MODULAR REPRESENTATIONS OF THE SUPERGROUP $Q(n)$, II

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We continue our study of the representations of the supergroup $Q(n)$ over a field of odd positive characteristic. The focus here is on the aspects of the theory that depend in some way on the interpretation of induction in terms of sheaf cohomology of certain equivariant vector bundles on the associated flag superschemes.

1. Introduction

This is the second of two articles investigating the representation theory of the supergroup $G = Q(n)$ over an algebraically closed field k of characteristic $p > 2$. In the first article [3], we extended many of the basic algebraic properties (e.g. the results of Penkov [16]) from characteristic 0 to characteristic p. The present article is concerned instead with results which depend in some way on the interpretation of induction from a Borel subgroup B in terms of sheaf cohomology on the flag superscheme G/B. These flag superschemes, and their analogues G/P for arbitrary parabolic subgroups of G, were introduced originally by Manin [12, 13], and have already played a fundamental role in the work of Penkov and Serganova [18, 19, 20] on the representation theory of $Q(n)$ over $\mathbb{C}$; see also [1].

To sketch the results proved here in more detail, recall from [3] that the irreducible representations of G are parametrized by *p-strict dominant weights*, i.e. the set

$$X_p^+(n) := \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \,\middle|\, \begin{array}{l} \lambda_1 \geq \cdots \geq \lambda_n \text{ with} \\ \lambda_i = \lambda_{i+1} \text{ only if } p|\lambda_i \end{array} \right\}.$$

The irreducible representation $L(\lambda)$ corresponding to $\lambda \in X_p^+(n)$ can be realized as the unique irreducible submodule of the induced module

$$H^0(\lambda) := H^0(G/B, \mathscr{L}(\mathfrak{u}(\lambda))),$$

where $\mathfrak{u}(\lambda)$ is a certain irreducible B-supermodule of dimension a power of 2 corresponding to the weight λ. For $\lambda \in X_p^+(n)$, we now study the higher cohomology modules

$$H^i(\lambda) := H^i(G/P, \mathscr{L}(\mathfrak{u}(\lambda)))$$

where P is the largest parabolic subgroup that the B-supermodule $\mathfrak{u}(\lambda)$ can be inflated to. Let

$$E_\lambda := \sum_{i \geq 0} (-1)^i \mathrm{ch}\, H^i(\lambda)$$

denote the corresponding Euler characteristic. In Theorem 4.3, we show that E_λ is equal to the classical symmetric function known as *Schur's P-function*, scaled by $\dim \mathfrak{u}(\lambda)$. The method used to prove this goes back at least to Penkov [**17**, §2.3], but we have attempted here to fill in some of the details, as promised in [**1**, **2**].

The E_λ's are important because they form a natural basis for the character group of G. We would of course like to be able compute the *decomposition numbers* $d_{\lambda,\mu}$ defined from the equation

$$E_\lambda = \sum_{\mu \in X_p^+(n)} d_{\lambda,\mu} L_\mu,$$

where $L_\mu := \mathrm{ch}\, L(\mu)$. Note that $d_{\lambda,\mu} \in \mathbb{Z}$, $d_{\lambda,\lambda} = 1$ and $d_{\lambda,\mu} = 0$ unless $\mu \leq \lambda$ in the usual dominance ordering. It appears to be the case in all the examples we have computed that the $d_{\lambda,\mu}$ are always non-negative. This is not obvious, since one can definitely have that $H^i(\lambda) \neq 0$ for $i > 0$, unlike the analogous situation when G is a reductive algebraic group; see Example 5.3. There is one further piece of information about the decomposition numbers $d_{\lambda,\mu}$ proved in Theorem 6.3: they are zero unless λ and μ have the same *residue content*, a purely combinatorial notion which appeared originally in the work of Leclerc and Thibon [**10**] in their study of the canonical bases of the Fock space of type $A_{p-1}^{(2)}$, see also [**3**, §8]. We refer to this result as the linkage principle for Euler characteristics.

At the end of the article, we have included by way of further examples tables of the decomposition matrices $D = (d_{\lambda,\mu})$ for the polynomial representations of $Q(n)$ of degree $d \leq 12$ in characteristic $p = 3$ (with n large). In that case, the dominant weights we are considering satisfy $\lambda_i \geq 0$ for all i, so we can represent them simply as partitions. The calculations here were made in part using a computer.

2. Geometric interpretation of induction

Throughout the article, we will work over a fixed algebraically closed field k of characteristic $p \neq 2$. For our general conventions regarding superalgebras, supergroups and superschemes, we refer to [**3**, §2] and [**12**, ch.3, §1–2, ch.4, §1]. We will simultaneously use geometric and functorial languages when talking about superschemes, in the spirit of [**7**, **9**].

For a superscheme X, an $\mathscr{O}_X$-*supermodule* means a sheaf $\mathscr{M}$ of abelian groups on the topological space underlying X such that $\mathscr{M}(U)$ has the additional structure of an $\mathscr{O}_X(U)$-supermodule for each open subset $U \subseteq$

X. Moreover, for open subsets $U \subseteq V$, the restriction $\mathscr{M}(V) \to \mathscr{M}(U)$ is required to be an *even* supermodule homomorphism. Given superschemes X, Y and an $\mathscr{O}_X$- resp. $\mathscr{O}_Y$-supermodule $\mathscr{M}$ resp. $\mathscr{N}$, $\mathscr{M} \otimes \mathscr{N}$ will denote their "outer" tensor product, i.e. the $\mathscr{O}_{X \times Y}$-supermodule $\mathrm{pr}_X^* \mathscr{M} \otimes_{\mathscr{O}_{X \times Y}} \mathrm{pr}_Y^* \mathscr{N}$. If X, Y are noetherian and $\mathscr{M}, \mathscr{N}$ are quasi-coherent, then $\mathscr{M} \otimes \mathscr{N}$ is again quasi-coherent and moreover $(\mathscr{M} \otimes \mathscr{N})(U \times V) = \mathscr{M}(U) \otimes_k \mathscr{N}(V)$ for affine open subsets $U \subseteq X, V \subseteq Y$.

Let $\mathfrak{mod}_{\mathscr{O}_X}$ (resp. $\mathfrak{qcoh}_{\mathscr{O}_X}$) denote the category of $\mathscr{O}_X$-supermodules (resp. quasi-coherent $\mathscr{O}_X$-supermodules). Note we allow arbitrary (not necessarily homogeneous) morphisms: a morphism $f : \mathscr{M} \to \mathscr{N}$ of $\mathscr{O}_X$-supermodules satisfies

$$f(am) = (-1)^{\bar{f}\bar{a}} a f(m)$$

for each $m \in \mathscr{M}(U)$, each $a \in \mathscr{O}_X(U)$ and each open subset U of X. Here, $\bar{f}, \bar{a} \in \mathbb{Z}_2$ denote parity assuming f and a are homogeneous, and the formula should be interpreted by extending additively from the homogeneous case if they are not. The categories $\mathfrak{mod}_{\mathscr{O}_X}$ and $\mathfrak{qcoh}_{\mathscr{O}_X}$ are not abelian categories, but the *underlying even categories* consisting of the same objects and only even morphisms are. This allows us to make use of all the usual machinery of homological algebra. We also have the *parity change functor*

$$(1) \qquad\qquad \Pi : \mathfrak{mod}_{\mathscr{O}_X} \to \mathfrak{mod}_{\mathscr{O}_X}$$

defined on objects by letting $(\Pi \mathscr{M})(U)$ equal $\mathscr{M}(U)$ (for $U \subseteq X$ open) as an abelian group, but with the opposite $\mathbb{Z}_2$-grading and new $\mathscr{O}_X(U)$-action defined by $a \cdot m := (-1)^{\bar{a}} a m$.

We have the usual sheaf cohomology functors $H^i(X, ?)$ from the category of sheaves of abelian groups on the underlying topological space to the category of abelian groups. If $\mathscr{M}$ is an $\mathscr{O}_X$-supermodule, each $H^i(X, \mathscr{M})$ has a canonical structure as an $\mathscr{O}_X(X)$-supermodule (hence in particular as a vector superspace). This follows because the category $\mathfrak{mod}_{\mathscr{O}_X}$ has enough injectives and, as in [**8**, III.2.4], injective $\mathscr{O}_X$-supermodules are acyclic for $H^0(X, ?)$; so we can compute $H^i(X, \mathscr{M})$ using an resolution of $\mathscr{M}$ by injective $\mathscr{O}_X$-supermodules. We recall [**8**, III.2.7]:

Grothendieck's vanishing theorem. *For a noetherian superscheme X, we have that $H^i(X, ?) = 0$ for all $i > \dim X$, where $\dim X$ denotes dimension as a topological space.*

To generalize other basic results, one can exploit the *canonical filtration* of an $\mathscr{O}_X$-supermodule $\mathscr{M}$, namely, the filtration

$$(2) \qquad\qquad \mathscr{M} \supseteq \mathscr{J}_X \mathscr{M} \supseteq \mathscr{J}_X^2 \mathscr{M} \supseteq \dots.$$

Here, $\mathscr{J}_X$ denotes the quasi-coherent sheaf of superideals on X defined as the sheaf associated to the presheaf $U \mapsto \mathscr{O}_X(U)\mathscr{O}_X(U)_{\bar{1}}$. The *underlying purely even scheme* X_{ev} is the scheme over k equal to X as a topological

space, with structure sheaf $\mathscr{O}_X/\mathscr{J}_X$. The factors $\mathscr{J}_X^i\mathscr{M}/\mathscr{J}_X^{i+1}\mathscr{M}$ in the canonical filtration are $\mathscr{O}_{X_{\mathrm{ev}}}$-modules, so since

$$H^i(X, \mathscr{J}_X^i\mathscr{M}/\mathscr{J}_X^{i+1}\mathscr{M}) = H^i(X_{\mathrm{ev}}, \mathscr{J}_X^i\mathscr{M}/\mathscr{J}_X^{i+1}\mathscr{M})$$

as the underlying topological spaces of X and X_{ev} are equal, we can obtain information about $H^i(X, \mathscr{M})$ using the purely even theory, the long exact sequence of cohomology and induction on the length of the canonical filtration (which is always finite for noetherian X). In particular, using [**8**, III.3.5] and [**8**, III.5.2], one obtains:

Serre's vanishing theorem. *For a noetherian superscheme X with X_{ev} affine, we have that $H^i(X, \mathscr{M}) = 0$ for all quasi-coherent $\mathscr{O}_X$-supermodules $\mathscr{M}$ and all $i > 0$.*

