A NEW APPROACH TO THE REPRESENTATION THEORY OF THE PARTITION CATEGORY

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Abstract. We explain a new approach to the representation theory of the partition category based on a reformulation of the definition of the Jucys-Murphy elements introduced originally by Halverson and Ram and developed further by Enyang. Our reformulation involves a new graphical monoidal category, the *affine partition category*, which is defined here as a certain monoidal subcategory of Khovanov's Heisenberg category. We use the Jucys-Murphy elements to construct some special projective functors, then apply these functors to give self-contained proofs of results of Comes and Ostrik on blocks of Deligne's category $\underline{\mathrm{Rep}}(S_t)$.

1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic zero and $t \in \Bbbk$ be a parameter. The *partition category* $\mathcal{P}ar_t$ is the free strict $\Bbbk$-linear symmetric monoidal category generated by a special commutative Frobenius object of categorical dimension t. Its additive Karoubi envelope is the category $\underline{\mathrm{Rep}}(S_t)$ introduced by Deligne [D], which interpolates the categories of representations of the symmetric groups S_t $(t \in \mathbb{N})$ to non-integer values of t. When $t \notin \mathbb{N}$, Deligne's category is a semisimple tensor category which is not of sub-exponential growth, hence, it does not admit a fiber functor; see [EGNO, Sec. 9.12] for further background here. When $t \in \mathbb{N}$, the category $\underline{\mathrm{Rep}}(S_t)$ is not semisimple, and its semisimplification is the usual tensor category $\Bbbk S_t\text{-Mod}_{\mathrm{fd}}$ of representations of the symmetric group.

The objects of the partition category are indexed by the natural numbers. For $n \in \mathbb{N}$, the endomorphism algebra $\mathrm{End}_{\mathcal{P}ar_t}(n)$ is the *partition algebra* $P_n(t)$ introduced by Martin [M1] and Jones [J]. The representation theory of this finite-dimensional algebra has been well studied. In [M2], Martin showed that $P_n(t)\text{-Mod}_{\mathrm{fd}}$ is a highest weight category except when $t = 0$, and he determined the precise structure of the standard modules; see also [DW]. When $t = 0$, $P_n(t)$ still has the structure of a cellular algebra, as established in [DW, X], and its representation theory is also well understood. The partition algebras form a tower $P_0(t) < P_1(t) < \cdots$, but the cell modules do not restrict along this tower in a multiplicity-free way, so that standard techniques like the Jones basic construction cannot be applied directly. To address this, Martin [M3] and Halverson and Ram [HR] consider an intermediate family of "half partition algebras" fitting into a tower

$$P_0(t) < P_{\frac{1}{2}}(t) < P_1(t) < P_{\frac{3}{2}}(t) < \cdots .$$

Halverson and Ram also defined analogs $L_0, L_{\frac{1}{2}}, L_1, L_{\frac{3}{2}}, \ldots$ of Jucys-Murphy elements in these partition algebras, which were studied further by Enyang [E1, E2]. Enyang worked out a recursive definition for the Jucys-Murphy elements and used them to construct an analog of Young's orthogonal form for the irreducible $P_n(t)$-modules. His definition involves a complicated five term recurrence relation, making the Jucys-Murphy elements for partition algebras considerably harder to work with than the classical Jucys-Murphy elements of the symmetric groups. Recently, Creedon [Cr] has revisited Enyang's work,

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showing that supersymmetric polynomials in a renormalization of the Jucys-Murphy elements give a family of central elements which is large enough to separate blocks.

In this article, we give a new treatment of the representation theory of $\mathcal{P}ar_t$. Let

$$Par_t := \bigoplus_{m,n\in\mathbb{N}} \mathrm{Hom}_{\mathcal{P}ar_t}(n,m)$$

be the path algebra of this $\Bbbk$-linear category, denoting the idempotents arising from the identity endomorphisms of the objects of $\mathcal{P}ar_t$ by $\{1_n \mid n \in \mathbb{N}\}$. Since the partition algebra $P_n(t)$ is the idempotent truncation $1_n Par_t 1_n$, most of the known results about the representation theory of the algebras $P_n(t)$ can be deduced from that of the partition category in a standard way. In fact, as well as producing more general results, we are convinced that it is easier to study the representation theory of the partition category $\mathcal{P}ar_t$, instead of working with the tower of partition algebras. To start with, $\mathcal{P}ar_t$ has an efficient monoidal presentation encoding its universal property, with generating morphisms (“crossing”), (“merge”), (“split”), (“cap”), (“cup”), (“downward leaf”) and (“upward leaf”); see Definition 3.1. This means that one can make calculations in $\mathcal{P}ar_t$ using the string calculus for strict monoidal categories, which seems more flexible than the traditional algebraic expressions used when working in $P_n(t)$. But the key reason we prefer to work with $\mathcal{P}ar_t$ is that its path algebra has a *triangular decomposition* in the sense of [BS, Def. 5.31], hence, the category Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$ of locally finite-dimensional Par_t-modules is an *upper finite highest weight category* as in [BS, Def. 3.34]. The Cartan subalgebra in this triangular decomposition is the locally unital algebra

$$Sym := \bigoplus_{n\geqslant 0} \Bbbk S_n,$$

with its irreducible modules being the Specht modules $\{S(\lambda) \mid \lambda \in \mathcal{P}\}$ indexed by the set $\mathcal{P}$ of all partitions. The *standard modules* $\{\Delta(\lambda) \mid \lambda \in \mathcal{P}\}$ for Par_t are the modules defined by parabolically inducing the Specht modules. Then we obtain a full set of pairwise inequivalent irreducible Par_t-modules $\{L(\lambda) \mid \lambda \in \mathcal{P}\}$ from the irreducible heads of the standard modules. This gives a quick proof of the classification of irreducible Par_t-modules, which was established originally by Deligne [D] and Comes and Ostrik [CO].

The highest weight approach to the representation theory of combinatorial monoidal categories such as the partition category as just outlined has been developed systematically by Sam and Snowden [SS2]. In their language, $\mathcal{P}ar_t$ is a *monoidal triangular category*. There are many other interesting examples of this structure, including several that are actually monoidal subcategories of $\mathcal{P}ar_t$: the Brauer category (cups, caps and crossings but no splits and merges), the Temperley-Lieb category (just cups and caps), and the category studied by Khovanov and Sazdanovic in [KS] (just leaves). In their earlier work [SS1], Sam and Snowden had already exposed the importance of the structure of the Borel subcategories of these and other such categories, although at that time they did not work out the details fully in the case of the partition category. In §3, we fill this gap by giving an exposition of some of their ideas in this case, exploiting the structure of the *upper partition category*, i.e., the positive Borel subcategory, to determine the Grothendieck ring $K_0(Par_t)$ of the category of finitely generated projective Par_t-modules. In fact, as a ring, this is identified with the ring Λ of symmetric functions, but the isomorphism classes of the standard modules $\Delta(\lambda)$ produce an interesting inhomogeneous basis $\{\tilde{s}_\lambda | \lambda \in \mathcal{P}\}$ for Λ of *deformed Schur functions*. These also appeared implicitly in [D, CO] and again in [SS1], and were rediscovered from a slightly different perspective by Orellana and Zabrocki [OZ]. They are interesting because the structure constants for multiplication in Λ with respect to this basis are the *reduced Kronecker coefficients*.

Although there are many important examples of monoidal triangular categories, and the work of Sam and Snowden has revealed many common features, this still seems to be a subject where the more intricate combinatorics needs to be studied separately in each case. For example, one wants to understand the *center* of the underlying category, and the induced decomposition of the irreducible representations into blocks which comes from considering central characters. This can be framed as a question about an analog of *Harish-Chandra homomorphism* for monoidal triangular categories; see §5.1. However, to answer it, one needs some way to construct sufficiently many central elements, and we do not know any uniform way to approach this. After understanding the block decomposition, the next step is to consider the combinatorics of *special projective functors*, which are functors on the module category induced by tensoring with generating objects of the underlying monoidal category.

The crucial new ingredient in our approach to $\mathcal{P}ar_t$ is the definition of another graphical monoidal category, the *affine partition category* $\mathcal{AP}ar$. This is obtained from the partition category by adjoining two new generating morphisms $\bullet\!\!|$ ("left dot") and $|\!\!\bullet$ ("right dot") in such a way that $\mathcal{P}ar_t$ can be recovered as the quotient of $\mathcal{AP}ar$ by a certain left tensor ideal, with the left and right dots mapping to renormalized versions of the Jucys-Murphy elements; see Theorem 4.15 and Corollary 4.19. However, it is not easy to do this without making additional choices. The actual definition of $\mathcal{AP}ar$ given in Definition 4.6 below adopts a quite different point of view based on an observation due to Likeng and Savage [LSR]: we construct $\mathcal{AP}ar$ initially as a monoidal subcategory of Khovanov's Heisenberg category $\mathcal{H}eis$ from [K]. This allows complicated relations in $\mathcal{AP}ar$ to be derived rather quickly by elementary calculations using the string calculus for $\mathcal{H}eis$; e.g., see Lemma 4.10 which recovers Enyang's five term recurrence relation for the Jucys-Murphy elements.

In the affine partition category, there is an obvious way to construct a large family of central elements; see Theorem 4.23. These map to central elements in $\mathcal{P}ar_t$ which turn out to be closely related to Creedon's central elements of the partition algebras from [Cr]. After that, we consider the self-adjoint projective functor

$$D : Par_t\text{-Mod} \to Par_t\text{-Mod}$$

induced by tensoring with the generating object 1 of $\mathcal{P}ar_t$. This plays an analogous role in our approach to induction and restriction along the tower of partition algebras in the work of Martin and others discussed earlier. We use the action of the left and right dots from $\mathcal{AP}ar$ to decompose D into summands $D = \bigoplus_{a,b\in\Bbbk} D_{b|a}$; see Theorem 5.18. There is a close analogy here to the way Jucys-Murphy elements were used to give a new approach to the representation theory of the symmetric groups in [OV]. In fact, the Jucys-Murphy elements of Par_t generate a large commutative subalgebra, and the resulting "Gelfand-Tsetlin characters" of the standard modules $\Delta(\lambda)$ can be computed explicitly using the branching rules from Theorem 5.18, although we do not pursue this further here. Finally, we use the combinatorial properties of the special projective functors $D_{b|a}$ to reprove the main structural result about the representation theory of $\mathcal{P}ar_t$ for $t \in \mathbb{N}$. This was established originally by Comes and Ostrik [CO].

Theorem. *When $t \in \mathbb{N}$, i.e., Par_t is not semisimple, the non-simple blocks of Par_t are in bijection with isomorphism classes of irreducibles in the semisimplification* $\Bbbk S_t$-Mod$_{\mathrm{fd}}$. *All of the non-simple blocks are Morita equivalent. These blocks have infinitely many isomorphism classes of irreducible modules parametrized by $\mathbb{N}$, and the structure of the corresponding indecomposable projectives is as follows:*

$$P(0) = \begin{tikzcd} 0 \arrow[d, squiggly, dash] \\ 1 \end{tikzcd}, \quad P(1) = \begin{tikzcd} & 1 \arrow[dl, dash] \arrow[dr, squiggly, dash] & \\ 0 \arrow[dr, squiggly, dash] & & 2 \arrow[dl, dash] \\ & 1 & \end{tikzcd}, \quad P(2) = \begin{tikzcd} & 2 \arrow[dl, dash] \arrow[dr, squiggly, dash] & \\ 1 \arrow[dr, squiggly, dash] & & 3 \arrow[dl, dash] \\ & 2 & \end{tikzcd}, \quad P(3) = \begin{tikzcd} & 3 \arrow[dl, dash] \arrow[dr, squiggly, dash] & \\ 2 \arrow[dr, squiggly, dash] & & 4 \arrow[dl, dash] \\ & 3 & \end{tikzcd}, \dots$$

For a more formal statement, see Theorem 5.24. It is a straightforward exercise to deduce from this that each non-simple block is Morita equivalent to the path algebra of the infinite quiver

$$0 \underset{y_0}{\overset{x_0}{\rightleftarrows}} 1 \underset{y_1}{\overset{x_1}{\rightleftarrows}} 2 \underset{y_2}{\overset{x_2}{\rightleftarrows}} 3 \cdots \text{ with relations } y_0 x_0 = 0, x_{i+1} x_i = y_i y_{i+1} = x_i y_i - y_{i+1} x_{i+1} = 0.$$

This quiver is well known in representation theory, for example, it also describes the non-trivial block of the Temperley-Lieb category, as noted in [CO, Rem. 6.5].

It is interesting to compare the general strategy developed here with the original arguments of Comes and Ostrik. There are many parallels. For example, they also construct a large family of central elements, although different from ours, and they also use summands of the functor D to construct equivalences between blocks; see Remark 4.27 and Theorem 5.21. Another technique which is crucial in [CO] is the idea of lifting projectives to the (semisimple) generic partition category. In our approach, this is replaced everywhere with arguments involving standard modules and the BGG reciprocity coming from the highest weight structure. In fact, largely due to the fact that they did not think in these module-theoreic terms, Comes and Ostrik were forced in the end to refer to some of Martin's results from [M2] to obtain the precise submodule structure of projectives in the above theorem, whereas our proof is independent of *loc. cit.*, indeed, Martin's results can now be deduced from here. One more key idea used by Comes and Ostrik involves an explicit formula for categorical dimensions derived ultimately from the hook formula, although we have avoided such considerations entirely by exploiting the functors $D_{b|a}$ for $a = b$. The definition of these diagonal components of D cannot be formulated without using Jucys-Murphy elements, so no counterpart for this part of our argument appears in [CO].

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2. Monoidal categories and representations

In the opening section, we explain our general conventions for representations of $\mathbb{k}$-linear (monoidal) categories. Always in this article $\mathbb{k}$ will be an algebraically closed field of characteristic zero, although all of the generalities recorded make sense more generally. Then we briefly recall some classical results about $\mathcal{Sym}$, the free strict $\mathbb{k}$-linear symmetric monoidal category on one object, which categorifies the ring of symmetric functions.

2.1. **Path algebras and modules.** Let $\mathcal{A}$ be a $\mathbb{k}$-linear category. Its *path algebra* is the associative algebra

$$A := \bigoplus_{X,Y \in \mathrm{ob}\,\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(X,Y)$$

with multiplication induced by composition in $\mathcal{A}$, so that $gf = g \circ f$ for $f : X \to Y$, $g : Y \to Z$. Note that A is not necessarily unital, but it is always a *locally unital algebra*, i.e., there is a distinguished family $\{1_X \mid X \in \mathbb{O}_A\}$ of mutually orthogonal idempotents such that $A = \bigoplus_{X,Y \in \mathbb{O}_A} 1_Y A 1_X$. In this case, $\mathbb{O}_A$ is the object set $\mathrm{ob}\,\mathcal{A}$ of the category $\mathcal{A}$, with 1_X being the identity endomorphism of X. If $\mathcal{A}$ is a *finite-dimensional category*, i.e., its morphism spaces are finite-dimensional, then the path algebra is *locally finite-dimensional* in the sense that $\dim 1_Y A 1_X < \infty$ for all $X, Y \in \mathbb{O}_A$.

The category A-Mod of left A-modules is the category $\mathcal{Hom}_{\mathbb{k}}(\mathcal{A}, \mathcal{Vec})$ of $\mathbb{k}$-linear functors from $\mathcal{A}$ to the category $\mathcal{Vec}$ of vector spaces, morphisms being natural transformations. Equivalently, using the language we systematically adopt below, a left A-module V is a left module in the usual sense of associative algebras such that $V = \bigoplus_{X \in \mathbb{O}_A} 1_X V$; this corresponds to the $\mathbb{k}$-linear functor $V : \mathcal{A} \to \mathcal{Vec}$ taking object X to the vector space $V(X) = 1_X V$ and morphism $f \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$ to the linear map

$V(f) : 1_X V \to 1_Y V$ defined by left multiplication by $f \in 1_Y A 1_X$. There is also the category Mod-A of right A-modules, which is just the same as the category $\mathcal{H}om_{\Bbbk}(\mathcal{A}^{\mathrm{op}}, \mathcal{V}ec)$.

We say that $V \in A$-Mod is *locally finite-dimensional* if $\dim 1_X V < \infty$ for all $X \in \mathbb{O}_A$; equivalently, the associated functor goes from $\mathcal{A}$ to the category $\mathcal{V}ec_{\mathrm{fd}}$ of finite-dimensional vector spaces. Let A-Mod$_{\mathrm{lfd}}$ be the full subcategory of A-Mod consisting of the locally finite-dimensional A-modules. For more background material about the structure of the category A-Mod$_{\mathrm{lfd}}$ in the case that A is locally finite-dimensional, we refer to [BS, §2.2–§2.3], where Abelian categories of this form are called *Schurian categories*.

We also let A-Mod$_{\mathrm{fd}}$ be the full subcategory of A-Mod consisting of the globally finite-dimensional modules, i.e., the V with $\dim V < \infty$, and A-Proj be the full subcategory of A-Mod consisting of the finitely generated projective modules. If A is a locally finite-dimensional locally unital algebra then $A1_X$ is a locally finite-dimensional module for each $X \in \mathbb{O}_A$, hence, A-Proj is a subcategory of A-Mod$_{\mathrm{lfd}}$. The category A-Proj can also be obtained in equivalent form directly from the $\Bbbk$-linear category $\mathcal{A}$ since the Yoneda embedding $h^* : \mathcal{A} \to \mathcal{H}om_{\Bbbk}(\mathcal{A}, \mathcal{V}ec)$ induces a contravariant $\Bbbk$-linear equivalence between $\mathrm{Kar}(\mathcal{A})$ and A-Proj. Here, $\mathrm{Kar}(\mathcal{A})$ denotes the *additive Karoubi envelope* of $\mathcal{A}$, that is, the idempotent completion of its additive envelope $\mathrm{Add}(\mathcal{A})$.

We let $K_0(A)$ be the split Grothendieck group of the category A-Proj. Assuming that A is locally finite-dimensional, every finitely generated module has a projective cover in A-Proj. Moreover, $K_0(A)$ is a free Abelian group with canonical basis coming from the projective covers of the irreducible A-modules.

2.2. **Pull-back and push-forward.** Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two $\Bbbk$-linear categories. Let

$$A = \bigoplus_{X,Y \in \mathbb{O}_A} 1_Y A 1_X, \qquad B = \bigoplus_{X,Y \in \mathbb{O}_B} 1_Y B 1_X$$

be their path algebras. To a $\Bbbk$-linear functor $F : \mathcal{A} \to \mathcal{B}$, we associate an exact functor

$$\mathrm{res}_F : B\text{-Mod} \to A\text{-Mod} \tag{2.1}$$

which we call *restriction along F*. It is just the functor $\mathcal{H}om_{\Bbbk}(\mathcal{B}, \mathcal{V}ec) \to \mathcal{H}om_{\Bbbk}(\mathcal{A}, \mathcal{V}ec)$ defined by composing on the right with F. In elementary terms, and introducing a shorthand which will be ubiquitous later on, the functor res_F takes $V \in B$-Mod to

$$1_F V := \bigoplus_{X \in \mathbb{O}_A} 1_{FX} V \in A\text{-Mod}, \tag{2.2}$$

with the left module structure defined so that $f \in 1_Y A 1_X$ acts on the X-th summand $1_{FX} V$ as the linear map $Ff : 1_{FX} V \to 1_{FY} V$, and it acts as zero on all other summands. It takes a B-module homomorphism $\phi : V \to W$ to the A-module homomorphism $\mathrm{res}_F(\phi) : 1_F V \to 1_F W$ defined by $\phi_{FX} : 1_{FX} V \to 1_{FX} W$ for each $X \in \mathbb{O}_A$. Similarly, there is the exact restriction functor we denote by

$${}_F\mathrm{res} : \text{Mod-}B \to \text{Mod-}A \tag{2.3}$$

between the categories of right modules taking $V \in$ Mod-B to

$$V 1_F := \bigoplus_{X \in \mathbb{O}_A} V 1_{FX} \in \text{Mod-}A. \tag{2.4}$$

The functors res_F and ${}_F\mathrm{res}$ may also be denoted F^* and $(F^{\mathrm{op}})^*$; e.g., see [SS1, (2.1.4)], [SS2, §3.6].

The restriction $B1_F = \bigoplus_{X \in \mathbb{O}_A} B1_{FX}$ is a (B, A)-bimodule. The functor $\mathrm{res}_F : B\text{-Mod} \to A\text{-Mod}$ is isomorphic to $\bigoplus_{X \in \mathbb{O}_A} \mathrm{Hom}_B(B1_{FX}, ?)$. Then adjointness of tensor and hom in the locally unital setting (e.g., see [BS, Lem. 2.7]) implies that the functor

$$\mathrm{ind}_F := B1_F \otimes_A : A\text{-Mod} \to B\text{-Mod} \tag{2.5}$$

is left adjoint to res_F. We call this *induction along F*. Since it is left adjoint to an exact functor, ind_F is right exact and takes projectives to projectives. In fact, we have that

$$\mathrm{ind}_F\ A1_X = B1_F \otimes_A A1_X \cong B1_{FX}, \tag{2.6}$$

i.e., ind_F can be viewed as an extension of F to arbitrary modules. From this, it is clear that ind_F preserves finite generation. Likewise, the restriction $1_F B = \bigoplus_{X\in\mathbb{O}_A} 1_{FX}B$ is an (A,B)-bimodule. The functor res_F is also isomorphic to $1_F B\otimes_B$, hence, it has a right adjoint given by the functor

$$\mathrm{coind}_F := \bigoplus_{Y\in\mathbb{O}_B} \mathrm{Hom}_A(1_F B1_Y, ?) : A\text{-Mod} \to B\text{-Mod}. \tag{2.7}$$

We call this *coinduction along F*. Since it is right adjoint to an exact functor, coind_F is left exact and takes injectives to injectives. The functors ind_F and coind_F are also called left and right Kan extensions and may be denoted $F_!$ and F_*, respectively; e.g., see [SS1, (2.1.4)], [SS2, §3.6]. There are also analogs of ind_F and coind_F with left modules replaced by right modules, which we denote by ${}_F\mathrm{ind}$ and ${}_F\mathrm{coind}$; in [SS2], these are denoted $(F^{\mathrm{op}})_!$ and $(F^{\mathrm{op}})_*$.

Lemma 2.1. *Let $F : \mathcal{A} \to \mathcal{B}$ be a $\Bbbk$-linear functor as above.*

(1) If $B1_F$ is a projective right A-module then ind_F *and* ${}_F\mathrm{coind}$ *are exact functors.*

(2) If $1_F B$ is a projective left A-module then ${}_F\mathrm{ind}$ *and* coind_F *are exact functors.*

Proof. This is obvious from the definitions of these functors. □

Suppose that $F,G : \mathcal{A} \to \mathcal{B}$ are $\Bbbk$-linear functors. A natural transformation $\alpha : F \Rightarrow G$ induces natural transformations $\mathrm{res}_\alpha : \mathrm{res}_F \Rightarrow \mathrm{res}_G$, $\mathrm{ind}_\alpha : \mathrm{ind}_G \Rightarrow \mathrm{ind}_F$ and $\mathrm{coind}_\alpha : \mathrm{coind}_G \Rightarrow \mathrm{coind}_F$. We leave the detailed definitions of these to the reader, just noting that ind_α and coind_α are the left and right mates of res_α. Similarly, α induces natural transformations ${}_\alpha\mathrm{res} : {}_G\mathrm{res} \Rightarrow {}_F\mathrm{res}$, ${}_\alpha\mathrm{ind} : {}_F\mathrm{ind} \Rightarrow {}_G\mathrm{ind}$ and ${}_\alpha\mathrm{coind} : {}_F\mathrm{coind} \Rightarrow {}_G\mathrm{coind}$. Assuming for simplicity[1] that $\mathcal{A} = \mathcal{B}$, so that F and G are $\Bbbk$-linear endofunctors of $\mathcal{A}$, these constructions define $\Bbbk$-linear monoidal functors

$$\mathrm{res}_* : \mathcal{E}nd_\Bbbk(\mathcal{A}) \to \mathcal{E}nd_\Bbbk(A\text{-Mod})^{\mathrm{rev}}, \qquad \mathrm{ind}_*, \mathrm{coind}_* : \mathcal{E}nd_\Bbbk(\mathcal{A})^{\mathrm{op}} \to \mathcal{E}nd_\Bbbk(A\text{-Mod}), \tag{2.8}$$

$${}_*\mathrm{res} : \mathcal{E}nd_\Bbbk(\mathcal{A})^{\mathrm{op}} \to \mathcal{E}nd_\Bbbk(\text{Mod-}A)^{\mathrm{rev}}, \qquad {}_*\mathrm{ind}, {}_*\mathrm{coind} : \mathcal{E}nd_\Bbbk(\mathcal{A}) \to \mathcal{E}nd_\Bbbk(\text{Mod-}A). \tag{2.9}$$

Here, $\mathcal{E}nd_\Bbbk(\mathcal{A})$ denotes the strict monoidal $\Bbbk$-linear category of $\Bbbk$-linear endofunctors and natural transformations, "op" means the opposite category with the same monoidal product, and "rev" means the same category with the reversed monoidal product.

2.3. **Duality.** Continue with A and B be the path algebras of $\mathcal{A}$ and $\mathcal{B}$, respectively. There is a contravariant functor

$$?^{\circledast} : A\text{-Mod} \to \text{Mod-}A \tag{2.10}$$

taking $V = \bigoplus_{X\in\mathbb{O}_A} 1_X V$ to $V^{\circledast} := \bigoplus_{X\in\mathbb{O}_A} (1_X V)^*$, the direct sum of the linear duals of the "weight spaces" $1_X V$. The restriction of this to locally finite-dimensional modules is an equivalence, with quasi-inverse given by the restriction of the analogously-defined duality functor

$${}^{\circledast}? : \text{Mod-}A \to A\text{-Mod} \tag{2.11}$$

in the other direction. To obtain a duality (= contravariant auto-equivalence) on $A\text{-Mod}_{\mathrm{lfd}}$ from (2.10) and (2.11), one also needs a $\Bbbk$-linear equivalence $\sigma : \mathcal{A} \to \mathcal{A}^{\mathrm{op}}$. Restriction along σ gives equivalences $\mathrm{res}_\sigma : \mathrm{Mod}_{\mathrm{lfd}}\text{-}A \to A\text{-Mod}_{\mathrm{lfd}}$ and ${}_\sigma\mathrm{res} : A\text{-Mod}_{\mathrm{lfd}} \to \mathrm{Mod}_{\mathrm{lfd}}\text{-}A$, hence, we obtain the duality functor

$$?^{\circledcirc} := \mathrm{res}_\sigma \circ ?^{\circledast} = {}^{\circledast}? \circ {}_\sigma\mathrm{res} : A\text{-Mod}_{\mathrm{lfd}} \to A\text{-Mod}_{\mathrm{lfd}}. \tag{2.12}$$

[1]To formulate analogs of (2.8) and (2.9) without this assumption, one needs to work in the strict 2-category of $\Bbbk$-linear categories.

