res}_n^{n+1} : H_{n+1}^f\text{-mod} \to H_n^f\text{-mod}\,. \tag{6.4}$$

We are going to make the Abelian category $\bigoplus_{n\geq 0} H_n^f$-mod into a left $\mathcal{H}eis_{-l}(z, f_0^{-1})$-module category, with $\uparrow$ and $\downarrow$ acting as induction and restriction, respectively. In order to do this, we need the *Mackey theorem* for H_n^f: there is an isomorphism of functors

$$\mathrm{ind}_{n-1}^n \circ \mathrm{res}_{n-1}^n \oplus \mathrm{Id}^{\oplus l} \xrightarrow{\sim} \mathrm{res}_n^{n+1} \circ \mathrm{ind}_n^{n+1}\,. \tag{6.5}$$

The standard proof shows that the map

$$H_n^f \otimes_{H_{n-1}^f} H_n^f \oplus \bigoplus_{r=0}^{l-1} H_n^f \to H_{n+1}^f, \qquad (u \otimes v, w_0, \dots, w_{l-1}) \mapsto u\tau_n v + \sum_{r=0}^{l-1} w_r x_{n+1}^r \tag{6.6}$$

is an isomorphism of (H_n^f, H_n^f)-bimodules. This implies that there is a unique (H_n^f, H_n^f)-bimodule homomorphism

$$\mathrm{tr}_n^f : H_{n+1}^f \to H_n^f \tag{6.7}$$

such that $\mathrm{tr}_n^f(\tau_n) = 0$ and $\mathrm{tr}_n^f(x_{n+1}^r) = \delta_{r,0}$ for $0 \leq r < l$.

Lemma 6.1. *For any $n \geq 0$, we have that $\mathrm{tr}_n^f\,(f(x_{n+1})) = 0$.*

Proof. For $u, v \in H_{n+1}^f$, write $u \equiv_n v$ as shorthand for $u = v$ in case $n = 0$, or $u - v \in H_n^f \tau_n H_n^f$ in case $n > 0$. We first show by induction on $n = 0, 1, \dots$ that

$$\tau_n \cdots \tau_1 x_1^a \tau_1 \cdots \tau_n \equiv_n \begin{cases} \displaystyle\sum_{\substack{b+c_1+\cdots+c_n=a \\ b>0, c_1,\dots,c_n\geq 0}} \left(\prod_{i \text{ with } c_i\neq 0} (-z^2 c_i)\right) x_{n+1}^b x_n^{c_n} \cdots x_1^{c_1} & \text{if } a > 0, \\ \displaystyle\sum_{\substack{b+c_1+\cdots+c_n=a \\ b\leq 0, c_1,\dots,c_n\leq 0}} \left(\prod_{i \text{ with } c_i\neq 0} (z^2 c_i)\right) x_{n+1}^b x_n^{c_n} \cdots x_1^{c_1} & \text{if } a \leq 0. \end{cases} \tag{6.8}$$

We just explain this in detail in the case $a > 0$, since the case $a \leq 0$ is similar. The base case is trivial. For the induction step, we have that

$$\begin{aligned} \tau_n x_n^a \tau_n &= \tau_n \tau_n^{-1} x_{n+1}^a - z \sum_{\substack{b+c=a \\ b,c>0}} \tau_n x_{n+1}^b x_n^c = x_{n+1}^a - z \sum_{\substack{b+c=a \\ b,c>0}} \tau_n^{-1} x_{n+1}^b x_n^c - z^2 \sum_{\substack{b+c=a \\ b,c>0}} x_{n+1}^b x_n^c \\ &\equiv_n x_{n+1}^a - z^2 \sum_{\substack{b+c+d=a \\ b,c,d>0}} x_{n+1}^b x_n^{c+d} - z^2 \sum_{\substack{b+c=a \\ b,c>0}} x_{n+1}^b x_n^c = x_{n+1}^a - z^2 \sum_{\substack{b+c=a \\ b,c>0}} c x_{n+1}^b x_n^c. \end{aligned}$$

Now take the expression for $\tau_{n-1} \cdots \tau_1 x_1^a \tau_1 \cdots \tau_{n-1}$ given by the induction hypothesis, multiply on left and right by τ_n, and use the above identity plus the observation

$$\tau_n \left(H_{n-1}^f \tau_{n-1} H_{n-1}^f\right) \tau_n = H_{n-1}^f \tau_n \tau_{n-1} \tau_n H_{n-1}^f = H_{n-1}^f \tau_{n-1} \tau_n \tau_{n-1} H_{n-1}^f \subseteq H_n^f \tau_n H_n^f.$$

Finally, to deduce the lemma, we multiply (6.8) by f_{l-a} and sum over $a = 0, 1, \dots, l$ to show

$$\tau_n \cdots \tau_1 f(x_1) \tau_1 \cdots \tau_n \equiv_n f_l + \sum_{a=1}^{l} f_{l-a} \sum_{\substack{b+c_1+\cdots+c_n=a \\ b>0, c_1,\dots,c_n\geq 0}} \left(\prod_{i \text{ with } c_i\neq 0} (-z^2 c_i)\right) x_{n+1}^b x_n^{c_n} \cdots x_1^{c_1}.$$

The left hand side is zero by the cyclotomic relation in H_{n+1}^f. The right hand side is equal to $f(x_{n+1})$ plus terms in the kernel of tr_n^f. □

Theorem 6.2. *There is a unique strict $\Bbbk$-linear monoidal functor*

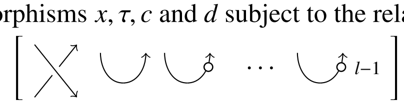
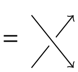
$$\Psi_f : \mathcal{H}eis_{-l}(z, f_0^{-1}) \to \mathcal{E}nd\left(\bigoplus_{n\geq 0} H_n^f\text{-mod}\right)$$

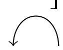
sending the generating object $\uparrow$ *(resp.* $\downarrow$*) to the additive endofunctor that takes an* H_n^f*-module* M *to* $\operatorname{ind}_n^{n+1} M$ *(resp.* $\operatorname{res}_{n-1}^n M$*), and the generating morphisms* x, τ, c *and* d *to the natural transformations defined on the* H_n^f*-module* M *as follows:*

- $\Psi_f(x)_M : H_{n+1}^f \otimes_{H_n^f} M \to H_{n+1}^f \otimes_{H_n^f} M$, $u \otimes v \mapsto u x_{n+1} \otimes v$;
- $\Psi_f(\tau)_M : H_{n+2}^f \otimes_{H_n^f} M \to H_{n+2}^f \otimes_{H_n^f} M$, $u \otimes v \mapsto u \tau_{n+1} \otimes v$ *(where we have identified* $\operatorname{ind}_{n+1}^{n+2} \circ \operatorname{ind}_n^{n+1}$ *with* $\operatorname{ind}_n^{n+2}$ *in the obvious way);*
- $\Psi_f(c)_M : M \to \operatorname{res}_n^{n+1} \left(H_{n+1}^f \otimes_{H_n^f} M\right)$, $v \mapsto 1 \otimes v$, *i.e., it is the unit of the canonical adjunction making* $(\operatorname{ind}_n^{n+1}, \operatorname{res}_n^{n+1})$ *into an adjoint pair of functors;*
- $\Psi_f(d)_M : H_n^f \otimes_{H_{n-1}^f} (\operatorname{res}_{n-1}^n M) \to M$, $u \otimes v \mapsto uv$, *i.e., it is the counit of the canonical adjunction making* $(\operatorname{ind}_{n-1}^n, \operatorname{res}_{n-1}^n)$ *into an adjoint pair of functors.*

Proof. We use the presentation for $\mathcal{H}eis_{-l}(z, f_0^{-1})$ from Definition 2.2. Let us first treat the degenerate case $l = 0$. In this case, $\bigoplus_{n \geq 0} H_n^f$-mod is the category of left $\Bbbk$-modules, and all of the induction and restriction functors are zero. Consequently, almost of the relations are trivially true. The only one that requires any thought is the relation  $= (f_0^{-1} z^{-1} - f_0 z^{-1}) 1_{\mathbb{1}}$ from (2.12). To see that this holds, one needs to observe that the scalar on the right hand side is zero, which follows from the assumption that $f_0^2 = 1$.

Henceforth, we assume that $l > 0$. Then $\mathcal{H}eis_{-l}(z, f_0^{-1})$ is generated by the objects $\uparrow$ and $\downarrow$ and morphisms x, τ, c and d subject to the relations (1.6)–(1.9), plus two more relations:

(1)  is invertible where $\sigma :=$  is defined by (2.2);

(2)  $= f_0^{-1} z^{-1} 1_{\mathbb{1}}$ where $\gamma :=$  is defined by (2.13), i.e., it is $-f_0 z^{-1}$ times the $(2,1)$-entry of the inverse of the matrix in (1).