Serre's finiteness theorem. *For a noetherian superscheme X with X_{ev} projective, $H^i(X, \mathscr{M})$ is finite dimensional for all coherent $\mathscr{O}_X$-supermodules $\mathscr{M}$ and all $i \geq 0$.*

We say that X is *decomposable* if $\mathscr{O}_X$ is isomorphic to the symmetric superalgebra of the $\mathscr{O}_{X_{\mathrm{ev}}}$-supermodule $\mathscr{J}_X/\mathscr{J}_X^2$. If instead X has an open cover $(U_i)_{i\in I}$ such that each $(U_i, \mathscr{O}_X|_{U_i})$ is decomposable, then X is called *locally decomposable.*

Lemma 2.1. *If X is locally decomposable and $\mathscr{M}$ is a locally free $\mathscr{O}_X$-supermodule, there is a natural isomorphism*

$$\mathscr{J}_X^i\mathscr{M}/\mathscr{J}_X^{i+1}\mathscr{M} \simeq S^i(\mathscr{J}_X/\mathscr{J}_X^2) \otimes_{\mathscr{O}_{X_{\mathrm{ev}}}} \mathscr{M}/\mathscr{J}_X\mathscr{M}.$$

Proof. Without any assumptions on X or $\mathscr{M}$, there is a natural map

$$S^i(\mathscr{J}_X/\mathscr{J}_X^2) \otimes_{\mathscr{O}_{X_{\mathrm{ev}}}} \mathscr{M}/\mathscr{J}_X\mathscr{M} \longrightarrow \mathscr{J}_X^i\mathscr{M}/\mathscr{J}_X^{i+1}\mathscr{M}$$

induced by multiplication. To check that it is an isomorphism for $\mathscr{M}$ locally free and X locally decomposable, reduce to the case that X is affine and decomposable and $\mathscr{M}$ is free, and then argue directly. □

Now let G be an algebraic supergroup in the sense of [**3**, §2]. Thus G is a functor from the category of commutative superalgebras to the category of groups which is an affine superscheme when viewed just as a functor to sets, such that in addition the coordinate ring $k[G]$ is finitely generated as a k-superalgebra. Suppose we are given a right action $\rho : X \times G \to X$ of G on a noetherian superscheme X. There is a standard notion of a G-*equivariant $\mathscr{O}_X$-supermodule* ($\mathscr{O}_X G$-*supermodule* for short), for example as in [**4**, 5.1.6] or [**15**, 1.6]. We present here a somewhat different formulation of the definition in the quasi-coherent case, which we found easier to work with.

Let $\mu : G \times G \to G$ be multiplication, $e : \operatorname{Spec} k \to G$ be the identity. The comorphisms are denoted $\mu^\# : \mathscr{O}_G \to \mu_*(\mathscr{O}_G \otimes \mathscr{O}_G)$ and $e^\# : \mathscr{O}_G \to e_*k$. A *quasi-coherent $\mathscr{O}_X G$-supermodule* means a quasi-coherent $\mathscr{O}_X$-supermodule

$\mathscr{M}$ equipped with an even $\mathscr{O}_X$-supermodule map $\eta : \mathscr{M} \to \rho_*(\mathscr{M} \otimes \mathscr{O}_G)$ such that the following diagrams of sheaves on X commute:

$$(3) \quad \begin{array}{ccccc} \mathscr{M} & \xrightarrow{\eta} & \rho_*(\mathscr{M} \otimes \mathscr{O}_G) & \xrightarrow{\rho_*(\mathrm{id}_{\mathscr{M}} \otimes \mu^{\#})} & \rho_*(\mathrm{id}_X \times \mu)_*(\mathscr{M} \otimes \mathscr{O}_G \otimes \mathscr{O}_G) \\ \| & & & & \| \\ \mathscr{M} & \xrightarrow{\eta} & \rho_*(\mathscr{M} \otimes \mathscr{O}_G) & \xrightarrow{\rho_*(\eta \otimes \mathrm{id}_{\mathscr{O}_G})} & \rho_*(\rho \times \mathrm{id}_G)_*(\mathscr{M} \otimes \mathscr{O}_G \otimes \mathscr{O}_G) \end{array}$$

$$(4) \quad \begin{array}{ccc} \mathscr{M} & \xrightarrow{\eta} & \rho_*(\mathscr{M} \otimes \mathscr{O}_G) \\ \uparrow \sim & & \downarrow \rho_*(\mathrm{id}_{\mathscr{M}} \otimes e^{\#}) \\ \mathscr{M} \otimes k & =\!=\!= & \rho_*(\mathrm{id}_X \times e)_*(\mathscr{M} \otimes k) \end{array}$$

(Observe if X is a point these are just the usual comodule axioms.)

There is a functor Spec_X which defines a contravariant equivalence from the category of quasi-coherent $\mathscr{O}_X$-superalgebras to the category of superschemes over X which are affine over X; its definition is the same as in the purely even case, see e.g. [**7**, I, §2, no.3]. For a quasi-coherent $\mathscr{O}_X$-supermodule $\mathscr{M}$, we can consider the associated fibration

$$\mathbb{V}(\mathscr{M}) := \mathrm{Spec}_X S(\mathscr{M}) \xrightarrow{\sigma} X,$$

where $S(\mathscr{M})$ denotes the symmetric superalgebra. If $\mathscr{M}$ is a quasi-coherent $\mathscr{O}_X G$-supermodule, the structure map η extends in a unique way to an $\mathscr{O}_X$-superalgebra map

$$\tilde{\eta} : S(\mathscr{M}) \to \rho_*(S(\mathscr{M}) \otimes \mathscr{O}_G).$$

This in turn defines a morphism $\mathrm{Spec}_X(\rho_*(S(\mathscr{M}) \otimes \mathscr{O}_G)) \to \mathrm{Spec}_X(S(\mathscr{M}))$ of superschemes over X. Composing with the canonical map

$$\mathbb{V}(\mathscr{M}) \times G \cong \mathrm{Spec}_{X \times G}(S(\mathscr{M}) \otimes \mathscr{O}_G) \to \mathrm{Spec}_X(\rho_*(S(\mathscr{M}) \otimes \mathscr{O}_G)),$$

we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{V}(\mathscr{M}) \times G & \xrightarrow{\Sigma} & \mathbb{V}(\mathscr{M}) \\ \sigma \times \mathrm{id}_G \downarrow & & \downarrow \sigma \\ X \times G & \xrightarrow{\rho} & X \end{array}$$

Now (3) implies that Σ satisfies associativity, i.e. there is a commutative cube analogous to that on [**15**, p.31], while (4) gives that Σ is unital in the obvious sense. So, Σ is a "linear" right action of G on the fibration $\mathbb{V}(\mathscr{M})$. Conversely, we can recover the original structure map η of $\mathscr{M}$ from such an action Σ since $\tilde{\eta} = \sigma_* \Sigma^{\#}$.

There is a natural notion of morphism between two quasi-coherent $\mathscr{O}_X G$-supermodules, giving us the category $\mathfrak{qcoh}_{\mathscr{O}_X G}$. Its underlying even category is abelian, and moreover:

Lemma 2.2. (i) *The category* $\mathfrak{qcoh}_{\mathscr{O}_X G}$ *has enough injectives.*

(ii) *Injective objects in* $\mathfrak{qcoh}_{\mathscr{O}_X G}$ *are acyclic for* $H^0(X, ?)$.

Proof. (i) One first checks that the forgetful functor res : $\mathfrak{qcoh}_{\mathscr{O}_X G} \to \mathfrak{qcoh}_{\mathscr{O}_X}$ has a right adjoint, namely, the functor ind := $\rho_*(? \otimes \mathscr{O}_G)$. Here for any quasi-coherent $\mathscr{O}_X$-supermodule $\mathscr{M}$, ind $\mathscr{M} = \rho_*(\mathscr{M} \otimes \mathscr{O}_G)$ is viewed as an $\mathscr{O}_X G$-supermodule with structure map $\rho_*(\mathrm{id}_{\mathscr{M}} \otimes \mu^{\#}) : \mathrm{ind}\mathscr{M} \to \rho_*(\mathrm{ind}\mathscr{M} \otimes \mathscr{O}_G)$.

Now take any $\mathscr{O}_X G$-supermodule $\mathscr{M}$ with structure map $\eta : \mathscr{M} \to \rho_*(\mathscr{M} \otimes \mathscr{O}_G)$. Embed res $\mathscr{M}$ into an injective quasi-coherent $\mathscr{O}_X$-supermodule $\mathscr{N}$ (which we can do since $\mathfrak{qcoh}_{\mathscr{O}_X}$ has enough injectives, cf. [**8**, III.3, ex.6]). Applying the left exact functor ind, we get a monomorphism ind∘res$(\mathscr{M}) \to$ ind$(\mathscr{N})$, with ind$(\mathscr{N})$ being injective. It remains to see that $\mathscr{M}$ embeds into ind ∘ res$(\mathscr{M})$; but this is immediate since $\eta : \mathscr{M} \to$ ind ∘ res$(\mathscr{M})$ is a monomorphism thanks to axiom (4).

(ii) By the proof of (i), any injective object in $\mathfrak{qcoh}_{\mathscr{O}_X G}$ embeds into (hence is a summand of) $\rho_*(\mathscr{M} \otimes \mathscr{O}_G)$ for an injective object $\mathscr{M} \in \mathfrak{qcoh}_{\mathscr{O}_X}$. So it suffices to check that $H^i(X, \rho_*(\mathscr{M} \otimes \mathscr{O}_G)) = 0$ for all $i > 0$. Note ρ is the composite

$$X \times G \xrightarrow{f} X \times G \xrightarrow{\mathrm{pr}_X} X,$$

where f is the isomorphism defined by $(x, g) \mapsto (xg, g)$. Hence, ρ is an affine morphism, so $(R^i \rho_*)(\mathscr{M} \otimes \mathscr{O}_G) = 0$ for all $i > 0$. Hence the Leray spectral sequence degenerates to give $H^i(X, \rho_*(\mathscr{M} \otimes \mathscr{O}_G)) \simeq H^i(X \times G, \mathscr{M} \otimes \mathscr{O}_G)$. Now by Serre's vanishing theorem and the Kunneth formula, this is isomorphic to $H^i(X, \mathscr{M}) \otimes k[G]$, which is zero for $i > 0$ by the injectivity of $\mathscr{M}$. □

Let $\mathscr{M}$ be a quasi-coherent $\mathscr{O}_X G$-supermodule with structure map η. The space $H^0(X, \mathscr{M})$ of global sections has the structure of a G-supermodule, with structure map obtained as the composite

$$H^0(X, \mathscr{M}) \xrightarrow{\eta} H^0(X, \rho_*(\mathscr{M} \otimes \mathscr{O}_G)) \simeq H^0(X, \mathscr{M}) \otimes k[G].$$

It follows from this and Lemma 2.2 that each $H^i(X, \mathscr{M})$ carries a canonical G-supermodule structure. Indeed, we can compute $H^i(X, \mathscr{M})$ using a resolution of $\mathscr{M}$ by injective $\mathscr{O}_X G$-supermodules, these being acyclic for $H^0(X, ?)$. In other words, we can regard each $H^i(X, ?)$ as a functor from $\mathfrak{qcoh}_{\mathscr{O}_X G}$ to $\mathfrak{mod}_G$.