Given a $\Bbbk$-linear functor $F : \mathcal{A} \to \mathcal{B}$, we obviously have that

$$?^{\circledast} \circ \operatorname{res}_F \cong {}_F\operatorname{res} \circ ?^{\circledast} \tag{2.13}$$

as functors from B-Mod to Mod-A. We deduce that

$$^{\circledast}? \circ {}_F\operatorname{ind} \cong \operatorname{coind}_F \circ {}^{\circledast}?, \qquad {}^{\circledast}? \circ {}_F\operatorname{coind} \cong \operatorname{ind}_F \circ {}^{\circledast}? \tag{2.14}$$

as functors from Mod-A to B-Mod.

2.4. **Induction product.** The $\Bbbk$-linear categories of interest later on will usually have some additional monoidal structure. In fact, they will be strict $\Bbbk$-linear monoidal categories defined by generators and relations. We use the symbol $\star$ for the monoidal product in such categories, reserving $\otimes$ for the tensor product $\otimes_\Bbbk$ of vector spaces over $\Bbbk$. We adopt the usual string calculus for morphisms in strict monoidal categories, our convention being that $f \circ g$, the composition of f and g, is drawn as f on top of g and $f \star g$, the monoidal product of f and g, is drawn as f to the left of g.

Let $\mathcal{C}$ be a strict $\Bbbk$-linear monoidal category with path algebra $C = \bigoplus_{X,Y \in \mathbb{O}_C} 1_Y C 1_X$. The monoidal product $\star$ on $\mathcal{C}$ extends canonically to $\operatorname{Kar}(\mathcal{C})$. There is also a monoidal product $\circledast$ making C-Proj into a (no longer strict) $\Bbbk$-linear monoidal category such that the contravariant Yoneda equivalence from $\operatorname{Kar}(\mathcal{C})$ to C-Proj is monoidal. This functor is the restriction of a tensor product functor $\circledast$ on the Abelian category C-Mod. We call this the *induction product*. Category theorists refer to this instead as *Day convolution* and define it via the coend expression:

$$V_1 \circledast V_2 = \int^{X_1, X_2 \in \mathcal{C}} \operatorname{Hom}_{\mathcal{C}}(X_1 \star X_2, ?) \otimes V_1(X_1) \otimes V_2(X_2).$$

We give the algebraist's formulation of the definition in the next paragraph; see also [SS1, (2.1.14)], [SS2, §3.10]. Using $\circledast$, we can make the split Grothendieck group $K_0(C)$ into a ring with multiplication

$$[P][Q] := [P \circledast Q]. \tag{2.15}$$

Its identity element is the isomorphism class of the distinguished projective module $C1_{\mathbb{1}}$, where $\mathbb{1} \in \mathbb{O}_C$ is the unit object.

Here is the detailed definition of $\circledast$. Let $\mathcal{C} \boxtimes \mathcal{C}$ be the $\Bbbk$-linearization of the Cartesian product $\mathcal{C} \times \mathcal{C}$. The objects in $\mathcal{C} \boxtimes \mathcal{C}$ are pairs $(X_1, X_2) \in \mathbb{O}_C \times \mathbb{O}_C$, and the morphism space from (X_1, X_2) to (Y_1, Y_2) is $\operatorname{Hom}_{\mathcal{C}}(X_1, Y_1) \otimes \operatorname{Hom}_{\mathcal{C}}(X_2, Y_2)$. We denote its path algebra by

$$C \boxtimes C = \bigoplus_{X_1, X_2, Y_1, Y_2 \in \mathbb{O}_C} 1_{Y_1} C 1_{X_1} \otimes 1_{Y_2} C 1_{X_2}.$$

Multiplication in $C \boxtimes C$ is the obvious "tensor-wise" product just like for a tensor product of algebras. If C is locally finite-dimensional, so too is $C \boxtimes C$. Given $V_1, V_2 \in C$-Mod, let

$$V_1 \boxtimes V_2 = \bigoplus_{X_1, X_2 \in \mathbb{O}_C} 1_{X_1} V_1 \otimes 1_{X_2} V_2$$

be their tensor product over $\Bbbk$ viewed as a left $C \boxtimes C$-module in the obvious way. In fact, this defines a functor $\boxtimes : C\text{-Mod} \boxtimes C\text{-Mod} \to C \boxtimes C\text{-Mod}$. The monoidal product on $\mathcal{C}$ is a $\Bbbk$-linear functor $\star : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$. Let

$$C1_\star = \bigoplus_{X_1, X_2 \in \mathbb{O}_C} C1_{X_1 \star X_2}$$

be the $(C, C \boxtimes C)$-bimodule obtained by restricting the right C-module C along this functor. Induction along $\star$, that is, the functor $\operatorname{ind}_\star = C1_\star \otimes_{C \boxtimes C} : C \boxtimes C\text{-Mod} \to C\text{-Mod}$ from (2.5), is left adjoint to the restriction functor $\operatorname{res}_\star$ from (2.1). Then the induction product is the composition

$$\circledast := \operatorname{ind}_\star \circ \boxtimes : C\text{-Mod} \boxtimes C\text{-Mod} \to C\text{-Mod}. \tag{2.16}$$

Thus, for $V_1, V_2 \in C$-Mod, we have that $V_1 \circledast V_2 = C1_\star \otimes_{C \boxtimes C} (V_1 \boxtimes V_2)$. Associativity of $\circledast$ (up to natural isomorphism) follows from "transitivity of induction", i.e., the associativity of tensor products of modules over locally unital algebras. We obviously have that

$$C1_{X_1} \circledast C1_{X_2} \cong C1_{X_1 \star X_2} \tag{2.17}$$

for $X_1, X_2 \in \mathbb{O}_C$. This justifies our earlier assertion that $\circledast$ extends the monoidal product $\star$ on $\mathrm{Kar}(\mathcal{C})$. It also follows that $V_1 \circledast V_2$ is finitely generated if both V_1 and V_2 are finitely generated.

The induction product $\circledast$ is right exact in both arguments, but in general it is not left exact. We denote the ith left derived functor of $\circledast$ on C-modules V, W by $\mathrm{Tor}_i^C(V, W)$. This can be computed from a projective resolution of either V or W.

Lemma 2.2. *If $C1_\star$ is a projective right $C \boxtimes C$-module then the induction product $\circledast$ is biexact.*

Proof. This follows from Lemma 2.1. □

Finally suppose that $\mathcal{C}$ is a strict $\Bbbk$-linear *symmetric* monoidal category, so that there is given a symmetric braiding $R : \star \overset{\sim}{\Rightarrow} \star^{\mathrm{rev}}$. From this, we obtain a braiding $\mathrm{ind}_R \circ \boxtimes : \circledast^{\mathrm{rev}} \overset{\sim}{\Rightarrow} \circledast$ making C-Mod into a $\Bbbk$-linear symmetric monoidal category too.

Remark 2.3. There is a second convolution product $\underline{\circledast}$ which we call the *coinduction product*. This is defined by replacing $\mathrm{ind}_\star$ with $\mathrm{coind}_\star$ in (2.16). It is easy to understand on injective rather than projective modules. It will not often be used subsequently, but note that the induction and coinduction products are interchanged by duality.

2.5. **Projective functors.** Suppose that $\mathcal{C}$ is a strict $\Bbbk$-linear monoidal category and $\mathcal{A}$ is a $\Bbbk$-linear category, denoting their path algebras by C and A as usual. We say that $\mathcal{A}$ is a *strict $\mathcal{C}$-module category* if there is a strictly associative and unital $\Bbbk$-linear monoidal functor $\star : \mathcal{C} \boxtimes \mathcal{A} \to \mathcal{A}$. Equivalently, this is the data of a strict $\Bbbk$-linear monoidal functor $\Psi : \mathcal{C} \to \mathcal{E}nd_\Bbbk(\mathcal{A})$. For $f \in \mathrm{Hom}_\mathcal{C}(X, X')$, we sometimes denote the evaluation of the natural transformation $\Psi(f)$ on $Y \in \mathbb{O}_A$ simply by $f_Y : X \star Y \to X' \star Y$.

The definition of the induction product $\star$ from (2.16) extends naturally to this setting, thereby defining a $\Bbbk$-linear functor

$$\circledast := \mathrm{ind}_\star \circ \boxtimes : C\text{-Mod} \boxtimes A\text{-Mod} \to A\text{-Mod} \tag{2.18}$$

which makes A-Mod into a (no longer strict) C-Mod-module category. For objects $X \in \mathbb{O}_C$ and $Y \in \mathbb{O}_A$, we have that

$$C1_X \circledast A1_Y \cong A1_{X \star Y}, \tag{2.19}$$

i.e., $\circledast$ extends $\star : \mathcal{C} \boxtimes \mathcal{A} \to \mathcal{A}$. Using $\circledast$ to define the action as in (2.15), the split Grothendieck group $K_0(A)$ becomes a left module over the split Grothendieck ring $K_0(C)$.

Now fix $X \in \mathbb{O}_C$ and consider the functor $X\star : \mathcal{A} \to \mathcal{A}$. There is an adjoint pair of endofunctors $(\mathrm{ind}_{X\star}, \mathrm{res}_{X\star})$ of A-Mod defined by induction and restriction along $X\star$:

$$\mathrm{ind}_{X\star} := A1_{X\star} \otimes_A \qquad \text{where} \qquad A1_{X\star} := \bigoplus_{Y \in \mathbb{O}_A} A1_{X \star Y}, \tag{2.20}$$

$$\mathrm{res}_{X\star} := 1_{X\star} A \otimes_A \qquad \text{where} \qquad 1_{X\star} A := \bigoplus_{Y \in \mathbb{O}_A} 1_{X \star Y} A. \tag{2.21}$$

The general properties discussed earlier give that $\mathrm{res}_{X\star}$ is exact, and $\mathrm{ind}_{X\star}$ is right exact and sends (finitely generated) projectives to (finitely generated) projectives. Thus, $\mathrm{ind}_{X\star}$ restricts to a well-defined functor $\mathrm{ind}_{X\star} : A\text{-Proj} \to A\text{-Proj}$. Note also that

$$\mathrm{ind}_{X\star}(A1_Y) \cong A1_{X \star Y} \tag{2.22}$$

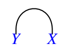
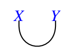
for all $Y \in \mathbb{O}_A$. One can also interpret $\mathrm{ind}_{X\star}$ as a special induction product, thanks to the following lemma.

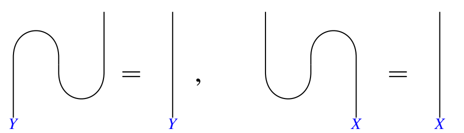
Lemma 2.4. *For any $X \in \mathbb{O}_C$, we have that* $\operatorname{ind}_{X\star} \cong C1_X\circledast$.

Proof. This follows from the chain of isomorphisms

$$\begin{aligned} A1_{X\star} \otimes_A V &\cong (A1_\star \otimes_{C\boxtimes A} (C1_X \boxtimes A)) \otimes_A V \\ &\cong A1_\star \otimes_{C\boxtimes A} ((C1_X \boxtimes A) \otimes_A V) \\ &\cong A1_\star \otimes_{C\boxtimes A} (C1_X \boxtimes V) = C1_X \circledast V \end{aligned}$$

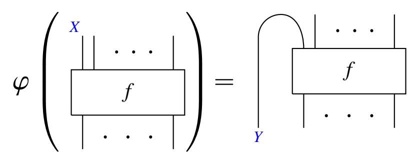
for $V \in A$-Mod. □

Let X, Y be objects of $\mathcal{C}$. Recall that Y is a *left dual* of X (equivalently, X is a *right dual* of Y) if there are evaluation and coevaluation morphisms $\mathrm{ev} : Y \star X \to \mathbb{1}$ and $\mathrm{coev} : \mathbb{1} \to X \star Y$ satisfying the zig-zag identities. In string diagrams, we denote ev and coev by the cap  and the cup , respectively, so that the zig-zag identities become



(2.23)

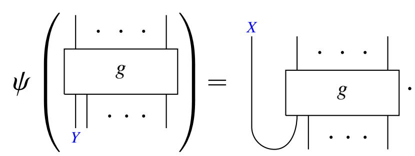
Lemma 2.5. *If X has a left dual Y in $\mathcal{C}$ then there is an isomorphism $\phi : 1_{X\star}A \xrightarrow{\sim} A1_{Y\star}$ of (A, A)-bimodules given explicitly by*



(2.24)

Hence, the functors $\operatorname{res}_{X\star}$ *and* $\operatorname{ind}_{Y\star}$ *are isomorphic.*

Proof. It is easily checked that ϕ is a bimodule homomorphism. It is an isomorphism because it has a two-sided inverse ψ defined by

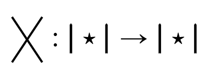


□

Corollary 2.6. *If X has a left dual Y in $\mathcal{C}$ then* $(\operatorname{ind}_{X\star}, \operatorname{ind}_{Y\star})$ *and* $(\operatorname{res}_{X\star}, \operatorname{res}_{Y\star})$ *are adjoint pairs of functors.*

From the corollary, we deduce that if X is *rigid*, i.e., it has both a left and a right dual, then both of the functors $\operatorname{ind}_{X\star}$ and $\operatorname{res}_{X\star}$ have both a right and a left adjoint. Moreover, as discussed earlier, both of these functors are exact and they preserve finitely generated projectives. We will refer to finite direct sums of direct summands of endofunctors of A-Mod of this sort as *projective functors*.

2.6. **The symmetric category.** For a basic example, we have the *symmetric category* $\mathcal{S}ym$, which is the free strict $\mathbb{k}$-linear symmetric monoidal category on one object. In string diagrams, we denote this generating object simply by $|$; then an arbitrary object is the monoidal product $|^{\star n}$ for some $n \geqslant 0$. Morphisms in $\mathcal{S}ym$ are generated by a single morphism depicted by the crossing

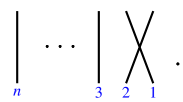


(2.25)

subject to the relations

(2.26)

Sometimes it is convenient to identify objects in $\mathcal{S}ym$ with natural numbers, so that the object set $\{|^{\star n} | n \in \mathbb{N}\}$ of $\mathcal{S}ym$ is identified with $\mathbb{N}$. For $m, n \geqslant 0$, the morphism space $\mathrm{Hom}_{\mathcal{S}ym}(n, m)$ is $\{0\}$ if $m \neq n$, while if $m = n$ it consists of $\mathbb{k}$-linear combinations of string diagrams representing permutations in the symmetric group S_n, i.e., we have that $\mathrm{End}_{\mathcal{S}ym}(n) = \mathbb{k}S_n$. Note our general convention here is to number strings by $1, \dots, n$ from *right to left*, so that the transposition $(1\ 2) \in S_n$ is represented by the string diagram



Let Sym be the path algebra of $\mathcal{S}ym$. Thus, we have that

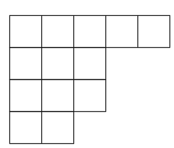
$$Sym = \bigoplus_{n \geqslant 0} \mathbb{k}S_n. \tag{2.27}$$

Since $\mathbb{k}$ is of characteristic zero, we deduce from Maschke's theorem that Sym is a *semisimple* locally unital algebra. In this case, the induction product $\circledast$ making Sym-Mod into a monoidal category is nothing more than the usual induction product on representations of the symmetric groups: we have that

$$V \circledast W = \mathrm{ind}_{S_n \times S_m}^{S_{n+m}} (V \boxtimes W)$$

for $V \in \mathbb{k}S_n$-Mod and $W \in \mathbb{k}S_m$-Mod. In fact, the induction product $\circledast$ and the coinduction product $\underline{\circledast}$ on Sym-Mod are isomorphic as $\mathrm{ind}_{S_n \times S_m}^{S_{n+m}} \cong \mathrm{coind}_{S_n \times S_m}^{S_{n+m}}$ (as always for finite groups).

Recall that the irreducible $\mathbb{k}S_n$-modules are the *Specht modules* $S(\lambda)$ parametrized by the set $\mathcal{P}_n$ of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of n. Hence, the irreducible Sym-modules are the Specht modules $S(\lambda)$ parametrized by all partitions $\lambda \in \mathcal{P} = \bigsqcup_{n \geqslant 0} \mathcal{P}_n$. We sometimes write $|\lambda|$ for the size $\lambda_1 + \lambda_2 + \cdots$ of a partition $\lambda \in \mathcal{P}$, and $\ell(\lambda)$ for its length, that is, the number of non-zero parts. We will often identify $\lambda \in \mathcal{P}$ with its Young diagram. For example, the partition $(5, 3^2, 2)$ is identified with



The Grothendieck ring $K_0(Sym)$ of the symmetric category is positively graded with degree n component being $K_0(\mathbb{k}S_n)$. It is well known that $K_0(Sym)$ is canonically isomorphic as a graded ring to the *ring of symmetric functions* $\Lambda = \bigoplus_{n \geqslant 0} \Lambda_n$, with the class $[S(\lambda)]$ of the Specht module corresponding under the isomorphism to the *Schur function* $s_\lambda \in \Lambda$. In Λ, we have that

$$s_\mu s_\nu = \sum_{\lambda \in \mathcal{P}} LR_{\mu,\nu}^{\lambda} s_\lambda \tag{2.28}$$

where $LR_{\mu,\nu}^{\lambda}$ is the Littlewood-Richardson coefficient. Since Sym is semisimple, this is equivalent to the existence of an isomorphism

$$S(\mu) \circledast S(\nu) \cong \bigoplus_{\lambda \in \mathcal{P}} S(\lambda)^{\oplus LR_{\mu,\nu}^{\lambda}} \tag{2.29}$$

at the level of modules. Later on, we will also need the "triple" Littlewood-Richardson coefficient

$$LR_{\lambda,\mu,\nu}^{\kappa} := \sum_{\gamma \in \mathcal{P}} LR_{\lambda,\mu}^{\gamma} LR_{\gamma,\nu}^{\kappa} = [S(\lambda) \circledast S(\mu) \circledast S(\nu) : S(\kappa)]. \tag{2.30}$$

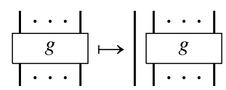
The *content* of the node in row i and column j of a Young diagram is the integer $c = j - i$. Let $\mathrm{add}(\lambda)$ be the set consisting of the contents of the *addable nodes* of λ, that is, the places in the Young diagram where a node can be added to the diagram to obtain a new Young diagram. Similarly, let $\mathrm{rem}(\lambda)$ be the set of contents of the *removable nodes* of λ, that is, the places in the Young diagram where a node can be removed from the diagram to obtain a new Young diagram. Note that all of the addable and removable nodes of a Young diagram are of different contents (another of the benefits of working in characteristic zero). For $a \in \mathrm{add}(\lambda)$, let $\lambda + \boxed{a}$ be the partition obtained by adding the unique addable node of content a to the diagram. For $b \in \mathrm{rem}(\lambda)$, let $\lambda - \boxed{b}$ be the partition obtained by removing the unique removable node of content b from the diagram.

The combinatorial notions just introduced arise naturally on considering *branching rules* for the symmetric group. In our setup, the sums over all $n \geqslant 0$ of the usual restriction and induction functors $\mathrm{res}^{S_{n+1}}_{S_n}$ and $\mathrm{ind}^{S_{n+1}}_{S_n} = \Bbbk S_{n+1} \otimes_{\Bbbk S_n}$ are isomorphic to the functors

$$F := \mathrm{res}_{|\star} : \mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}} \to \mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}}, \qquad E := \mathrm{ind}_{|\star} : \mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}} \to \mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}}, \tag{2.31}$$

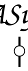
notation as in (2.20) and (2.21). This follows because the functor

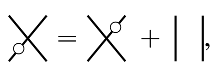
$$|\star : \mathcal{S}ym \to \mathcal{S}ym, \tag{2.32}$$

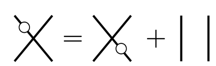


coincides with the natural inclusion $S_n \hookrightarrow S_{n+1}$ on permutations $g \in S_n \subset \mathrm{End}_{\mathcal{S}ym}(n)$. The canonical adjunction makes (E, F) into an adjoint pair of functors. In fact, these functors are are *biadjoint*, i.e., there is also an adjunction making (F, E) into an adjoint pair. The effect of the functors F and E on the Specht module $S(\lambda)$ is well known: we have that

$$FS(\lambda) \cong \bigoplus_{b \in \mathrm{rem}(\lambda)} S(\lambda - \boxed{b}), \qquad ES(\lambda) \cong \bigoplus_{a \in \mathrm{add}(\lambda)} S(\lambda + \boxed{a}). \tag{2.33}$$

We finally recall a bit about the Jucys-Murphy elements in $\mathcal{S}ym$. One natural way to obtain these is to start from the *affine symmetric category* $\mathcal{AS}ym$, which is the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{S}ym$ by adjoining an extra generator  subject to the equivalent relations



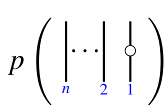


(2.34)

The path algebra $ASym$ is isomorphic to $\bigoplus_{n \geqslant 0} AH_n$ where AH_n is the nth *degenerate affine Hecke algebra*. There is an obvious faithful strict $\Bbbk$-linear monoidal functor $i : \mathcal{S}ym \to \mathcal{AS}ym$. There is also a unique (non-monoidal) full $\Bbbk$-linear functor

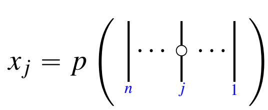
$$p : \mathcal{AS}ym \to \mathcal{S}ym \tag{2.35}$$

such that $p \circ i = \mathrm{Id}_{\mathcal{S}ym}$ and



$$= 0 \tag{2.36}$$

for all $n \geqslant 1$. For $1 \leqslant j \leqslant n$, the jth Jucys-Murphy element of the symmetric group S_n is



$$= \sum_{i=1}^{j-1} (i\ j) \in \Bbbk S_n, \tag{2.37}$$

i.e., it is the sum of the transpositions "ending" in j. Whenever we use this notation, it should be clear from context exactly which symmetric group we have in mind. Note $x_1 = 0$ always. We may also occasionally write x_0, which should be interpreted as zero by convention.

The Jucys-Murphy elements $x_1, \dots, x_n$ generate a commutative subalgebra of $\Bbbk S_n$ known as the *Gelfand-Tsetlin subalgebra*. As concisely explained by [OV], for $\lambda \in \mathcal{P}_n$, each Jucys-Murphy element acts diagonalizably on the Specht module $S(\lambda)$, and the *Gelfand-Tsetlin character* of $S(\lambda)$ recording the dimensions of the simultaneous generalized eigenspaces of $x_1, \dots, x_n$ may be obtained from the contents of standard λ-tableaux. Indeed, Young's orthonormal basis $\{v_{\mathrm{T}}\}$ for $S(\lambda)$ indexed by standard λ-tableaux T is a basis of simultaneous eigenvectors for $x_1, \dots, x_n$, with x_j acting on v_{T} as the content $\mathrm{cont}_j(\mathrm{T})$ of the node labelled by j in T. We will assume the reader is familiar with these ideas without giving any further explanation.

The functor p induces an isomorphism $\mathcal{ASym}/\mathcal{I} \xrightarrow{\sim} \mathcal{Sym}$ where $\mathcal{I}$ is the left tensor ideal[2] of $\mathcal{ASym}$ generated by the morphism $\mid\!\!\!\circ$. It follows that $\mathcal{Sym}$ is a strict $\mathcal{ASym}$-module category. The functors E and F from (2.31) are also the induction and restriction functors $\mathrm{ind}_{\mid\star}$ and $\mathrm{res}_{\mid\star}$ defined using this categorical action of $\mathcal{ASym}$ on $\mathcal{Sym}$. The advantage of passing from $\mathcal{Sym}$ to $\mathcal{ASym}$ here is that the object $\mid$ of $\mathcal{ASym}$ has the endomorphism defined by the dot, giving us a natural transformation

$$\alpha := \ \mid\!\!\!\circ\,\star\ : \mid\star \Rightarrow \mid\star .$$

Applying the general construction from (2.8) to this, we obtain endomorphisms

$$x := \mathrm{res}_\alpha : F \Rightarrow F, \qquad\qquad x^\vee := \mathrm{ind}_\alpha : E \Rightarrow E. \tag{2.38}$$

Explicitly, on a $\Bbbk S_n$-module V, x_V is the endomorphism of $FV = \mathrm{res}^{S_n}_{S_{n-1}} V$ defined by multiplying on the left by $x_n \in \Bbbk S_n$, while $x^\vee_V$ is the endomorphism of $EV = \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V$ defined by multiplying $\Bbbk S_{n+1}$ on the right by $x_{n+1} \in \Bbbk S_{n+1}$. For $c \in \Bbbk$, let F_c and E_c be the c eigenspaces of $x : F \Rightarrow F$ and $x^\vee : E \Rightarrow E$, respectively. Since $x^\vee$ is the mate of x and E and F are biadjoint, it follows that E_c and F_c are biadjoint endofunctors of $\mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}}$ for each $c \in \Bbbk$. The description of Gelfand-Tsetlin characters of Specht modules from the previous paragraph is equivalent to the assertion that the functors E_a and F_b take the Specht module $S(\lambda)$ to exactly the summands $S(\lambda + \boxed{a})$ and $S(\lambda - \boxed{b})$ in (2.33), or to zero if $a \notin \mathrm{add}(\lambda)$ or $b \notin \mathrm{rem}(\lambda)$, respectively. It follows that

$$F = \bigoplus_{b \in \mathbb{Z}} F_b, \qquad\qquad E = \bigoplus_{a \in \mathbb{Z}} E_a. \tag{2.39}$$

3. The partition category and its triangular decomposition

Next we introduce the partition category $\mathcal{Par}_t$, which we define by generators and relations. We then make some basic observations about its representation theory. Most of the results here are due to Sam and Snowden [SS2, Sec. 6], but we have tried to give a self-contained account since our general notation and other conventions are often different. The most important point is that the path algebra Par_t of the category $\mathcal{Par}_t$ has a triangular decomposition, hence, the category of locally finite-dimensional Par_t-modules is an upper finite highest weight category in the sense of [BS, §3.3]. In fact, $\mathcal{Par}_t$ is a monoidal triangular category in the sense of Sam and Snowden.