The relations (1.6)–(1.9) are straightforward to check. On H_n^f-modules, $\Psi_f(\sigma)$ comes from the (H_n^f, H_n^f)-bimodule homomorphism $H_n^f \otimes_{H_{n-1}^f} H_n^f \to H_{n+1}^f, u \otimes v \mapsto u \tau_n v$. So we get the relation (1) since (6.6) is invertible by the proof of the Mackey theorem. Moreover, we see from (6.6) and the definition that $\Psi_f(\gamma)$ comes from the (H_n^f, H_n^f)-bimodule homomorphisms $-f_0 z^{-1} \operatorname{tr}_n^f : H_{n+1}^f \to H_n^f$ for all $n \geq 0$. So for (2) we must show that $-f_0 z^{-1} \operatorname{tr}_n^f \left(x_{n+1}^\ell\right) = f_0^{-1} z^{-1}$. This follows from Lemma 6.1 and the definition of tr_n^f, remembering that $f_l = f_0^{-1}$. $\square$

If we switch the roles of induction and restriction, we can reformulate Theorem 6.2 in terms of Heisenberg categories of positive central charge. We prefer for this to replace the induction functor $\operatorname{ind}_n^{n+1}$ from before (which is the canonical left adjoint to restriction) with the *coinduction functor*

$$\operatorname{coind}_n^{n+1} := \operatorname{Hom}_{H_n^f}(H_{n+1}^f, -) : H_n^f\text{-mod} \to H_{n+1}^f\text{-mod} \tag{6.9}$$

which is its canonical right adjoint.

Theorem 6.3. *There is a unique strict* $\Bbbk$*-linear monoidal functor*

$$\Psi_f^\vee : \mathcal{H}eis_l(z, f_0) \to \mathcal{E}nd\left(\bigoplus_{n \geq 0} H_n^f\text{-mod}\right)$$

sending the generating object $\uparrow$ *(resp.* $\downarrow$*) to the additive endofunctor that takes an* H_n^f*-module* M *to* $\operatorname{res}_{n-1}^n M$ *(resp.* $\operatorname{coind}_n^{n+1} M$*), and the generating morphisms* x, τ, c *and* d *to the natural transformations defined on the* H_n^f*-module* M *as follows:*

- $\Psi_f^\vee(x)_M : \operatorname{res}_{n-1}^n M \to \operatorname{res}_{n-1}^n M$, $v \mapsto x_n v$;
- $\Psi_f^\vee(\tau)_M : \operatorname{res}_{n-2}^n M \to \operatorname{res}_{n-2}^n M$, $v \mapsto -\tau_{n-1}^{-1} v$;
- $\Psi_f^\vee(c)_M : M \to \operatorname{Hom}_{H_{n-1}^f}(H_n^f, \operatorname{res}_{n-1}^n M)$, $v \mapsto (u \mapsto uv)$, *i.e., it is the unit of the canonical adjunction making* $(\operatorname{res}_{n-1}^n, \operatorname{coind}_{n-1}^n)$ *into an adjoint pair of functors;*

- $\Psi_f^\vee(d)_M : \mathrm{res}_n^{n+1}\left(\mathrm{Hom}_{H_n^f}(H_{n+1}^f, M)\right) \to M$, $\theta \mapsto \theta(1)$, *i.e., it is the counit of the canonical adjunction making* $(\mathrm{res}_n^{n+1}, \mathrm{coind}_n^{n+1})$ *into an adjoint pair of functors.*

Proof. This may be proved directly in a similar way to the proof of Theorem 6.2. One uses the presentation for $\mathcal{H}eis_l(z, f_0)$ from Definition 3.1 instead of the one from Definition 2.2, plus the Mackey isomorphism (6.6) and Lemma 6.1 as before. We leave the details to the reader. □

In fact, we have that $\mathrm{ind}_n^{n+1} \cong \mathrm{coind}_n^{n+1}$. This follows by the uniqueness of adjoints, since Lemma 3.7 and Theorem 6.2 (resp. Theorem 6.3) implies that ind_n^{n+1} is right adjoint to restriction as well as being left adjoint (resp. coind_n^{n+1} is left adjoint to restriction as well as being right adjoint). It follows that all three functors (induction, coinduction and restriction) send finitely generated projective modules to finitely generated projective modules. Hence:

Lemma 6.4. *The restrictions of the functors Ψ_f and $\Psi_f^\vee$ constructed in Theorems 6.2–6.3 give strict $\Bbbk$-linear monoidal functors*

$$\Psi_f : \mathcal{H}eis_{-l}(z, f_0^{-1}) \to \mathcal{E}nd\left(\bigoplus_{n\geq 0} H_n^f\text{-pmod}\right), \quad \Psi_f^\vee : \mathcal{H}eis_l(z, f_0) \to \mathcal{E}nd\left(\bigoplus_{n\geq 0} H_n^f\text{-pmod}\right),$$

where H_n^f-pmod denotes the category of finitely generated projective left H_n^f-modules.

7. Action on category $\mathcal{O}$ for rational Cherednik algebras

The Heisenberg action on $\bigoplus_{n\geq 0} H_n^f$-mod from Theorem 6.2 can also be extended to an action on the category $\mathcal{O}$ for rational Cherednik algebras, following an argument of Shan. To explain this in more detail, assume that $\Bbbk = \mathbb{C}$, and consider the complex reflection group $G(l, 1, n) \cong \mathfrak{S}_n \wr \mathbb{Z}/l\mathbb{Z}$ for $l \geq 1$, with reflection representation $\Bbbk^n$ defined as in [Sh, §3.1]. Defining a rational Cherednik algebra requires a choice of parameters, for which there are a bewildering number of different parameterizations. We have:

- a single parameter $\kappa \in \Bbbk$, which is the parameter $k_{H,1}$ in [GGOR, Remark 3.2] for a reflecting hyperplane H on which the difference of two coordinates vanish;
- an l-tuple $(\kappa_1, \dots, \kappa_l) \in \Bbbk^l$ of parameters, which corresponds to the family $\{k_{H,i}\}_{0\leq i\leq l}$ of parameters in [GGOR, Remark 3.2] associated to a reflecting hyperplane H on which a single coordinate vanishes so that $\kappa_i = k_{H,i}$. In *loc. cit.*, it is assumed that $k_{H,0} = k_{H,l} = 0$, but adding a constant to all $k_{H,i}$ leaves the algebra unchanged. It is useful for us to incorporate an additional degree of freedom, so we drop the vanishing condition here: our parameter κ_l may be non-zero.

Let H_n be the rational Cherednik algebra attached to these parameters as in [GGOR, §3].

Let $q := \exp(\sqrt{-1}\pi\kappa)$ and $q_i := \exp(\sqrt{-1}\pi(\kappa_i - i/\ell))$ for $i = 1, \dots, l$. One can relate these to the parameters in [Sh] by choosing integers $e \geq 2$ and $(s_1, \dots, s_l)$ then letting $\kappa := \frac{1}{e}$ and $\kappa_i := \kappa s_i + i/\ell$, so $q_i = q^{s_i}$, for $i = 1, \dots, l$; note that the parameter q in *loc. cit.* is our q^2. Let $\mathcal{O} = \mathcal{O}_{\kappa;\kappa_1,\dots,\kappa_l} := \bigoplus_{n\geq 0} \mathcal{O}_n$ where $\mathcal{O}_n$ is the category of H_n-modules introduced in [GGOR, §3]. Also define

$$f(w) := \prod_{i=1}^{l}\left(q_i^{-1}w + q_i\right), \qquad t := q_1 \dots q_l.$$

By [GGOR, Theorem 5.16], there is an exact functor

$$\mathsf{KZ} : \mathcal{O} \to \bigoplus_{n\geq 0} H_n^f\text{-mod}. \tag{7.1}$$

Matching with the formulae in [GGOR, Sh] requires using the isomorphism from the cyclotomic Hecke algebra in [Sh, §3.1] to ours that sends the generators $T_0, T_1, \dots, T_{n-1}$ to $-x_1, q\tau_1, \dots, q\tau_{n-1}$. The Hecke algebra generators T_i $(i = 1, \dots, n-1)$ in [Sh] are of the form $-T$ for Hecke algebra generators T from [GGOR, §5.2.5] associated to reflections in the first type of hyperplane above. Also, T_0 is a scalar multiple (depending on the choice of κ_l)

of the Hecke algebra generator T in [GGOR, §5.2.5] associated to a reflection of the second type. The key point in all of this is that the minimal polynomials for x_1 and τ_i $(i = 1, \dots, n-1)$ arising from the key formula in [GGOR, §5.2.5] are $f(w)$ and $(w-q)(w+q^{-1})$ (up to scalars), i.e., we do indeed get defining relations of H_n^f.

The functor KZ is fully faithful on projectives [GGOR, Theorem 5.16]. Moreover, it intertwines the Bezrukavnikov-Etingof induction and restriction functors denoted $\mathrm{ind}_{b_{n+1}}$ and $\mathrm{res}_{b_{n+1}}$ in [Sh, §3.2] with the functors ind_n^{n+1} and res_n^{n+1} thanks to [Sh, Theorem 2.1].