Suppose next that H is a closed subgroup of our fixed algebraic supergroup G. By a *quotient of G by H* ("$H \backslash G$") we mean here a noetherian superscheme X together with a morphism $\pi : G \to X$ such that:

(Q1) π is constant on the right $H(A)$-cosets $H(A)g$ in $G(A)$ for each commutative superalgebra A and each $g \in G(A)$;

(Q2) given any other morphism $f : G \to Y$ of superschemes that is constant on the right $H(A)$-cosets in $G(A)$ for each commutative superalgebra A, there is a unique $\tilde{f} : X \to Y$ such that $f = \tilde{f} \circ \pi$;

(Q3) $\pi : G \to X$ is a faithfully flat, affine morphism.

Assume that X is a quotient of G by H. Let $\rho : X \times G \to X$ be the right action of G on X induced by multiplication in G. We have commutative diagrams

$$
\begin{array}{ccc}
G \times G & \xrightarrow{\ \mu\ } & G \\
{\scriptstyle \pi \times \mathrm{id}_G} \downarrow & & \downarrow {\scriptstyle \pi} \\
X \times G & \xrightarrow{\ \rho\ } & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
H \times G & \xrightarrow{\ \bar{\mu}\ } & G \\
{\scriptstyle \mathrm{id}_H \times \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\
H \times X & \xrightarrow{\ \mathrm{pr}_X\ } & X
\end{array}
$$

where μ is the multiplication in G, and $\bar{\mu}$ is its restriction to $H \times G$.

The first diagram gives that $\pi_* \mathscr{O}_G$ is a quasi-coherent $\mathscr{O}_X G$-supermodule with structure map $\pi_* \mu^{\#} : \pi_* \mathscr{O}_G \to \rho_*(\pi_* \mathscr{O}_G \otimes \mathscr{O}_G)$. Hence, for any vector superspace M, $M \otimes \pi_* \mathscr{O}_G$ is also a quasi-coherent $\mathscr{O}_X G$-supermodule with structure map $\mathrm{id}_M \otimes \pi_* \mu^{\#}$. We will usually denote this by $M_{tr} \otimes \pi_* \mathscr{O}_G$ to indicate that the action on M is trivial. In particular, $k[H]_{tr} \otimes \pi_* \mathscr{O}_G$ is a quasi-coherent $\mathscr{O}_X G$-supermodule in this way. From the second diagram, we get a natural $\mathscr{O}_X G$-supermodule map $\delta := \pi_* \bar{\mu}^{\#} : \pi_* \mathscr{O}_G \to k[H]_{tr} \otimes \pi_* \mathscr{O}_G$. If M is an H-supermodule with structure map $\eta : M \to M \otimes k[H]$ (see [**3**, §2]), we define the "induced" quasi-coherent $\mathscr{O}_X G$-supermodule $\mathscr{L}(M)$ to be the kernel of the map $\partial = \eta \otimes \mathrm{id}_{\pi_* \mathscr{O}_G} - \mathrm{id}_M \otimes \delta$ in the following exact sequence of quasi-coherent $\mathscr{O}_X G$-supermodules:

$$
(5) \qquad 0 \longrightarrow \mathscr{L}(M) \longrightarrow M_{tr} \otimes \pi_* \mathscr{O}_G \xrightarrow{\ \partial\ } M_{tr} \otimes k[H]_{tr} \otimes \pi_* \mathscr{O}_G.
$$

If $f : M \to M'$ is a morphism of H-supermodules, $\mathscr{L}(f) : \mathscr{L}(M) \to \mathscr{L}(M')$ can be defined as the restriction of $f \otimes \mathrm{id}_{\pi_* \mathscr{O}_G}$, giving that

$$
(6) \qquad\qquad\qquad \mathscr{L} : \mathfrak{mod}_H \to \mathfrak{qcoh}_{\mathscr{O}_X G}
$$

is a functor, where $\mathfrak{mod}_H$ denotes the category of all H-supermodules.

Conversely, recall that $e : \mathrm{Spec}\, k \to G$ denotes the identity of G; we let $\bar{e} : \mathrm{Spec}\, k \to X$ denote $\pi \circ e$. There is a natural H-supermodule structure on $\mathscr{M}(\bar{e}) := \mathscr{M}_{\bar{e}} \otimes_{\mathscr{O}_{X,\bar{e}}} k$. In other words, "evaluation at $\bar{e}$" gives a functor

$$
(7) \qquad\qquad\qquad \mathscr{E} : \mathfrak{qcoh}_{\mathscr{O}_X G} \to \mathfrak{mod}_H.
$$

In terms of the associated action of G on the fibration $\mathbb{V}(\mathscr{M})$, the restriction of the action to H induces a right action of H on the fiber $\mathbb{V}(\mathscr{M})_{\bar{e}}$. There is a natural isomorphism $\mathbb{V}(\mathscr{M})_{\bar{e}} = \mathrm{Spec}_X(S(\mathscr{M})) \times_X \mathrm{Spec}\, k \cong \mathrm{Spec}\, S(\mathscr{M}(\bar{e}))$ (cf. [**7**, I, §2, 3.6] and [**8**, II, ex.5.16(e)]), so that H acts on the right on $\mathrm{Spec}\, S(\mathscr{M}(\bar{e}))$. Now the structure map on $\mathscr{M}(\bar{e})$ is precisely the restriction to $\mathscr{M}(\bar{e})$ of the comorphism of this right action of H on $\mathrm{Spec}\, S(\mathscr{M}(\bar{e}))$.

We are ready state the following fundamental result, cf. [**6**, p.249] or [**5**, Theorem (2.7)]. We omit the proof, but note that it depends in an essential way on the property (Q3) of quotients as formulated above.

Theorem 2.3. *The functors $\mathscr{L}$ and $\mathscr{E}$ are mutually inverse equivalences. Moreover, for an H-supermodule M, $\mathscr{L}(M)$ is locally free of rank equal to the superdimension of the vector superspace M.*

There are induction and restriction functors

$$\mathrm{ind}_H^G : \mathfrak{mod}_H \to \mathfrak{mod}_G, \qquad \mathrm{res}_H^G : \mathfrak{mod}_G \to \mathfrak{mod}_H$$

defined in [**3**, §6]. We recall that restriction is exact and ind_H^G is right adjoint to res_H^G. Comparing the formula [**3**, (6.1)] with (5), it is immediate that there is an isomorphism $\mathrm{ind}_H^G \simeq H^0(X, ?) \circ \mathscr{L}$ of functors from $\mathfrak{mod}_H$ to $\mathfrak{mod}_G$. By the theorem, the functor $\mathscr{L}$ is exact and maps injectives to injectives. So, letting $R^i\mathrm{ind}_H^G$ denote the ith right derived functor of induction, we have:

Corollary 2.4. *For each $i \geq 0$, we have that*

$$R^i\mathrm{ind}_H^G \simeq H^i(X, ?) \circ \mathscr{L}.$$

The goal in the remainder of the section is to sketch the proof of a result from [**17**, §2.3] about Euler characteristics. Let $\pi : G \to X$ be a quotient of G by H as before. We now make the following additional assumptions:

(Q4) the restriction $\pi_{\mathrm{ev}} : G_{\mathrm{ev}} \to X_{\mathrm{ev}}$ of π to the underlying purely even schemes is a quotient of G_{ev} by H_{ev};

(Q5) X is locally decomposable.

Write $i : X_{\mathrm{ev}} \to X$ for the canonical closed immersion. The map i is G_{ev}-equivariant, so taking direct and inverse images as $\mathscr{O}_X$-supermodules give functors

$$i_* : \mathfrak{mod}_{\mathscr{O}_{X_{\mathrm{ev}}}G_{\mathrm{ev}}} \to \mathfrak{mod}_{\mathscr{O}_X G_{\mathrm{ev}}}, \qquad i^* : \mathfrak{mod}_{\mathscr{O}_X G_{\mathrm{ev}}} \to \mathfrak{mod}_{\mathscr{O}_{X_{\mathrm{ev}}}G_{\mathrm{ev}}}.$$

There is a natural restriction functor $\mathrm{res}_{G_{\mathrm{ev}}}^G : \mathfrak{qcoh}_{\mathscr{O}_X G} \to \mathfrak{qcoh}_{\mathscr{O}_X G_{\mathrm{ev}}}$, and obviously

$$(8) \qquad\qquad H^i(X, \mathrm{res}_{G_{\mathrm{ev}}}^G \mathscr{M}) \simeq \mathrm{res}_{G_{\mathrm{ev}}}^G H^i(X, \mathscr{M})$$

for any quasi-coherent $\mathscr{O}_X G$-supermodule $\mathscr{M}$. Finally, by a special case of the above theory, applied to the quotient $\pi_{\mathrm{ev}} : G_{\mathrm{ev}} \to X_{\mathrm{ev}}$ of G_{ev} by H_{ev}, we have inverse equivalences

$$\mathscr{L}_{\mathrm{ev}} : \mathfrak{mod}_{H_{\mathrm{ev}}} \to \mathfrak{qcoh}_{\mathscr{O}_{X_{\mathrm{ev}}}G_{\mathrm{ev}}}, \qquad\qquad \mathscr{E}_{\mathrm{ev}} : \mathfrak{qcoh}_{\mathscr{O}_{X_{\mathrm{ev}}}G_{\mathrm{ev}}} \to \mathfrak{mod}_{H_{\mathrm{ev}}}.$$

Lemma 2.5. *For any H-supermodule M, we have that*

$$i^*(\mathrm{res}_{G_{\mathrm{ev}}}^G \mathscr{L}(M)) \simeq \mathscr{L}_{\mathrm{ev}}(\mathrm{res}_{H_{\mathrm{ev}}}^H M).$$

Proof. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X G$-supermodule. Then,

$$\mathbb{V}(i^*\mathcal{M})_{\bar{e}} \simeq (\mathbb{V}(\mathcal{M}) \times_X X_{\mathrm{ev}})_{\bar{e}} = \mathbb{V}(\mathcal{M}) \times_X X_{\mathrm{ev}} \times_{X_{\mathrm{ev}}} \mathrm{Spec}\, k$$
$$\simeq \mathbb{V}(\mathcal{M}) \times_X \mathrm{Spec}\, k = \mathbb{V}(\mathcal{M})_{\bar{e}}.$$

Hence, $\mathscr{E}_{\mathrm{ev}}(i^*\mathcal{M}) \simeq \mathscr{E}(\mathcal{M})$ as H_{ev}-supermodules. The lemma follows on applying the functor $\mathscr{L}_{\mathrm{ev}}$ to both sides. $\qquad\square$

Suppose that $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X G$-supermodule. The canonical filtration (2) of $\mathcal{M}$ is a filtration as an $\mathcal{O}_X G_{\mathrm{ev}}$-supermodule (for instance, $\mathscr{J}_X \mathcal{M}$ is the kernel of the canonical map $\mathcal{M} \to i_* i^*\mathcal{M}$). The factors are $\mathcal{O}_{X_{\mathrm{ev}}} G_{\mathrm{ev}}$-supermodules.