3.1. The partition category.

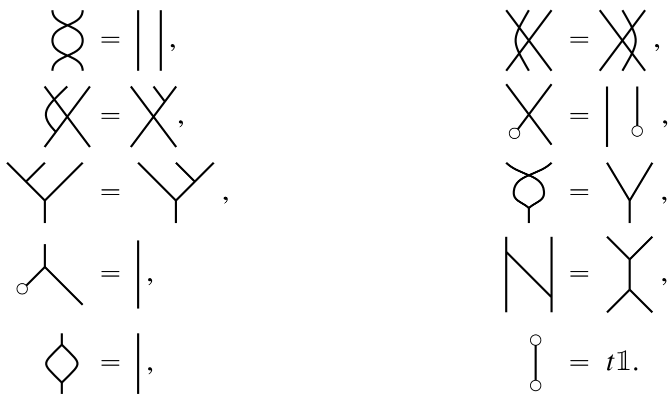
Let $t \in \Bbbk$ be a parameter. According to the following definition, the partition category $\mathcal{Par}_t$ is the free strict $\Bbbk$-linear symmetric monoidal category generated by a commutative Frobenius object which is special of categorical dimension t.

[2] A left tensor ideal $\mathcal{I}$ of a $\Bbbk$-linear monoidal category $\mathcal{C}$ is the data of subspaces $\mathcal{I}(X, Y) \leqslant \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathrm{ob}\,\mathcal{C}$, such that these subspaces are closed in the obvious sense under vertical composition either on top of bottom and under horizontal composition on the left with any morphism. Then $\mathcal{C}/\mathcal{I}$ is the $\mathcal{C}$-module category with the same objects as $\mathcal{C}$ and morphisms that are the quotient spaces $\mathrm{Hom}_{\mathcal{C}}(X, Y)/\mathcal{I}(X, Y)$.

Definition 3.1. The *partition category* $\mathcal{P}ar_t$ is the strict $\Bbbk$-linear monoidal category generated by one object $\mid$ and the morphisms

 $: \mid\star\mid \to \mid\star\mid$,  $: \mid\star\mid \to \mid$,  $: \mid \to \mid\star\mid$,  $: \mid \to \mathbb{1}$,  $: \mathbb{1} \to \mid$ (3.1)

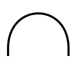
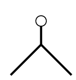
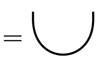
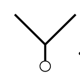
subject to the following relations, as well as the ones obtained from these by horizontal and vertical flips:



(3.2)
(3.3)
(3.4)
(3.5)
(3.6)

The object set of $\mathcal{P}ar_t$ is $\{\mid^{\star n} \mid n \in \mathbb{N}\}$. We will sometimes denote $\mid^{\star n}$ simply by n, so that the object set is identified with $\mathbb{N}$.

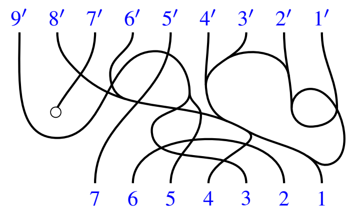
The relations (3.2) and (3.3) imply that $\mathcal{P}ar_t$ is a *symmetric* monoidal category, (3.4) and (3.5) imply that the generating object is a *commutative Frobenius object*, and the first relation from (3.6) means that this object is actually a *special* Frobenius object. The symmetric monoidal category $\mathcal{P}ar_t$ is *rigid* with every object being self-dual. To justify this, it is enough to specify the evaluation and coevaluation morphisms $\mathrm{ev} : \mid\star\mid \to \mathbb{1}$ and $\mathrm{coev} : \mathbb{1} \to \mid\star\mid$ for the generating object, which we represent graphically by the cap and cup:

ev =  := , coev =  := . (3.7)

These satisfy the zig-zag identities as in (2.23), as may easily be checked using (3.5). Now the relations in (3.6) imply that the *categorical dimension* of the generating object is t.

By an $m\times n$ *partition diagram*, we mean a string diagram f representing a morphism in $\mathrm{Hom}_{\mathcal{P}ar_t}(n, m)$ obtained by horizontally and vertically composing the generating morphisms (3.1), such that every connected component of f has at least one endpoint, i.e., is not a "floating bubble". In view of the dimension relation in (3.6), floating bubbles can be contracted then removed, multiplying the result by the scalar t each time this occurs. It follows that every morphism in $\mathrm{Hom}_{\mathcal{P}ar_t}(n, m)$ can be written as a $\Bbbk$-linear combination of $m \times n$ partition diagrams. Let $\sigma : \mathcal{P}ar_t \to (\mathcal{P}ar_t)^{\mathrm{op}}$ be the strict $\Bbbk$-linear monoidal functor that is the identity on objects and sends the generating morphisms to their flips in a horizontal axis. More generally, σ sends an $m \times n$ partition diagram to the $n \times m$ partition diagram that is its flip in a horizontal axis.

The above definition of $\mathcal{P}ar_t$ by generators and relations is not the most common definition found in the literature. It was first formulated in this way by Comes in [C, Th. 2.1]; see also [LSR, Prop. 2.1]. In the more traditional approach (e.g., see [D, §8] and [CO, Def. 2.11]), one instead defines the morphism space $\mathrm{Hom}_{\mathcal{P}ar_t}(n, m)$ to be the vector space with basis labelled by set partitions of $\{1, \dots, n, 1', \dots, m'\}$, giving explicit combinatorial rules for the horizontal and vertical compositions in terms of these partitions. Suppose that f is an $m \times n$ partition diagram. Labelling the endpoints of f from right to left by $1, \dots, n$ on the bottom bounadry and $1', \dots, m'$ on the top boundary as in the following example, the diagram f determines a partition of the set $\{1, \ldots, n, 1', \ldots, m'\}$ with parts arising from the labels at the endpoints of the connected components in the diagram. For example, the 9×7 partition diagram



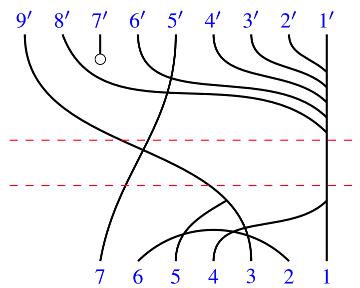
(3.8)

determines the partition

$$\{1, 4, 1', 2', 3', 4', 6', 8'\} \sqcup \{2, 6\} \sqcup \{3, 5, 9'\} \sqcup \{7, 5'\} \sqcup \{7'\}.$$

In this way, one obtains a strict $\Bbbk$-linear monoidal functor from the category $\mathcal{P}ar_t$ defined by generators and relations as above to the category $\mathcal{P}ar_t$ as defined via the more traditional combinatorial approach. Then the result of Comes just mentioned asserts that this functor is an isomorphism.

The discussion in the previous paragraph shows that two $m \times n$ partition diagrams represent the same morphism in $\mathrm{Hom}_{\mathcal{P}ar_t}(n, m)$ if and only if the diagrams are equivalent in the sense that they determine the same partition of the set $\{1, \ldots, n, 1', \ldots, m'\}$ labelling their endpoints. For example, the morphism represented by (3.8) is equal to the one represented by the tidier diagram

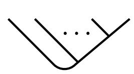
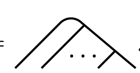


(3.9)

because this determines the same partition of the set labelling the endpoints. In fact, Comes' result implies that any set of representatives for the equivalence classes $m \times n$ partition diagrams give a *basis* for the morphism space $\mathrm{Hom}_{\mathcal{P}ar_t}(n, m)$. In particular, $\dim \mathrm{Hom}_{\mathcal{P}ar_t}(n, m)$ is equal to the the $(m + n)$th *Bell number* which counts set partitions of $m + n$. Taking $m = n = 0$, this implies that

$$\mathrm{End}_{\mathcal{P}ar_t}(\mathbb{1}) = \Bbbk. \tag{3.10}$$

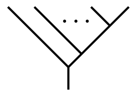
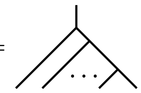
3.2. **Triangular decomposition.** Let c be a connected component in some partition diagram representing a morphism in $\mathcal{P}ar_t$. We call c an *upward branch* if c has at least two endpoints on its top boundary and no endpoints on its bottom boundary, and a *downward branch* if it has at least two endpoints on its bottom boundary but no endpoints at the top:

$c =$  or $c =$ .

We call c an *upward leaf* if it has exactly one endpoint at the top and no endpoints at the bottom, and a *downward leaf* if it has no endpoints at the top and exactly one at the bottom:

$c =$  or $c =$ .

We refer to c as an *upward tree* if it has more than one endpoint at the top and exactly one endpoint at the bottom, and a *downward tree* if it has exactly one endpoint at the top and more than one endpoint at the bottom:

$c =$  or $c =$ 

We say that c is a *double tree* if c has more than one endpoint at the top and more than one endpoint at the bottom. In that case, it is equivalent to the composition of an upward tree and a downward tree; for example, the rightmost connected component in (3.9) is a double tree. Finally we say that c is a *trunk* if c has exactly one endpoint both at the top and at the bottom:

$$c = \Big| \,.$$

Any connected component of a partition diagram can be represented either as an upward branch, an upward leaf, an upward tree, a downward branch, a downward leaf, a downward tree, a double tree, or a trunk.

Let f be an $m \times n$ partition diagram. We say f is

- a *permutation diagram* if all of its connected components are trunks, in which case we must have that $m = n$;
- an *upward partition diagram* if its connected components are trunks, upward branches, upward leaves and upward trees, in which case we must have that $m \geqslant n$;
- a *downward partition diagram* if its connected components are trunks, downward branches, downward leaves and downward trees, in which case we must have that $m \leqslant n$.

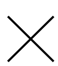
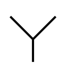
Let f be an upward $m \times n$ partition diagram. We say that it is *strictly upward* if $m > n$. Let $c_1, \dots, c_k$ be the connected components of f that are either trunks or upward trees, indexing them so that their bottom endpoints are in order from right to left in f. We say that f is *normally ordered* if the rightmost of the top endpoints of each of $c_1, \dots, c_k$ are also in order from right to left in f. In other words, f is normally ordered if it can be drawn so that the right edges of all of the upward trees and trunks in f are non-crossing. Similarly, we define *strictly downward* and *normally ordered* downward partition diagrams.

Now we can define some monoidal subcategories of $\mathcal{Par}_t$. Let $\mathcal{Sym}$ be the symmetric category as defined in §2.6. There is a strict $\Bbbk$-linear symmetric monoidal functor

$$i_t^\circ : \mathcal{Sym} \to \mathcal{Par}_t \tag{3.11}$$

sending the generating object and the generating morphism of $\mathcal{Sym}$ to the generating object and the generating morphism of $\mathcal{Par}_t$ that is represented by the crossing. Using the basis theorem for morphism spaces in $\mathcal{Par}_t$, it follows that this functor is faithful. We use it to *identify* $\mathcal{Sym}$ with a monoidal subcategory of $\mathcal{Par}_t$. In other words, $\mathcal{Sym}$ is identified with the subcategory of $\mathcal{Par}_t$ consisting of all objects and all the morphisms which can be written as linear combinations of permutation diagrams.

Next, let $\mathcal{Par}^\flat$ be the strict $\Bbbk$-linear monoidal category generated by one object $|$ and the morphisms

 $: | \star | \to | \star | ,$  $: | \to | \star | ,$  $: \mathbb{1} \to |$ (3.12)

subject to the relations (3.2) to (3.4) and their flips in a vertical axis. We call this the *upward partition category*. The cup can also be defined in $\mathcal{Par}^\flat$ as in (3.7). Any upward partition diagram can be interpreted as a string diagram representing a morphism in $\mathcal{Par}^\flat$. Moreover, the defining relations in $\mathcal{Par}^\flat$ imply that two upward $m \times n$ partition diagrams which are equivalent in the sense that they define the same partition of the set $\{1, \dots, n, 1', \dots, m'\}$ labelling the endpoints are also equal as morphisms in $\mathrm{Hom}_{\mathcal{Par}^\flat}(n, m)$. There is a strict $\Bbbk$-linear monoidal functor

$$i_t^\flat : \mathcal{Par}^\flat \to \mathcal{Par}_t \tag{3.13}$$

sending the generating morphisms of $\mathcal{P}ar^\flat$ to the corresponding ones in $\mathcal{P}ar_t$. As equivalence classes of upward $m \times n$ partition diagrams span $\mathrm{Hom}_{\mathcal{P}ar^\flat}(n,m)$ and their images in $\mathrm{Hom}_{\mathcal{P}ar_t}(n,m)$ are linearly independent, this functor is faithful. We use it to *identify* $\mathcal{P}ar^\flat$ with a monoidal subcategory of $\mathcal{P}ar_t$. In other words, $\mathcal{P}ar^\flat$ is identified with the monoidal subcategory of $\mathcal{P}ar_t$ consisting of all objects and all of the morphisms which can be written as linear combinations of upward partition diagrams. Also let $\mathcal{P}ar^-$ be the monoidal subcategory of $\mathcal{P}ar^\flat$ consisting of all objects and all of the morphisms which can be written as linear combinations of normally ordered upward partition diagrams.

Similarly to the previous paragraph, we define $\mathcal{P}ar^\sharp$, the *downward partition category*, to be the strict $\Bbbk$-linear monoidal category generated by one object $|$ and the morphisms that are the flips of (3.12) in a horizontal axis, subject to the relations that are the flips of the ones for $\mathcal{P}ar^\flat$. The cap can also be defined in $\mathcal{P}ar^\sharp$ as in (3.7). Evidently, $\mathcal{P}ar^\sharp \cong (\mathcal{P}ar^\flat)^{\mathrm{op}}$ with isomorphism being defined by the flip σ in a horizontal axis. There is a strict $\Bbbk$-linear monoidal functor

$$i_t^\sharp : \mathcal{P}ar^\sharp \to \mathcal{P}ar_t \tag{3.14}$$

sending the generating morphisms of $\mathcal{P}ar^\sharp$ to the corresponding ones in $\mathcal{P}ar_t$. We have that $i_t^\sharp = \sigma \circ i_t^\flat \circ \sigma$, so we deduce from the previous paragraph that $i_t^\sharp$ is faithful too. We use it to *identify* $\mathcal{P}ar^\sharp$ with a monoidal subcategory of $\mathcal{P}ar_t$. In other words, $\mathcal{P}ar^\sharp$ is identified with the monoidal subcategory of $\mathcal{P}ar_t$ consisting of all objects and all of the morphisms which can be written as linear combinations of downward partition diagrams. Also let $\mathcal{P}ar^+$ be the monoidal subcategory of $\mathcal{P}ar^\sharp$ consisting of all objects and all of the morphisms which can be written as linear combinations of normally ordered downward partition diagrams.

Finally we let Par_t be the path algebra of $\mathcal{P}ar_t$. It is a locally unital algebra with distinguished idempotents $\{1_n \mid n \in \mathbb{N}\}$ arising from the identity endomorphisms of the objects of $\mathcal{P}ar_t$. We also have the path algebras $Par^\flat, Par^-, Sym, Par^+, Par^\sharp$ of $\mathcal{P}ar^\flat, \mathcal{P}ar^-, \mathcal{S}ym, \mathcal{P}ar^+, \mathcal{P}ar^\sharp$, which we may view as locally unital subalgebras of Par_t via the embeddings (3.11), (3.13) and (3.14). The following theorem is the *triangular decomposition* of Par_t.

Theorem 3.2. *Let $\mathbb{K} := \bigoplus_{n\geqslant 0} \Bbbk 1_n$ viewed as a locally unital subalgebra of Par_t. Multiplication defines a linear isomorphism*

$$Par^- \otimes_{\mathbb{K}} Sym \otimes_{\mathbb{K}} Par^+ \xrightarrow{\sim} Par_t. \tag{3.15}$$

Hence, we also have isomorphisms

$$Par^- \otimes_{\mathbb{K}} Sym \xrightarrow{\sim} Par^\flat, \tag{3.16}$$

$$Sym \otimes_{\mathbb{K}} Par^+ \xrightarrow{\sim} Par^\sharp, \tag{3.17}$$

$$Par^\flat \otimes_{Sym} Par^\sharp \xrightarrow{\sim} Par_t. \tag{3.18}$$

Proof. Any partition diagram is equivalent to a diagram that is the composition of a normally ordered upward partition diagram, a permutation diagram, and a normally ordered downward partition diagram; see (3.9) for an example of such a decomposition. Moreover, equivalence classes of these sorts of diagrams give bases for Par_t, Par^-, Sym and Par^+. This implies that (3.15) is an isomorphism. Then (3.16) to (3.18) follow as in [BS, Rem. 5.32]. □

Theorem 3.2 is all that is needed to see that the locally finite-dimensional locally unital algebra

$$Par_t = \bigoplus_{m,n\in\mathbb{N}} 1_m Par_t 1_n$$

has a *split triangular decomposition* in the sense of [BS, Rem. 5.32]. Its negative and positive Borel subalgebras are $Par^\flat$ and $Par^\sharp$, and its Cartan subalgebra is $Sym = Par^\flat \cap Par^\sharp$. The set I in the notation of [BS] is the set $\mathbb{N}$ indexing the distinguished idempotents $\{1_n \mid n \in \mathbb{N}\}$. The upper finite poset $(\Lambda, \leqslant)$ in the setup of [BS] is $(\mathbb{N}, \geqslant)$; we stress that the ordering is reversed here, as it has to be in order to have an upper finite poset, thereby conforming to the general conventions of [BS]. The function ∂ from [BS] is the identity function.

As *Sym* is semisimple, this discussion shows equivalently that $\mathcal{P}ar_t$ is a triangular category in the sense of [SS2, Def. 4.1]. Its upward and downward subcategories $\mathcal{U}$ and $\mathcal{D}$ in the setup of *loc. cit.* are $\mathcal{P}ar^\flat$ and $\mathcal{P}ar^\sharp$, respectively. In fact, $\mathcal{P}ar_t$ is a *monoidal triangular category* as defined in [SS2, §4.11], as was established already by Sam and Snowden in [SS2, Prop. 6.3]. This means that induction commutes with induction product: we have that

$$\mathrm{ind}_{Sym}^{Par_t}(V \circledast W) \cong (\mathrm{ind}_{Sym}^{Par_t} V) \circledast (\mathrm{ind}_{Sym}^{Par_t} W), \tag{3.19}$$

for *Sym*-modules V and W. This is easily seen directly from the definition of the induction product using $i_t^\circ \circ \star \cong \star \circ (i_t^\circ \boxtimes i_t^\circ)$. Similarly,

$$\mathrm{ind}_{Sym}^{Par^\sharp}(V \circledast W) \cong (\mathrm{ind}_{Sym}^{Par^\sharp} V) \circledast (\mathrm{ind}_{Sym}^{Par^\sharp} W), \qquad \mathrm{ind}_{Par^\sharp}^{Par_t}(V \circledast W) \cong (\mathrm{ind}_{Par^\sharp}^{Par_t} V) \circledast (\mathrm{ind}_{Par^\sharp}^{Par_t} W). \tag{3.20}$$

3.3. **Classification of irreducible modules and highest weight structure.** As Par_t has a triangular decomposition with Cartan subalgebra *Sym* being semisimple, we can appeal to the general results of [BS, §5.5] to obtain the classification of irreducible Par_t-modules. Alternatively, this follows from the results in [SS2, §5.5], but note that Sam and Snowden use the language of *lowest weight* rather than *highest weight* categories. Since isomorphism classes of irreducible Par_t-modules are in bijection with isomorphism classes of indecomposable projective Par_t-modules, and the latter are identified with isomorphism classes of indecomposable objects in $\mathrm{Kar}(\mathcal{P}ar_t)$, the results discussed in this subsection are equivalent to the classification obtained originally in [CO, Th. 3.7].

The algebra Par_t is $\mathbb{Z}$-graded with $1_m Par_t 1_n$ being in degree $m - n$. The induced gradings on the subalgebras $Par^\flat$ and $Par^\sharp$ make these into positively and negatively graded algebras, respectively, with degree zero components in both cases being the semisimple algebra *Sym*. It follows that the Jacobson radicals of $Par^\flat$ and $Par^\sharp$ are the direct sums of their non-zero graded components. Moreover, the quotients by their Jacobson radicals are naturally identified with *Sym*, i.e., there are locally unital algebra homomorphisms

$$\pi^\flat : Par^\flat \twoheadrightarrow Sym, \qquad \pi^\sharp : Par^\sharp \twoheadrightarrow Sym. \tag{3.21}$$

Let $\mathrm{infl}^\sharp : Sym\text{-}\mathrm{Mod}_{\mathrm{fd}} \to Par^\sharp\text{-}\mathrm{Mod}_{\mathrm{fd}}$ and $\mathrm{infl}^\flat : Sym\text{-}\mathrm{Mod}_{\mathrm{fd}} \to Par^\flat\text{-}\mathrm{Mod}_{\mathrm{fd}}$ be the functors defined by restriction along these homomorphisms. The modules

$$\left\{S^\flat(\lambda) := \mathrm{infl}^\flat S(\lambda) \,\middle|\, \lambda \in \mathcal{P}\right\}, \qquad \left\{S^\sharp(\lambda) := \mathrm{infl}^\sharp S(\lambda) \,\middle|\, \lambda \in \mathcal{P}\right\} \tag{3.22}$$

give full sets of pairwise inequivalent irreducible modules for $Par^\flat$ and $Par^\sharp$, respectively.

As in [BS, (5.13)–(5.14)], we define the *standardization* and *costandardization functors*

$$j_! := \mathrm{ind}_{Par^\sharp}^{Par_t} \circ \mathrm{infl}^\sharp : Sym\text{-}\mathrm{Mod}_{\mathrm{fd}} \to Par_t\text{-}\mathrm{Mod}_{\mathrm{lfd}}, \tag{3.23}$$

$$j_* := \mathrm{coind}_{Par^\flat}^{Par_t} \circ \mathrm{infl}^\flat : Sym\text{-}\mathrm{Mod}_{\mathrm{fd}} \to Par_t\text{-}\mathrm{Mod}_{\mathrm{lfd}}, \tag{3.24}$$

where $\mathrm{ind}_{Par^\sharp}^{Par_t} := Par_t \otimes_{Par^\sharp}$ and $\mathrm{coind}_{Par^\flat}^{Par_t} := \bigoplus_{n \in \mathbb{N}} \mathrm{Hom}_{Par^\flat}(Par_t 1_n, ?)$. From (3.15) to (3.17) it follows that Par_t is projective both as a right $Par^\sharp$-module and as a left $Par^\flat$-module, hence, these functors are exact. Then we define the *standard* and *costandard modules* for Par_t by

$$\Delta(\lambda) := j_! S(\lambda) = \mathrm{ind}_{Par^\sharp}^{Par_t} S^\sharp(\lambda), \qquad \nabla(\lambda) = j_* S(\lambda) = \mathrm{ind}_{Par^\flat}^{Par_t} S^\flat(\lambda), \tag{3.25}$$

respectively.