Theorem 7.1. *There is a strict $\Bbbk$-linear monoidal functor*

$$\widehat{\Psi}_f : \mathcal{H}eis_{-l}(z,t) \to \mathcal{E}nd\,(\mathcal{O})\,. \tag{7.2}$$

that makes $\mathcal{O}$ into a module category over $\mathcal{H}eis_{-l}(z,t)$, with $\uparrow$ and $\downarrow$ acting as Bezrukavnikov-Etingof induction and restriction functors, respectively. This can be done in such a way that KZ *is a morphism of $\mathcal{H}eis_{-l}(z,t)$-module categories, viewing $\bigoplus_{n\geq 0} H_n^f$*-mod *as a module category via the functor Ψ_f from Theorem 6.2.*

Proof. Our argument is exactly as in the proof of [Sh, Theorem 5.1] using [Sh, Lemma 2.4]. We need to show that there are certain natural transformations of functors satisfying specific relations. Theorem 6.2 allows us to define these on the image of the functor KZ via the action of $\mathcal{H}eis_{-l}(z,t)$. The fully-faithfulness of KZ allows us to transfer this to an action on the full subcategory of projectives in $\mathcal{O}$, and this action can extended to an arbitrary object X by presenting X as the cokernel of a map between projectives. □

Remark 7.2. This quantum Heisenberg action is in many ways more convenient for working with category $\mathcal{O}$ over Cherednik algebras than a Kac-Moody 2-category action, since the Heisenberg action requires no special assumptions on parameters. In fact, this action is still well defined if $\Bbbk$ is replaced by a complete local ring, so one can extend the Heisenberg action to deformed category $\mathcal{O}$.

8. Categorical comultiplication

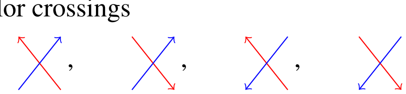
In this section, we construct the quantum analog of the categorical comultiplication from [BSW1, Theorem 5.3]. As well as the quantum Heisenberg category $\mathcal{H}eis_k(z,t)$, we will work with $\mathcal{H}eis_l(z,u)$ and $\mathcal{H}eis_m(z,v)$ for $l, m \in \mathbb{Z}$ and $u, v \in \Bbbk^\times$ chosen so that

$$k = l + m, \qquad\qquad t = uv. \tag{8.1}$$

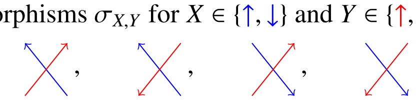
To avoid confusion between these different categories, the reader will want to view the material in this section in color. Let $\mathcal{H}eis_l(z,u) \odot \mathcal{H}eis_m(z,v)$ be the symmetric product of $\mathcal{H}eis_l(z,u)$ and $\mathcal{H}eis_m(z,v)$. As in [BSW1, §3], this is the strict $\Bbbk$-linear monoidal category obtained by first taking the free product $\mathcal{H}eis_l(z,u) \circledast \mathcal{H}eis_m(z,v)$ in the category of strict $\Bbbk$-linear monoidal categories, then adjoining isomorphisms $\sigma_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ for each pair of objects $X \in \mathcal{H}eis_l(z,u)$ and $Y \in \mathcal{H}eis_m(z,v)$, subject to the relations

$$\sigma_{X_1 \otimes X_2, Y} = (\sigma_{X_1,Y} \otimes 1_{X_2}) \circ (1_{X_1} \otimes \sigma_{X_2,Y}), \qquad \sigma_{X_2,Y} \circ (f \otimes 1_Y) = (1_Y \otimes f) \circ \sigma_{X_1,Y},$$
$$\sigma_{X, Y_1 \otimes Y_2} = (1_{Y_1} \otimes \sigma_{X,Y_2}) \circ (\sigma_{X,Y_1} \otimes 1_{Y_2}), \qquad \sigma_{X,Y_2} \circ (1_X \otimes g) = (g \otimes 1_X) \circ \sigma_{X,Y_1}$$

for all $X, X_1, X_2 \in \mathcal{H}eis_l(z,u)$, $Y, Y_1, Y_2 \in \mathcal{H}eis_m(z,v)$ and $f : X_1 \to X_2, g : Y_1 \to Y_2$. Morphisms in $\mathcal{H}eis_l(z,u) \odot \mathcal{H}eis_m(z,v)$ are linear combinations of diagrams colored both blue and red. In these diagrams, as well as the generating morphisms of $\mathcal{H}eis_l(z,u)$ and $\mathcal{H}eis_m(z,v)$, we have the additional two-color crossings



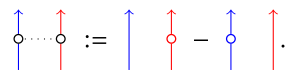
which represent the isomorphisms $\sigma_{X,Y}$ for $X \in \{\uparrow, \downarrow\}$ and $Y \in \{\uparrow, \downarrow\}$, and their inverses



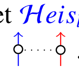
Definition 8.1. Given a diagram D representing a morphism in $\mathcal{H}eis_l(z,u) \odot \mathcal{H}eis_m(z,v)$ and two generic points in this diagram, one on a red string and the other on a blue string, we will denote the morphism represented by

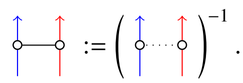
$$(D \text{ with an extra dot at the red point}) - (D \text{ with an extra dot at the blue point})$$

by labelling the points with dots joined by a dotted line. For example:



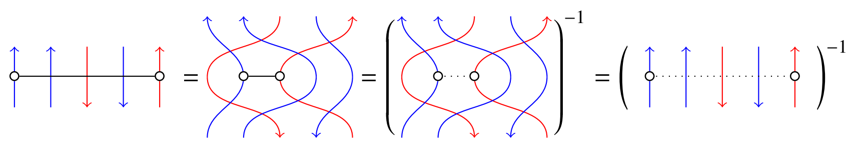
(8.2)

Let $\mathcal{H}eis_l(z,u) \mathbin{\overline{\odot}} \mathcal{H}eis_m(z,v)$ be the strict $\Bbbk$-linear monoidal category obtained by localizing at  . This means that we adjoin a two-sided inverse to this morphism, which we denote as a dumbbell

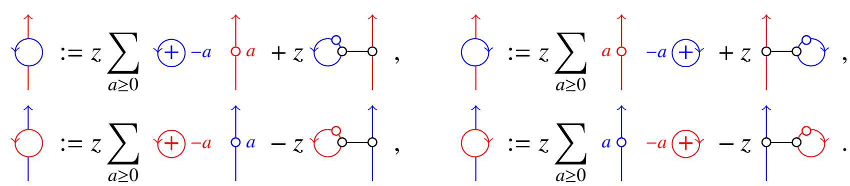


(8.3)

Just as explained in the degenerate case in [BSW1, §§4–5], all morphisms whose string diagram is that of an identity morphism with a horizontal dotted line joining two points of different colors are also automatically invertible in the localized category. We also denote the inverses of such morphisms by using a solid dumbbell in place of the dotted one. For instance:



We also need the following morphisms, which we refer to as *internal bubbles*:



(8.4)

(8.5)

The category $\mathcal{H}eis_l(z,u) \mathbin{\overline{\odot}} \mathcal{H}eis_m(z,v)$ possesses various symmetries which are often useful. Derived from (3.3), we have the strict $\Bbbk$-linear monoidal isomorphism

$$\Omega_{l|m} : \mathcal{H}eis_l(z,u) \mathbin{\overline{\odot}} \mathcal{H}eis_m(z,v) \xrightarrow{\sim} \left(\mathcal{H}eis_{-l}(z,u^{-1}) \mathbin{\overline{\odot}} \mathcal{H}eis_{-m}(z,v^{-1})\right)^{\mathrm{op}}, \tag{8.6}$$

which takes a diagram to its mirror image in a horizontal plane multiplied by $(-1)^{x+y}$ where x is the number of one-colored crossings and y is the number of leftward cups and caps (including ones in +-, −- and internal bubbles). Also, we have

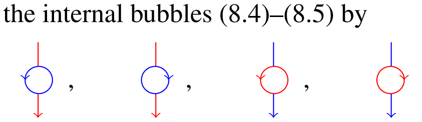
$$\eta : \mathcal{H}eis_l(z,u) \mathbin{\overline{\odot}} \mathcal{H}eis_m(z,v) \xrightarrow{\sim} \mathcal{H}eis_m(z,v) \mathbin{\overline{\odot}} \mathcal{H}eis_l(z,u) \tag{8.7}$$

defined on diagrams by switching the colors blue and red then multiplying by $(-1)^z$ where z is the total number of dumbbells (both solid and dotted) in the picture. Finally, the category $\mathcal{H}eis_l(z,u) \mathbin{\overline{\odot}} \mathcal{H}eis_m(z,v)$ is strictly pivotal, with duality functor

$$* : \mathcal{H}eis_l(z,u) \mathbin{\overline{\odot}} \mathcal{H}eis_m(z,v) \xrightarrow{\sim} \left(\left(\mathcal{H}eis_l(z,u) \mathbin{\overline{\odot}} \mathcal{H}eis_m(z,v)\right)^{\mathrm{op}}\right)^{\mathrm{rev}} \tag{8.8}$$

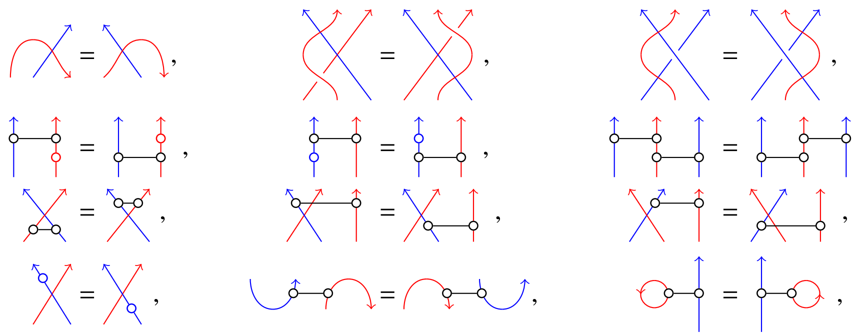
defined by rotating diagrams through 180° just like in (3.21).