Lemma 2.6. *There is an isomorphism*

$$\mathscr{J}_X / \mathscr{J}_X^2 \simeq \mathscr{L}_{\mathrm{ev}}((\mathrm{Lie} G / \mathrm{Lie} H)_{\bar{1}}^*)$$

as $\mathcal{O}_{X_{\mathrm{ev}}} G_{\mathrm{ev}}$-supermodules.

Proof. The sheaf Ω_X of (super) Kähler differentials on X (over k) has a natural G-structure, and the closed immersion $i : X_{\mathrm{ev}} \to X$ induces a standard exact sequence

$$\mathscr{J}_X / \mathscr{J}_X^2 \xrightarrow{\delta} \Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{\mathrm{ev}}} \longrightarrow \Omega_{X_{\mathrm{ev}}} \longrightarrow 0$$

of $\mathcal{O}_{X_{\mathrm{ev}}} G_{\mathrm{ev}}$-supermodules (cf. [**8**, II.8.12] with appropriate modifications in the super case). Using the assumption that X is locally decomposable, one checks by reducing to the case that X is affine and decomposable that the map δ is a monomorphism. Now we note that $\mathscr{E}(\Omega_X) \simeq (\mathrm{Lie}(G)/\mathrm{Lie}(H))^*$ and $\mathscr{E}_{\mathrm{ev}}(\Omega_{X_{\mathrm{ev}}}) \simeq (\mathrm{Lie}(G_{\mathrm{ev}})/\mathrm{Lie}(H_{\mathrm{ev}}))^*$. Applying the equivalence of categories $\mathscr{E}_{\mathrm{ev}}$ and Lemma 2.5, we obtain an exact sequence

$$0 \longrightarrow \mathscr{E}_{\mathrm{ev}}(\mathscr{J}_X / \mathscr{J}_X^2) \longrightarrow (\mathrm{Lie}(G)/\mathrm{Lie}(H))^* \longrightarrow (\mathrm{Lie}(G_{\mathrm{ev}})/\mathrm{Lie}(H_{\mathrm{ev}}))^* \longrightarrow 0$$

of H_{ev}-supermodules. Since $\mathrm{Lie}(G_{\mathrm{ev}}) = \mathrm{Lie}(G)_{\bar{0}}$ and similarly for H, we deduce that

$$\mathscr{E}_{\mathrm{ev}}(\mathscr{J}_X / \mathscr{J}_X^2) \simeq (\mathrm{Lie}(G)/\mathrm{Lie}(H))_{\bar{1}}^*$$

and the lemma follows. $\qquad\square$

Theorem 2.7. *Let M be an H-supermodule. Then, the factors in the canonical filtration of $\mathscr{L}(M)$ satisfy*

$$\mathscr{J}_X^i \mathscr{L}(M) / \mathscr{J}_X^{i+1} \mathscr{L}(M) \simeq \mathscr{L}_{\mathrm{ev}}(S^i[(\mathrm{Lie} G / \mathrm{Lie} H)_{\bar{1}}^*] \otimes \mathrm{res}_{H_{ev}}^H M)$$

as $\mathcal{O}_{X_{\mathrm{ev}}} G_{\mathrm{ev}}$-supermodules.

Proof. Note $\mathscr{L}(M)$ is locally free by Theorem 2.3. So we obtain from Lemma 2.1 an isomorphism

$$\mathscr{J}_X^i \mathscr{L}(M) / \mathscr{J}_X^{i+1} \mathscr{L}(M) \simeq S^i(\mathscr{J}_X / \mathscr{J}_X^2) \otimes_{\mathcal{O}_{X_{\mathrm{ev}}}} \mathcal{M} / \mathscr{J}_X \mathcal{M}.$$

This is actually an isomorphism as $\mathscr{O}_{X_{\mathrm{ev}}}G_{\mathrm{ev}}$-supermodules. So the theorem follows using Lemmas 2.5 and 2.6. (We have noted that the usual operations of tensor algebra commute with the functor $\mathscr{L}_{\mathrm{ev}}$).   □

For the final corollary, we make one further assumption:

(Q6) X_{ev} is projective.

Corollary 2.8. *For any finite dimensional H-supermodule M, we have that*

$$\sum_{i \geq 0}(-1)^i \left[\mathrm{res}^G_{G_{\mathrm{ev}}} R^i \mathrm{ind}^G_H M\right] =$$

$$\sum_{i \geq 0}(-1)^i \left[R^i \mathrm{ind}^{G_{\mathrm{ev}}}_{H_{\mathrm{ev}}} \left\{S((\mathrm{Lie}(G)/\mathrm{Lie}(H))^*_{\bar{1}}) \otimes M\right\}\right]$$

where the equality is written in the Grothendieck group of finite dimensional G_{ev}-supermodules.

Proof. By Theorem 2.3, Corollary 2.4 and Serre's finiteness theorem, the functor $R^i\mathrm{ind}^G_H$ sends finite dimensional H-supermodules to finite dimensional G-supermodules, and similarly for $R^i\mathrm{ind}^{G_{\mathrm{ev}}}_{H_{\mathrm{ev}}}$. So all modules appearing in the formula are finite dimensional. Moreover, only finitely many terms in either summation are non-zero by Grothendieck vanishing. So the formula at least makes sense. Now combine (8), Corollary 2.4, Theorem 2.7 and additivity of Euler characteristics.   □

3. Flag superschemes of type $Q(n)$

Now we are ready to introduce the supergroup $G = Q(n)$ into the picture. Recall from [**3**, §3] that G is the functor from the category of commutative superalgebras to the category of groups defined on a superalgebra A so that $G(A)$ is the group of all invertible $2n \times 2n$ matrices of the form

$$(9) \qquad\qquad g = \left(\begin{array}{c|c} S & S' \\ \hline -S' & S \end{array}\right)$$

where S is an $n \times n$ matrix with entries in $A_{\bar{0}}$ and S' is an $n \times n$ matrix with entries in $A_{\bar{1}}$. The underlying even group G_{ev} is isomorphic to $GL(n)$.

We also need the closed subgroup H of G defined on a commutative superalgebra A so that $H(A)$ consists of all matrices of the form (9) with S, S' being diagonal matrices, and the standard Borel subgroup B of G defined so that $B(A)$ consists of matrices with S, S' being lower triangular. Let $T = H_{\mathrm{ev}}$ be the standard n-dimensional maximal torus of G_{ev}. Let $X(T)$ be the character group of T, the free abelian group on generators $\varepsilon_1, \ldots, \varepsilon_n$ where $\varepsilon_i : T \to \mathbb{G}_m$ picks out the ith diagonal entry. The root system associated to G_{ev} is denoted $R = R^+ \cup (-R^+)$, where $R^+ = \{\varepsilon_i - \varepsilon_j \,|\, 1 \leq i < j \leq n\} \subset X(T)$. We partially order $X(T)$ by the usual dominance order, so $\lambda \leq \mu$ if and only if $\mu - \lambda$ is a sum of positive roots.

For $\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i \in X(T)$, we write $x^\lambda = x_1^{\lambda_1} \ldots x_n^{\lambda_n} \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The Weyl group $W \cong S_n$ associated to G_{ev} acts naturally on $X(T)$ hence on $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The character

$$\text{ch } M := \sum_{\lambda \in X(T)} (\dim M_\lambda) x^\lambda$$

of a finite dimensional G-supermodule M is naturally W-invariant, so is an element of the ring $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^W$ of symmetric functions.

The irreducible G-supermodules are classified in [**3**, Theorem 6.11] by their highest weights. For every $\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i \in X(T)$, there is by [**3**, Lemma 6.4] a unique irreducible H-supermodule denoted $\mathfrak{u}(\lambda)$ with character $2^{\lfloor (h_{p'}(\lambda)+1)/2 \rfloor} x^\lambda$, where $h_{p'}(\lambda)$ denotes the number of $i = 1, \ldots, n$ for which $p \nmid \lambda_i$. Let $X_p^+(T)$ denote the set of all $\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i \in X(T)$ such that $\lambda_1 \geq \cdots \geq \lambda_n$ and moreover $\lambda_i = \lambda_{i+1}$ implies $p | \lambda_i$ for each $i = 1, \ldots, n-1$. Then, according to [**3**, Theorem 6.11], the induced module

$$(10) \qquad H^0(\lambda) := \text{ind}_B^G \mathfrak{u}(\lambda)$$

is non-zero if and only if $\lambda \in X_p^+(T)$, and in that case $H^0(\lambda)$ has a unique irreducible submodule denoted $L(\lambda)$. The $\{L(\lambda) \mid \lambda \in X_p^+(T)\}$ form a complete set of pairwise non-isomorphic irreducible G-supermodules. Also by [**3**, Lemma 6.10], $L(\lambda)_\lambda \simeq \mathfrak{u}(\lambda)$ and the lowest weight of $L(\lambda)$ is $w_0 \lambda$, where w_0 is the longest element of W.