Theorem 3.3. *The Par_t-modules $\{L(\lambda) \mid \lambda \in \mathcal{P}\}$ defined from*

$$L(\lambda) := \operatorname{hd}\Delta(\lambda) \cong \operatorname{soc}\nabla(\lambda)$$

give a complete set of pairwise inequivalent irreducible left Par_t-modules. Moreover, Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$ is an upper finite highest weight category in the sense of [BS, Def. 3.34] *with weight poset $(\mathcal{P}, \preceq)$, where $\preceq$ is the partial order on $\mathcal{P}$ defined by $\lambda \preceq \mu$ if and only if either $\lambda = \mu$ or $|\lambda| > |\mu|$. Its standard and costandard objects are the modules $\Delta(\lambda)$ and $\nabla(\lambda)$, respectively.*

Proof. This follows immediately from [BS, Cor. 5.39] using the triangular decomposition from Theorem 3.2 and the semisimplicity of *Sym*; see also [SS2, §5.5]. □

The fact established in Theorem 3.3 that Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$ is an upper finite highest weight category has several significant consequences. As for any Schurian category, $L(\lambda)$ has a projective cover we denote by $P(\lambda) \in Par_t$-$\mathrm{Mod}_{\mathrm{lfd}}$. Let Par_t-Mod_Δ be the exact subcategory of Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$ consisting of all modules with a Δ-flag, that is, a finite filtration whose sections are of the form $\Delta(\lambda)$ for $\lambda \in \mathcal{P}$. For any $V \in Par_t$-Mod_Δ, the multiplicity $(V : \Delta(\mu))$ of $\Delta(\mu)$ as a section of some Δ-flag in V is well-defined independent of the flag, indeed, it can be calculated from

$$(V : \Delta(\mu)) = \dim \operatorname{Hom}_{Par_t}(V, \nabla(\mu)). \tag{3.26}$$

This follows from the fundamental Ext-vanishing property of highest weight categories, namely, that

$$\dim \operatorname{Ext}^i_{Par_t}(\Delta(\lambda), \nabla(\mu)) = \delta_{i,0}\delta_{\lambda,\mu} \tag{3.27}$$

for any $\lambda, \mu \in \mathcal{P}$ and $i \geqslant 0$; see [BS, Lem. 3.48]. The definition of highest weight category gives that $P(\lambda)$ has a Δ-flag, so that Par_t-Proj is a full subcategory of Par_t-Mod_Δ. Moreover, from (3.26), one obtains the usual *BGG reciprocity formula*

$$(P(\lambda) : \Delta(\mu)) = [\nabla(\mu) : L(\lambda)]. \tag{3.28}$$

The functor $\sigma : \mathcal{P}ar_t \xrightarrow{\sim} (\mathcal{P}ar_t)^{\mathrm{op}}$ defined by flipping diagrams in a horizontal axis can also be viewed as a locally unital anti-involution of the algebra Par_t. It interchanges the subalgebras $Par^\flat$ and $Par^\sharp$, and restricts to an anti-involution also denoted σ on the subalgebra *Sym*. Let $?^{\circledast}$ be the duality on *Sym*-$\mathrm{Mod}_{\mathrm{fd}}$ taking a finite-dimensional left *Sym*-module to its linear dual viewed again as a left module using the anti-automorphism σ. Since $\sigma(g) = g^{-1}$ for a permutation $g \in S_n \subset Sym$, this is the usual duality on each of the subcategories $\Bbbk S_n$-$\mathrm{Mod}_{\mathrm{fd}}$. It is well known that the irreducible $\Bbbk S_n$-modules are self-dual, hence,

$$S(\lambda)^{\circledast} \cong S(\lambda) \tag{3.29}$$

for all $\lambda \in \mathcal{P}$. There is also a duality $?^{\circledast}$ on Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$ defined as in (2.12). Similarly, as σ interchanges $Par^\flat$ and $Par^\sharp$, we get contravariant equivalences also denoted $?^{\circledast}$ between $Par^\sharp$-$\mathrm{Mod}_{\mathrm{lfd}}$ and $Par^\flat$-$\mathrm{Mod}_{\mathrm{lfd}}$. Similarly to (2.13) and (2.14), we have that

$$?^{\circledast} \circ \operatorname{infl}^\flat \cong \operatorname{infl}^\sharp \circ ?^{\circledast}, \qquad \operatorname{ind}^{Par_t}_{Par^\sharp} \circ ?^{\circledast} \cong ?^{\circledast} \circ \operatorname{coind}^{Par_t}_{Par^\flat}. \tag{3.30}$$

Hence:

$$j_! \circ ?^{\circledast} \cong ?^{\circledast} \circ j_*, \qquad j_* \circ ?^{\circledast} \cong ?^{\circledast} \circ j_! \tag{3.31}$$

as functors from *Sym*-$\mathrm{Mod}_{\mathrm{fd}}$ to Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$. Then from (3.29) and (3.31), we deduce that

$$\Delta(\lambda)^{\circledast} \cong \nabla(\lambda), \qquad \nabla(\lambda)^{\circledast} \cong \Delta(\lambda), \qquad L(\lambda)^{\circledast} \cong L(\lambda) \tag{3.32}$$

for $\lambda \in \mathcal{P}$.

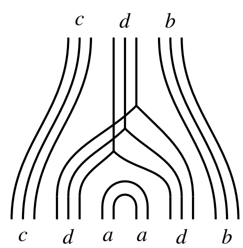
Remark 3.4. In fact, the duality $?^{\circledcirc}$ is a *Chevalley duality* of Par_t-Mod$_{\mathrm{lfd}}$ in the sense of [BS, Def. 4.49]. The general construction from [BS, Cor. 5.36, Rem. 5.40] can be used to show that Par_t admits a basis making it into an upper finite symmetrically based quasi-hereditary algebra in the sense of [BS, Def. 5.1]. Equivalently, $\mathcal{P}ar_t$ is an object-adapted cellular category in the sense of [EL, Def. 2.1].

3.4. **The downward partition category and reduced Kronecker coefficients.** The results in this subsection are the analogs for the downward partition category of the results proved in [SS1, §§7.5–7.6] for the downward Brauer and downward walled Brauer categories. The methods are similar. In the first lemma, our standing assumption that char $\Bbbk = 0$ is essential in order to define the idempotent $1^d_{l,m,n}$.

Lemma 3.5 (cf. [SS2, Prop. 6.5])**.** *The right $Par^\sharp \boxtimes Par^\sharp$-module $Par^\sharp 1_\star$ is projective. Consequently, by Lemma 2.2, the induction product $\circledast : Par^\sharp$-Mod $\boxtimes$ $Par^\sharp$-Mod $\to Par^\sharp$-Mod is biexact. More precisely, for integers $l, m, n \geqslant 0$ and $0 \leqslant d \leqslant \min(l+m-n, l+n-m, m+n-l)$ with $d \equiv l+m+n \pmod 2$, let*

$$a := (m+n-l-d)/2, \qquad b := (l+n-m-d)/2, \qquad c := (l+m-n-d)/2 \tag{3.33}$$

and define

$$f^d_{l,m,n} :=$$


$$\in 1_l Par^\sharp 1_{m\star n} \tag{3.34}$$

so that there are a nested caps at the bottom, b parallel trunks on the right, c parallel trunks on the left, and d nested downward binary trees in the middle. Also let $1^d_{l,m,n}$ be the image of the idempotent $\frac{1}{a!}\sum_{w\in S_a} w$ under the embedding $\Bbbk S_a \hookrightarrow \Bbbk(S_m \times S_n) < 1_m Par^\sharp \boxtimes 1_n Par^\sharp$ sending $w \in S_a$ to the diagram representing the permutation of $m+n, \dots, 1+n, n, \dots, 1$ defined by $1+n-i \mapsto 1+n-w(i)$, $i+n \mapsto w(i)+n$ for $i = 1, \dots, a$, and fixing all other points (i.e., it arises from a permutation of the a nested caps in the above picture). Finally, let $S_l/S_c \times S_d \times S_b$ be a set of coset representatives viewed as a subset of Sym via the usual embedding (in particular, S_c and S_b are permuting the leftmost c and rightmost b strings, respectively). Then there is a right $Par^\sharp \boxtimes Par^\sharp$-module isomorphism

$$\bigoplus_{l,m,n\geqslant 0}\ \bigoplus_{\substack{d=0\\ d\equiv l+m+n\ (2)}}^{\min(l+m-n,l+n-m,m+n-l)}\ \bigoplus_{g\in S_l/S_c\times S_d\times S_b} 1^d_{l,m,n}(Par^\sharp \boxtimes Par^\sharp) \xrightarrow{\sim} Par^\sharp 1_\star$$

taking the idempotent $1^d_{l,m,n}$ in the g-th summand on the left hand side to $g \circ f^d_{l,m,n}$.

Proof. This follows on considering the bases for $1_l Par^\sharp$ and $1_m Par^\sharp \boxtimes 1_n Par^\sharp$ given by equivalence classes of downward partition diagrams. □

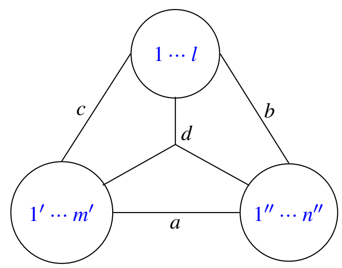
Recall for $\lambda, \mu, \nu \in \mathcal{P}_n$ that the *Kronecker coefficients* are defined to be the structure constants for the internal Kronecker product on representations of the symmetric group S_n:

$$G^\lambda_{\mu,\nu} = [S(\mu) \otimes S(\nu) : S(\lambda)] = \dim\left(S(\lambda) \otimes S(\mu) \otimes S(\nu)\right)^{S_n}. \tag{3.35}$$

Obviously from the second equality, $G^\lambda_{\mu,\nu}$ is invariant under permuting the partitions λ, μ, ν. For a partition λ and $n \geqslant |\lambda| + \lambda_1$, let $\lambda(n)$ denote $(n - |\lambda|, \lambda_1, \lambda_2, \dots) \in \mathcal{P}_n$. By a classical result of Murnaghan, for $\lambda, \mu, \nu \in \mathcal{P}$, the value of the Kronecker coeffcient $G^{\lambda(n),}_{\mu(n),\nu(n)}$ stabilizes as $n \to \infty$; see [BOR] for more background. The stable value is the *reduced Kronecker coefficient*, denoted $\overline{G}^{\lambda}_{\mu,\nu}$. Like the Kronecker coefficients themselves, $\overline{G}^{\lambda}_{\mu,\nu}$ is invariant under permutations of λ,μ and ν.

Example 3.6. For any $\lambda,\mu \in \mathcal{P}$, the reduced Kronecker coefficient $\overline{G}^{\mu}_{\square,\lambda}$ is equal to zero unless $\mu = \lambda + \boxed{a}$ for some $a \in \mathrm{add}(\lambda)$, $\mu = \lambda - \boxed{b}$ for some $b \in \mathrm{rem}(\lambda)$, or $\mu = (\lambda - \boxed{b}) + \boxed{a}$ for some $b \in \mathrm{rem}(\lambda)$ and $a = \mathrm{add}(\lambda - \boxed{b})$. In these cases, $\overline{G}^{\mu}_{\square,\lambda}$ is 1 if $\mu \neq \lambda$ and $|\mathrm{rem}(\lambda)|$ if $\mu = \lambda$. To see this using the definition just given, consider the natural n-dimensional permutation module for the symmetric group $U_n \cong S((n-1,1)) \oplus S((n))$ and note that the functor $U_n \otimes$ is isomorphic to $\mathrm{ind}^{S_n}_{S_{n-1}} \circ \mathrm{res}^{S_n}_{S_{n-1}}$.

Lemma 3.7. *Take $l,m,n \geqslant 0$ and $0 \leqslant d \leqslant \min(l+m-n, l+n-m, m+n-l)$ with $d \equiv l+m+n \pmod 2$, and define a,b,c according to (3.33). Let $\Omega^d_{l,m,n}$ be the set of partitions of $\{1,\dots,l\} \sqcup \{1',\dots,m'\} \sqcup \{1'',\dots,n''\}$ such that exactly a of the parts are subsets of the form $\{j',k''\}$, exactly b of the parts are subsets of the form $\{i,k''\}$, exactly c of the parts are subsets of the form $\{i,j'\}$, and the remaining d parts are subsets of the form $\{i,j',k''\}$ for $i \in \{1,\dots,l\}$, $j' \in \{1',\dots,m'\}$ and $k'' \in \{1'',\dots,n''\}$ as suggested by the picture:*



The group $S_l \times S_m \times S_n$ acts on the left on $\Omega^d_{l,m,n}$ so that S_l permutes $\{1,\dots,l\}$, S_m permutes $\{1',\dots,m'\}$ and S_n permutes $\{1'',\dots,n''\}$. Let $\Bbbk\Omega^d_{l,m,n}$ be the linearization, which is a $\Bbbk(S_l \times S_m \times S_n)$-module. For $\lambda \in \mathcal{P}_l, \mu \in \mathcal{P}_m, \nu \in \mathcal{P}_n$ we have that

$$[\Bbbk\Omega^d_{l,m,n} : S(\lambda) \boxtimes S(\mu) \boxtimes S(\nu)] = \sum_{\substack{\alpha\in\mathcal{P}_a,\beta\in\mathcal{P}_b,\gamma\in\mathcal{P}_c\\ \delta,\delta',\delta''\in\mathcal{P}_d}} LR^{\lambda}_{\beta,\gamma,\delta} LR^{\mu}_{\alpha,\gamma,\delta'} LR^{\nu}_{\alpha,\beta,\delta''} G^{\delta}_{\delta',\delta''}.$$

Proof. Let $G := S_l \times S_m \times S_n$, P be the parabolic subgroup $(S_b \times S_d \times S_c) \times (S_c \times S_d \times S_a) \times (S_a \times S_d \times S_b) \leqslant G$ and L be the subgroup $S_a \times S_b \times S_c \times S_d \leqslant P$ embedded diagonally via the map

$$(x,y,z,w) \mapsto (y,w,z;z,w,x;x,w,y).$$

The action of G on $\Omega^d_{l,m,n}$ is transitive and the subgroup L is a point stabilizer. Hence, the $\Bbbk G$-module $\Bbbk\Omega^d_{l,m,n}$ is induced from triv_L, the trivial $\Bbbk L$-module. By Frobenius reciprocity, it follows that

$$[\Bbbk\Omega^d_{l,m,n} : S(\lambda) \boxtimes S(\mu) \boxtimes S(\nu)] = \dim(S(\lambda) \boxtimes S(\mu) \boxtimes S(\nu))^L.$$

The restriction of $S(\lambda) \boxtimes S(\mu) \boxtimes S(\nu)$ to the parabolic subgroup P is isomorphic to

$$\bigoplus_{\substack{\alpha,\alpha',\beta,\beta',\gamma,\gamma'\\ \delta,\delta',\delta''\in\mathcal{P}_d}} \Big(\big(S(\beta) \boxtimes S(\delta) \boxtimes S(\gamma')\big) \boxtimes \big(S(\gamma) \boxtimes S(\delta') \boxtimes S(\alpha')\big) \boxtimes \big(S(\alpha) \boxtimes S(\delta'') \boxtimes S(\beta')\big)\Big)^{\oplus N}.$$

where $N := LR^{\lambda}_{\beta,\delta,\gamma'} LR^{\mu}_{\gamma,\delta',\alpha'} LR^{\nu}_{\alpha,\delta''',\beta'}$. By Schur's lemma, this contributes to the L-fixed points only from the summands with $\alpha = \alpha', \beta = \beta'$ and $\gamma = \gamma'$, and for each of those summands the contribution is $G^{\delta}_{\delta',\delta''}$ by (3.35). $\square$

Since $Par^\sharp$ is negatively graded, any $Par^\sharp$-module V has the *degree filtration* $0 = V_{-1} < V_0 < \cdots < V_n < \cdots$ defined from $V_n := \bigoplus_{m \leqslant n} 1_m V$. The section V_n/V_{n-1} in this filtration is isomorphic to $\mathrm{infl}^\sharp(1_n V)$, viewing $1_n V$ as a *Sym*-module by restriction. It follows that

$$[V : S^\sharp(\lambda)] = [1_n V : S(\lambda)] \tag{3.36}$$

for $\lambda \in \mathcal{P}_n$.

Theorem 3.8. *For $\lambda \in \mathcal{P}_l, \mu \in \mathcal{P}_m$ and $\nu \in \mathcal{P}_n$, we have that $[S^\sharp(\mu) \circledast S^\sharp(\nu) : S^\sharp(\lambda)] = \overline{G}^\lambda_{\mu,\nu}$.*

Proof. Call d *admissible* if $0 \leqslant d \leqslant \min(l+m-n, l+n-m, m+n-l)$ and $d \equiv m+n+n \pmod 2$. For such a d, let $M^d_{l,m,n}$ be the sub-bimodule of the $(\Bbbk S_l, \Bbbk S_m \boxtimes \Bbbk S_n)$-bimodule

$$M_{l,m,n} := 1_l \left(\left(\mathrm{infl}^\sharp \Bbbk S_m \right) \circledast \left(\mathrm{infl}^\sharp \Bbbk S_n \right) \right) = 1_l Par^\sharp 1_\star \otimes_{Par^\sharp \boxtimes Par^\sharp} \left(\left(\mathrm{infl}^\sharp \Bbbk S_m \right) \boxtimes \left(\mathrm{infl}^\sharp \Bbbk S_n \right) \right)$$

generated by the vector $f^d_{l,m,n} \otimes (1 \otimes 1)$, where $f^d_{l,m,n}$ is as in (3.34). Lemma 3.5 implies that

$$M_{l,m,n} = \bigoplus_{\text{admissible } d} M^d_{l,m,n}.$$

Note also that $1_l \left(S^\sharp(\mu) \circledast S^\sharp(\nu) \right) \cong M_{l,m,n} \otimes_{\Bbbk S_m \boxtimes \Bbbk S_n} (S(\mu) \boxtimes S(\nu))$ as $\Bbbk S_l$-modules. In view of this and (3.36), the number we are trying to compute is equal to

$$\left[1_l \left(S^\sharp(\mu) \circledast S^\sharp(\nu) \right) : S(\lambda) \right] = \sum_{\text{admissible } d} \left[M^d_{l,m,n} \otimes_{\Bbbk S_m \boxtimes \Bbbk S_n} (S(\mu) \boxtimes S(\nu)) : S(\lambda) \right].$$

Let $\widetilde{M}^d_{l,m,n}$ be the left $\Bbbk S_l \boxtimes \Bbbk S_m \boxtimes \Bbbk S_n$-module obtained from the $(\Bbbk S_l, \Bbbk S_m \boxtimes \Bbbk S_n)$-bimodule $M^d_{l,m,n}$ by twisting the right actions of S_m and S_n into left actions using $\sigma : g \mapsto g^{-1}$. By the self-duality of Specht modules, we have that

$$\left[M^d_{l,m,n} \otimes_{\Bbbk S_m \boxtimes \Bbbk S_n} (S(\mu) \boxtimes S(\nu)) : S(\lambda) \right] = \left[\widetilde{M}^d_{l,m,n} : S(\lambda) \boxtimes S(\mu) \boxtimes S(\nu) \right].$$

Now we claim that $\widetilde{M}^d_{l,m,n}$ is isomorphic to the module $\Bbbk \Omega^d_{l,m,n}$ from Lemma 3.7. To see this, recall from the proof of that lemma that $\Bbbk \Omega^d_{l,m,n}$ is the permutation module induced from the trivial representation of the subgroup $L = S_a \times S_b \times S_c \times S_d < S_l \times S_m \times S_n$. It is easy to see from (3.34) that L acts trivially on the generating vector $f^d_{l,m,n} \otimes (1 \otimes 1) \in \widetilde{M}^d_{l,m,n}$. Hence, there is a surjective homomorphism $\Bbbk \Omega^d_{l,m,n} \twoheadrightarrow \widetilde{M}^d_{l,m,n}$. It is an isomorphism because both of these modules are of dimension $(l!m!n!)/(a!b!c!d!)$. From the claim, the previous displayed equation and Lemma 3.7, the problem is reduced to computing

$$\sum_{\text{admissible } d} [\Bbbk \Omega^d_{l,m,n} : S(\lambda) \boxtimes S(\mu) \boxtimes S(\nu)] = \sum_{\alpha,\beta,\gamma,\delta,\delta',\delta'' \in \mathcal{P}} LR^\lambda_{\beta,\gamma,\delta} LR^\mu_{\alpha,\gamma,\delta'} LR^\nu_{\alpha,\beta,\delta''} G^\delta_{\delta',\delta''}.$$

This expression is equal to the reduced Kronecker coefficient $\overline{G}^\lambda_{\mu,\nu}$ by a theorem of Littlewood [L]. $\square$

3.5. **Grothendieck rings.** Next we describe the Grothendieck rings $K_0(Par^\sharp)$ and $K_0(Par_t)$. For $\lambda \in \mathcal{P}$, let

$$P^\sharp(\lambda) := \mathrm{ind}^{Par^\sharp}_{Sym} S(\lambda). \tag{3.37}$$

This is a finite-dimensional projective $Par^\sharp$-module. In fact, it is the projective cover of the irreducible $Par^\sharp$-module $S^\sharp(\lambda)$. This follows because $P^\sharp(\lambda) \twoheadrightarrow S^\sharp(\lambda)$, and $\mathrm{End}_{Par^\sharp}(P^\sharp(\lambda)) \cong \mathrm{End}_{Sym}(S(\lambda)) \cong \Bbbk$ so that $P^\sharp(\lambda)$ is indecomposable. Thus, $K_0(Par^\sharp)$ is the free $\mathbb{Z}$-module with basis $\left\{ [P^\sharp(\lambda)] \mid \lambda \in \mathcal{P} \right\}$, and we see that the monoidal functor $\mathrm{ind}^{Par^\sharp}_{Sym}$ induces a ring isomorphism

$$K_0(Sym) \xrightarrow{\sim} K_0(Par^\sharp). \tag{3.38}$$

Recall from §2.6 that $K_0(Sym)$ is identified with the graded ring Λ of symmetric functions. Using (3.38), it follows that we can also identify $K_0(Par^\sharp)$ with Λ so that $[P^\sharp(\lambda)]$ corresponds to the Schur function s_λ. Let

$$B_{\lambda,\mu} := \dim \operatorname{Hom}_{Par^\sharp}(P^\sharp(\mu), P^\sharp(\lambda)) = [P^\sharp(\lambda) : S^\sharp(\mu)], \tag{3.39}$$

i.e., $(B_{\lambda,\mu})_{\lambda,\mu\in\mathcal{P}}$ is the Cartan matrix of $Par^\sharp$.

Lemma 3.9. *We have that $B_{\lambda,\mu} = \dim e_\mu Par^\sharp e_\lambda$ where $e_\lambda \in Sym$ denotes Young's idempotent. Hence:*

(i) $B_{\lambda,\lambda} = 1$ for every $\lambda \in \mathcal{P}$.
(ii) $B_{\lambda,\mu} = 0$ if $|\mu| > |\lambda|$ or if $|\mu| = |\lambda|$ and $\mu \neq \lambda$.
(iii) If $\lambda = (1^n)$ then $B_{\lambda,\mu} = \begin{cases} 1 & \text{if } \mu = (1^n) \text{ or } \mu = (1^{n-1}) \\ 0 & \text{for all other } \mu \in \mathcal{P}. \end{cases}$

Proof. For $\lambda \in \mathcal{P}_n$, the primitive idempotent $e_\lambda \in \Bbbk S_n$ has the property that $S(\lambda) \cong \Bbbk S_n e_\lambda$. Hence, $P^\sharp(\lambda) \cong Par^\sharp e_\lambda$. We deduce that

$$B_{\lambda,\mu} \overset{(3.39)}{=} \dim \operatorname{Hom}_{Par^\sharp}(P^\sharp(\mu), P^\sharp(\lambda)) = \dim \operatorname{Hom}_{Par^\sharp}(Par^\sharp e_\mu, Par^\sharp e_\lambda) = \dim e_\mu Par^\sharp e_\lambda.$$

Parts (i) and (ii) follow easily using this formula. For (iii), let $\lambda := (1^n)$. Then $e_\lambda = \sum_{g\in S_n}(-1)^{\ell(g)}g$, so that any crossing composed with e_λ equals $-e_\lambda$. The space $Par^\sharp e_\lambda$ is spanned by terms of the form $f e_\lambda$ for downward partition diagrams $f \in Par^\sharp 1_n$. When $\lambda = (1^n)$, it follows using the anti-symmetry that $f e_\lambda = 0$ if some connected component of f is a downward branch or a downward tree, or if f has two components that are downward leaves. So in this case $Par^\sharp e_\lambda$ is spanned just by the vectors e_λ and $(1_{n-1} \star c)e_\lambda$ where c is a single downward leaf. Since these two elements of $Par^\sharp$ lie in different weight spaces, they are linearly independent, so $Par^\sharp e_\lambda$ is exactly two-dimensional. It remains to observe that $\Bbbk e_\lambda$ is the sign representation $S(\lambda)$ of S_n and $\Bbbk(1_{n-1} \star c)e_\lambda$ is the sign representation $S(\mu)$ of S_{n-1} for $\mu = (1^{n-1})$. ∎

Lemma 3.9(i)–(ii) shows that the Cartan matrix is unitriangular, hence, invertible. It follows that the inclusion $Par^\sharp$-Proj $\to Par^\sharp$-Mod$_{\mathrm{fd}}$ induces an isomorphism

$$K_0(Par^\sharp) \xrightarrow{\sim} K_0'(Par^\sharp), \tag{3.40}$$

where $K_0'(Par^\sharp)$ denotes the Grothendieck group of the Abelian category $Par^\sharp$-Mod$_{\mathrm{fd}}$. By Lemma 3.5, we know that $\circledast$ is biexact on $Par^\sharp$-Mod$_{\mathrm{fd}}$, so it induces a multiplication making $K_0'(Par^\sharp)$ into a ring in such a way that (3.40) is a ring isomorphism. Using the isomorphisms (3.38) and (3.40), the canonical basis $\{[S^\sharp(\lambda)] \mid \lambda \in \mathcal{P}\}$ of $K_0'(Par^\sharp)$ gives us another basis $\{\tilde{s}_\lambda \mid \lambda \in \Lambda\}$ for the ring Λ. We call these the *deformed Schur functions*. We have that

$$s_\lambda = \sum_{\mu\in\mathcal{P}} B_{\lambda,\mu}\tilde{s}_\mu, \qquad \tilde{s}_\lambda = \sum_{\mu\in\mathcal{P}} A_{\lambda,\mu}s_\mu, \tag{3.41}$$

where $(B_{\lambda,\mu})_{\lambda,\mu\in\mathcal{P}}$ is the Cartan matrix from (3.39) and $(A_{\lambda,\mu})_{\lambda,\mu\in\mathcal{P}}$ is the inverse matrix. From the unitriangularity of the latter matrix, it follows that $\tilde{s}_\lambda$ is equal to s_λ plus a linear combination of s_μ of strictly lower degree. In other words, viewing the graded algebra Λ as a filtered algebra with filtration induced by the grading, the deformed Schur function $\tilde{s}_\lambda$ is in filtered degree $n := |\lambda|$ and $\operatorname{gr}_n \tilde{s}_\lambda = s_\lambda$. This justifies the name "deformed Schur function". By Theorem 3.8 we have that

$$\tilde{s}_\mu\tilde{s}_\nu = \sum_{\lambda\in\mathcal{P}} \overline{G}^\lambda_{\mu,\nu}\tilde{s}_\lambda, \tag{3.42}$$

i.e., the reduced Kronecker coefficients are the structure constants of Λ in its inhomogeneous basis arising from deformed Schur functions. Comparing with (2.28), we deduce that $\overline{G}^\lambda_{\mu,\nu} = LR^\lambda_{\mu,\nu}$ if $|\lambda| =$

$|\mu| + |\nu|$ and $\overline{G}^{\lambda}_{\mu,\nu} = 0$ if $|\lambda| > |\mu| + |\nu|$, both of which are well known properties of reduced Kronecker coefficients.

Remark 3.10. The deformed Schur function $\tilde{s}_\lambda \in \Lambda$ as just defined is equal to the symmetric function $\tilde{s}_\lambda$ defined in a different way [OZ]. This follows from Lemma 3.9(iii), (3.42) and the characterization given in [OZ, Th. 1(3)].