We denote the duals of the internal bubbles (8.4)–(8.5) by



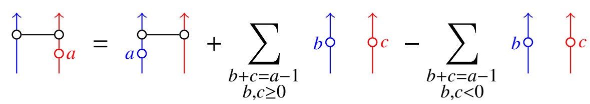
This definition ensures that internal bubbles commute past cups and caps in all possible configurations. For example:

Again as in [BSW1, §§4–5], there are many other obvious commuting relations, such as



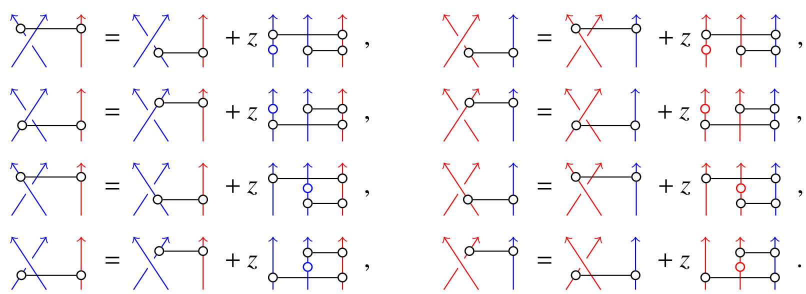
as well as the mirror images of these under the symmetries $\Omega_{l|m}$, η and $*$. We will appeal to all such relations below without further mention.

Here are some more interesting relations. The first shows how to "teleport" dots across dumbbells (plus a correction term):



(8.9)

for any $a \in \mathbb{Z}$. We also have the following relations to commute dumbbells past one-color crossings:



(8.10)

(8.11)

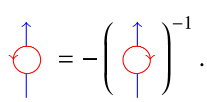
(8.12)

(8.13)

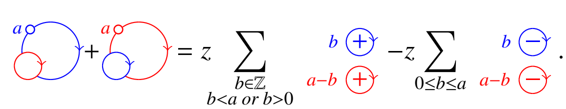
These are all straightforward to prove: one first cancels the solid dumbbells by composing on the top and bottom with their inverses then uses the affine Hecke algebra relations (1.6)–(1.7) to commute dots past crossings in the result.

The following seven lemmas are the quantum analogs of [BSW1, Lemmas 5.5–5.11]. Their proofs are quite similar to the degenerate case.

Lemma 8.2. *We have that*

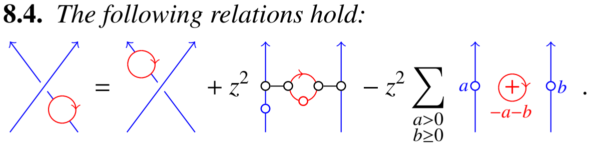


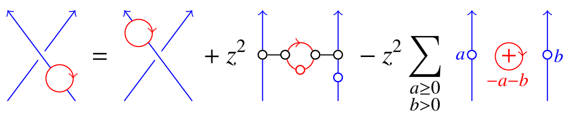
Lemma 8.3. *For any $a \in \mathbb{Z}$, we have that*

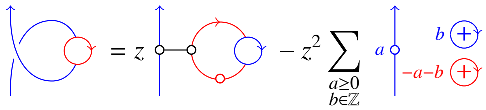


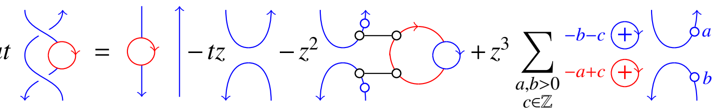
Lemma 8.4. *The following relations hold:*

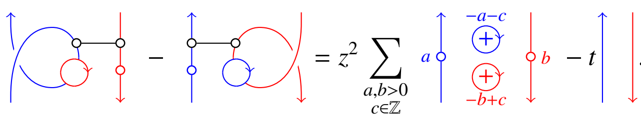
(i)

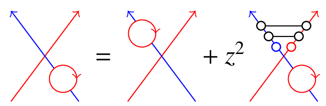


(ii) 

Lemma 8.5. *We have that* 

Lemma 8.6. *We have that* 

Lemma 8.7. *We have that* 

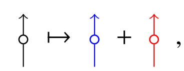
Lemma 8.8. *We have that* 

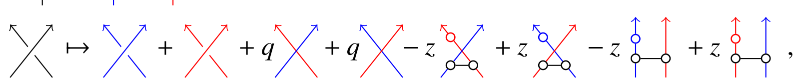
Using these, we can prove the main theorem of the section:

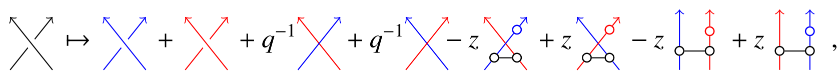
Theorem 8.9. *For $k = l + m$ and $t = uv$, there is a unique strict $\Bbbk$-linear monoidal functor*

$$\Delta_{l|m} : \mathcal{H}eis_k(z,t) \to \mathrm{Add}\left(\mathcal{H}eis_l(z,u) \mathbin{\bar{\odot}} \mathcal{H}eis_m(z,v)\right)$$

such that $\uparrow \mapsto \uparrow \oplus \uparrow$, $\downarrow \mapsto \downarrow \oplus \downarrow$, and on morphisms

 (8.14)

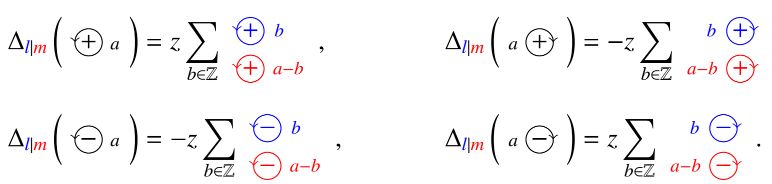
 (8.15)

 (8.16)

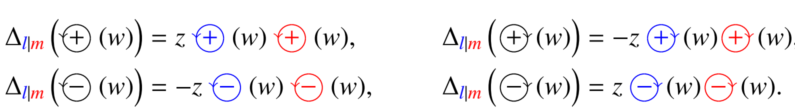
 (8.17)

 (8.18)

Moreover, the following hold for all $a \in \mathbb{Z}$:

 (8.19) (8.20)

Equivalently, in terms of the generating functions (3.14)–(3.17) and their analogs in $\mathcal{H}eis_l(z,u)$ and $\mathcal{H}eis_m(z,v)$:

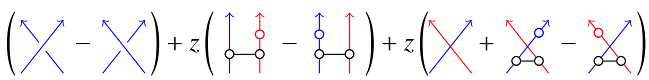
 (8.21) (8.22)

Remark 8.10. For the proof, it is helpful to notice that $\eta \circ \Delta_{l|m} = \Delta_{m|l}$ (on extending η to the additive envelopes in the obvious way). However, $\Delta_{l|m}$ does not commute with either of the other symmetries Ω or $*$. In fact, the map $\Omega_{-l|-m} \circ \Delta_{-l|-m} \circ \Omega_k$ would be an equally good

alternative choice for the categorical comultiplication map. The only change to the above formulae if one uses this alternative is that one needs to replace q with $-q^{-1}$ in (8.15)–(8.16); this is the "Galois symmetry" in the choice of the root q of the equation $x^2 - zx - 1 = 0$.

Proof. Using the presentation from Definition 4.1, we need to check that the images under $\Delta_{l|m}$ of the relations (1.6)–(1.9) and (4.1)–(4.4) are valid in $\mathrm{Add}\big(\mathcal{H}eis_l(z,u) \mathbin{\overline{\odot}} \mathcal{H}eis_m(z,v)\big)$, plus we must also check (8.19)–(8.20). The details are sufficiently similar to the degenerate case from the proof of [BSW1, Theorem 5.3] that we only sketch the steps needed below.