Next we introduce the standard Levi subgroups of G. Just as for $GL(n)$, these can be parametrized by *compositions of n*, i.e. tuples $\gamma = (\gamma_1, \ldots, \gamma_s)$ of positive integers summing to n. Given such a γ, the *standard Levi subgroup* $G_\gamma \cong Q(\gamma_1) \times \cdots \times Q(\gamma_s)$ of G is defined on a commutative superalgebra A so that $G_\gamma(A)$ is the subgroup of $G(A)$ consisting of all elements of the form (9) with S, S' being block diagonal matrices, block sizes $\gamma_1, \ldots, \gamma_s$ down the diagonal. The *standard parabolic subgroup P_γ* with Levi factor G_γ is defined similarly, so $P_\gamma(A)$ consists of all elements of the form (9) with S, S' being lower triangular block matrices, block sizes $\gamma_1, \ldots, \gamma_s$ down the diagonal. Also let U_γ^+ denote the unipotent radical of the opposite parabolic subgroup to P_γ. Thus, $U_\gamma^+(A)$ consists of all matrices of the form (9) with S, S' being upper unitriangular block matrices, block sizes $\gamma_1, \ldots, \gamma_s$ down the diagonal. The root system associated to G_γ will be denoted $R_\gamma \subseteq R$, and its Weyl group is $W_\gamma \leq W$, with longest element denoted w_γ.

Lemma 3.1. *Let $\lambda \in X_p^+(T)$ and γ be a composition of n satisfying $w_\gamma \lambda = \lambda$. Then, $\text{ind}_B^{P_\gamma} \mathfrak{u}(\lambda) \simeq \mathfrak{u}(\lambda)$, i.e. the H-action on $\mathfrak{u}(\lambda)$ extends uniquely to a P_γ-action.*

Proof. As in [**3**, Lemma 6.10(i)], the lowest weight of $\text{ind}_B^{P_\gamma} \mathfrak{u}(\lambda)$ is $w_\gamma \lambda = \lambda$. $\qquad\square$

We need the parabolic analogue of the big cell. For $1 \leq i, j \leq n$, let $s_{i,j}$ resp. $s'_{i,j}$ be the coordinate function picking out the ij-entry of the matrix S resp. S' of $g \in G(A)$ written in the form (9). As in [**3**, §3], the coordinate ring $k[G]$ is the free commutative superalgebra on even generators $s_{i,j}$ and odd generators $s'_{i,j}$ localized at $\det := \det((s_{i,j})_{1 \leq i,j \leq n})$. Instead, for $1 \leq m \leq n$, define $\det_m$ to be the determinant of the $m \times m$ matrix $(s_{i,j})_{1 \leq i,j \leq m}$. Given a composition $\gamma = (\gamma_1, \ldots, \gamma_s)$ of n, set

$$\det{}_\gamma = \det{}_{\gamma_1} \det{}_{\gamma_1 + \gamma_2} \ldots \det{}_{\gamma_1 + \gamma_2 + \cdots + \gamma_s}.$$

We will denote the principal open subset of G defined by $\det_\gamma$ by Ω_γ; so Ω_γ is an affine superscheme with coordinate ring $k[\Omega_\gamma]$ being the localization of $k[G]$ at $\det_\gamma$. Proceeding by induction on the number s of blocks, one shows as in [**3**, Theorem 3.5] that:

Lemma 3.2. *Multiplication defines an isomorphism of affine superschemes between $P_\gamma \times U_\gamma^+$ and Ω_γ.*

Let V be the natural G-supermodule. Thus, V is the vector superspace on basis $v_1, \ldots, v_n, v'_1, \ldots, v'_n$, where v_i is even and v'_i is odd. For a superalgebra A, we identify elements of $V \otimes A$ with column vectors

$$\sum_{i=1}^n (v_i \otimes a_i + v'_i \otimes a'_i) \longleftrightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ a'_1 \\ \vdots \\ a'_n \end{pmatrix}.$$

Then, the action of $G(A)$ on $V \otimes A$ defining the supermodule structure is the obvious action on column vectors by left multiplication. Moreover, the map $J : V \to V, v_i \mapsto v'_i, v'_i \mapsto -v_i$ is an odd automorphism of V as a G-supermodule.

Instead, let $\widetilde{V}$ denote the affine superscheme defined on a superalgebra A by $\widetilde{V}(A) := \mathrm{Hom}_k(V, A)$ and on a morphism $\theta : A \to B$ of superalgebras by $\widetilde{V}(\theta) : \widetilde{V}(A) \to \widetilde{V}(B), f \mapsto \theta \circ f$. Note $\widetilde{V}(A)$ is a free left A-supermodule of rank $n|n$, with action defined by $(a \cdot f)(v) = a(f(v))$. Indeed, we can identify functions in $\widetilde{V}(A)$ with row vectors so that

$$f \longleftrightarrow (f(v_1), \ldots, f(v_n), f(v'_1), \ldots, f(v'_n)).$$

Now right multiplication defines a right action of $G(A)$ on $\tilde{V}(A)$ by A-supermodule automorphisms for each A. Hence we have a right action $\rho : \tilde{V} \times G \to \tilde{V}$ of G on the superscheme $\tilde{V}$. There is a G-equivariant morphism $\widetilde{J} : \widetilde{V} \to \widetilde{V}$, defined by $\widetilde{J}(f)(v) = (-1)^{\bar{f}} f(J(v))$ for each $v \in V, f \in \widetilde{V}(A)$ and each superalgebra A.

At last we can introduce the flag superscheme X_γ corresponding to a composition $\gamma = (\gamma_1, \ldots, \gamma_s)$ of n. Let $d_i = \gamma_1 + \cdots + \gamma_i$ for short. Given a superalgebra A, a *J-invariant γ-flag* in $\widetilde{V}(A)$ means a chain $(f_1 \subseteq \cdots \subseteq f_s)$ of J-invariant direct summands of the free A-supermodule $\widetilde{V}(A)$, where each f_i has rank $d_i|d_i$ as a projective A-supermodule. Let X_γ denote the functor mapping a superalgebra A to the set $X_\gamma(A)$ of all such J-invariant γ-flags in $\widetilde{V}(A)$. For a morphism $\theta : A \to B$, $X_\gamma(\theta) : X_\gamma(A) \to X_\gamma(B)$ is the map induced by composing with θ.

The right action of G on $\widetilde{V}$ induces a right action $\rho : X_\gamma \times G \to X_\gamma$ of G on X_γ. Let $f_\gamma = (f_1 \subseteq \cdots \subseteq f_s) \in X_\gamma(k)$ denote the *standard γ-flag*, where f_i is the direct summand of $\widetilde{V}(k) = \mathrm{Hom}_k(V, k)$ consisting of all functions annihilating $v_{d_i+1}, \ldots, v_n, v'_{d_i+1}, \ldots, v'_n$. Note that $\mathrm{stab}_G f_\gamma = P_\gamma$, i.e. $P_\gamma(A)$ is the stabilizer in $G(A)$ of the canonical image of f_γ in $X_\gamma(A)$ for each superalgebra A. We get a corresponding orbit map

$$(11) \qquad\qquad \pi_\gamma : G \to X_\gamma$$

defined for each superalgebra A and $g \in G(A)$ by $\pi_\gamma(g) := f_\gamma \cdot g$, which is constant on right $P_\gamma(A)$-cosets in $G(A)$ for each A. In fact, according to a result of Manin, $\pi_\gamma : G \to X_\gamma$ is the quotient of G by the parabolic subgroup P_γ:

Theorem 3.3. $\pi_\gamma : G \to X_\gamma$ *satisfies the properties (Q1)–(Q6) from the previous section.*

The properties (Q4) and (Q6) hold because the underlying even scheme $X_{\gamma,\mathrm{ev}}$ is precisely the usual flag variety of $GL(n)$. For the remaining properties, the main step is to show that X_γ really is a superscheme, for which we refer to [**12**, ch.4, §3] and [**13**, §3.1]. One can also give a proof directly in the functorial language that X_γ is a superscheme along the lines of the arguments in [**7**, I, §1, 3.13], by constructing an affine open cover and checking directly that X_γ is local in the sense of [**7**, I, §1, 3.11]. We just describe the construction of the affine open cover here; all the other properties claimed follow easily given this.

Let $e_\gamma = (e_1 \subseteq \cdots \subseteq e_s)$ be the J-invariant γ-flag in the vector superspace V with $e_i = \mathrm{span}\{v_1, \ldots, v_{d_i}, v'_1, \ldots, v'_{d_i}\}$. Let U_1 denote the subfunctor of X_γ defined by

$$U_1(A) = \left\{ (f_1 \subseteq \cdots \subseteq f_s) \in X_\gamma(A) \;\middle|\; \begin{array}{l} f_i \text{ is a complement to} \\ \mathrm{ann}_{\widetilde{V}(A)}(e_i) \text{ in } \widetilde{V}(A) \end{array} \right\}$$

for each superalgebra A. Then, U_1 is an open subfunctor of X_γ, $\pi_\gamma^{-1}(U_1)$ is the big cell $P_\gamma U_\gamma^+ = \Omega_\gamma \subset G$ (see Lemma 3.2), and the restriction of π_γ to U_γ^+ gives an isomorphism $U_\gamma^+ \xrightarrow{\sim} U_1$. It follows that for any $w \in W$, the translate $U_w := U_1 \cdot w^{-1}$ is isomorphic to U_γ^+, hence it is a decomposable

open affine subfunctor of X_γ. For each A that is a field, the $\{U_w(A)\,|\,w \in W\}$ cover $X_\gamma(A)$. Moreover, $U_w = U_{w'}$ if and only if $wW_\gamma = w'W_\gamma$. Hence

(12) $$\{U_w \mid w \in W^\gamma\}$$

is the desired affine open cover of X_γ, where W^γ denotes the set of minimal length W/W_γ-coset representatives.

4. Euler characteristics and Schur's P-functions

Now we combine the results of the previous two sections. Continue with the notation of §3. Let $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i \in X_p^+(T)$. Gathering together equal parts, we can write $(\lambda_1, \ldots, \lambda_n) = (a_1^{\gamma_1}, a_2^{\gamma_2}, \ldots, a_s^{\gamma_s})$ with $a_1 > a_2 > \cdots > a_s$ and $\sum_{i=1}^s \gamma_i = n$. We thus obtain from λ a composition $\gamma(\lambda) := (\gamma_1, \ldots, \gamma_s)$ of n. *Schur's P-function* p_λ is defined by:

(13) $$p_\lambda = \sum_{w \in W^{\gamma(\lambda)}} w \left\{ x^\lambda \prod_{\substack{1 \le i < j \le n \\ \lambda_i > \lambda_j}} \frac{x_i + x_j}{x_i - x_j} \right\}.$$

This is the definition from [**11**, III(2.2)] with t there equal to -1, compare [**11**, III.8] (actually, Macdonald only describes the case when all $\lambda_i \ge 0$, but everything easily extends to $\lambda_i \in \mathbb{Z}$). For any $\lambda \in X(T)$, let $a_\lambda = \sum_{w \in W} \text{sgn}(w) w x^\lambda$. Writing $\rho = \sum_{i=1}^n (n - i)\varepsilon_i \in X^+(T)$, a_ρ is the *Weyl denominator* and equals $\prod_{1 \le i < j \le n}(x_i - x_j)$ ([**11**, I.3]). The classical *Schur function* s_λ can then be defined for arbitrary $\lambda \in X(T)$ by $s_\lambda := a_{\lambda+\rho}/a_\rho$.