It remains to pass from $K_0(Par^\sharp)$ to $K_0(Par_t)$. Let $K_0''(Par_t)$ be the Grothendieck group of the exact category Par_t-Mod$_\Delta$. It is the free Abelian group on basis $\{[\Delta(\lambda)]\,|\,\lambda \in \mathcal{P}\}$. The following result implies that $K_0''(Par_t)$ is a ring with multiplication induced by $\circledast$.

Theorem 3.11. *For $V, W \in Par_t$-Mod$_\Delta$, we have that $\mathrm{Tor}_i^{Par_t}(V, W) = 0$ for all $i > 0$, hence, $\circledast$ is biexact on Par_t-Mod$_\Delta$. For $\mu, \nu \in \mathcal{P}$, there is a filtration $0 = V_{-1} < V_0 < \cdots < V_{|\mu|+|\nu|} = \Delta(\mu)\circledast\Delta(\nu)$ such that*

$$V_i/V_{i-1} \cong \bigoplus_{\lambda\in\mathcal{P}_i} \Delta(\lambda)^{\oplus \overline{G}^{\lambda}_{\mu,\nu}}$$

where $\overline{G}^{\lambda}_{\mu,\nu}$ is the reduced Kronecker coefficient.

Proof. The first statement is [SS2, Cor. 6.6], which is deduced from [SS2, Props. 4.31–4.32] using also the exactness of $\circledast$ on $Par^\sharp$-Mod$_{\mathrm{fd}}$ established in Lemma 3.5 together with exactness of the monoidal functor $\mathrm{ind}_{Par^\sharp}^{Par_t}$. The second statement follows by applying $\mathrm{ind}_{Par^\sharp}^{Par_t}$ to the degree filtation of $S^\sharp(\mu)\circledast S^\sharp(\nu)$, using Theorem 3.8 which computes the multiplicities. $\square$

Now consider the following commutative diagram of rings and ring homomorphisms, with maps induced by the indicated biexact monoidal functors:

$$\begin{tikzcd} & K_0(Par^\sharp) \arrow[r, "\mathrm{inc}"] \arrow[d, "\mathrm{ind}_{Par^\sharp}^{Par_t}"] & K_0'(Par^\sharp) \arrow[d, "\mathrm{ind}_{Par^\sharp}^{Par_t}"] \\ \Lambda \cong K_0(Sym) \arrow[ur, "\mathrm{ind}_{Sym}^{Par^\sharp}"] \arrow[dr, "\mathrm{ind}_{Sym}^{Par_t}"'] & & \\ & K_0(Par_t) \arrow[r, "\mathrm{inc}"'] & K_0''(Par_t). \end{tikzcd} \tag{3.43}$$

(Recall K_0 is the split Grothendieck group of finitely generated projectives, $K_0'(Par^\sharp)$ is the Grothendieck group of the Abelian category $Par^\sharp$-Mod$_{\mathrm{fd}}$, $K_0''(Par_t)$ is the Grothendieck group of the exact category Par_t-Mod$_\Delta$, and all of these Grothendieck groups are actually rings with multiplication induced by $\circledast$.)

Theorem 3.12. *All of the arrows in (3.43) are isomorphisms, so that all of the Grothendieck rings in this diagram are identified with Λ.*

Proof. We already established this for the top two arrows in (3.38) and (3.40). It is immediate for the arrow on the right since it takes basis element $[S^\sharp(\lambda)]$ to basis element $[\Delta(\lambda)]$. The fact that the bottom arrow is an isomorphism is a general property of upper finite highest weight categories. Indeed, we have that

$$[P(\lambda)] = [\Delta(\lambda)] + (\text{a sum of } [\Delta(\mu)] \text{ for } \mu \text{ with } |\mu| < |\lambda|), \tag{3.44}$$

equality in $K_0''(Par_t)$. Hence, the transition matrix between the image of the canonical basis for $K_0(Par_t)$ and the standard basis for $K_0''(Par_t)$ is invertible, as required to see that the bottom map is an isomorphism. We deduce that the other two arrows are isomorphisms too using the commutativity of the diagram. $\square$

From Theorem 3.12, we see that there are *three* natural basis for $K_0(\mathcal{P}ar_t)$:

- The canonical basis $\{[P(\lambda)] \mid \lambda \in \mathcal{P}\}$ arising from the indecomposable projectives.
- The basis $\{[\Delta(\lambda) \mid \lambda \in \mathcal{P}\}$ arising from the standard basis for $K_0''(\mathcal{P}ar_t)$ via the isomorphism that is the bottom arrow of (3.43).
- The basis $\{[Q(\lambda)] \mid \lambda \in \mathcal{P}\}$ where $Q(\lambda) := \mathrm{ind}_{\mathcal{S}ym}^{\mathcal{P}ar_t} S(\lambda) \cong \mathrm{ind}_{\mathcal{P}ar^\sharp}^{\mathcal{P}ar_t} S^\sharp(\lambda)$.

Note $Q(\lambda)$ is a finitely generated projective $\mathcal{P}ar_t$-module which is usually decomposable. In fact

$$Q(\lambda) \cong P(\lambda) \oplus (\text{a finite direct sum of } P(\mu) \text{ for } \mu \text{ with } |\mu| < |\lambda|), \tag{3.45}$$

as follows from (3.44) and the following lemma.

Lemma 3.13. *For $\lambda \in \mathcal{P}_n$, the $\mathcal{P}ar_t$-module $V := Q(\lambda)$ has a filtration $0 = V_{-1} < V_0 < \cdots < V_n = V$ such that*

$$V_i/V_{i-1} \cong \bigoplus_{\mu \in \mathcal{P}_i} \Delta(\mu)^{\oplus B_{\lambda,\mu}}.$$

In particular, $V_n/V_{n-1} \cong \Delta(\lambda)$.

Proof. Recalling the definition (3.39), this follows by applying the exact functor $\mathrm{ind}_{\mathcal{P}ar^\sharp}^{\mathcal{P}ar_t}$ to the degree filtration of $P^\sharp(\lambda)$, also using Lemma 3.9. □

Under the identification of $K_0(\mathcal{P}ar_t)$ with Λ, the isomorphism classes $[Q(\lambda)]$ correspond to the Schur functions $s_\lambda \in \Lambda$, and the isomorphism classes $[\Delta(\lambda)]$ correspond to the deformed Schur functions $\tilde{s}_\lambda$. These statements are both clear from our previous discussion of $K_0(\mathcal{P}ar^\sharp)$. The Q and Δ bases for $K_0(\mathcal{P}ar_t)$ are independent of the value of the parameter t, whereas the P basis coming from indecomposable projectives undoubtedly does depend on t. For values of t such that $\mathcal{P}ar_t$ is semisimple (see Corollary 5.11 below), we have that $P(\lambda) = \Delta(\lambda) = \nabla(\lambda) = L(\lambda)$, and Theorem 3.11 implies that

$$\Delta(\mu) \circledast \Delta(\nu) \cong \bigoplus_{\lambda \in \mathcal{P}} \Delta(\lambda)^{\oplus \overline{G}_{\mu,\nu}^{\lambda}}. \tag{3.46}$$

This was established before in [E-A]; see also [CO, Lem 5.14] and [BDVO, Cor. 3.2.2].

4. Jucys-Murphy elements via the affine partition category

Next, we introduce an auxiliary monoidal category $\mathcal{AP}ar$, the *affine partition category*. We define this as a certain monoidal subcategory of the Heisenberg category $\mathcal{H}eis$, exploiting an observation of Likeng and Savage from [LSR]. We then use $\mathcal{AP}ar$ to give a new approach to the definition of the Jucys-Murphy elements of $\mathcal{P}ar_t$. These were first defined in the context of the partition algebra by Halverson and Ram [HR] and computed recursively by Enyang [E1]. We also construct more general central elements.

4.1. **Schur-Weyl duality.** Recall the generators and relations for the partition category from Definition 3.1. The following theorem of Deligne will play a key role in this section; see e.g. [C, Th. 2.3] for a proof.

Theorem 4.1. *Suppose that $t \in \mathbb{N}$. Let U_t be the natural permutation representation of the symmetric group S_t with standard basis $u_1, \dots, u_t$. Viewing $\Bbbk S_t\text{-}\mathrm{Mod}_{\mathrm{fd}}$ as a symmetric monoidal category via the usual Kronecker tensor product $\otimes$, there is full $\Bbbk$-linear symmetric monoidal functor $\psi_t : \mathcal{P}ar_t \to \Bbbk S_t\text{-}\mathrm{Mod}_{\mathrm{fd}}$ sending the generating object $\mid$ to U_t and defined on generating morphisms by*

$$\begin{aligned}
\psi_t(\times) &: U_t \otimes U_t \to U_t \otimes U_t, & u_i \otimes u_j &\mapsto u_j \otimes u_i, \\
\psi_t(\curlywedge) &: U_t \otimes U_t \to U_t, & u_i \otimes u_j &\mapsto \delta_{i,j} u_i, \\
\psi_t(\curlyvee) &: U_t \to U_t \otimes U_t, & u_i &\mapsto u_i \otimes u_i,
\end{aligned}$$

$\psi_t($  $) : U_t \to \Bbbk\,, \qquad\qquad u_i \mapsto 1\,,$

$\psi_t($  $) : \Bbbk \to U_t \qquad\qquad 1 \mapsto u_1 + \cdots + u_t\,.$

Furthermore, the linear map $\mathrm{Hom}_{\mathcal{P}ar_t}(n,m) \to \mathrm{Hom}_{\Bbbk S_t}(U_t^{\otimes n}, U_t^{\otimes m}), f \mapsto \psi_t(f)$ *is an isomorphism whenever* $t \geqslant m+n$.

For the next corollary, we assume some basic facts about semisimplification of monoidal categories; e.g., see [BEEO, Sec. 2] which gives a concise summary of everything needed here.

Corollary 4.2. *When* $t \in \mathbb{N}$*, the functor* ψ_t *induces a monoidal equivalence* $\overline{\psi}_t$ *between the semisimplification of* $\mathrm{Kar}(\mathcal{P}ar_t)$ *and* $\Bbbk S_t\text{-Mod}_{\mathrm{fd}}$. *In particular,* Par_t *is* not *a semisimple locally unital algebra in these cases.*

Proof. The functor ψ_t extends canonically to a functor $\mathrm{Kar}(\mathcal{P}ar_t) \to \Bbbk S_t\text{-Mod}_{\mathrm{fd}}$. It is well known that every irreducible $\Bbbk S_t$-module appears as a constituent of some tensor power of U_t, hence, this functor is dense. Now the first statement follows from the fullness of the functor ψ_t using [BEEO, Lem. 2.6]; see also [D, Th. 2.18] and [CO, Th. 3.24]. Since $\mathrm{Kar}(\mathcal{P}ar_t)$ has infinitely many isomorphism classes of irreducible objects, it is definitely not equivalent to its semisimplification $\Bbbk S_t\text{-Mod}_{\mathrm{fd}}$. This shows that $\mathrm{Kar}(\mathcal{P}ar_t)$ is not a semisimple Abelian category as it contains non-zero negligible morphisms. Equivalently, the path algebra Par_t is not semisimple in these cases. □

Remark 4.3. Continue to assume that $t \in \mathbb{N}$. By the general theory of semisimplification, the irreducible objects in the semisimplification of $\mathrm{Kar}(\mathcal{P}ar_t)$ correspond to the indecomposable projective Par_t-modules $P(\lambda)$ of non-zero categorical dimension. In [D, Prop. 6.4], Deligne showed that $P(\lambda)$ has non-zero categorical dimension if and only if $t - |\lambda| > \lambda_1 - 1$, in which case the irreducible object of the semisimpliciation arising from $P(\lambda)$ corresponds under the equivalence $\overline{\psi}_t$ to the irreducible $\Bbbk S_t$-module $S(\kappa)$ where $\kappa := (t - |\lambda|, \lambda_1, \lambda_2, \dots)$.

The *generic partition category* $\mathcal{P}ar$ is the strict $\Bbbk$-linear monoidal category with the same generating object and generating morphisms as $\mathcal{P}ar_t$ subject to all of the same relations except for the final relation in (3.6), which is omitted. The morphism

$T :=$  $\in \mathrm{End}_{\mathcal{P}ar}(\mathbb{1}) \qquad (4.1)$

is strictly central in $\mathcal{P}ar$, so that $\mathcal{P}ar$ can be viewed as a $\Bbbk[T]$-linear monoidal category. For $t \in \Bbbk$, let

$$\mathrm{ev}_t : \mathcal{P}ar \to \mathcal{P}ar_t \tag{4.2}$$

be the canonical functor taking T to $t1_{\mathbb{1}}$. Using the basis theorem for $\mathcal{P}ar_t$ for infinitely many values of t, one obtains a basis theorem for the generic partition category: each morphism space $\mathrm{Hom}_{\mathcal{P}ar}(n,m)$ is free as a $\Bbbk[T]$-module with basis given by a set of representatives for the equivalence classes of $m \times n$ partition diagrams. From this, we see that ev_t induces an isomorphism $\Bbbk \otimes_{\Bbbk[T]} \mathcal{P}ar \cong \mathcal{P}ar_t$, where on the left hand side we are viewing $\Bbbk$ as a $\Bbbk[T]$-module so that T acts as t. This point of view is often useful since it can be used to prove a statement involving relations in $\mathcal{P}ar_t$ for all values of t just by checking it for all sufficiently large positive integers, in which case Theorem 4.1 can often be applied to reduce to a question about symmetric groups. To make a precise statement, let

$$\phi_t := \psi_t \circ \mathrm{ev}_t : \mathcal{P}ar \to \Bbbk S_t\text{-Mod}_{\mathrm{fd}}, \tag{4.3}$$

assuming $t \in \mathbb{N}$.

Lemma 4.4. *If* $f \in \mathrm{Hom}_{\mathcal{P}ar}(n,m)$ *satifies* $\phi_t(f) = 0$ *for infinitely many values of* $t \in \mathbb{N}$ *then* $f = 0$.

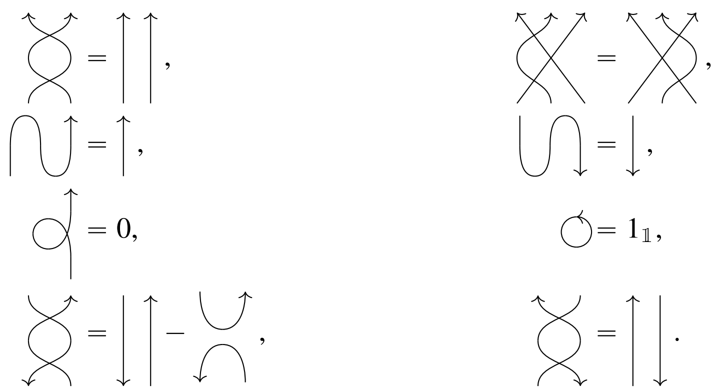
Proof. We can write $f = \sum_i p_i(T) f_i$ for polynomials $p_i(T) \in \Bbbk[T]$ and f_i running over a set of representatives for the equivalence classes of $m \times n$ partition diagrams. Since $\phi_t(f) = 0$ we have that $\sum_i p_i(t)\phi_t(f_i) = 0$ for infinitely many values of t. By the final assertion in Theorem 4.1, this implies that $\sum_i p_i(t)\,\mathrm{ev}_t(f_i) = 0$ for infinitely many values of $t \geqslant m+n$. By the basis theorem in $\mathcal{P}ar_t$, this means for each i that $p_i(t) = 0$ for infinitely many values of t. Hence, $p_i(T) = 0$ for each i. □

We note that the proof of Lemma 4.4 depends on our standing assumption that the ground field $\Bbbk$ is of characteristic zero.

4.2. **Heisenberg category.** Next we recall the definition of the Heisenberg category $\mathcal{H}eis$ which was introduced by Khovanov in [K]. We follow the approach of [B]; Khovanov's category is denoted $\mathcal{H}eis_{-1}(0)$ in the more general setup developed there.

Definition 4.5 ([B, Rem. 1.5(2)]). The Heisenberg category $\mathcal{H}eis$ is the strict $\Bbbk$-linear monoidal category with two generating objects $\uparrow$ and $\downarrow$ and five generating morphisms

subject to the following relations:



(4.4)

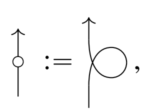
(4.5)

(4.6)

(4.7)

Here, we have used the the sideways crossings which are defined from

It is also convenient to introduce the shorthand

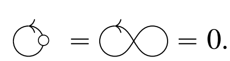


(4.8)

which automatically satisfies the degenerate affine Hecke algebra relation as in (2.34):

(4.9)

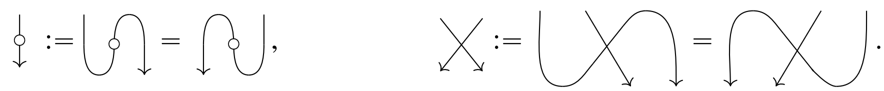
Note by (4.6) that



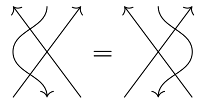
(4.10)

In addition, the following relations hold, so that $\mathcal{H}eis$ is strictly pivotal with duality functor defined by rotating diagrams through 180°:

(4.11)

 (4.12)

Then we obtain further variations on (4.9) by rotating through 90° or 180° using this strictly pivotal structure. One more useful consequence of the defining relations is that

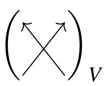
 (4.13)

There is also a symmetry $\sigma : \mathcal{H}eis \to \mathcal{H}eis^{\mathrm{op}}$, which is the strict $\Bbbk$-linear monoidal functor that is the identity on objects and sends a morphism to the morphism obtained by reflecting in a horizontal axis and then reversing all orientations of strings.

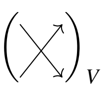
Khovanov constructed a categorical action of $\mathcal{H}eis$ on $Sym\text{-Mod}_{\mathrm{fd}} = \bigoplus_{n\geqslant 0} \Bbbk S_n\text{-Mod}_{\mathrm{fd}}$, i.e., a strict $\Bbbk$-linear monoidal functor

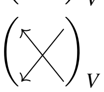
$$\Theta : \mathcal{H}eis \to \mathcal{E}nd_{\Bbbk}(Sym\text{-Mod}_{\mathrm{fd}}). \tag{4.14}$$

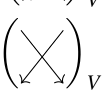
Explicitly, this takes the generating objects $\uparrow$ and $\downarrow$ to the induction functor E and the restriction functor F, respectively, notation as in (2.31), and Θ takes generating morphisms for $\mathcal{H}eis$ to the natural transformations defined on a $\Bbbk S_n$-module V as follows (where g is an element of the appropriate symmetric group):

 $: \Bbbk S_{n+2} \otimes_{\Bbbk S_{n+1}} \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V \to \Bbbk S_{n+2} \otimes_{\Bbbk S_{n+1}} \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V,$

$$g \otimes 1 \otimes v \mapsto g(n{+}1\; n{+}2) \otimes 1 \otimes v,$$

 $: \Bbbk S_n \otimes_{\Bbbk S_{n-1}} V \to \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V, \quad g \otimes v \mapsto g(n\; n{+}1) \otimes v,$

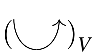
 $: \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V \to \Bbbk S_n \otimes_{\Bbbk S_{n-1}} V, \quad g \otimes v \mapsto \begin{cases} g_2 \otimes g_1 v & \text{if } g = g_2(n\; n{+}1)g_1 \text{ for } g_i \in S_n, \\ 0 & \text{otherwise,} \end{cases}$

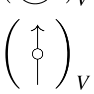
 $: V \to V, \quad v \mapsto (n{-}1\; n)v,$

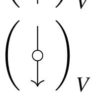
 $: \Bbbk S_n \otimes_{\Bbbk S_{n-1}} V \to V, \quad g \otimes v \mapsto gv,$

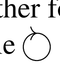
 $: V \to \Bbbk S_n \otimes_{\Bbbk S_{n-1}} V \quad v \mapsto \sum_{i=1}^{n} (i\; n) \otimes (i\; n)v,$

 $: \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V \to V, \quad g \otimes v \mapsto \begin{cases} gv & \text{if } g \in S_n, \\ 0 & \text{otherwise,} \end{cases}$

 $: V \to \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V \quad v \mapsto 1 \otimes v,$

 $: \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V \to \Bbbk S_{n+1} \otimes_{\Bbbk S_n} V, \quad g \otimes v \mapsto g x_{n+1} \otimes v,$

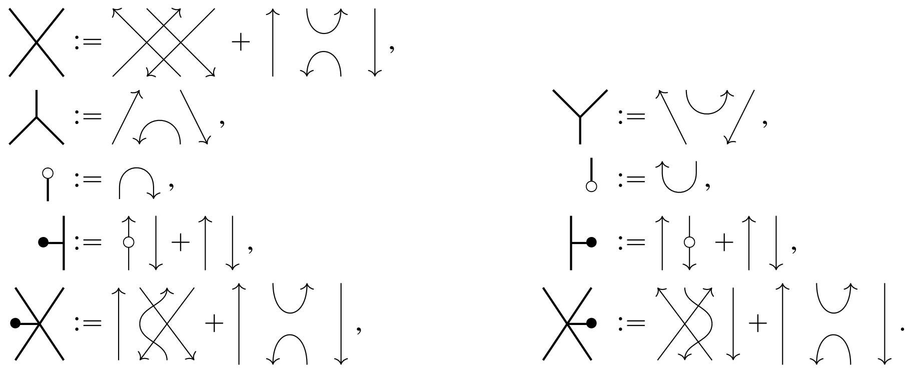
 $: V \to V, \quad v \mapsto x_n v.$

In the last two formulae, we have used the Jucys-Murphy elements $x_{n+1} \in \Bbbk S_{n+1}$ and $x_n \in \Bbbk S_n$ from (2.37), respectively; the natural transformations here are the endomorphisms of E and F denoted x and $x^\vee$ just before (2.39). All of the other formulae displayed here can also be found in [LSR, §3]. Note in particular that the clockwise bubble  acts as multiplication by n on any $V \in \Bbbk S_n\text{-Mod}_{\mathrm{fd}}$.

It is known moreover that the functor Θ is faithful. Indeed, in [K], Khovanov uses the functor Θ to prove a basis theorem for morphism spaces in $\mathcal{H}eis$, and the argument implicitly establishes the faithfulness of Θ over fields of characteristic zero. We will not use this here in any essential way.

4.3. **The affine partition category.** Now the background is in place and we can make a new definition.

Definition 4.6. The *affine partition category* $\mathcal{AP}ar$ is the monoidal subcategory of $\mathcal{H}eis$ generated by the object $| := \uparrow \star \downarrow$ and the following morphisms



(4.15)

(4.16)

(4.17)

(4.18)

(4.19)

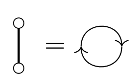
We refer to the morphisms in (4.18) as the *left dot* and the *right dot*, and the morphisms in (4.19) as the *left crossing* and the *right crossing*, respectively. The other shorthands for the generating morphisms of $\mathcal{AP}ar$ introduced in Definition 4.6 are the same as the symbols used for generators of the partition category. This is deliberate, indeed, the morphisms (4.15) to (4.17) generate a copy of the generic partition category $\mathcal{P}ar$ as a monoidal subcategory of $\mathcal{H}eis$. This important observation is due to Likeng and Savage; see Corollary 4.16 below. For now, we just need the following, which is proved in [LSR] by a direct calculation using the defining relations in $\mathcal{H}eis$.

Lemma 4.7 ([LSR, Th. 4.1])**.** *There is a strict $\Bbbk$-linear monoidal functor*

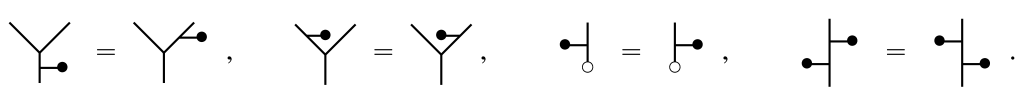
$$i : \mathcal{P}ar \to \mathcal{AP}ar \tag{4.20}$$

sending the generating object and generating morphisms of $\mathcal{P}ar$ to the generating object and generating morphisms in $\mathcal{AP}ar$ denoted by the same diagrams.

Because of the symmetry of the generators of $\mathcal{AP}ar$ under rotation through 180°, the strictly pivotal structure on $\mathcal{H}eis$ restricts to a strictly pivotal structure on $\mathcal{AP}ar$. The left and right dots are duals, as are the left and right crossings. Moreover, the cap and the cup making $|$ into a self-dual object are given by the same formula (3.7) as we had before in $\mathcal{P}ar$, hence, i is a pivotal monoidal functor. Note also that

$T :=$  . (4.21)

Also, the symmetry σ on $\mathcal{H}eis$ restricts to $\sigma : \mathcal{AP}ar \to \mathcal{AP}ar^{\mathrm{op}}$. This just reflects affine partition diagrams in a horizontal axis, just like the earlier anti-automorphism σ on $\mathcal{P}ar_t$. Here are some further relations, all of which are easily proved using the defining relations in $\mathcal{H}eis$:



(4.22)

Of course, the horizontal and vertical flips of all of these also hold. The next two lemmas establish some less obvious relations.