First one checks (1.6)–(1.8). For example, to check the skein relation, the image under $\Delta_{l|m}$ of $-$ is $A + \eta(A)$ where

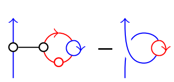
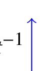
$A :=$ 

Using the skein relation in $\mathcal{H}eis_l(z,u)$ plus (8.9), A simplifies to $B := z$ $+ z$. This is what is required since the image under $\Delta_{l|m}$ of z is $B + \eta(B)$. The other relations here are checked by similarly explicit calculations. The one for the braid relation is rather long.

The relation (1.9) is easy.

To check (8.19)–(8.20), we assume to start with that $k \geq 0$. Consider the clockwise plus-bubble . When $a \leq 0$, this is just a scalar (usually zero) due to (3.11) and the assumption $k \geq 0$, and the relation to be checked is trivial. So assume that $a > 0$. Then $=$, hence, its image under $\Delta_{l|m}$ is $-$ $-$, which is indeed equal to $-z\sum_{b\in\mathbb{Z}}$ by Lemma 8.3. This establishes the left hand identity in (8.19), hence, the left hand identity in (8.21). The right hand identity in (8.21) then follows using (3.13), thereby establishing the right hand identity in (8.19) as well. Next, consider the clockwise minus-bubble . This time the relation to be checked is trivial when $a \geq 0$, so assume that $a < 0$. Then, using the assumption $k \geq 0$ again, we have that $=$, hence, its image under $\Delta_{l|m}$ is $-$ $-$, which is equal to $z\sum_{b\in\mathbb{Z}}$ by Lemma 8.3 (noting when $a < 0 \leq k$ that the term involving plus-bubbles is zero). Then we complete the proof of (8.20) using the equivalent form (8.22) and (3.13) once again. It remains to treat $k \leq 0$. This follows by similar arguments; one starts by considering the counterclockwise plus- and minus-bubbles using the identities obtained by applying $\Omega_{l|m}$ to Lemma 8.3, then gets the clockwise ones using (3.13).

Consider (4.3)–(4.4). The relations involving bubbles follow easily from (8.19)–(8.20). Next consider the right curl relation in (4.3), so $k \geq 0$. Applying $\Delta_{l|m}$ to the relation reveals that we must show that $A + \eta(A) = B + \eta(B)$ where

$A := z$  , $\qquad B := \delta_{k,0}t^{-1}$  .

This follows from Lemma 8.5, noting that the only non-zero term in the summation on the right hand side of that identity is the one with $a = b = 0$ due to the assumption that $k \geq 0$. The argument for the left curl in (4.4) is entirely similiar; it uses the identity obtained by applying $* \circ \Omega_{l|m}$ to Lemma 8.5.

Finally, one must check (4.1)–(4.2). This is a calculation just like in the final paragraph of the proof of [BSW1, Theorem 5.3]; ultimately one uses Lemmas 8.6–8.8. ∎

9. Generalized cyclotomic quotients

In this section, we define some $\mathbb{k}$-linear categories, namely, the generalized cyclotomic quotients of $\mathcal{H}eis_k(z,t)$. Recall that $x =$ and $y =$.

Definition 9.1. Suppose we are given polynomials

$$f(w) = f_0 w^l + f_1 w^{l-1} + \cdots + f_l \in \Bbbk[w], \tag{9.1}$$

$$g(w) = g_0 w^m + g_1 w^{m-1} + \cdots + g_m \in \Bbbk[w] \tag{9.2}$$

of degrees $l, m \geq 0$, respectively, such that $f_0 f_l = 1 = g_0 g_m$. Let $k := m - l$ and $t := g_0 f_0^{-1}$. Define series $A^\pm(w) = \sum_{n\in\mathbb{Z}} A_n^\pm w^{-n}$ and $B^\pm(w) = \sum_{n\in\mathbb{Z}} B_n^\pm w^{-n}$ by

$$A^+(w) := z^{-1} g(w)/f(w) \in tz^{-1}w^k + w^{k-1}\Bbbk[\![w^{-1}]\!], \tag{9.3}$$

$$B^+(w) := -z^{-1} f(w)/g(w) \in -t^{-1}z^{-1}w^{-k} + w^{-k-1}\Bbbk[\![w^{-1}]\!], \tag{9.4}$$

$$A^-(w) := -z^{-1} g(w)/f(w) \in -t^{-1}z^{-1} + w\Bbbk[\![w]\!], \tag{9.5}$$

$$B^-(w) := z^{-1} f(w)/g(w) \in tz^{-1} + w\Bbbk[\![w]\!]; \tag{9.6}$$

cf. (3.14)–(3.17). Let $\mathcal{I}(f|g)$ be the left tensor ideal generated by the morphisms

$$\Big\{ f(x),\ \oplus_n - A_n^+ 1\!\!1 \ \Big|\ -k < n < l \Big\}. \tag{9.7}$$

The *generalized cyclotomic quotient* associated to the polynomials $f(w)$ and $g(w)$ is the quotient category

$$\mathcal{H}(f|g) := \mathcal{H}eis_k(z,t)/\mathcal{I}(f|g). \tag{9.8}$$

It is a module category over $\mathcal{H}eis_k(z,t)$.

The following is the quantum analog of [B1, Lemma 1.8]; see also [BD, Lemma 4.14] for the analog in the setting of Kac-Moody 2-categories.

Lemma 9.2. *In the setup of Definition 9.1, $\mathcal{I}(f|g)$ may be defined equivalently as the left tensor ideal generated by*

$$\Big\{ g(y),\ {}_n\oplus - B_n^+ 1\!\!1 \ \Big|\ k < n < m \Big\}. \tag{9.9}$$

Moreover, it contains $\oplus_n - A_n^+1\!\!1$, $\ominus_n - A_n^-1\!\!1$, ${}_n\oplus - B_n^+1\!\!1$ and ${}_n\ominus - B_n^-1\!\!1$ for all $n \in \mathbb{Z}$.

Proof. For morphisms $\theta, \phi : X \to Y$, we will write $\theta \equiv \phi$ as shorthand for $\theta - \phi \in \mathcal{I}(f|g)$. By (3.11)–(3.12), we have automatically that $\oplus_n \equiv A_n^+1\!\!1$ when $n \leq -k$, ${}_n\oplus \equiv B_n^+1\!\!1$ when $n \leq k$, $\ominus_n \equiv A_n^-1\!\!1$ when $n \geq 0$, and ${}_n\ominus \equiv B_n^-1\!\!1$ when $n \geq 0$.

In this paragraph, we use ascending induction on n to show that $\oplus_n \equiv A_n^+1\!\!1$ for all $n \in \mathbb{Z}$. This is immediate from (9.7) if $n < l$, so assume that $n \geq l$. The fact that $f(x) \equiv 0$ implies that $\sum_{a=0}^l f_a \oplus_{n-a} + \sum_{a=0}^l f_a \ominus_{n-a} = \sum_{a=0}^l f_a \bigcirc_{n-a} \equiv 0$. On the left hand side of this, the only nonzero minus-bubble arises when $n = a = l$, so it shows that $\sum_{a=0}^l f_a \oplus_{n-a} \equiv \delta_{l,n} f_l z^{-1} t^{-1} 1\!\!1$. Using the induction hypothesis, we deduce that $f_0 \oplus_n + \sum_{a=1}^l f_a A_{n-a}^+ 1\!\!1 \equiv \delta_{l,n} z^{-1} g_0^{-1} 1\!\!1$. Equating w^{l-n}-coefficients in $f(w)A^+(w) = z^{-1}g(w)$, we get that $\sum_{a=0}^l f_a A_{n-a}^+ = \delta_{l,n} z^{-1} g_0^{-1}$. Hence, $\oplus_n \equiv A_n^+1\!\!1$ as claimed.

Next, we show by descending induction on n that $\ominus_n \equiv A_n^-1\!\!1$ for all $n \in \mathbb{Z}$. We may assume that $n < 0$. Equating w^{-n}-coefficients in $f(w)A^+(w) = -f(w)A^-(w)$ gives that $\sum_{a=0}^l f_{l-a} A_{a+n}^+ = -\sum_{a=0}^l f_{l-a} A_{a+n}^-$. Using the induction hypothesis plus the previous paragraph, we deduce that $\sum_{a=0}^l f_{l-a} \oplus_{a+n} + f_l A_n^- + \sum_{a=1}^l f_{l-a} \ominus_{a+n} \equiv 0$. But also from $f(x) \equiv 0$ we get that $\sum_{a=0}^l f_{l-a} \oplus_{a+n} + \sum_{a=0}^l f_{l-a} \ominus_{a+n} = \sum_{a=0}^l f_{l-a} \bigcirc_{a+n} \equiv 0$. Combining these two identities establishes the induction step.

Using the notation of (3.14)–(3.17), we have now shown that $\pm(w) \equiv A^\pm(w)1\!\!1$. Taking inverses using (3.13), we deduce that $\pm(w) \equiv B^\pm(w)1\!\!1$. Hence, ${}_n\pm \equiv B_n^\pm 1\!\!1$ for all $n \in \mathbb{Z}$. So we have established the last assertion from the lemma.