Lemma 4.1. *For $\lambda \in X_p^+(T)$,*

(i) $p_\lambda = \sum_{w \in W} \text{sgn}(w) w \left\{ x^{\lambda+\rho} \prod_{\substack{1 \le i < j \le n \\ \lambda_i > \lambda_j}} (1 + x_i^{-1}x_j) \right\} \Big/ \prod_{1 \le i < j \le n}(x_i - x_j);$

(ii) $p_\lambda = \sum_{S \subseteq R^+ - R_{\gamma(\lambda)}^+} s_{\lambda - \sum S}.$

Proof. (i) The given expression equals

$$\sum_{w \in W} w \left\{ x^{\lambda+\rho} \frac{\prod_{\lambda_i > \lambda_j}(1 + x_i^{-1}x_j)}{\prod_{i<j}(x_i - x_j)} \right\} =$$

$$\sum_{w \in W} w \left\{ x^\lambda \prod_{\substack{1 \le i < j \le n \\ \lambda_i > \lambda_j}} \frac{x_i + x_j}{x_i - x_j} \prod_{\substack{1 \le i < j \le n \\ \lambda_i = \lambda_j}} \frac{x_i}{x_i - x_j} \right\}.$$

Now [**11**, III(1.4)] with $t = 0$ shows that $\sum_{w \in W} w \left\{ \prod_{i<j} \frac{x_i}{x_i - x_j} \right\} = 1$. The conclusion now follows on observing that $w \in W_{\gamma(\lambda)}$ fixes $x^\lambda \prod_{\lambda_i > \lambda_j} \frac{(x_i + x_j)}{(x_i - x_j)}$.

(ii) We note that

$$\prod_{\lambda_i > \lambda_j} (1 + x_i^{-1} x_j) = \sum_{S \subseteq R^+ - R^+_{\gamma(\lambda)}} x^{-\sum S}.$$

Now (ii) follows immediately using (i). $\qquad\square$

Schur's P-functions arise naturally for us as certain Euler characteristics. Suppose that $\lambda \in X_p^+(T)$, and define $\gamma = \gamma(\lambda)$ as above. By Lemma 3.1, we can view $\mathfrak{u}(\lambda)$ as a P_γ-supermodule. For any $i \geq 0$, define

$$(14) \qquad H^i(\lambda) := R^i \mathrm{ind}_{P_\gamma}^G \mathfrak{u}(\lambda).$$

We note that $\mathrm{ind}_B^G \mathfrak{u}(\lambda) \simeq \mathrm{ind}_{P_\gamma}^G (\mathrm{ind}_B^{P_\gamma} \mathfrak{u}(\lambda)) = \mathrm{ind}_{P_\gamma}^G \mathfrak{u}(\lambda)$ by transitivity of induction, i.e. the new definition of $H^0(\lambda)$ agrees with the old one from (10).

By Theorem 3.3, the flag superscheme X_γ is the quotient of G by P_γ. So we also have that

$$(15) \qquad H^i(\lambda) = H^i(X_\gamma, \mathscr{L}(\mathfrak{u}(\lambda))),$$

invoking Corollary 2.4. Moreover, as in the proof of Corollary 2.8, Grothendieck's vanishing theorem and Serre's finiteness theorem imply:

Lemma 4.2. *Each $H^i(\lambda)$ is finite dimensional, and is 0 for $i > \dim X_{\gamma(\lambda)}$.*

So we obtain a well-defined element of the character group of G:

$$(16) \qquad E_\lambda := \sum_{i \geq 0} (-1)^i \mathrm{ch}\, H^i(\lambda) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm n}]^W.$$

We can compute these explicitly using Corollary 2.8:

Theorem 4.3. *For $\lambda \in X_p^+(T)$, $E_\lambda = 2^{\lfloor (h_{p'}(\lambda)+1)/2 \rfloor} p_\lambda$.*

Proof. Let $\gamma = \gamma(\lambda), P = P_\gamma$. For $G = Q(n)$, it is obvious that

$$(\mathrm{Lie}(G)/\mathrm{Lie}(P))_{\bar{1}}^* \simeq \Pi(\mathrm{Lie}(G_{\mathrm{ev}})/\mathrm{Lie}(P_{\mathrm{ev}}))^*$$

as a G_{ev}-supermodule. So Corollary 2.8 tells us that

$$\sum_{i \geq 0} (-1)^i \mathrm{ch}\, H^i(\lambda) = \sum_{i \geq 0} (-1)^i \mathrm{ch}\, R^i \mathrm{ind}_{P_{\mathrm{ev}}}^{G_{\mathrm{ev}}} \left\{ \bigwedge (\mathrm{Lie}(G_{\mathrm{ev}})/\mathrm{Lie}(P_{\mathrm{ev}}))^* \otimes \mathfrak{u}(\lambda) \right\}.$$

We note that for any P_{ev}-module M, $R^i \mathrm{ind}_{P_{\mathrm{ev}}}^{G_{\mathrm{ev}}} M \cong R^i \mathrm{ind}_{B_{\mathrm{ev}}}^{G_{\mathrm{ev}}} M$ by [**9**, II.4.11] and the tensor identity. Since Euler characteristic is additive on short exact

sequences, it therefore suffices to determine the composition factors of the B_{ev}-module $\bigwedge(\mathrm{Lie}(G_{\mathrm{ev}})/\mathrm{Lie}(P_{\mathrm{ev}}))^* \otimes \mathfrak{u}(\lambda)$: its character is

$$\dim \mathfrak{u}(\lambda) \sum_{S \subseteq R^+ - R_\gamma^+} x^{\lambda - \sum S}.$$

By Weyl's character formula [**9**, II.5.10], $\sum_{i \geq 0} \mathrm{ch}\, R^i \mathrm{ind}_{B_{\mathrm{ev}}}^{G_{\mathrm{ev}}} k_\mu = s_\mu$. So

$$\sum_{i \geq 0} \mathrm{ch}\, H^i(\lambda) = \dim \mathfrak{u}(\lambda) \sum_{S \subseteq R^+ - R_\gamma^+} s_{\lambda - \sum S}.$$

Now we are done using Lemma 4.1(ii), recalling finally that $\dim \mathfrak{u}(\lambda) = 2^{\lfloor (h_{p'}(\lambda)+1)/2 \rfloor}$. $\qquad\square$

For $\lambda \in X_p^+(T)$, set $L_\lambda := \mathrm{ch}\, L(\lambda)$. The

(17) $$\{L_\lambda \mid \lambda \in X_p^+(T)\}$$

form a $\mathbb{Z}$-basis for the character group of G. So we can write

(18) $$E_\lambda = \sum_{\mu \in X_p^+(T)} d_{\lambda,\mu} L_\mu$$

for $d_{\lambda,\mu} \in \mathbb{Z}$. The resulting integer matrix $D = (d_{\lambda,\mu})_{\lambda,\mu \in X_p^+(T)}$ is the *decomposition matrix* of G. By Theorem 4.3, E_λ equals $x^\lambda + (*)$ where $(*)$ is a linear combination of x^μ for $\mu < \lambda$, and similarly L_λ has this form. So we have that $d_{\lambda,\lambda} = 1$ and $d_{\lambda,\mu} = 0$ if $\mu \not\leq \lambda$. Hence, D is a unitriangular matrix if rows and columns are ordered in some way refining dominance, and in particular

(19) $$\{E_\lambda \mid \lambda \in X_p^+(T)\}$$

gives us another natural basis for the character group of G. If $p = 0$, the decomposition matrices are known explicitly, see [**1**]. For some examples of decomposition matrices in positive characteristic, see the tables at the end of the article.

We record here one other consequence of Theorem 2.7, giving complete information about the characters of all $R^i \mathrm{ind}_B^G \mathfrak{u}(\lambda)$ in the case $G = Q(2)$:

Lemma 4.4. *Suppose that $G = Q(2)$ and $\lambda \in X(T)$.*

(i) *For $\lambda \in X_p^+(T)$,*

$$\mathrm{ch}\, \mathrm{ind}_B^G \mathfrak{u}(\lambda) = 2^{\lfloor (h_{p'}(\lambda)+1)/2 \rfloor}(x^\lambda + 2x^{\lambda - \alpha} + \cdots + 2x^{w_0\lambda + \alpha} + x^{w_0\lambda}).$$

(ii) *For $\lambda \notin X_p^+(T)$, $\mathrm{ch}\, \mathrm{ind}_B^G \mathfrak{u}(\lambda) = 0$.*

(iii) $\mathrm{ch}\, R^1 \mathrm{ind}_B^G \mathfrak{u}(\lambda) = \mathrm{ch}\, \mathrm{ind}_B^G \mathfrak{u}(w_0\lambda).$

In particular, $\mathrm{ind}_B^G k \simeq k$, $R^1 \mathrm{ind}_B^G k \simeq \Pi k$.

Proof. Let X denote the quotient of G by B. By Theorem 2.7, the canonical filtration gives rise to a short exact sequence of quasi-coherent $\mathscr{O}_X G_{\mathrm{ev}}$-supermodules

$$(20) \qquad 0 \longrightarrow \mathscr{L}_{\mathrm{ev}}(\Pi k_{-\alpha} \otimes \mathfrak{u}(\lambda)) \longrightarrow \mathscr{L}(\mathfrak{u}(\lambda)) \longrightarrow \mathscr{L}_{\mathrm{ev}}(\mathfrak{u}(\lambda)) \longrightarrow 0,$$

where $\alpha = \varepsilon_1 - \varepsilon_2$. Also we already know from the classification [**3**, Theorem 6.11] that $\mathrm{ind}_B^G \mathfrak{u}(\lambda) = 0$ unless λ is p-strict and by Lemma 3.1 that $\mathrm{ind}_B^G \mathfrak{u}(\lambda) \simeq \mathfrak{u}(\lambda)$ in case $\lambda_1 = \lambda_2$ with $p | \lambda_i$. The lemma follows on combining these facts, the long exact sequence arising from (20) and the known cohomology of equivariant line bundles on $\mathbb{P}^1$. $\qquad\square$

5. Serre duality and some examples

In this section, we wish to give a few simple examples. To do this, we need to make use in addition of Serre duality for the flag superschemes X_γ. We could not find any satisfactory reference for this, but see at least the discussion in [**14**, §2]. Let us state the theorem:

Theorem 5.1. *Let γ be a composition of n and $N = |R_\gamma^+|$. There is a natural isomorphism*

$$(R^i \mathrm{ind}_{P_\gamma}^G M)^* \cong R^{N-i} \mathrm{ind}_{P_\gamma}^G (M^*)$$

for each finite dimensional P_γ-supermodule M.