Lemma 4.8. *The following relations hold in* $\mathcal{APar}$*:*

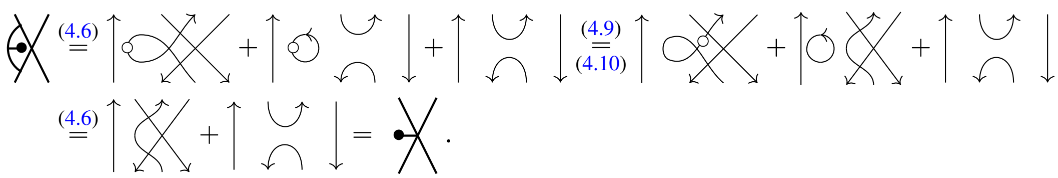

(4.23)


(4.24)

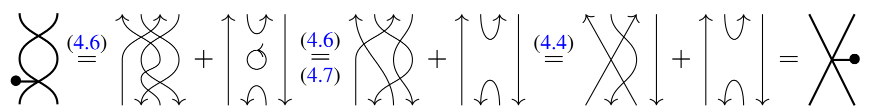

(4.25)


(4.26)

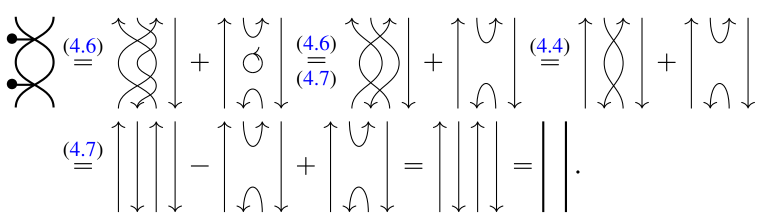
Proof. For each of (4.23) to (4.25), it suffices just to prove the first equality, and then all the others follow using σ and duality to reflect in horizontal and/or vertical axes. For (4.23), use (4.22) and (3.5). To prove (4.24), we expand as morphisms in $\mathcal{Heis}$ to see that



For (4.25), we again expand the left hand side as a morphism in $\mathcal{Heis}$:



Finally, to prove (4.26), the second set of relations follows from the first set of relations by composing on the bottom with a crossing and using (4.25). For the first set of relations, it suffices to prove the first equality, the second then follows by duality. Expanding both of the left crossings as morphisms in $\mathcal{Heis}$ produces a sum of four terms, two of which are zero, so we obtain:

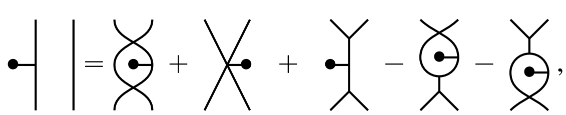


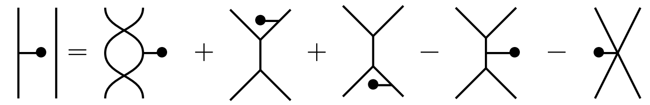
□

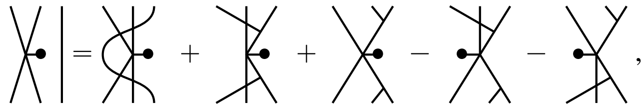
Corollary 4.9. *As a* $\Bbbk$*-linear monoidal category, the subcategory* $\mathcal{APar}$ *of* $\mathcal{Heis}$ *is generated by the object* $|$*, the five undotted generators* (4.15) *to* (4.17)*, and the left dot.*

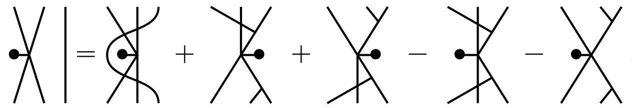
Proof. The relations (4.23) and (4.24) together show that the right dot and the left and right crossings may be written in terms of the left dot and the other undotted generating morphisms. □

Lemma 4.10. *The following relations hold:*

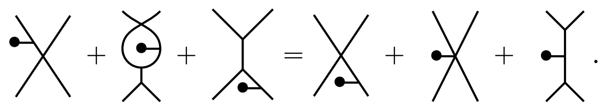

(4.27)

 (4.28)

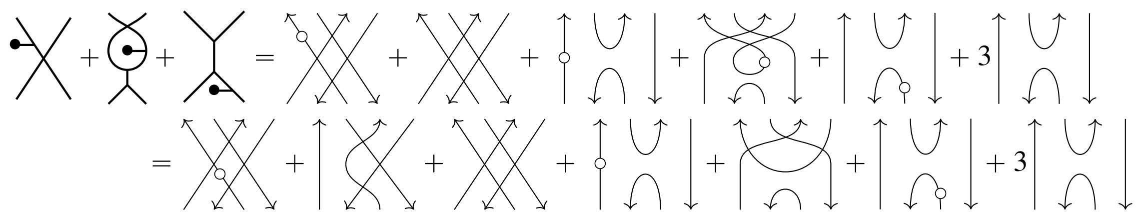
 (4.29)

 (4.30)

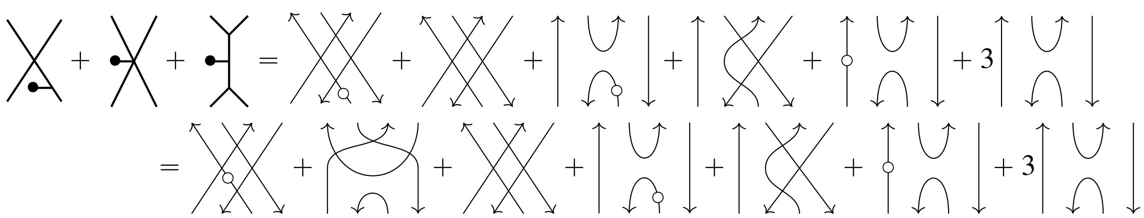
Proof. To prove (4.27), we observe by composing on the bottom with the crossing and using relations from $\mathcal{P}ar$ plus (4.25) that the relation we are trying to prove is equivalent to



Now we expand the left hand side in terms of morphisms in $\mathcal{H}eis$ using (4.6) and (4.10), then we use (4.9) to commute the dot past a crossing in the first and fourth terms:

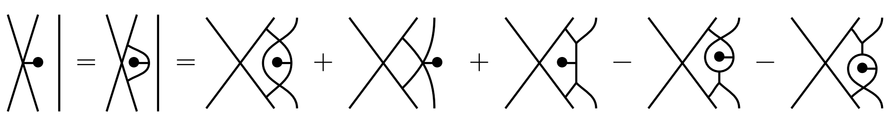


Similarly, the expansion of the right hand side is

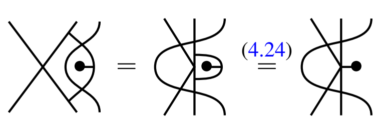


where we commuted the dot past a crossing just in the first term. These are equal. To deduce (4.28), first apply duality to (4.27), i.e., rotate through 180°. Then compose on the top and bottom with a crossing and simplify using relations in $\mathcal{P}ar$ together with (4.25).

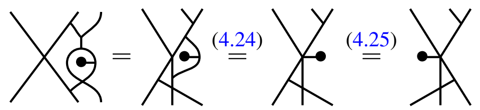
To prove (4.29), we rewrite its left hand side, replacing the right crossing with a left dot using (4.24), then we apply (4.27) to push this left dot past the right hand string:



Now we simplify the five terms on the right hand side of the equation just displayed to obtain the five terms on the right hand side of (4.29) (there is no need here to expand in terms of morphisms in $\mathcal{H}eis$). The following treats the first term:

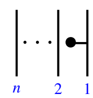


The second and third terms are easy to handle, we omit the details. For the fourth and the fifth terms, it suffices by symmetry to consider the fifth term, which we rewrite as follows:



Finally (4.30) follows easily from (4.29) on composing on the bottom with the crossing of the leftmost two strings and using (4.25). ◻

Corollary 4.11. *As a $\Bbbk$-linear category, $\mathcal{APar}$ has object set $\{|^{\star n} \mid n \in \mathbb{N}\}$ (which we often identify simply with $\mathbb{N}$) and morphisms that are linear combinations of vertical compositions of morphisms in the image of $i : \mathcal{Par} \to \mathcal{Heis}$ together with the morphisms*



(4.31)

for all $n \geqslant 1$.

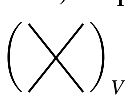
Proof. In view of Corollary 4.9, we just need to show that one can obtain the endomorphism of n defined by the left dot on the mth string ($m = 1, \dots, n$) by taking a linear combinations of compositions of morphisms in the image of i and the given morphism (4.31) (in which the left dot is on the first string). This follows by induction on m using relations (4.27) and (4.24). ◻

Remark 4.12. We have not attempted to formulate or prove a basis theorem for the morphism spaces in $\mathcal{APar}$. This is closely related to the problem of finding a complete monoidal presentation for $\mathcal{APar}$.

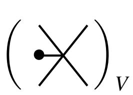
4.4. **Action of $\mathcal{APar}$ on $\Bbbk S_t$-Mod$_{\text{fd}}$.** Suppose that $t \in \mathbb{N}$. The restriction of the functor Θ from (4.14) to the subcategory $\mathcal{APar}$ is a strict $\Bbbk$-linear monoidal functor $\mathcal{APar} \to \mathcal{End}_\Bbbk(\mathcal{Sym}\text{-Mod}_{\text{fd}})$ sending the generating object $|$ to the endofunctor $E \circ F$ (induction after restriction). Since $E \circ F$ takes $\Bbbk S_t$-modules to $\Bbbk S_t$-modules, the restriction of Θ gives strict $\Bbbk$-linear monoidal functors

$$\Theta_t : \mathcal{APar} \to \mathcal{End}_\Bbbk(\Bbbk S_t\text{-Mod}_{\text{fd}}). \tag{4.32}$$

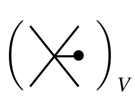
The functor Θ_t takes $|$ to the endofunctor $\operatorname{ind}_{S_{t-1}}^{S_t} \circ \operatorname{res}_{S_{t-1}}^{S_t} = \Bbbk S_t \otimes_{\Bbbk S_{t-1}}$ of $\Bbbk S_t$-Mod$_{\text{fd}}$; this should be interpreted as the zero functor in the case $t = 0$. The natural transformations arising by applying Θ_t to the other generating morphisms of $\mathcal{APar}$ may be computed using the formulae after (4.14) (taking $n := t$). Explicitly, one obtains the following for $V \in \Bbbk S_t$-Mod$_{\text{fd}}$ and $g, h \in S_t$:



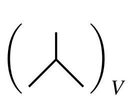
$$: \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V,$$
$$g \otimes h \otimes v \mapsto gh \otimes h^{-1} \otimes hv,$$



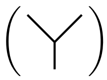
$$: \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V,$$
$$g \otimes h \otimes v \mapsto g \otimes h \otimes (h^{-1}(t)\ \ t)v,$$

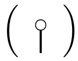


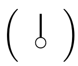
$$: \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V,$$
$$g \otimes h \otimes v \mapsto gh \otimes (h^{-1}(t)\ \ t) \otimes v,$$

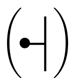


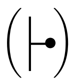
$$: \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V, \qquad g \otimes h \otimes v \mapsto \delta_{h(t),t}\, gh \otimes v,$$

 $_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V, \qquad g \otimes v \mapsto g \otimes 1 \otimes v,$

 $_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to V, \qquad g \otimes v \mapsto gv,$

 $_V : V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V, \qquad v \mapsto \sum_{i=1}^{t} (i\, t) \otimes (i\, t)v,$

 $_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V, \qquad g \otimes v \mapsto \sum_{j=1}^{t} g(j\, t) \otimes v,$

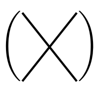
 $_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \to \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V, \qquad g \otimes v \mapsto \sum_{j=1}^{t} g \otimes (j\, t)v.$

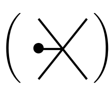
Recall the natural $\Bbbk S_d$-module U_t from Theorem 4.1; in particular, U_0 is the zero module. Using the Kronecker product, we can consider $U_t \otimes$ as an endofunctor of $\Bbbk S_t$-Mod$_{\mathrm{fd}}$. Also let triv_{S_t} be the trivial module.

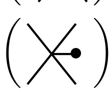
Lemma 4.13. *The functor Θ_t is monoidally isomorphic to the strict $\Bbbk$-linear monoidal functor*

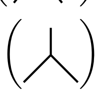
$$\Phi_t : \mathcal{APar} \to \mathcal{End}_{\Bbbk}(\Bbbk S_t\text{-Mod}_{\mathrm{fd}}) \tag{4.33}$$

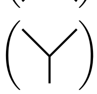
which sends the generating object $\mid$ to the endofunctor $U_t \otimes$ and taking the generating morphisms for $\mathcal{APar}$ to the natural transformations defined as follows on $V \in \Bbbk S_t$-Mod$_{\mathrm{fd}}$ and $1 \leqslant i, j \leqslant t$:

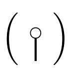
 $_V : U_t \otimes U_t \otimes V \to U_t \otimes U_t \otimes V, \qquad u_i \otimes u_j \otimes v \mapsto u_j \otimes u_i \otimes v,$

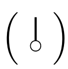
 $_V : U_t \otimes U_t \otimes V \to U_t \otimes U_t \otimes V, \qquad u_i \otimes u_j \otimes v \mapsto u_i \otimes u_j \otimes (i\, j)v,$

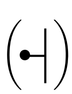
 $_V : U_t \otimes U_t \otimes V \to U_t \otimes U_t \otimes V, \qquad u_i \otimes u_j \otimes v \mapsto u_j \otimes u_i \otimes (i\, j)v,$

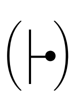
 $_V : U_t \otimes U_t \otimes V \to U_t \otimes V, \qquad u_i \otimes u_j \otimes v \mapsto \delta_{i,j} u_i \otimes v,$

 $_V : U_t \otimes V \to U_t \otimes U_t \otimes V, \qquad u_i \otimes v \mapsto u_i \otimes u_i \otimes v,$

 $_V : U_t \otimes V \to V, \qquad u_i \otimes v \mapsto v,$

 $_V : V \to U_t \otimes V, \qquad v \mapsto \sum_{i=1}^{t} u_i \otimes v,$

 $_V : U_t \otimes V \to U_t \otimes V, \qquad u_i \otimes v \mapsto \sum_{j=1}^{t} u_j \otimes (i\, j)v,$

 $_V : U_t \otimes V \to U_t \otimes V, \qquad u_i \otimes v \mapsto \sum_{j=1}^{t} u_i \otimes (i\, j)v.$

Proof. There is an isomorphism $\Bbbk S_t \otimes_{\Bbbk S_{t-1}} \mathrm{triv}_{S_{t-1}} \xrightarrow{\sim} U_t, g \otimes 1 \mapsto g u_t$. Combining this with the tensor identity, we obtain a natural $\Bbbk S_t$-module isomorphism

$$(\alpha_1^{(t)})_V : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} V \xrightarrow{\sim} U_t \otimes V, \qquad g \otimes v \mapsto g u_t \otimes g v \tag{4.34}$$

for $V \in \Bbbk S_t\text{-Mod}_{\mathrm{fd}}$. This defines an isomorphism $\alpha_1^{(t)} : \Bbbk S_t \otimes_{\Bbbk S_{t-1}} \overset{\sim}{\Rightarrow} U_t\otimes$. Let $\alpha_n^{(t)} := \alpha_1^{(t)} \cdots \alpha_1^{(t)}$ be the n-fold horizontal composition of $\alpha_1^{(t)}$. This is a natural isomorphism $\alpha_n^{(t)} : (\Bbbk S_t \otimes_{\Bbbk S_{t-1}})^{\circ n} \overset{\sim}{\Rightarrow} (U\otimes)^{\circ n}$ whose value on a $\Bbbk S_t$-module V is given explicitly by the map

$$\left(\alpha_n^{(t)}\right)_V : g_n \otimes \cdots \otimes g_1 \otimes v \mapsto g_n u_t \otimes g_n g_{n-1} u_t \otimes \cdots \otimes g_n g_{n-1} \cdots g_1 u_t \otimes g_n g_{n-1} \cdots g_1 v.$$

Now *define* $\Phi_t : \mathcal{APar} \to \mathcal{End}_{\Bbbk}(\Bbbk S_t\text{-Mod}_{\mathrm{fd}})$ to be the strict $\Bbbk$-linear monoidal functor taking the object n to $(U_t\otimes)^{\circ n}$, and defined on a morphism $f \in \mathrm{Hom}_{\mathcal{APar}}(n,m)$ by $\Phi_t(f) := \alpha_m^{(t)} \circ \Theta_t(f) \circ \left(\alpha_n^{(t)}\right)^{-1}$. It is immediate from this definition that $\alpha^{(t)} = \left(\alpha_n^{(t)}\right)_{n\geqslant 0} : \Theta_t \Rightarrow \Phi_t$ is an isomorphism of strict $\Bbbk$-linear monoidal functors.

It remains to check that Φ_t as defined in the previous paragraph is equal to the functor Φ_t defined on generating morphisms in the statement of the lemma. So we need to check for each generating morphism $f \in \mathrm{Hom}_{\mathcal{APar}}(n,m)$ that the formula for $\Phi_t(f)_V$ written in the statement of the lemma is equal to $\left(\alpha_m^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_n^{(t)}\right)_V^{-1}$ for $V \in \Bbbk S_t\text{-Mod}_{\mathrm{fd}}$ and $t \in \mathbb{N}$. This is a routine but lengthy calculation. We just go through a couple of the cases.

If f is the crossing, we need to show that $\left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_2^{(t)}\right)_V^{-1}\right)(u_i \otimes u_j \otimes v) = u_j \otimes u_i \otimes v$. Now we consider four cases. If $t \neq i \neq j \neq t$ we have that

$$\begin{aligned}\left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_2^{(t)}\right)_V^{-1}\right)(u_i \otimes u_j \otimes v) &= \left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V\right)((i\,t) \otimes (j\,t) \otimes (j\,t)(i\,t)v)\\ &= \left(\alpha_2^{(t)}\right)_V ((i\,t)(j\,t) \otimes (j\,t) \otimes (i\,t)v) = u_j \otimes u_i \otimes v.\end{aligned}$$

If $i = j$ we have that

$$\begin{aligned}\left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_2^{(t)}\right)_V^{-1}\right)(u_i \otimes u_j \otimes v) &= \left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V\right)((i\,t) \otimes 1 \otimes (i\,t)v)\\ &= \left(\alpha_2^{(t)}\right)_V ((i\,t) \otimes 1 \otimes (i\,t)v) = u_j \otimes u_i \otimes v.\end{aligned}$$

If $i = t \neq j$ we have that

$$\begin{aligned}\left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_2^{(t)}\right)_V^{-1}\right)(u_i \otimes u_j \otimes v) &= \left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V\right)(1 \otimes (j\,t) \otimes (j\,t)v)\\ &= \left(\alpha_2^{(t)}\right)_V ((j\,t) \otimes (j\,t) \otimes v) = u_j \otimes u_i \otimes v.\end{aligned}$$

Finally if $i \neq t = j$ we have that

$$\begin{aligned}\left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_2^{(t)}\right)_V^{-1}\right)(u_i \otimes u_j \otimes v) &= \left(\left(\alpha_2^{(t)}\right)_V \circ \Theta_t(f)_V\right)((i\,t) \otimes (i\,t) \otimes v)\\ &= \left(\alpha_2^{(t)}\right)_V (1 \otimes (i\,t) \otimes (i\,t)v) = u_j \otimes u_i \otimes v.\end{aligned}$$

This completes the check in this case.

If f is the left dot, we have that

$$\begin{aligned}\left(\left(\alpha_1^{(t)}\right)_V \circ \Theta_t(f)_V \circ \left(\alpha_1^{(t)}\right)_V^{-1}\right)(u_i \otimes v) &= \left(\left(\alpha_1^{(t)}\right)_V \circ \Theta_t(f)_V\right)((i\,t) \otimes (i\,t)v)\\ &= \sum_{j=1}^{t} \left(\alpha_1^{(t)}\right)_V((i\,t)(j\,t) \otimes (i\,t)v) = \sum_{j=1}^{t} (i\,t)(j\,t)u_t \otimes (i\,t)(j\,t)(i\,t)v.\end{aligned}$$

If $i = t$ this is $\sum_{j=1}^{t} u_j \otimes (j\,t)v$ which is right. If $i \neq t$ we pull out the $j = i$ and $j = t$ terms of the sum, simplify the three types of terms separately, then recombine to get the desired expression $\sum_{j=1}^{t} u_j \otimes (i\,j)v$. ∎

We now have in our hands monoidal functors ϕ_t from (4.3), i from (4.20), and Φ_t from (4.33). Let

$$\mathrm{Act} : \Bbbk S_t\text{-Mod}_{\mathrm{fd}} \to \mathcal{E}nd_{\Bbbk}(\Bbbk S_t\text{-Mod}_{\mathrm{fd}}) \tag{4.35}$$

be the $\Bbbk$-linear monoidal functor induced by the Kronecker product, i.e., $\mathrm{Act}(V) = V\otimes$ for a $\Bbbk S_t$-module V and $\mathrm{Act}(f) = f\otimes$ for a homomorphism $f : V \to V'$.

Lemma 4.14. *For every $t \in \mathbb{N}$, the following diagram commutes up to the obvious canonical isomorphism of monoidal functors:*

$$\begin{tikzcd} \mathcal{APar} \arrow[r, "\Phi_t"] & \mathcal{E}nd_{\Bbbk}(\Bbbk S_t\text{-Mod}_{\mathrm{fd}}) \\ \mathcal{P}ar \arrow[u, "i"] \arrow[r, "\phi_t"] & \Bbbk S_t\text{-Mod}_{\mathrm{fd}}. \arrow[u, "\mathrm{Act}"'] \end{tikzcd} \tag{4.36}$$

Proof. The composition $\Phi_t \circ i$ takes the nth object of $\mathcal{APar}$ to $(U_t\otimes)^{\circ n}$, while $\mathrm{Act} \circ\phi_t$ takes it to $U_t^{\otimes n}\otimes$. Let

$$\beta_n^{(t)} : (U_t\otimes)^{\circ n} \xRightarrow{\sim} U_t^{\otimes n}\otimes$$

be the canonical isomorphism between these functors defined by associativity of tensor product. Then $\beta^{(t)} = \big(\beta_n^{(t)}\big)_{n\geqslant 0} : \Phi_t \circ i \Rightarrow \mathrm{Act} \circ\phi_t$ is an isomorphism of monoidal functors. To see this, we need to check naturality. This follows because the five formulae defining ϕ_t from Theorem 4.1 tensored on the right with a vector v are exactly the same as the formulae defining Φ_t on these five generating morphisms from Lemma 4.13. □

Now we can prove the main theorem justifying the significance of the affine partition category. Let

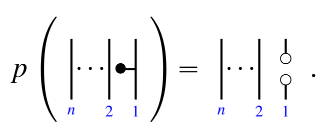
$$\mathrm{Ev} : \mathcal{E}nd_{\Bbbk}(\Bbbk S_t\text{-Mod}_{\mathrm{fd}}) \to \Bbbk S_t\text{-Mod}_{\mathrm{fd}} \tag{4.37}$$

be the (non-monoidal) $\Bbbk$-linear functor defined by evaluating on triv_{S_t}. There is an obvious isomorphism of functors $\mathrm{Ev} \circ \mathrm{Act} \xRightarrow{\sim} \mathrm{Id}_{\Bbbk S_t\text{-Mod}_{\mathrm{fd}}}$ defined on V by the isomorphism $V \otimes \mathrm{triv}_{S_t} \to V, v \otimes 1 \mapsto v$.

Theorem 4.15. *There is a unique (non-monoidal) $\Bbbk$-linear functor*

$$p : \mathcal{APar} \to \mathcal{P}ar \tag{4.38}$$

such that $p \circ i = \mathrm{Id}_{\mathcal{P}ar}$ and

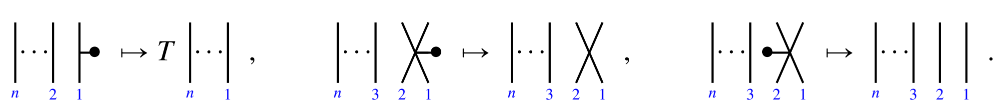


(4.39)

Moreover, for any $t \in \mathbb{N}$, the following diagram of functors commutes up to natural isomorphism:

$$\begin{tikzcd} \mathcal{APar} \arrow[r, "\Phi_t"] \arrow[d, "p"'] & \mathcal{E}nd_{\Bbbk}(\Bbbk S_t\text{-Mod}_{\mathrm{fd}}) \arrow[d, "\mathrm{Ev}"] \\ \mathcal{P}ar \arrow[r, "\phi_t"] & \Bbbk S_t\text{-Mod}_{\mathrm{fd}}. \end{tikzcd} \tag{4.40}$$

The functor p also maps



(4.41)

Proof. For $t \in \mathbb{N}$, let $\gamma_n^{(t)} : U_t^{\otimes n} \otimes \mathrm{triv}_{S_t} \xrightarrow{\sim} U_t^{\otimes n}$ be the obvious isomorphism sending $u_{i_n} \otimes \cdots \otimes u_{i_1} \otimes 1 \mapsto u_{i_n} \otimes \cdots \otimes u_{i_1}$. We say that $f \in \mathrm{Hom}_{\mathcal{APar}}(n,m)$ is *good* if there exists a morphism $\bar{f} \in \mathrm{Hom}_{\mathcal{Par}}(n,m)$ such that

$$\phi_t(\bar{f}) = \gamma_m^{(t)} \circ \mathrm{Ev}(\Phi_t(f)) \circ \left(\gamma_n^{(t)}\right)^{-1} \tag{4.42}$$

for all $t \in \mathbb{N}$. If f is good, there is a *unique* $\bar{f}$ such that (4.42) holds for all t. To see this, suppose that $\bar{f}$ and $\bar{f}'$ both satisfy (4.42) for all $t \in \mathbb{N}$. Then $\phi_t(\bar{f}) = \gamma_m^{(t)} \circ \mathrm{Ev}(\Phi_t(f)) \circ \left(\gamma_n^{(t)}\right)^{-1} = \phi_t(\bar{f}')$, so that $\phi_t(\bar{f} - \bar{f}') = 0$ for all $t \in \mathbb{N}$. In view of Lemma 4.4 this implies that $\bar{f} = \bar{f}'$ as claimed.

Suppose that $f \in \mathrm{Hom}_{\mathcal{APar}}(n,m)$ and $g \in \mathrm{Hom}_{\mathcal{APar}}(l,m)$ are both good. Then $f \circ g$ is good with $\overline{f \circ g} := \bar{f} \circ \bar{g}$. This follows because

$$\phi_t(\bar{f} \circ \bar{g}) = \gamma_m^{(t)} \circ \mathrm{Ev}(\Phi_t(f)) \circ \left(\gamma_m^{(t)}\right)^{-1} \circ \gamma_m^{(t)} \circ \mathrm{Ev}(\Phi_t(f)) \circ \left(\gamma_l^{(t)}\right)^{-1} = \gamma_m^{(t)} \circ \mathrm{Ev}(\Phi_t(f \circ g)) \circ \left(\gamma_l^{(t)}\right)^{-1}.$$

Similarly, sums of good morphisms are good with $\overline{f+g} := \bar{f} + \bar{g}$.