Equating w^b-coefficients in $g(w) = zf(w)A^+(w)$ shows that $g_{m-b} = z\sum_{a=0}^l f_{l-a} A_{a-b}^+$. Hence:

$$g(y) = \sum_{b=0}^m g_{m-b} y^b \equiv z \sum_{b=0}^m \sum_{a=0}^l f_{l-a} y^b \oplus_{a-b} = \sum_{a=0}^l f_{l-a} \left(z \sum_{b\geq 0} {}^{b}\!\downarrow \oplus_{a-b} \right) \stackrel{(4.17)}{=} \sum_{a=0}^l f_{l-a} \bigcirc_a \equiv 0.$$

We have now shown that $\mathcal{I}(f|g)$, the left tensor ideal generated by (9.7), contains (9.9). Similarly, the left tensor ideal generated by (9.9) contains (9.7). This completes the proof. $\square$

We assume for the the rest of the section that $\Bbbk$ is a field. The following is well known.

Lemma 9.3. *Let V be a finite-dimensional AH_2-module. All eigenvalues of x_2 on V are of the form $\lambda, q^2\lambda$ or $q^{-2}\lambda$ for eigenvalues λ of x_1 on V.*

Proof. Suppose that $v \in V$ is a simultaneous eigenvector for the commuting operators x_1 and x_2 of eigenvalues λ_1 and λ_2, respectively. If $\tau_1 v = qv$ (resp. $\tau_1 v = -q^{-1}v$) then $\lambda_2 = q^2\lambda_1$ (resp. $\lambda_2 = q^{-2}\lambda_1$), as follows easily from the relation $x_2(\tau_1 - z)v = \tau_1 x_1 v$. Otherwise, v and $\tau_1 v$ are linearly independent, in which case the matrix describing the action of x_1 on the subspace with basis $\{v, \tau_1 v\}$ is $\begin{pmatrix} \lambda_1 & -z\lambda_2 \\ 0 & \lambda_2 \end{pmatrix}$. So λ_2 is another eigenvalue of x_1 on V. $\square$

For $f(w)$ and $g(w)$ as in Definition 9.1, let $\mathcal{V}(f)$ and $\mathcal{V}(g)^\vee$ denote $\bigoplus_{n\geq 0} H_n^f$-pmod and $\bigoplus_{n\geq 0} H_n^g$-pmod viewed as a module categories over $\mathcal{H}eis_{-l}(z, f_0^{-1})$ and $\mathcal{H}eis_m(z, g_0)$ via the monoidal functors Ψ_f and $\Psi_g^\vee$ from Lemma 6.4. Let

$$\mathcal{V}(f|g) := \mathcal{V}(f) \boxtimes \mathcal{V}(g)^\vee \tag{9.10}$$

be their linearized Cartesian product, i.e., the $\Bbbk$-linear category with objects that are pairs (X, Y) for $X \in \mathcal{V}(f)$, $Y \in \mathcal{V}(g)^\vee$, and morphisms

$$\mathrm{Hom}_{\mathcal{V}(f|g)}((X, Y), (U, V)) := \mathrm{Hom}_{\mathcal{V}(f)}(X, U) \otimes \mathrm{Hom}_{\mathcal{V}(g)^\vee}(Y, V)$$

with the obvious composition law. There is an equivalence of categories

$$\mathcal{V}(f|g) \to \bigoplus_{r,s\geq 0} \left(H_r^f \otimes H_s^g\right)\text{-pmod},$$

hence, $\mathcal{V}(f|g)$ is additive Karoubian. Moreover, $\mathcal{V}(f|g)$ is a module category over the symmetric product $\mathcal{H}eis_{-l}(z, f_0^{-1}) \odot \mathcal{H}eis_m(z, g_0)$.

Lemma 9.4. *Suppose that $\Bbbk$ is a field over which both $f(w)$ and $g(w)$ factor as products of linear factors, and that $\lambda\mu^{-1} \notin \left\{q^{2i} \,\middle|\, i \in \mathbb{Z}\right\}$ for all roots λ of $f(w)$ and μ of $g(w)$. Then the categorical action of $\mathcal{H}eis_{-l}(z, f_0^{-1}) \odot \mathcal{H}eis_m(z, g_0)$ on $\mathcal{V}(f|g)$ extends to an action of the localization $\mathcal{H}eis_{-l}(z, f_0^{-1}) \mathbin{\bar{\odot}} \mathcal{H}eis_m(z, g_0)$ from Definition 8.1.*

Proof. Lemma 9.3 implies that the eigenvalues of $x_1, \dots, x_n$ on any finite-dimensional H_n^f-module are of the form $q^{2i}\lambda$ for $i \in \mathbb{Z}$ and a root λ of $f(w)$. Consequently, the commuting endomorphisms defined by evaluating and on an object of $\mathcal{V}(f|g)$ have eigenvalues contained in the sets $\left\{q^{2i}\lambda \,\middle|\, i \in \mathbb{Z}, \lambda \text{ a root of } f(w)\right\}$ and $\left\{q^{2j}\mu \,\middle|\, j \in \mathbb{Z}, \mu \text{ a root of } g(w)\right\}$, respectively. By the genericity assumption, these sets are disjoint, hence, all eigenvalues of the endomorphism defined by $=$ $-$ lie in $\Bbbk^\times$. Consequently, this endomorphism is invertible. $\square$

Lemma 9.4 shows for suitably generic $f(w), g(w)$ that there is a strict $\Bbbk$-linear monoidal functor $\Psi_f \mathbin{\bar{\odot}} \Psi_g^\vee : \mathcal{H}eis_{-l}(z, f_0^{-1}) \mathbin{\bar{\odot}} \mathcal{H}eis_m(z, g_0) \to \mathcal{E}nd(\mathcal{V}(f|g))$. Composing this functor with the functor from Theorem 8.9, we obtain a strict $\Bbbk$-linear monoidal functor

$$\Psi_{f|g} := \Delta_{-l|m} \circ \Psi_f \mathbin{\bar{\odot}} \Psi_g^\vee : \mathcal{H}eis_k(z, t) \to \mathcal{E}nd\left(\mathcal{V}(f|g)\right), \tag{9.11}$$

where $t := g_0 f_0^{-1}$ and $k := m - l$ as in Definition 9.1. Thus, we have made $\mathcal{V}(f|g)$ into a module category over $\mathcal{H}eis_k(z, t)$.

Theorem 9.5. *Assume that $\Bbbk$ is a field and $f(w), g(w)$ satisfy the genericity assumption from Lemma 9.4. Let* $\mathrm{Ev} : \mathcal{E}nd\left(\mathcal{V}(f|g)\right) \to \mathcal{V}(f|g)$ *be the $\Bbbk$-linear functor defined by evaluation on*

$S := (H_0^f, H_0^g) \in \mathcal{V}(f|g)$. *The composition* $\mathrm{Ev} \circ \Psi_{f|g}$ *factors through the generalized cyclotomic quotient* $\mathcal{H}(f|g)$ *to induce an equivalence of* $\mathcal{H}eis_k(z,t)$*-module categories*

$$\psi_{f|g} : \mathrm{Kar}\,(\mathcal{H}(f|g)) \to \mathcal{V}(f|g).$$

Proof. We first show that $\Psi_{f|g}\left(\oplus\,(w)\right)_S \in w^k \operatorname{End}(S)[\![w^{-1}]\!]$ equals $A^+(w)1_S$. Recalling that $A^+(w)$ is the expansion at $w = \infty$ of the rational function $z^{-1}g(w)/f(w)$, this follows because

$$\Psi_{f|g}\left(\oplus\,(w)\right)_S = z\,\Psi_f\left(\oplus\,(w)\right)_{H_0^f} \otimes \Psi_g^\vee\left(\oplus\,(w)\right)_{H_0^g}$$

thanks to (8.21), and also $\Psi_f\left(\oplus\,(w)\right)_{H_0^f} = z^{-1}/f(w)$ and $\Psi_g^\vee\left(\oplus\,(w)\right)_{H_0^g} = z^{-1}g(w)$. To see the last two assertions, we first apply Lemma 9.2 to see that $\mathcal{I}(f|1)$, the left tensor ideal of $\mathcal{H}eis_{-l}(z, f_0^{-1})$ generated by $f(x)$, contains all coefficients of the series $\oplus\,(w) - z^{-1}/f(w)\mathbb{1}$; all elements of this ideal act as zero on H_0^f since its generator $f(x)$ acts as zero. Then we apply Lemma 9.2 again to see that $\mathcal{I}(1|g)$, the left tensor ideal of $\mathcal{H}eis_m(z, g_0)$ generated by $g(y)$, contains all coefficients of $\oplus\,(w) - z^{-1}g(w)\mathbb{1}$; all elements of this act as zero on H_0^g.