Example 5.2. For $\lambda \in X_p^+(T)$, define $V(\lambda) := R^N \mathrm{ind}_B^G \mathfrak{u}(w_0 \lambda)$, where $N = \frac{1}{2} n(n-1)$. Then by Serre duality, $V(\lambda) \cong H^0(-w_0 \lambda)^*$, i.e. $V(\lambda)$ is the universal highest weight module of [**3**, (6.14), (10.14)]. In particular, $V(\lambda)$ has a unique irreducible quotient isomorphic to $L(\lambda)$, and $\mathrm{ch}\, V(\lambda) = \mathrm{ch}\, H^0(\lambda)$.

Example 5.3. Let us compute $\mathrm{ch}\, H^0(\varepsilon_1 - \varepsilon_3)$ for $G = Q(3)$. Let P_1, P_2 denote the minimal parabolic subgroups of G corresponding to the simple roots $\varepsilon_1 - \varepsilon_2$ and $\varepsilon_2 - \varepsilon_3$ respectively. Denote the irreducible P_i-supermodule of highest weight μ by $L_i(\mu)$. We proceed in a number of steps.

(i) To start with, Serre duality gives $H^3(\varepsilon_1 - \varepsilon_3) \cong H^0(-\varepsilon_1 + \varepsilon_3)^*$ which is zero since $-\varepsilon_1 + \varepsilon_3 \notin X_p^+(T)$.

(ii) The $Q(2)$-supermodule $H^0(\varepsilon_1)$ is irreducible, so $\mathrm{ind}_B^{P_1} \mathfrak{u}(\varepsilon_1 - \varepsilon_3) \cong L_1(\varepsilon_1 - \varepsilon_3)$. Hence,

$$(\mathrm{ind}_B^{P_1} \mathfrak{u}(\varepsilon_1 - \varepsilon_3))^* \cong L_1(-\varepsilon_2 + \varepsilon_3) \cong \mathrm{ind}_B^{P_1} \mathfrak{u}(-\varepsilon_2 + \varepsilon_3).$$

Applying Serre duality, we deduce that

$$R^1 \mathrm{ind}_B^{P_1} \mathfrak{u}(\varepsilon_2 - \varepsilon_3) \cong \mathrm{ind}_B^{P_1} \mathfrak{u}(\varepsilon_1 - \varepsilon_3).$$

Since $R^i \mathrm{ind}_B^{P_1} \mathfrak{u}(\varepsilon_2 - \varepsilon_3) = 0$ for $i \neq 1$, we have that

$$R^i \mathrm{ind}_{P_1}^G (R^1 \mathrm{ind}_B^{P_1} \mathfrak{u}(\varepsilon_2 - \varepsilon_3)) \cong R^{i+1} \mathrm{ind}_B^G \mathfrak{u}(\varepsilon_2 - \varepsilon_3).$$

Similarly,

$$R^i \mathrm{ind}_{P_1}^G (\mathrm{ind}_B^{P_1} \mathfrak{u}(\varepsilon_1 - \varepsilon_3)) \cong R^i \mathrm{ind}_B^G \mathfrak{u}(\varepsilon_1 - \varepsilon_3).$$

So we have shown that

$$R^{i+1} \mathrm{ind}_B^G \mathfrak{u}(\varepsilon_2 - \varepsilon_3) \cong R^i \mathrm{ind}_B^G \mathfrak{u}(\varepsilon_1 - \varepsilon_3)$$

for each $i \geq 0$. As in step (i), the left hand side is zero for $i = 2$, since $-\varepsilon_2 + \varepsilon_3 \notin X_p^+(T)$. Hence, $H^2(\varepsilon_1 - \varepsilon_3) = 0$ and $H^1(\varepsilon_1 - \varepsilon_3) \cong R^2 \mathrm{ind}_B^G \mathfrak{u}(\varepsilon_2 - \varepsilon_3)$.

(iii) By Lemma 4.4, the character of the $Q(2)$-supermodule $H^0(\varepsilon_1 - \varepsilon_2)$ is $2e^{\varepsilon_1 - \varepsilon_2} + 4 + 2e^{\varepsilon_2 - \varepsilon_1}$. By decomposing the adjoint representation of $Q(2)$, the irreducible $Q(2)$-supermodule $L(\varepsilon_1 - \varepsilon_2)$ has character $2e^{\varepsilon_1 - \varepsilon_2} + 2 + 2e^{\varepsilon_2 - \varepsilon_1}$. Hence, we have the short exact sequence

$$0 \longrightarrow L_2(\varepsilon_2 - \varepsilon_3) \longrightarrow \mathrm{ind}_B^{P_2} \mathfrak{u}(\varepsilon_2 - \varepsilon_3) \longrightarrow C \longrightarrow 0$$

of P_2-supermodules, where C has character 2. Dualizing and applying Serre duality gives the short exact sequence

$$0 \longrightarrow D \longrightarrow R^1 \mathrm{ind}_B^{P_2} \mathfrak{u}(-\varepsilon_2 + \varepsilon_3) \longrightarrow L_2(\varepsilon_2 - \varepsilon_3) \longrightarrow 0$$

where D has character 2. Considering the resulting long exact sequences gives exact sequences

$$R^2 \mathrm{ind}_{P_2}^G L_2(\varepsilon_2 - \varepsilon_3) \longrightarrow R^2 \mathrm{ind}_B^G \mathfrak{u}(\varepsilon_2 - \varepsilon_3) \longrightarrow R^2 \mathrm{ind}_{P_2}^G C$$
$$\longrightarrow R^3 \mathrm{ind}_{P_2}^G L_2(\varepsilon_2 - \varepsilon_3) = 0$$

and

$$0 = R^3 \mathrm{ind}_B^G \mathfrak{u}(-\varepsilon_2 + \varepsilon_3) \longrightarrow R^2 \mathrm{ind}_{P_2}^G L_2(\varepsilon_2 - \varepsilon_3) \longrightarrow R^3 \mathrm{ind}_{P_2}^G D = 0.$$

Hence $R^2 \mathrm{ind}_B^G \mathfrak{u}(\varepsilon_2 - \varepsilon_3) \cong R^2 \mathrm{ind}_{P_2}^G C$. Serre duality gives that the latter has character 2, so we have shown ch $H^1(\varepsilon_1 - \varepsilon_3) = 2$.

(iv) Hence, by the definition (18), ch $H^0(\varepsilon_1 - \varepsilon_3) = E_{\varepsilon_1 - \varepsilon_3} + 2$.

Example 5.4. Continue with $G = Q(3)$ and let $\lambda = 4\varepsilon_1 + 3\varepsilon_2 + 2\varepsilon_3$. In characteristic 0, $H^0(\lambda)$ is irreducible by the linkage principle [**3**, Theorem 8.10], so its character is known by [**21**]: ch $H^0(\lambda) = E_\lambda$. But in characteristic 3, the Frobenius twist of determinant gives us a one dimensional G-supermodule $\det^p$ of character $x^{3(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)}$. So by the tensor identity [**3**, (6.3)] and the previous example, we have that

$$\mathrm{ch}\, H^0(\lambda) = \mathrm{ch}\, H^0(\varepsilon_1 - \varepsilon_3) x^{3(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)} = E_\lambda + 2x^{3(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)}.$$

This example shows that the characters of the induced modules $H^0(\lambda)$ are not stable under reduction modulo p, even for strict λ.

6. Linkage principle for Euler characteristics

In [**3**, §8], we introduced the notion of the *residue content* $\text{cont}(\lambda)$ of $\lambda \in X(T)$. This is a tuple $(c_0, c_1, \ldots, c_\ell)$ of integers, where $\ell = (p-1)/2$ or ∞ in case $p = 0$. Rather than recall the definition here, we just give one example: let $n = 6$, $\lambda = (9, 7, 4, 0, -5, -8)$ and $p = 5$. We represent λ pictorially as

									0	1	2	1	0	0	1	2	1
									0	1	2	1	0	0	1		
$-$									0	1	2	1				$+$	
			0	1	2	1	0										
	2	1	0	0	1	2	1	0									

Note in this example that $\ell = 2$. The content of λ is $(2, 4, 1)$, 2 counting the total number of entries in the diagram equal to 0 (there being 7 in the positive half of the diagram and 5 in the negative half), 4 counting the total number entries equal to 1 and 1 counting the total number of entries equal to 2.

In [**3**, Theorem 8.10], we showed that the composition multiplicity $[H^0(\lambda) : L(\mu)]$ is zero unless $\text{cont}(\mu) = \text{cont}(\lambda)$. We wish now to show moreover that the decomposition number $d_{\lambda,\mu}$ is zero unless $\text{cont}(\mu) = \text{cont}(\lambda)$. For the proof, we will work in a rather larger category. Recall the definition of the superalgebra of distributions $\text{Dist}(G)$ from [**3**, §4]. It was constructed in *loc. cit.* by reduction modulo p from the Kostant $\mathbb{Z}$-form for the enveloping superalgebra of the Lie superalgebra $\mathfrak{q}(n, \mathbb{C})$, proving that it has a PBW basis given by all monomials

$$\prod_{1 \leq i \neq j \leq n} e_{i,j}^{(a_{i,j})} (e'_{i,j})^{d_{i,j}} \prod_{1 \leq i \leq n} \binom{h_i}{a_{i,i}} (h'_i)^{d_{i,i}},$$

for all $a_{i,j} \in \mathbb{Z}_{\geq 0}$ and $d_{i,j} \in \{0, 1\}$ (the product being taken in some arbitrary but fixed order). For a $\text{Dist}(G)$-supermodule M and $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$, we let

$$M_\lambda = \left\{ m \in M \,\middle|\, \binom{h_i}{r} m = \binom{\lambda_i}{r} m \text{ for all } i = 1, \ldots, n, r \geq 0 \right\}.$$

Also let $\text{Dist}(B^+)$ be the subalgebra of $\text{Dist}(G)$ generated by $\text{Dist}(H)$ and all $e_{i,j}^{(r)}, e'_{i,j}$ for $1 \leq i < j \leq n, r \geq 1$. Let $\mathcal{O}_n$ be the category of all $\text{Dist}(G)$-supermodules M satisfying the properties

(1) M is locally finite when viewed as a $\text{Dist}(B^+)$-supermodule;
(2) $M = \bigoplus_{\lambda \in X(T)} M_\lambda$.