In this paragraph, we show that every morphism in $\mathcal{APar}$ is good. In view of the previous paragraph, it suffices to show that some family of generating morphisms for $\mathcal{APar}$ are all good. Hence, in view of Corollary 4.11, it is enough to show that $i(f)$ is good for every morphism f in $\mathcal{Par}$ and that the morphisms (4.31) are good for all n. For $f \in \mathrm{Hom}_{\mathcal{Par}}(n,m)$, the morphism $i(f)$ is good with $\overline{i(f)} := f$. This follows from the following calculation using Lemma 4.14:

$$\gamma_m^{(t)} \circ \mathrm{Ev}(\Phi_t(i(f))) \circ \left(\gamma_m^{(t)}\right)^{-1} = \gamma_m^{(t)} \circ \mathrm{Ev}(\mathrm{Act}(\phi_t(f))) \circ \left(\gamma_m^{(t)}\right)^{-1} = \phi_t(f).$$

Also the morphism f from (4.31) is good for every n. To see this, let $\bar{f}$ be the morphism on the right hand side of (4.39). Using the definition in Theorem 4.1, $\phi_t(\bar{f})$ is the map $u_{i_n} \otimes \cdots \otimes u_{i_1} \mapsto \sum_{j=1}^t u_{i_n} \otimes \cdots \otimes u_{i_2} \otimes u_j$. Also using the definition in Lemma 4.13, $\mathrm{Ev}(\Phi_t(f))$ is the map $u_{i_n} \otimes \cdots \otimes u_{i_1} \otimes 1 \mapsto \sum_{j=1}^t u_{i_n} \otimes \cdots \otimes u_{i_2} \otimes u_j \otimes 1$. On contracting the final $\otimes 1$ using $\gamma_n^{(t)}$, these are equal, as required to prove that f is good.

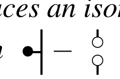
Now we can define a $\Bbbk$-linear functor p making (4.40) commute (up to natural isomorphism) for all $t \in \mathbb{N}$. On objects, define p by declaring that $p(n) = n$ for each $n \geqslant 0$. On a morphism $f \in \mathrm{Hom}_{\mathcal{APar}}(n,m)$, we define $p(f) := \bar{f}$. The checks made so far imply that this is a well-defined $\Bbbk$-linear functor satisfying (4.39). The equation (4.42) shows that $\gamma^{(t)} = \left(\gamma_n^{(t)}\right)_{n \geqslant 1} : \mathrm{Ev} \circ \Phi_t \Rightarrow \phi_t \circ p$ is a natural isomorphism. We have also already shown that $p \circ i = \mathrm{Id}_{\mathcal{Par}}$ and that (4.39) holds. Thus, we have established the existence of a $\Bbbk$-linear functor $p : \mathcal{APar} \to \mathcal{Par}$ satisfying all of the properties in the statement of the theorem. The uniqueness of p follows from Corollary 4.11.

It remains to check the three properties (4.41). These can be checked using the commutativity of (4.40) in the same way as we just established (4.39). Alternatively, and possibly quicker, they can be deduced directly from (4.39) using the relations (4.23) to (4.25), respectively. We leave the details to the reader. □

The faithfulness of i in the following corollary was already proved in two different ways in [LSR]. Our approach is similar in spirit to the first proof given in *loc. cit.*, i.e., the argument used to prove [LSR, Th. 5.2].

Corollary 4.16. *The functor $i : \mathcal{Par} \to \mathcal{APar}$ is faithful and the functor $p : \mathcal{APar} \to \mathcal{Par}$ is full.*

Proof. This follows because $p \circ i = \mathrm{Id}_{\mathcal{Par}}$. □

Corollary 4.17. *The functor p induces an isomorphism $\mathcal{APar}/\mathcal{I} \xrightarrow{\sim} \mathcal{Par}$ where $\mathcal{I}$ is the left tensor ideal of $\mathcal{APar}$ generated by the morphism*  .

Proof. The left tensor ideal $\mathcal{I}$ is the data of subspaces $\mathcal{I}(n,m)$ of $\mathrm{Hom}_{\mathcal{APar}}(n,m)$ for each $m,n \geqslant 0$ which are closed under vertical composition on the top or bottom with any morphism and closed under horizontal composition on the left with any morphism. It is clear from (4.39) that p sends morphisms in $\mathcal{I}$ to zero, hence, p induces a $\Bbbk$-linear functor $\bar{p} : \mathcal{APar}/\mathcal{I} \to \mathcal{Par}$. This is surjective on objects and full. To see that it is faithful, suppose that $f + \mathcal{I}(n,m) \in \mathrm{Hom}_{\mathcal{APar}/\mathcal{I}}(n,m) = \mathrm{Hom}_{\mathcal{APar}}(n,m)/\mathcal{I}(n,m)$ is a morphism sent to zero by $\bar{p}$, hence, $p(f) = 0$. In view of Corollary 4.11 and the definition of $\mathcal{I}$, we may assume that $f = i(\bar{f})$ for some $\bar{f} \in \mathrm{Hom}_{\mathcal{Par}}(n,m)$. Then $\bar{f} = p(i(\bar{f})) = p(f) = 0$, so that $f = i(\bar{f}) = 0$. □

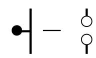
Composing the functor $p : \mathcal{APar} \to \mathcal{Par}$ with evaluation at any $t \in \Bbbk$ gives a full $\Bbbk$-linear functor

$$p_t := \mathrm{ev}_t \circ p : \mathcal{APar} \to \mathcal{Par}_t \tag{4.43}$$

such that


(4.44)


(4.45)

Like in Corollary 4.17, the functor p_t induces an isomorphism $\mathcal{APar}/\mathcal{I}_t \xrightarrow{\sim} \mathcal{Par}_t$ where $\mathcal{I}_t$ is the left tensor ideal of $\mathcal{APar}$ generated by $T - t1_{\mathbb{1}}$ and  .

4.5. **Jucys-Murphy elements for partition algebras.**

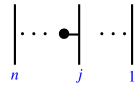
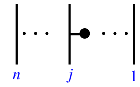
Now we can explain how affine partition category is related to the works of Enyang [E1] and Halverson-Ram [HR]. These are concerned with the *partition algebra*, which is the endomorphism algebra

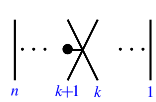
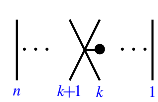
$$P_n(t) := \mathrm{End}_{\mathcal{Par}_t}(n) = 1_n Par_t 1_n. \tag{4.46}$$

By analogy, we define the *affine partition algebra* to be

$$AP_n := \mathrm{End}_{\mathcal{APar}}(n) = 1_n APar 1_n. \tag{4.47}$$

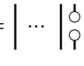
Let us denote the elements of AP_n defined by the left and right dots on the jth string by X_j^L and X_j^R, and the elements defined by the left and right crossings of the kth and $(k+1)$th strings by S_k^L and S_k^R:

$X_j^L :=$  , $X_j^R :=$  , (4.48)

$S_k^L :=$  , $S_k^R :=$  (4.49)

for $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant n-1$. We note that $\left\{X_j^L, X_j^R \mid j = 1,\dots,n\right\}$ are algebraically independent, so they generate a free polynomial algebra of rank $2n$ inside $AP_n(t)$; his follows easily from the basis theorem for morphism spaces $\mathcal{Heis}$ proved in [K]. Taking the images of the elements (4.48) and (4.49) under the functor p_t from (4.43) gives us elements of $P_n(t)$ denoted

$$x_j^L := p_t(X_j^L), \qquad x_j^R := p_t(X_k^R), \qquad s_k^L := p_t(S_k^L), \qquad s_k^R := p_t(S_k^R). \tag{4.50}$$

The notation here depends implicitly on the values of n and t, which should be clear from the context. By (4.44) and (4.45), we have that $x_1^L =$  , $x_1^R = t$, $s_1^L = 1$ and $s_1^R = (1\ 2) \in S_n \subset P_n(t)$.

Theorem 4.18. *Suppose that $t \in \mathbb{N}$ and let $\psi_t : P_n(t) \to \operatorname{End}_{\Bbbk S_t}(U_t^{\otimes n})$ be the homomorphism induced by the functor ϕ_t from Theorem 4.1. The elements $x_j^L, x_j^R, s_k^L, s_k^R \in P_n(t)$ satisfy*

$$\psi_t(x_j^L)(u_{i_n} \otimes \cdots \otimes u_{i_1}) = \sum_{i=1}^{t} u_{i_n} \otimes \cdots \otimes u_{i_{j+1}} \otimes (i\ \ i_j)\left[u_{i_j} \otimes \cdots \otimes u_{i_2} \otimes u_{i_1}\right], \tag{4.51}$$

$$\psi_t(x_j^R)(u_{i_n} \otimes \cdots \otimes u_{i_1}) = \sum_{i=1}^{t} u_{i_n} \otimes \cdots \otimes u_{i_j} \otimes (i\ \ i_j)\left[u_{i_{j-1}} \otimes \cdots \otimes u_{i_2} \otimes u_{i_1}\right], \tag{4.52}$$

$$\psi_t(s_k^L)(u_{i_n} \otimes \cdots \otimes u_{i_1}) = u_{i_n} \otimes \cdots \otimes u_{i_k} \otimes (i_k\ \ i_{k+1})\left[u_{i_{k-1}} \otimes \cdots \otimes u_{i_2} \otimes u_{i_1}\right], \tag{4.53}$$

$$\psi_t(s_k^R)(u_{i_n} \otimes \cdots \otimes u_{i_1}) = u_{i_n} \otimes \cdots \otimes u_{i_{k+2}} \otimes (i_k\ \ i_{k+1})\left[u_{i_{k+1}} \otimes \cdots \otimes u_{i_2} \otimes u_{i_1}\right] \tag{4.54}$$

for $1 \leqslant i_1, \dots, i_n \leqslant t$, where we are using the diagonal action of S_t on tensor powers of U_t.

Proof. This follows from the commutativity of (4.40), (4.50) and the formulae in Lemma 4.13. □

Corollary 4.19. *Identifying $P_n(t)$ with the partition algebra in* [E1] *by reflecting diagrams through a vertical axis to account for the fact that we number vertices from right to left rather than from left to right, the elements* (4.50) *are related to the elements $L_{\frac{1}{2}}, L_1, \dots$ and $\sigma_{\frac{3}{2}}, \sigma_2, \dots$ of the partition algebra $P_n(t)$ defined in* [E1] *according to the dictionary*

$$x_j^L \leftrightarrow L_j, \qquad t - x_j^R \leftrightarrow L_{j-\frac{1}{2}}, \qquad s_k^L \leftrightarrow \sigma_{k+\frac{1}{2}}, \qquad s_k^R \leftrightarrow \sigma_{k+1}. \tag{4.55}$$

Hence, by [E1, Th. 5.5]*, the elements x_j^L and $t - x_j^R$ are identified with the Jucys-Murphy elements introduced originally by Halverson and Ram in* [HR].

Proof. Enyang's elements are defined by a recurrence relation which is independent of the value of the parameter t. Hence, his elements can be viewed as specializations at $T = t$ of corresponding elements of the generic partition algebra $\operatorname{End}_{\mathcal{P}ar}(n)$. To identify them with our elements, we can use Lemma 4.4 to see that it suffices to check that they act in the same way on $U_t^{\otimes n}$ for infinitely many values of the parameter $t \in \mathbb{N}$. This follows on comparing (4.51) to (4.54) to the formulae in [E1, Prop. 5.2, Prop. 5.3]. □

Remark 4.20. Alternatively, one can prove Corollary 4.19 inductively, using the recurrence relations in Lemma 4.10 which are equivalent to Enyang's recurrence relations [E1, (3.1)–(3.4)]. In fact, all of the relations derived in *loc. cit.* can now be deduced easily using the relations in $\mathcal{APar}$ derived in the previous subsection.

Remark 4.21. Recently, Creedon [Cr] has introduced a renormalization of the Jucys-Murphy elements, which he denotes by $N_1, N_2, \dots, N_{2n} \in P_n(t)$. They are defined in terms of the Enyang-Halverson-Ram elements simply by $N_{2j-1} := L_{j-\frac{1}{2}} - \frac{t}{2}$ and $N_{2j} := L_j - \frac{t}{2}$. The dictionary between Creedon's elements and ours is

$$x_j^L - \tfrac{t}{2} \leftrightarrow N_{2j}, \qquad \tfrac{t}{2} - x_j^R \leftrightarrow N_{2j-1}. \tag{4.56}$$

The motivation for such a renormalization will be discussed further in Remark 4.26 below.

4.6. **Central elements.** By the *center* of a $\Bbbk$-linear category $\mathcal{A}$, we mean the (unital) commutative algebra $Z(\mathcal{A}) := \operatorname{End}_{\mathcal{A}}(\operatorname{Id}_{\mathcal{A}})$ of endomorphisms of the identity endofunctor of $\mathcal{A}$. Thus, an element $z \in Z(\mathcal{A})$ is a tuple $(z_X)_{X \in \operatorname{ob} \mathcal{A}}$ such that $z_Y \circ f = f \circ z_X$ for all morphisms $f : X \to Y$ in $\mathcal{A}$. Equivalently, in terms of the path algebra A, it is the algebra

$$Z(A) := \left\{ z = (z_X)_{X \in \mathbb{O}_A} \in \prod_{X \in \mathbb{O}_A} 1_X A 1_X \;\middle|\; za = az \text{ for all } a \in A \right\}, \tag{4.57}$$

interpreting the products in the obvious way. We note that there is an algebra isomorphism

$$\mathrm{End}_{A\boxtimes A^{\mathrm{op}}}(A) \xrightarrow{\sim} Z(A), \qquad \zeta \mapsto (\zeta(1_X))_{X\in\mathbb{O}_A} \in \prod_{X\in\mathbb{O}_A} 1_X A 1_X, \tag{4.58}$$

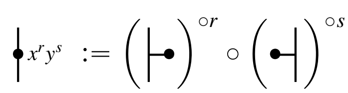
where the algebra on the left is the endomorphism algebra of the $A \boxtimes A^{\mathrm{op}}$-module associated to the (A, A)-bimodule A. If A is locally finite-dimensional, then it is a locally finite-dimensional $A \boxtimes A^{\mathrm{op}}$-module, hence, by [BS, Lem. 2.10], the endomorphism algebra $\mathrm{End}_{A\boxtimes A^{\mathrm{op}}}(A) \cong Z(A)$ is a pseudo-compact topological algebra with respect to the pro-finite topology (ideals of finite codimension form a base of neighborhoods of zero).

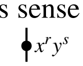
In the locally finite-dimensional case, $Z(A)$ is isomorphic to the algebra $C(A)^*$ that is the linear dual of the *cocenter* $C(A)$. The cocenter is a cocommutative coalgebra isomorphic to $\mathrm{Coend}_{A\boxtimes A^{\mathrm{op}}}(A)$ in the notation of [BS, (2.15)]. To define $C(A)$ explicitly, note that the space $D := \bigoplus_{X,Y\in\mathbb{O}_A}(1_X A 1_Y)^*$ is naturally an (A, A)-bimodule with $1_Y D 1_X = (1_X A 1_Y)^*$. Also each $1_X D 1_X$ is a coalgebra as it is the dual of the finite-dimensional algebra $1_X A 1_X$. Hence, $\bigoplus_{X\in\mathbb{O}_A} 1_X D 1_X$ is a coalgebra. Then the cocenter is

$$C(A) := \Big(\bigoplus_{X\in\mathbb{O}_A} 1_X D 1_X\Big)\Big/ J \tag{4.59}$$

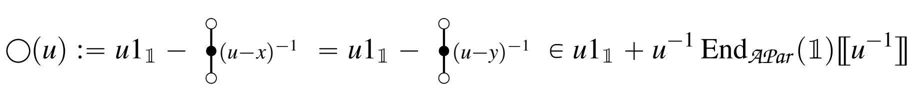
where J is the coideal spanned by the elements $\{af - fa \,|\, X, Y \in \mathbb{O}_A, a \in 1_X A 1_Y, f \in 1_Y D 1_X\}$. To identify $C(A)^*$ with $Z(A)$, note that the linear dual of the coalgebra $\bigoplus_{X\in\mathbb{O}_A} 1_X D 1_X$ is the algebra $\prod_{X\in\mathbb{O}_A} 1_X A 1_X$; the annihilator J° of the coideal J defines a subalgebra of $\prod_{X\in\mathbb{O}_A} 1_X A 1_X$ which is exactly the center $Z(A)$ according to the original definition (4.57).

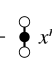
In this subsection, we are going to construct a family of elements $(z^{(r)})_{r\geqslant 1}$ in the center $Z(APar_t)$ of the affine partition category $\mathcal{APar}_t$. We start by introducing some convenient shorthand. Given a monomial $x^r y^s \in \Bbbk[x, y]$, we use the notation


(4.60)

to denote the element of $\mathrm{End}_{\mathcal{APar}}(|)$ on the right hand side, that is, it is the rth power of the right dot (represented by x) composed with the sth power of the left dot (represented by y). It then makes sense to label dots by polynomials $f(x) \in \Bbbk[x, y]$, meaning the linear combination of the morphisms  just as $f(x)$ is the linear combination of its monomials. We are also going to use generating functions in the same way as explained in the context of $\mathcal{H}eis$ in [BSW, §3.1]. For these, u will be a formal variable which should always be interpreted by expanding as formal Laurent series in $\Bbbk((u^{-1}))$, e.g., $(u - x)^{-1} = u^{-1} + u^{-2}x + u^{-3}x^2 + \cdots$.

Let


(4.61)

For $r \geqslant 0$, the coefficient of u^{-r-1} in this formal Laurent series is $-$  ; the x^r here can be replaced by y^r due to the third relation in (4.22). Also introduce the rational function

$$\alpha_x(u) := \frac{(u - (x + 1))(u - (x - 1))}{(u - x)^2} \in \Bbbk(x, u). \tag{4.62}$$

The expansion of this as a power series in $\Bbbk[x][[u^{-1}]]$ is

$$\alpha_x(u) = 1 - (u - x)^{-2} = 1 - u^{-2} - 2xu^{-3} - 3x^2u^{-4} - 4x^3u^{-5} - \cdots , \tag{4.63}$$

$$\alpha_x(u)^{-1} = 1 + u^{-2} + 2xu^{-3} + (3x^2 + 1)u^{-4} + (4x^3 + 4x)u^{-5} + \cdots . \tag{4.64}$$

The following elementary lemma will play a fundamental role in the rest of the article. It would be hard to formulate this without the aid of generating functions.

Lemma 4.22. *The following bubble slide relations hold in $\mathcal{APar}$:*

$$\bigcirc(u)\ \Big| = \frac{\alpha_y(u)}{\alpha_x(u)}\Big\bullet\ \ \bigcirc(u)\ , \qquad \Big|\ \bigcirc(u) = \bigcirc(u)\ \ \Big\bullet\frac{\alpha_x(u)}{\alpha_y(u)}\ . \tag{4.65}$$

Proof. The two equations are equivalent, so we just prove the first one. When working with $\mathcal{Heis}$, we adopt the notation of [BSW, §3.1]: an open dot labelled by x^r means the rth power of the open dot in $\mathcal{Heis}$, and $\circlearrowright(u)$ is the formal Laurent series from [BSW, (3.13)]. Under the embedding of $\mathcal{APar}$ into $\mathcal{Heis}$, we have that

$$\bigcirc(u+1) = (u+1)1_{\mathbb{1}} - \bullet\,(u-(y-1))^{-1} = (u+1)1_{\mathbb{1}} - (u-x)^{-1}\circlearrowright = 1_{\mathbb{1}} + \circlearrowright(u).$$

The bubble slide relation for $\mathcal{Heis}$ from [BSW, (3.18)] gives that

$$\circlearrowright(u)\ \uparrow\ \downarrow = \alpha_x(u)\!\uparrow\ \circlearrowright(u)\ \downarrow = \alpha_x(u)\!\uparrow\ \ \downarrow\!\alpha_x(u)^{-1}\ \circlearrowright(u)\ .$$

According to (4.18), the label x on the open dot on the $\downarrow$ string translates into the label $x-1$ on a closed dot in $\mathcal{APar}$, and the label x on the open dot on the $\uparrow$ string translates into the label $y-1$ on a closed dot in $\mathcal{APar}$. So the relation just recorded can be written equivalently as

$$\bigcirc(u+1)\ \Big| = \frac{\alpha_{y-1}(u)}{\alpha_{x-1}(u)}\Big\bullet\ \ \bigcirc(u+1) = \frac{\alpha_y(u+1)}{\alpha_x(u+1)}\Big\bullet\ \ \bigcirc(u+1)\ .$$

Replacing u by $u-1$ everywhere gives the desired relation. □

The rational function $\alpha_y(u)/\alpha_x(u) \in \Bbbk(x,y,u)$ will also be important later on. The low degree terms of its expansion as a power series in u^{-1} can be computed using (4.63) and (4.64):

$$\frac{\alpha_y(u)}{\alpha_x(u)} = 1 + 2(x-y)u^{-3} + 3(x^2-y^2)u^{-4} + \left[4(x^3-y^3) + 2(x-y)\right]u^{-5} + \cdots . \tag{4.66}$$

For $n \geqslant 0$, let

$$C_n(u) = \sum_{r\geqslant 0} C_n^{(r)}u^{-r} := \bigcirc(u) \star 1_n \star \bigcirc(u)^{-1} = \frac{\alpha_y(u)}{\alpha_x(u)}\underset{n}{\Big\bullet}\cdots\frac{}{}\underset{2}{\Big\bullet}\frac{\alpha_y(u)}{\alpha_x(u)}\ \underset{1}{\Big\bullet}\frac{\alpha_y(u)}{\alpha_x(u)} \in 1_n APar 1_n[\![u^{-1}]\!], \tag{4.67}$$

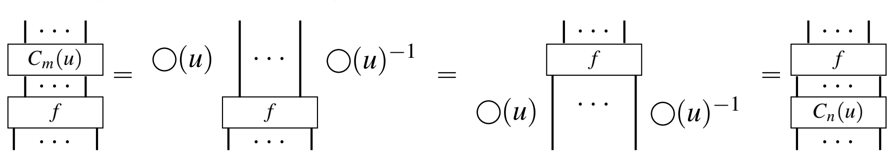
where the final equality follows by applying the bubble slide relation repeatedly. Then we define

$$C(u) = \sum_{r\geqslant 0} C^{(r)}u^{-r} := (C_n(u))_{n\geqslant 0} \in \prod_{n\geqslant 0} 1_n APar 1_n[\![u^{-1}]\!]. \tag{4.68}$$

Note by (4.66) that $C^{(0)} = 1$ and $C^{(1)} = C^{(2)} = 0$.

Theorem 4.23. $C(u) \in Z(APar)[\![u^{-1}]\!]$.

Proof. The interchange law immediately gives that



for any $f \in \mathrm{Hom}_{\mathcal{APar}}(n,m)$. □

Corollary 4.24. *For each $r \geqslant 1$, the element $Z^{(r)} = (Z_n^{(r)})_{n\geqslant 0} \in \prod_{n\geqslant 0} 1_n APar 1_n$ defined from*

$$Z_n^{(r)} := \sum_{i=1}^n \left((X_i^L)^r - (X_i^R)^r\right) = \left(X_1^L\right)^r + \cdots + \left(X_n^L\right)^r - \left(X_1^R\right)^r - \cdots - \left(X_n^R\right)^r$$

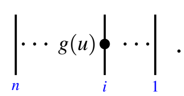
belongs to $Z(APar)$ (notation as in (4.48) and (4.49)). Moreover, the elements $Z^{(1)}, Z^{(2)}, \ldots$ generate the same subalgebra $Z_0(APar)$ of $Z(APar)$ as the elements $C^{(3)}, C^{(4)}, \ldots$.

Proof. Let $f(u) := \alpha_y(u)/\alpha_x(u)$ for short. Then define $g(u) := f'(u)/f(u) = \frac{d}{du}(\ln f(u))$ to be its logarithmic derivative. We have that

$$\begin{aligned} g(u) &= \left(-\frac{1}{u-(x-1)} + \frac{2}{u-x} - \frac{1}{u-(x+1)}\right) - \left(-\frac{1}{u-(y-1)} + \frac{2}{u-y} - \frac{1}{u-(y+1)}\right) \\ &= 2\cdot 3(y-x)u^{-4} + 2\cdot 6(y^2-x^2)u^{-5} + 2\cdot[10(y^3-x^3)+5(y-x)]u^{-6} + \cdots. \end{aligned}$$

We deduce for $r \geqslant 1$ that the u^{-r-3}-coefficient of $g(u)$ is equal to $2\binom{r+2}{2}(y^r - x^r)$ plus a linear combination of terms $(y^s - x^s)$ for $1 \leqslant s < r$ with $s \equiv r \pmod 2$.

The coefficients of the power series $C'(u)/C(u)$ are polynomials in the coefficients of the series $C(u)$. Hence, by the theorem, these coefficients are all central. To compute them, we take logarithmic derivatives of (4.67) to obtain the identity

$$C_n'(u)/C_n(u) = \sum_{i=1}^n$$ 

Using the previous paragraph and the definition of $Z^{(r)}$, we deduce for $r \geqslant 1$ that the central element defined by the u^{-r-3}-coefficient of $C'(u)/C(u)$ is equal to $2\binom{r+2}{2}Z^{(r)}$ plus a linear combination of $Z^{(s)}$ for $1 \leqslant s < r$ with $s \equiv r \pmod 2$. Finally, induction on r shows that each $Z^{(r)}$ is central.