The previous paragraph shows that $\oplus_n - A_n^+\mathbb{1}$ acts as zero on S for all $n \in \mathbb{Z}$. Also it is obvious that $f(x)$ acts as zero on S. So the left tensor ideal $\mathcal{I}(f|g)$ acts as zero on S, which proves that $\mathrm{Ev} \circ \Psi_{f|g}$ factors through the quotient $\mathcal{H}(f|g) = \mathcal{H}eis_k(z,t)/\mathcal{I}(f|g)$ to induce a $\Bbbk$-linear functor $\mathcal{H}(f|g) \to \mathcal{V}(f) \boxtimes \mathcal{V}(g)^\vee$. Since $\mathcal{V}(f|g)$ is additive Karoubian, this extends to the Karoubi envelope to induce the functor $\psi_{f|g}$ from the statement of the theorem. Moreover, it is automatic from the definition that $\psi_{f|g}$ is a morphism of $\mathcal{H}eis_k(z,t)$-module categories. It just remains to show that $\psi_{f|g}$ is an equivalence, which we do by showing that it is full, faithful and dense.

First we show that $\psi_{f|g}$ is full and faithful. It suffices to check this on objects $X = X_r \otimes \cdots \otimes X_1$ and $Y = Y_s \otimes \cdots \otimes Y_1$ that are words in $\uparrow$ and $\downarrow$. We assume moreover that $k \geq 0$; a similar argument with the roles of $\uparrow$ and $\downarrow$ interchanged does the job when $k \leq 0$ too. Let $X^* = X_1^* \otimes \cdots \otimes X_r^*$ be the dual object (here, $\uparrow^* = \downarrow, \downarrow^* = \uparrow$). By rigidity, we have a canonical isomorphism $\operatorname{Hom}_{\mathcal{H}(f|g)}(X, Y) \cong \operatorname{Hom}_{\mathcal{H}(f|g)}(\mathbb{1}, X^* \otimes Y)$, from which we get a commuting diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{H}(f|g)}(X, Y) \arrow[r, "\sim"] \arrow[d, "\psi_{f|g}"'] & \operatorname{Hom}_{\mathcal{H}(f|g)}(\mathbb{1}, X^* \otimes Y) \arrow[d, "\psi_{f|g}"] \\
\operatorname{Hom}_{\mathcal{V}(f|g)}(X \otimes S, Y \otimes S) \arrow[r, "\sim"] & \operatorname{Hom}_{\mathcal{V}(f|g)}(S, X^* \otimes Y \otimes S).
\end{tikzcd}$$

The left hand vertical map in this diagram is an isomorphism if and only if the right hand vertical map is one. Also the left hand vertical map is an isomorphism when $X = Y = \uparrow^{\otimes n}$. Proof: the canonical homomorphism $AH_n \to \operatorname{End}_{\mathcal{H}eis_k(z,t)}(\uparrow^{\otimes n})$ induces a homomorphism $H_n^f \to \operatorname{End}_{\mathcal{V}(f|g)}(\uparrow^{\otimes n})$ which is obviously surjective since bubbles on the right hand edge are scalars in the generalized cyclotomic quotient; on the other hand, $\operatorname{End}_{\mathcal{V}(f|g)}(\uparrow^{\otimes n} \otimes S) = \operatorname{End}_{H_n^f}\left(H_n^f\right) = H_n^f$. Hence, the right hand vertical map is an isomorphism when $X^* \otimes Y = \downarrow^{\otimes n} \otimes \uparrow^{\otimes n}$. Using this, we can show that the right hand vertical map is an isomorphism in general. All of the morphism spaces are zero unless $X^* \otimes Y$ has the same number of $\uparrow$'s as $\downarrow$'s. If all $\downarrow$'s are to the left of all $\uparrow$'s, we are done already, so we may assume that $X^* \otimes Y$ involves $\uparrow \otimes \downarrow$ as a subword. Let U be $X^* \otimes Y$ with the two letters in this subword interchanged and V be $X^* \otimes Y$ with these two letters deleted. Using the isomorphism $\uparrow \otimes \downarrow \cong \downarrow \otimes \uparrow \oplus \mathbb{1}^{\oplus k}$ from (2.10), we get a commuting diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{H}(f|g)}(\mathbb{1}, X^* \otimes Y) \arrow[r, "\sim"] \arrow[d, "\psi_{f|g}"'] & \operatorname{Hom}_{\mathcal{H}(f|g)}\left(\mathbb{1}, U \oplus V^{\oplus k}\right) \arrow[d, "\psi_{f|g}"] \\
\operatorname{Hom}_{\mathcal{V}(f|g)}(S, X^* \otimes Y \otimes S) \arrow[r, "\sim"] & \operatorname{Hom}_{\mathcal{V}(f|g)}\left(S, U \otimes S \oplus V \otimes S^{\oplus k}\right).
\end{tikzcd}$$

By induction, the right hand vertical map is an isomorphism, hence, so too is the left hand one.

Finally, we explain why $\psi_{f|g}$ is dense. Let Q be an indecomposable object in $\mathcal{V}(f|g)$. We have that $\downarrow^{\otimes m} \otimes \uparrow^{\otimes n} \otimes S = \downarrow^{\otimes m} \otimes (H_n^f, H_0^g) = (H_n^f, H_m^g) \oplus M$ where M is a direct sum of summands of $(H_{n'}^f, H_{m'}^g)$ with $n' < n$ and $m' < m$. It follows that Q is isomorphic to the image of some idempotent in $\mathrm{End}_{\mathcal{V}(f|g)}(\downarrow^{\otimes m} \otimes \uparrow^{\otimes n} \otimes S)$ for some $m, n \geq 0$. Since we have shown already that $\psi_{f|g}$ is full and faithful, there is a corresponding idempotent in $\mathrm{End}_{\mathcal{H}(f|g)}(\downarrow^{\otimes m} \otimes \uparrow^{\otimes n})$. The latter idempotent defines an object P of $\mathrm{Kar}(\mathcal{H}(f|g))$ such that $\psi_{f|g}(P) \cong Q$. □

Remark 9.6. If $g(w) = 1$ the genericity assumption is vacuous, so Theorem 9.5 gives us an equivalence of categories $\psi_{f|1} : \mathrm{Kar}(\mathcal{H}(f|1)) \to \mathcal{V}(f)$. In other words, the generalized cyclotomic quotient $\mathcal{H}(f|1)$ is Morita equivalent to the "usual" cyclotomic quotient defined by the cyclotomic Hecke algebras H_n^f for all $n \geq 0$. This statement is the quantum analog of [B1, Theorem 1.7]; see also [R, Theorem 4.25] for the analogous result in the setting of Kac-Moody 2-categories.

10. Basis theorem

Finally, we prove a basis theorem for the morphism spaces in $\mathcal{H}eis_k(z,t)$. Our proof of this is very similar to the argument in the degenerate case from [BSW1, Theorem 6.4]. Let $X = X_r \otimes \cdots \otimes X_1$ and $Y = Y_s \otimes \cdots \otimes Y_1$ be objects of $\mathcal{H}eis_k(z,t)$ for $X_i, Y_j \in \{\uparrow, \downarrow\}$. An (X,Y)-*matching* is a bijection between $\{i \mid X_i = \uparrow\} \sqcup \{j \mid Y_j = \downarrow\}$ and $\{i \mid X_i = \downarrow\} \sqcup \{j \mid Y_j = \uparrow\}$. A *reduced lift* of an (X,Y)-matching means a diagram representing a morphism $X \to Y$ such that

- the endpoints of each string are points which correspond under the given matching;
- there are no floating bubbles and no dots on any string;
- there are no self-intersections of strings and no two strings cross each other more than once.

Fix a set $B(X,Y)$ consisting of a choice of reduced lift for each of the (X,Y)-matchings. Let $B_\circ(X,Y)$ be the set of all morphisms that can be obtained from the elements of $B(X,Y)$ by adding dots labelled with integer multiplicities near to the terminus of each string. Also recall the homomorphism $\beta : \mathrm{Sym} \otimes \mathrm{Sym} \to \mathrm{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1})$ from (3.7). Using it, we can make the morphism space $\mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(X,Y)$ into a right $\mathrm{Sym} \otimes \mathrm{Sym}$-module: $\phi\theta := \phi \otimes \beta(\theta)$.

Theorem 10.1. *For any ground ring $\Bbbk$, parameters $z, t \in \Bbbk^\times$, and objects $X, Y \in \mathcal{H}eis_k(z,t)$, the morphism space* $\mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(X,Y)$ *is a free right* $\mathrm{Sym} \otimes \mathrm{Sym}$*-module with basis $B_\circ(X,Y)$.*

Proof. We just prove this when $k \leq 0$; the result for $k \geq 0$ then follows by applying Ω_k. Let $X = X_r \otimes \cdots \otimes X_1$ and $Y = Y_s \otimes \cdots \otimes Y_1$ be two objects.

We first observe that $B_\circ(X,Y)$ spans $\mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(X,Y)$ as a right $\mathrm{Sym} \otimes \mathrm{Sym}$-module. The defining relations and the additional relations derived in sections 2, 3 and 4 give Reidemeister-type relations modulo terms with fewer crossings, plus a skein relation and bubble and dot sliding relations. These relations allow diagrams for morphisms in $\mathcal{H}eis_k(z,t)$ to be transformed in a similar way to the way oriented tangles are simplified in skein categories, modulo diagrams with fewer crossings. Hence, there a straightening algorithm to rewrite any diagram representing a morphism $X \to Y$ as a linear combination of the ones in $B_\circ(X,Y)$.