By [**3**, Corollary 5.7], we can identify the category of all G-supermodules with the full subcategory $\mathcal{C}_n$ of $\mathcal{O}_n$ consisting of all $M \in \mathcal{O}_n$ that are locally finite as $\mathrm{Dist}(G)$-supermodules.

The basic objects in category $\mathcal{O}_n$ are the *Verma supermodules*

$$M(\lambda) := \mathrm{Dist}(G) \otimes_{\mathrm{Dist}(B^+)} \mathfrak{u}(\lambda)$$

for each $\lambda \in X(T)$. By standard arguments, $M(\lambda)$ has a unique irreducible quotient denoted $L(\lambda)$, and the supermodules $\{L(\lambda) \mid \lambda \in X(T)\}$ form a complete set of pairwise non-isomorphic irreducibles in category $\mathcal{O}_n$. Note $L(\lambda)$ is finite dimensional if and only if $\lambda \in X_p^+(T)$, when it agrees with the G-supermodule denoted $L(\lambda)$ earlier. The character of $M(\lambda)$ is given by

$$(21) \quad \mathrm{ch}\, M(\lambda) = x^\lambda \prod_{1 \leq i < j \leq n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \in \mathbb{Z}[x_1^{\pm 1}][[x_i^{-1} x_j \mid 1 \leq i < j \leq n]].$$

Lemma 6.1. *For $\lambda, \mu \in X(T)$, $[M(\lambda) : L(\mu)]$ is zero unless $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$.*

Proof. This follows immediately from [**3**, Lemmas 8.4, 8.9]. □

Lemma 6.2. $\prod_{1 \leq i < j \leq n} \frac{1 - x_i^{-1} x_j}{1 + x_i^{-1} x_j} \in \mathbb{Z}[[x_i^{-1} x_j \mid 1 \leq i < j \leq n]]$ *is an infinite linear combination of x^μ's for $\mu \leq 0$ with $\mathrm{cont}(\mu) = (0, \ldots, 0)$.*

Proof. Working in $\mathbb{Z}[x_1^{\pm 1}][[x_i^{-1} x_j \mid 1 \leq i < j \leq n]]$, we can write the character 1 of the trivial module as a linear combination of characters of Verma supermodules. In view of (21) and Lemma 6.1, this expression only involves Verma characters of the form $\mathrm{ch}\, M(\mu)$ for $\mu \leq 0$ with $\mathrm{cont}(\mu) = (0, \ldots, 0)$. Hence,

$$1 = \sum_{\substack{\mu \leq 0 \\ \mathrm{wt}(\mu) = 0}} \left(a_\mu x^\mu \prod_{1 \leq i < j \leq n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right)$$

for some coefficients $a_\mu \in \mathbb{Z}$. The lemma follows. □

Now we can prove the main result of the section.

Theorem 6.3. *For $\lambda, \mu \in X_p^+(T)$, $d_{\lambda,\mu}$ is zero unless $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$.*

Proof. Let $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i \in X_p^+(T)$. Working in $\mathbb{Z}[x_1^{\pm 1}][[x_i^{-1} x_j \mid 1 \leq i < j \leq n]]$, we will show that E_λ can be written as a (possibly infinite) linear combination of $\mathrm{ch}\, M(\mu)$'s for $\mu \in X(T)$ with $\mathrm{cont}(\mu) = \mathrm{cont}(\lambda)$. The theorem

follows from this and Lemma 6.1. By Theorem 4.3 and (13), we have that

$$E_\lambda = 2^{\lfloor (h(\lambda)+1)/2 \rfloor} \sum_{w \in W^{\gamma(\lambda)}} w \left(x^\lambda \prod_{\substack{1 \le i < j \le n \\ \lambda_i = \lambda_j}} \frac{1 - x_i^{-1} x_j}{1 + x_i^{-1} x_j} \prod_{1 \le i < j \le n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right)$$

$$= 2^{\lfloor (h(\lambda)+1)/2 \rfloor} \sum_{w \in W^{\gamma(\lambda)}} (-1)^{\ell(w)} x^{w\lambda} \prod_{\substack{1 \le i < j \le n \\ \lambda_i = \lambda_j}} \frac{1 - x_{wi}^{-1} x_{wj}}{1 + x_{wi}^{-1} x_{wj}} \prod_{1 \le i < j \le n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j}.$$

By Lemma 6.2, $x^{w\lambda} \prod_{1 \le i < j \le n, \lambda_i = \lambda_j} \frac{1 - x_{wi}^{-1} x_{wj}}{1 + x_{wi}^{-1} x_{wj}}$ is a (possibly infinite) linear combination of x^μ's for $\mu \le w\lambda$ with $\mathrm{cont}(\mu) = \mathrm{cont}(w\lambda) = \mathrm{cont}(\lambda)$. Hence recalling (21), E_λ is a (possibly infinite) linear combination of ch $M(\mu)$'s for $\mu \in X(T)$ with $\mathrm{cont}(\mu) = \mathrm{cont}(\lambda)$. $\square$

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Supplement: Some decomposition matrices for polynomial representations in characteristic $p = 3$, n large.

$d=3$	(21)	(3)
(21)	1	0
(3)	1	1

$d=4$	(31)	(4)
(31)	1	0
(4)	1	1

$d=5$	(32)	(41)	(5)
(32)	1	0	0
(41)	0	1	0
(5)	1	0	1

$d=6$	(321)	(3^2)	(42)	(51)	(6)
(321)	1	0	0	0	0
(3^2)	1	1	0	0	0
(42)	2	2	1	0	0
(51)	1	2	1	1	0
(6)	1	1	0	1	1

$d=7$	(3^21)	(421)	(43)	(52)	(61)	(7)
(3^21)	1	0	0	0	0	0
(421)	2	1	0	0	0	0
(43)	1	1	1	0	0	0
(52)	0	0	0	1	0	0
(61)	1	1	1	0	1	0
(7)	1	0	0	0	1	1

$d=8$	(3^22)	(431)	(521)	(53)	(62)	(71)	(8)
(3^22)	1	0	0	0	0	0	0
(431)	0	1	0	0	0	0	0
(521)	2	0	1	0	0	0	0
(53)	1	0	1	1	0	0	0
(62)	1	0	1	1	1	0	0
(71)	0	1	0	0	0	1	0
(8)	1	0	0	0	1	0	1

d=9	(3²21)	(3³)	(432)	(531)	(54)	(621)	(63)	(72)	(81)	(9)
(3²21)	1	0	0	0	0	0	0	0	0	0
(3³)	1	1	0	0	0	0	0	0	0	0
(432)	0	0	1	0	0	0	0	0	0	0
(531)	2	2	1	1	0	0	0	0	0	0
(54)	1	2	0	1	1	0	0	0	0	0
(621)	2	2	0	1	0	1	0	0	0	0
(63)	1	3	0	1	1	1	1	0	0	0
(72)	1	6	1	1	1	2	2	1	0	0
(81)	1	4	1	0	0	1	2	1	1	0
(9)	1	2	0	0	0	1	2	0	1	1

d=10	(3³1)	(4321)	(43²)	(532)	(541)	(631)	(64)	(721)	(73)	(82)	(91)	(10)
(3³1)	1	0	0	0	0	0	0	0	0	0	0	0
(4321)	0	1	0	0	0	0	0	0	0	0	0	0
(43²)	1	1	1	0	0	0	0	0	0	0	0	0
(532)	0	0	0	1	0	0	0	0	0	0	0	0
(541)	2	2	2	0	1	0	0	0	0	0	0	0
(631)	2	1	1	0	1	1	0	0	0	0	0	0
(64)	1	0	1	0	1	1	1	0	0	0	0	0
(721)	2	1	2	0	1	2	0	1	0	0	0	0
(73)	1	1	3	0	1	2	1	1	1	0	0	0
(82)	0	0	0	1	0	0	0	0	0	1	0	0
(91)	1	1	2	0	0	1	0	1	1	0	1	0
(10)	1	0	0	0	0	1	0	0	0	0	1	1

d=11	3³2	43²1	5321	53²	542	632	641	65	731	74	821	83	92	10 1	11
3³2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43²1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
5321	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
53²	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0
542	2	0	2	2	1	0	0	0	0	0	0	0	0	0	0
632	2	0	1	1	1	1	0	0	0	0	0	0	0	0	0
641	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
65	1	0	0	1	1	1	0	1	0	0	0	0	0	0	0
731	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0
74	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0
821	2	0	1	2	1	2	0	0	0	0	1	0	0	0	0
83	1	0	1	3	1	2	0	1	0	0	1	1	0	0	0
92	1	0	1	2	0	1	0	0	0	0	1	1	1	0	0
10 1	0	1	0	0	0	0	0	0	1	0	0	0	0	1	0
11	1	0	0	0	0	1	0	0	0	0	0	1	1	0	1

$d{=}12$	$3^3 21$	3^4	$43^2 2$	$53^2 1$	5421	543	6321	63^2	642	651	6^2	732	741	75	831	84	921	93	10 2	11 1	12
$3^3 21$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3^4	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$43^2 2$	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$53^2 1$	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5421	0	0	0	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
543	1	2	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6321	1	2	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
63^2	2	3	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
642	4	8	1	2	3	2	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0
651	1	4	1	0	1	1	1	2	1	1	0	0	0	0	0	0	0	0	0	0	0
6^2	1	4	0	0	1	1	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0
732	3	6	1	1	2	1	2	2	1	0	0	1	0	0	0	0	0	0	0	0	0
741	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
75	2	6	0	0	2	2	2	2	1	2	0	1	0	1	0	0	0	0	0	0	0
831	2	6	1	1	2	2	3	2	1	1	0	1	0	0	1	0	0	0	0	0	0
84	1	4	0	1	2	3	3	2	1	2	0	1	0	1	1	1	0	0	0	0	0
921	2	2	0	1	1	1	2	0	0	0	0	0	0	0	1	0	1	0	0	0	0
93	1	3	0	1	1	2	2	1	0	1	1	0	0	0	1	1	1	1	0	0	0
10 2	1	0	1	1	0	1	1	0	0	0	0	1	0	0	1	1	2	0	1	0	0
11 1	1	2	1	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	1	1	0
12	1	1	0	0	0	0	1	1	0	0	1	0	0	0	0	0	1	1	0	1	1

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