It remains to prove the linear independence. We say $\phi \in B_\circ(X,Y)$ is *positive* if it only involves non-negative powers of dots. It suffices to show just that the positive morphisms in $B_\circ(X,Y)$ are linearly independent. Indeed, given any linear relation of the form $\sum_{i=1}^N \phi_i \otimes \beta(\theta_i) = 0$ for morphisms $\phi_i \in B_\circ(X,Y)$ and coefficients $\theta_i \in \mathrm{Sym} \otimes \mathrm{Sym}$, we can "clear denominators" by multiplying the termini of the strings by sufficiently large positive powers of dots to reduce to the positive case.

The main step now is to prove the linear independence in the special case that $X = Y = \uparrow^{\otimes n}$. To do this, we need to allow the ground ring $\Bbbk$ to change, so we will add a subscript to our notation, denoting $\mathcal{H}eis_k(z,t), \mathcal{V}(f|g), \mathrm{Sym} \otimes \mathrm{Sym}, \ldots$ by ${}_\Bbbk\mathcal{H}eis_k(z,t), {}_\Bbbk\mathcal{V}(f|g), {}_\Bbbk\mathrm{Sym} \otimes_\Bbbk {}_\Bbbk\mathrm{Sym}, \ldots$ to avoid any confusion. It suffices to prove the linear independence of positive elements of

$B_\circ(X,Y)$ in the special case that $\Bbbk = \mathbb{Z}[z^{\pm 1}, t^{\pm 1}]$; one can then use the canonical $\Bbbk$-linear monoidal functor ${}_{\Bbbk}\mathcal{H}eis_k(z,t) \to \Bbbk \otimes_{\mathbb{Z}[z^{\pm1},t^{\pm1}]} {}_{\mathbb{Z}[z^{\pm1},t^{\pm1}]}\mathcal{H}eis_k(z,t)$ to deduce the linear independence over an arbitrary ground ring $\Bbbk$ and for arbitrary parameters.

So assume now that $\Bbbk = \mathbb{Z}[z^{\pm 1}, t^{\pm 1}]$ and take a linear relation $\sum_{i=1}^N \phi_i \otimes \beta(\theta_i) = 0$ for positive $\phi_i \in B_\circ(X,Y)$. Choose a so that the multiplicities of dots in all ϕ_i arising in this linear relation are $\leq a$. Also choose $b, c \geq 0$ so that all of the symmetric functions $\theta_i \in {}_{\Bbbk}\mathrm{Sym} \otimes_{\Bbbk} {}_{\Bbbk}\mathrm{Sym}$ are polynomials in the elementary symmetric functions $e_1 \otimes 1, \dots, e_b \otimes 1$ and $1 \otimes e_1, \dots, 1 \otimes e_c$. Then choose l, m so that $a < l$, $b + c < m$ and $k = m - l$. Note that $l \geq m$ due to our standing assumption that $k \leq 0$. Let $u_1, \dots, u_b$ and $v_1, \dots, v_c$ be indeterminates and $\mathbb{K}$ be the algebraic closure of the field $\mathbb{Q}(z, t, u_1, \dots, u_b, v_1, \dots, v_c)$. Pick $q \in \mathbb{K}^\times$ so that $z = q - q^{-1}$ and consider the cyclotomic Hecke algebras ${}_{\mathbb{K}}H_n^f$ and ${}_{\mathbb{K}}H_n^g$ over $\mathbb{K}$ associated to the polynomials

$$f(w) := t^{-1}w^l + t, \qquad g(w) = w^m + u_1 w^{m-1} + \cdots + u_b w^{m-b} + v_c w^c + \cdots + v_1 w + 1.$$

Note the formula for $g(w)$ makes sense because $b + c < m$. Consider the ${}_{\mathbb{K}}\mathcal{H}eis_k(z,t)$-module category ${}_{\mathbb{K}}\mathcal{V}(f|g)$ from (9.11). Since $\Bbbk \hookrightarrow \mathbb{K}$, there is a canonical $\Bbbk$-linear monoidal functor ${}_{\Bbbk}\mathcal{H}eis_k(z,t) \to {}_{\mathbb{K}}\mathcal{H}eis_k(z,t)$, allowing us to view ${}_{\mathbb{K}}\mathcal{V}(f|g)$ also as a module category over ${}_{\Bbbk}\mathcal{H}eis_k(z,t)$. Then we evaluate the relation $\sum \phi_i \otimes \beta(\theta_i) = 0$ on ${}_{\mathbb{K}}S := ({}_{\mathbb{K}}H_0^f, {}_{\mathbb{K}}H_0^g)$ to obtain a relation in ${}_{\mathbb{K}}H_n^f$. By the basis theorem for ${}_{\mathbb{K}}H_n^f$ from (6.2) and the assumption that $a < l$, the images of $\phi_1, \dots, \phi_N$ in ${}_{\mathbb{K}}H_n^f$ are linearly independent over $\mathbb{K}$, so we deduce that the image of $\beta(\theta_i)$ in $\mathbb{K}$ is zero for each i. To deduce from this that $\theta_i = 0$, recall that θ_i is a polynomial in $e_1 \otimes 1, \dots, e_b \otimes 1, 1 \otimes e_1, \dots, 1 \otimes e_c$. So we need to show that the images of $\beta(e_1 \otimes 1), \dots, \beta(e_b \otimes 1), \beta(1 \otimes e_1), \dots, \beta(1 \otimes e_c)$ in $\mathbb{K}$ are algebraically independent. In fact, we claim that these images are the indeterminates $u_1, \dots, u_b, v_1, \dots, v_c$, respectively. To prove the claim, we use the definition (3.9) and Lemma 9.2 to see that the image of $\beta(e_n \otimes 1)$ is $t^{-1}zA_{n-k}^+$ and the image of $\beta(1 \otimes e_n)$ is $-tzA_n^-$. Then we compute the low degree terms of $A^\pm(w)$:

$$t^{-1}zA^+(w) = t^{-1}g(w)/f(w) = w^k + u_1 w^{k-1} + \cdots + u_b w^{k-b} + \cdots \in w^k \mathbb{K}[\![w^{-1}]\!],$$
$$-tzA^-(w) = tg(w)/f(w) = 1 + v_1 w + \cdots + v_c w^c + \cdots \in \mathbb{K}[\![w]\!],$$

so indeed $\beta(e_n \otimes 1) = u_n$ for $n = 1, \dots, b$ and $\beta(1 \otimes e_n) = v_n$ for $n = 1, \dots, c$.

We have now proved the linear independence when $X = Y = \uparrow^{\otimes n}$. Returning to the general case, we can use the canonical isomorphism $\mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(X,Y) \cong \mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(\mathbb{1}, X^* \otimes Y)$ arising from the rigidity so see that the $\mathrm{Sym} \otimes \mathrm{Sym}$-linear independence of the positive morphisms in $B_\circ(X,Y)$ is equivalent to the $\mathrm{Sym} \otimes \mathrm{Sym}$-linear independence of the positive morphisms in $B_\circ(\mathbb{1}, X^* \otimes Y)$. Thus, we are reduced to the case that $X = \mathbb{1}$. Assume this from now on. The set $B_\circ(\mathbb{1}, Y)$ is empty unless Y has the same number n of $\uparrow$'s as $\downarrow$'s. Also we have already proved the linear independence in the case $Y = \downarrow^{\otimes n} \otimes \uparrow^{\otimes n}$. So we may assume that Y has a subword $\uparrow \otimes \downarrow$. Let Z be Y with the two letters in the subword interchanged. By induction, we may assume the linear independence has already been established for $B_\circ(\mathbb{1}, Z)$. Consider a linear relation $\sum_{i=1}^N \phi_i \otimes \beta(\theta_i)$ for positive $\phi_i \in B_\circ(\mathbb{1}, Y)$. Recalling the isomorphism $\uparrow \otimes \downarrow \oplus \mathbb{1}^{\oplus(-k)} \xrightarrow{\sim} \downarrow \otimes \uparrow$ from (2.25), multiplying the subword $\uparrow \otimes \downarrow$ on top by the sideways crossing defines a $\mathrm{Sym} \otimes \mathrm{Sym}$-linear map $s : \mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(\mathbb{1}, Y) \hookrightarrow \mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(\mathbb{1}, Z)$. Unfortunately, s does not send $B_\circ(\mathbb{1}, Y)$ into $B_\circ(\mathbb{1}, Z)$. However, the image of $B_\circ(\mathbb{1}, Y)$ is related to $B_\circ(\mathbb{1}, Z)$ in a triangular way, which is good enough to complete the argument. The full explanation of this is almost exactly the same as in the degenerate case, so we refer the reader to the last paragraph of the proof of [BSW1, Theorem 6.4] for the details. □

Corollary 10.2. $\mathrm{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1}) \cong \mathrm{Sym} \otimes \mathrm{Sym}$.

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