The argument just given shows that each $Z^{(r)}$ lies in the subalgebra generated by $C^{(3)}, C^{(4)}, \ldots$. Conversely, by exponentiating an anti-derivative of the series $C'(u)/C(u)$, one shows that each $C^{(r)}$ can be expressed as a polynomial in $Z^{(1)}, Z^{(2)}, \ldots$. Hence, the two families of elements generate the same subalgebra of $Z(APar)$. ∎

Taking the images of $C(u)$ and each $Z^{(r)}$ under the functor p_t from (4.43) give

$$c(u) = \sum_{r\geqslant 0} c^{(r)}u^{-r} := (c_n(u))_{n\geqslant 0} \in Z(Par_t)[\![u^{-1}]\!] \quad \text{where } c_n(u) = \sum_{r\geqslant 0} c_n^{(r)}u^{-r} := p_t(C_n(u)), \tag{4.69}$$

$$z^{(r)} := \left(z_n^{(r)}\right)_{n\geqslant 0} \in Z(Par_t) \qquad \text{where } z_n^{(r)} := p_t(Z_n^{(r)}). \tag{4.70}$$

The elements $c_n^{(r)}$ and $z_n^{(r)}$ belong to the center $Z(P_n(t))$ of the partition algebra $P_n(t)$. In terms of the Jucys-Murphy elements (4.50), we have that

$$z_n^{(r)} = \sum_{i=1}^n \left[\left(x_i^L\right)^r - \left(x_i^R\right)^r\right] = (x_1^L)^r + \cdots + (x_n^L)^r - (x_1^R)^r - \cdots - (x_n^R)^r. \tag{4.71}$$

From Corollary 4.19, it follows that $z_n^{(1)}$ equals $z_n - nt$ where z_n is the central element from [E1, Th. 3.10(2)]. In fact, $z_n^{(1)}$ is closely related to the central elements of the group algebras $\Bbbk S_t$ defined by sums of transpositions:

Lemma 4.25 ([E1, Prop. 5.4])**.** *If $t \in \mathbb{N}$ then $\psi_t(z_n^{(1)}) : U_t^{\otimes n} \to U_t^{\otimes n}$ is equal to the endomorphism defined by the action of $\sum_{1\leqslant i<j\leqslant t}((i\ j) - 1) \in Z(\Bbbk S_t)$.*

Remark 4.26. After constructing the elements $z_n^{(r)} \in Z(P_n(t))$ in the manner explained above, we came across a recent paper of Creedon which constructs similar central elements; see [Cr, Th. 3.2.6]. To explain the connection, recall that the *rth supersymmetric power sum* in variables $x_1, \dots, x_n, y_1, \dots, y_m$ is $p_r(x_1, \dots, x_n | y_1, \dots, y_m) = x_1^r + \cdots + x_n^r - y_1^r - \cdots - y_m^r$. The expression on the right hand side of (4.71) is $p_r(x_1^L, \dots, x_n^L | x_1^R, \dots, x_n^R)$. It is easy to see that these elements belong to $Z(P_n(t))$ for all $r \geqslant 1$ if and only if $p_r((x_1^L - t/2)^r, \dots, (x_n^L - t/2)^r | (x_1^R - t/2)^r, \dots, (x_n^R - t/2)) \in Z(P_n(t))$ for all $r \geqslant 1$. Moreover, $p_r((x_1^L - t/2)^r, \dots, (x_n^L - t/2)^r | (x_1^R - t/2)^r, \dots, (x_n^R - t/2)) \in Z(P_n(t))$ coincides with the rth supersymmetric power sum $p_r(N_2, N_4, \dots, N_{2n} | -N_1, -N_3, \dots, -N_{2n-1})$ in Creedon's renormalized Jucys-Murphy elements from (4.56). Creedon showed that his elements are central in $P_n(t)$ by a direct check of relations. This gives an independent way to verify that $z^{(r)} = (z_n^{(r)})_{n \geqslant 0}$ belong to $Z(\mathcal{P}ar_t)$: Creedon's calculations show that they commute with all crossings (a surprisingly hard calculation), and after that it is easy to see that they commute with all other generators of $\mathcal{P}ar_t$.

Remark 4.27. In [CO, Def. 4.5], Comes and Ostrik define another family of central elements $\omega^r(t) = (\omega_n^r(t))_{n \geqslant 0}$ which lift the central elements of the group algebras $\Bbbk S_t$ defined by the sums of all r-cycles. We expect that our elements $z_n^{(r)}$ and their elements $\omega_n^r(t)$ are closely related, but we do not know any explicit formula. In particular, the Comes-Ostrik elements should generate the same subalgebra of $Z(\mathcal{P}ar_t)$ as our elements.

5. Classification and structure of blocks

Now we return to the study of the representation theory of $\mathcal{P}ar_t$. By considering images of the central elements from §4.6 under an analog of the Harish-Chandra homomorphism, we decompose $\mathcal{P}ar_t$-Mod as a product of subcategories, which turn out to be precisely the blocks. In fact, Par_t is semisimple if and only if $t \notin \mathbb{N}$, while if $t \in \mathbb{N}$ the non-simple blocks are in bijection with partitions of t. We also determine the structure of the non-simple blocks and explicitly show that they are all equivalent, recovering the results of Comes and Ostrik [CO].

5.1. **Harish-Chandra homomorphism.** Although we just explain in the case of $\mathcal{P}ar_t$, the general development in this subsection is valid for any monoidal triangular category, replacing $\mathcal{S}ym$ with the (semisimple) Cartan subcategory and replacing the set $\mathcal{P}$ of partitions by a set parametrizing isomorphism classes of irreducible representations of the Cartan subcategory.

According to the general definition (4.57), the center of the partition category is a subalgebra of the unital algebra $\prod_{n \geqslant 0} 1_n Par_t 1_n$. Let K^+ (resp., K^-) be the left ideal (resp., right ideal) of Par_t generated by the strictly downward partition diagrams (resp., the strictly upward partition diagrams). From the triangular basis, it is easy to see that $1_n K^+ 1_n = 1_n K^- 1_n$. We denote this by K_n. It is a two-sided ideal of the finite-dimensional algebra $1_n Par_t 1_n$, and we have that

$$1_n Par_t 1_n = \Bbbk S_n \oplus K_n. \tag{5.1}$$

Equivalently, K_n is the two-sided ideal of $1_n Par_t 1_n$ spanned by morphisms that factor through objects $m < n$. By analogy with Lie theory, we define the *Harish-Chandra homomorphism*

$$\widehat{\mathrm{HC}} : \prod_{n \geqslant 0} 1_n Par_t 1_n \to \prod_{n \geqslant 0} \Bbbk S_n, \qquad (z_n)_{n \geqslant 0} \mapsto (\mathrm{HC}_n\, z_n)_{n \geqslant 0}, \tag{5.2}$$

where $\mathrm{HC}_n : 1_n Par_t 1_n \twoheadrightarrow \Bbbk S_n$ is the projection along the direct sum decomposition (5.1). It is obvious from (5.1) that the restriction of $\widehat{\mathrm{HC}}$ to $Z(Par_t)$ defines an algebra homomorphism

$$\mathrm{HC} : Z(Par_t) \to Z(Sym) = \prod_{n \geqslant 0} Z(\Bbbk S_n). \tag{5.3}$$

As each $\Bbbk S_n$ is semisimple with its isomorphism classes of irreducible representations parametrized by $\mathcal{P}_n$, we can identify the algebra appearing on the right hand side of (5.3) with the algebra $\Bbbk[\mathcal{P}]$ of all functions from the set $\mathcal{P}$ to the field $\Bbbk$ with pointwise operations. Under this identification, the tuple $(z_n)_{n\geqslant 0} \in \prod_{n\geqslant 0} Z(\Bbbk S_n)$ corresponds to the function $f : \mathcal{P} \to \Bbbk$ such that $f(\lambda)$ is the scalar that z_n acts by on the Specht module $S(\lambda)$ for each $\lambda \in \mathcal{P}_n$. Then the Harish-Chandra homorphism becomes a homomorphism

$$\mathrm{HC} : Z(\mathit{Par}_t) \to \Bbbk[\mathcal{P}]. \tag{5.4}$$

To describe HC more explicitly in these terms, let $\lambda \in \mathcal{P}_n$ be a partition. As we have that $\mathrm{End}_{\mathit{Par}_t}(\Delta(\lambda)) \cong \mathrm{End}_{\mathit{Sym}}(S(\lambda)) \cong \Bbbk$, an element $z = (z_n)_{n\geqslant 0} \in Z(\mathit{Par}_t)$ acts on the standard module $\Delta(\lambda)$ as multiplication by a scalar denoted $\chi_\lambda(z)$. This defines an algebra homomorphism

$$\chi_\lambda : Z(\mathit{Par}_t) \to \Bbbk. \tag{5.5}$$

To compute $\chi_\lambda(z)$, note that it is the scalar by which z_n acts on the highest weight space $1_n\Delta(\lambda)$, which is the scalar arising from the action of $\mathrm{HC}_n(z_n) \in Z(\Bbbk S_n)$ on $S(\lambda)$. It follows that

$$\chi_\lambda(z) = \mathrm{HC}_n(z_n)(\lambda) = \mathrm{HC}(z)(\lambda). \tag{5.6}$$

Recall that $Z(\mathit{Par}_t)$ is a commutative pseudo-compact topological algebra with respect to the profinite topology. Let $\mathrm{Spec}(Z(\mathit{Par}_t))$ be its set of open (= finite-codimensional) maximal ideals.

Lemma 5.1. $\mathrm{Spec}(Z(\mathit{Par}_t)) = \{\ker \chi_\lambda \mid \lambda \in \mathcal{P}\}$.

Proof. Points in $\mathrm{Spec}(Z(\mathit{Par}_t))$ parametrize isomorphism classes of finite-dimensional irreducible modules for $Z(\mathit{Par}_t)$ Let L_λ be the irreducible $Z(\mathit{Par}_t)$-module associated to $\chi_\lambda : Z(\mathit{Par}_t) \to \Bbbk$. Then we need to show that any finite-dimensional irreducible $Z(\mathit{Par}_t)$-module L is isomorphic to L_λ for some $\lambda \in \mathcal{P}$. To see this, we find it easiest to work equivalently in terms of irreducible comodules over the cocenter $C := C(\mathit{Par}_t)$ defined in (4.59). So let L be an irreducible C-comodule and L^* be the dual comodule, there being no need to distinguish between left or right since C is cocommutative. By definition, C is a quotient of the coalgebra D that is the direct sum of the coalgebras $(1_n \mathit{Par}_t 1_n)^*$ for all $n \geqslant 0$. Since L^* is isomorphic to a subcomodule of the regular C-comodule, it follows that L^* is isomorphic to a subquotient of the restriction of the regular D-comodule to C. Hence, L^* is isomorphic to a subquotient of $(1_n \mathit{Par}_t 1_n)^*$ for some n. So L is isomorphic to a subquotient of $1_n \mathit{Par}_t 1_n$. Now recall that the left Par_t-module $\mathit{Par}_t 1_n$ has a Δ-flag, and $z \in Z(\mathit{Par}_t)$ acts on $\Delta(\lambda)$ as multiplication by the scalar $\chi_\lambda(z)$. Hence, all composition factors of the finite-dimensional $Z(\mathit{Par}_t)$-module $1_n \mathit{Par}_t 1_n$ are of the form L_λ for $\lambda \in \mathcal{P}$. □

Let $\approx_t$ be the equivalence relation on $\mathcal{P}$ defined by

$$\lambda \approx_t \mu \Leftrightarrow \chi_\lambda = \chi_\mu. \tag{5.7}$$

From Lemma 5.1, we see that the equivalence classes $\mathcal{P}/\approx_t$ parametrize the points in $\mathrm{Spec}(Z(\mathit{Par}_t))$.

Lemma 5.2. *The image of* $\mathrm{HC} : Z(\mathit{Par}_t) \to \Bbbk[\mathcal{P}]$ *consists of of all functions* $f \in \Bbbk[\mathcal{P}]$ *which are constant on* $\approx_t$*-equivalence classes. Moreover, for each subset* S *of* $\mathcal{P}$ *that is a union of* $\approx_t$*-equivalence classes, there is a unique central idempotent* $1_S \in Z(\mathit{Par}_t)$ *such that*

$$\mathrm{HC}(1_S)(\lambda) = \begin{cases} 1 & \text{if } \lambda \in S, \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

If S *is a single equivalence class then* 1_S *is a primitive idempotent, and* $Z(\mathit{Par}_t) = \prod_{S\in\mathcal{P}/\approx_t} 1_S Z(\mathit{Par}_t)$.

Proof. It is clear from (5.6) that any function in the image of HC is constant on $\approx_t$-equivalence classes. Conversely, take a function $f \in \Bbbk[\mathcal{P}]$ which is constant on equivalence classes. For an equivalence class $S \in \mathcal{P}/\approx_t$, let L_S be the irreducible $Z(Par_t)$-module associated to the central character χ_λ ($\lambda \in S$). The previous lemma shows that these give a full set of pairwise inequivalent irreducible finite-dimensional $Z(Par_t)$-modules. It follows that the cocommutative coalgebra $C(Par_t)$ decomposes as a direct sum of indecomposable coideals

$$C(Par_t) = \bigoplus_{S \in \mathcal{P}/\approx_t} C_S,$$

where C_S is the injective hull of L_S. Then we consider the linear map $\theta : C(Par_t) \to C(Par_t)$ defined by multiplication by the scalar $f(\lambda)$ ($\lambda \in S$) on the summand C_S. This is a comodule homomorphism. Now we use that

$$Z(Par_t) = C(\mathcal{P}ar_t)^* \cong \operatorname{End}_{C(Par_t)}(C(Par_t))^{\mathrm{op}}$$

as holds for any coalgebra, e.g., see [BS, Lem. 2.2]. It implies that θ defines an element of $Z(Par_t)$. The image of this element under HC is the function $f \in \Bbbk[\mathcal{P}]$.

To prove the existence of the idempotent 1_S for any S that is a union of $\approx_t$-equivalence classes, we apply the construction in the previous paragraph to obtain $1_S \in Z(Par_t)$ such that 1_S acts as the identity on the indecomposable summands $C_{S'}$ of $C(Par_t)$ for all $\approx_t$-equivalence classes $S' \subseteq S$ and as zero on all other summands. This is an idempotent satisfying (5.8), and it is a primitive idempotent if and only if S is a single equivalence class. We then have that

$$Z(Par_t) = \prod_{S \in \mathcal{P}/\approx_t} 1_S Z(Par_t)$$

as this is the algebra decomposition that is dual to the decomposition of $C(Par_t)$ as the direct sum of its indecomposable coideals. □

For $S \in \mathcal{P}/\approx_t$, the primitive central idempotent $1_S \in Z(Par_t)$ from Lemma 5.2 is not an element of Par_t, but we have that $1_S = (1_{S,n})_{n \geqslant 0}$ for idempotents $1_{S,n} = 1_S 1_n = 1_n 1_S \in 1_n Par_t 1_n$. Moreover, for a fixed n the idempotent $1_{S,n}$ is zero for all but finitely many S, so that $1_n = \sum_{S \in \mathcal{P}/\approx_t} 1_{S,n}$. The locally unital algebras $1_S Par_t = \bigoplus_{m,n \geqslant 0} 1_{S,m} Par_t 1_{S,n}$ are the *blocks* of the partition algebra Par_t, and we have the block decompositions

$$Par_t = \bigoplus_{S \in \mathcal{P}/\approx_t} 1_S Par_t, \qquad Par_t\text{-Mod} = \prod_{S \in \mathcal{P}/\approx_t} 1_S Par_t\text{-Mod}. \tag{5.9}$$

Representatives for the isomorphism classes of irreducible $1_S Par_t$-modules are given by the modules $L(\lambda)$ for all $\lambda \in S$.

Lemma 5.3. *The following properties are equivalent:*

- *(i) Par_t is semisimple.*
- *(ii) All of the $\approx_t$-equivalence classes are singletons.*
- *(iii) $\mathrm{HC} : Z(Par_t) \to \Bbbk[\mathcal{P}]$ is surjective.*
- *(iv) $\mathrm{HC} : Z(Par_t) \to \Bbbk[\mathcal{P}]$ is an isomorphism.*

Proof. If (i) holds, then Par_t is a direct sum of locally unital matrix algebras indexed by the set $\mathcal{P}$ that labels its irreducible representations. Hence, its center is the direct product $\prod_{\lambda \in \mathcal{P}} \Bbbk_\lambda$. It follows easily that HC is an isomorphism, i.e., (iv) holds.

Obviously, (iv) implies (iii).

The equivalence of (ii) and (iii) follows from Lemma 5.2.

It remains to show that (ii) implies (i). Assuming (ii), Lemma 5.2 shows for any $\lambda \in \mathcal{P}$ that there is a primitive central idempotent in $Z(Par_t)$ which acts as the identity on $\Delta(\lambda)$ and as zero on $L(\mu)$ for all

$\mu \neq \lambda$. We deduce that all composition factors of $\Delta(\lambda)$ are isomorphic to $L(\lambda)$. Since this is a highest weight module we have that $[\Delta(\lambda) : L(\lambda)] = 1$, so actually $\Delta(\lambda)$ is irreducible. This is the case for all $\lambda \in \mathcal{P}$, so by BGG reciprocity we deduce that $P(\lambda) = \Delta(\lambda) = L(\lambda)$ for all λ, and (i) holds. □

Remark 5.4. When Par_t is semisimple, the standardization functor $j_! : Sym\text{-Mod}_{\mathrm{fd}} \to \mathcal{P}ar_t\text{-Mod}_{\mathrm{lfd}}$ sends the irreducible Sym-modules $S(\lambda)$ to the irreducible Par_t-modules $\Delta(\lambda) = L(\lambda)$ for all $\lambda \in \mathcal{P}$. It follows easily that $j_!$ is an equivalence of categories in the semisimple case (although it is not a monoidal equivalence). Since the center is a Morita invariant, it follows that $Z(Sym) \cong Z(Par_t)$ in the semisimple case. Recalling that $Z(Sym) \cong \Bbbk[\mathcal{P}]$, this gives another way to understand the equivalence (i)⇒(iv) of Lemma 5.3.

Remark 5.5. As Par_t-Mod$_{\mathrm{lfd}}$ is an upper finite highest weight category, there is also a canonical partial order on $\mathcal{P}$, called the *minimal order* in [BS, Rem. 3.68], which we denote here by $\succeq_t$. By definition, this is the partial order generated by the relation $\lambda \succeq_t \mu$ if $[\Delta(\lambda) : L(\mu)] \neq 0$. As always for highest weight categories, the equivalence relation $\approx_t$ defining the blocks of Par_t is the transitive closure of the minimal order $\succeq_t$. We will describe $\succeq_t$ explicitly in Corollary 5.26 below.

5.2. **"Blocks".** In the previous subsection, we introduced an equivalence relation $\approx_t$ on $\mathcal{P}$ whose equivalence classes parametrize the blocks of Par_t. The relation $\approx_t$ was defined in terms of the central characters $\chi_\lambda : Z(Par_t) \to \Bbbk$ arising from the irreducible Par_t-modules $L(\lambda)$; see (5.7). On the other hand, in (4.69) and (4.70), we constructed some explicit central elements of Par_t. Let $\sim_t$ be the equivalence relation on $\mathcal{P}$ defined from

$$\lambda \sim_t \mu \Leftrightarrow \chi_\lambda|_{Z_0(Par_t)} = \chi_\mu|_{Z_0(Par_t)} \tag{5.10}$$

where $Z_0(Par_t)$ is the subalgebra of $Z(Par_t)$ generated by the elements $\left\{c^{(r)} \;\middle|\; r \geqslant 3\right\}$ (equivalently, by the elements $\left\{z^{(r)} \;\middle|\; r \geqslant 1\right\}$). We refer to the $\sim_t$-equivalence classes as "blocks". We obviously have that

$$\lambda \approx_t \mu \Rightarrow \lambda \sim_t \mu, \tag{5.11}$$

i.e., "blocks" are unions of blocks. Defining 1_S as in Lemma 5.2, there are induced "block" decompositions

$$Par_t = \bigoplus_{S \in \mathcal{P}/\sim_t} 1_S\, Par_t, \qquad\qquad Par_t\text{-Mod} = \prod_{S \in \mathcal{P}/\sim_t} 1_S\, Par_t\text{-Mod}. \tag{5.12}$$

In this subsection, we are going to describe the relation $\sim_t$ in explicit combinatorial terms.

Lemma 5.6. *The images of the elements $x_j^L, x_j^R, s_k^L, s_k^R \in 1_n Par_t 1_n$ from (4.50) under the Harish-Chandra homomorphism $\widehat{\mathrm{HC}}$ from (5.2) are*

$$\mathrm{HC}_n(x_j^L) = x_j, \qquad\qquad \mathrm{HC}_n(x_j^R) = t - j + 1, \tag{5.13}$$

$$\mathrm{HC}_n(s_k^L) = 1, \qquad\qquad \mathrm{HC}_n(s_k^R) = (k\; k{+}1), \tag{5.14}$$

where $x_j \in \Bbbk S_n$ is the Jucys-Murphy element from (2.37).

Proof. Applying HC_n to the relations (4.29) and (4.30) (on the kth, $(k+1)$th and $(k+2)$th strings) we deduce that $\mathrm{HC}_n(s_{k+1}^L) = (k\;\; k{+}1\;\; k{+}2)\,\mathrm{HC}_n(s_k^L)(k{+}2\;\; k{+}1\;\; k)$ and $\mathrm{HC}_n(s_{k+1}^R) = (k\;\; k{+}1\;\; k{+}2)\,\mathrm{HC}_n(s_k^R)(k{+}2\;\; k{+}1\;\; k)$. Now (5.14) follows by induction on k, the base case $k = 1$ being immediate from (4.45). Note for this that $(k\;\; k{+}1\;\; k{+}2)(k\;\; k{+}1)(k{+}2\;\; k{+}1\;\; k) = (k{+}1\;\; k{+}2)$.

Applying HC_n to the relations (4.27) and (4.28) (on the jth and $(j{+}1)$th strings), using also (4.24), we deduce that $\mathrm{HC}_n(x_{j+1}^L) = (j\;\, j{+}1)\,\mathrm{HC}_n(x_j^L)(j\;\, j{+}1) + \mathrm{HC}_n(s_j^R)$ and $\mathrm{HC}_n(x_{j+1}^R) = (j\;\, j{+}1)\,\mathrm{HC}_n(x_j^R)(j\;\, j{+}1) - \mathrm{HC}_n(s_j^L)$. Now (5.13) follows using (5.14) and induction on j, the base case $j = 1$ being immediate from (4.44). Note for this that $(j\;\; j{+}1)x_j(j\;\; j{+}1) + (j\;\; j{+}1) = x_{j+1}$. □

Lemma 5.7. *For $\lambda \in \mathcal{P}_n$, we have that*

$$\chi_\lambda(c(u)) = \prod_{i=1}^{n} \frac{\alpha_{\mathrm{cont}_i(\mathrm{T})}(u)}{\alpha_{t-i+1}(u)} \tag{5.15}$$

where T *is some fixed standard λ-tableau and $\alpha_x(u)$ is as in* (4.62).

Proof. Note by (5.6) that $\chi_\lambda(c(u)) = \mathrm{HC}_n(c_n(u))(\lambda) \in \Bbbk[\![u^{-1}]\!]$. To compute this, we use Lemma 5.6 and the explicit formula for $c_n(u) = p_t(C_n(u))$ arising from (4.67) to deduce that

$$\mathrm{HC}_n(c_n(u)) = \prod_{i=1}^{n} \frac{\alpha_{x_i}(u)}{\alpha_{t-i+1}(u)} \in Z(\Bbbk S_n)[\![u^{-1}]\!].$$

To evaluate this at λ, we act on the basis vector v_T from Young's orthonormal basis for $S(\lambda)$, remembering that $x_i v_\mathrm{T} = \mathrm{cont}_i(\mathrm{T}) v_\mathrm{T}$. □

Lemma 5.6 suggests some combinatorics of weights. Let P be the free Abelian group on basis $\{\Lambda_c \mid c \in \Bbbk\}$. Let $\varepsilon_c := \Lambda_c - \Lambda_{c+1}$ and $\alpha_c := \varepsilon_c - \varepsilon_{c-1}$. We define the *weight* of a rational function $f(u) \in \Bbbk(u)$ to be

$$\mathrm{wt}\, f(u) := \sum_{c \in \Bbbk} \left[\begin{pmatrix} \text{Multiplicity of } c \\ \text{as a pole of } f(u) \end{pmatrix} - \begin{pmatrix} \text{Multiplicity of } c \\ \text{as a zero of } f(u) \end{pmatrix} \right] \Lambda_c \in P. \tag{5.16}$$

For example, $\mathrm{wt}\, \alpha_c(u) = -\Lambda_{c-1} + 2\Lambda_c - \Lambda_{c+1} = \alpha_c$. For $\lambda \in \mathcal{P}_n$, let $\mathrm{wt}_t(\lambda)$ be the weight of the rational function appearing on the right hand side of (5.15). As the coefficients of the power series $c(u)$ generate the subalgebra $Z_0(Par_t)$, the equivalence relation $\sim_t$ defined by (5.10) satisfies

$$\lambda \sim_t \mu \Leftrightarrow \mathrm{wt}_t(\lambda) = \mathrm{wt}_t(\mu). \tag{5.17}$$

This suggests using elements of P rather than $\sim_t$-equivalence classes to index the "blocks" from (5.12): for any $\gamma \in P$, let

$$S(\gamma) := \left\{ \lambda \in \mathcal{P} \,\middle|\, \mathrm{wt}_t(\lambda) = \gamma \right\}. \tag{5.18}$$

Then define

$$\mathrm{pr}_\gamma : Par_t\text{-Mod} \to Par_t\text{-Mod} \tag{5.19}$$

to be the projection functor defined by multiplication by the central idempotent $1_{S(\gamma)}$ from Lemma 5.2. In other words, pr_γ projects a Par_t-module V to its largest submodule all of whose irreducible subquotients are of the form $L(\lambda)$ for $\lambda \in \mathcal{P}$ with $\mathrm{wt}_t(\lambda) = \gamma$. The admissible $\gamma \in P$ which parametrize "blocks" are the ones with $S(\gamma) \neq \varnothing$; if $S(\gamma) = \varnothing$ then pr_γ is the zero functor.

Lemma 5.8. *For $\lambda \in \mathcal{P}_n$ and any standard λ-tableau T, we have that*

$$\mathrm{wt}_t(\lambda) = \sum_{i=1}^{n} (\alpha_{\mathrm{cont}_i(\mathrm{T})} - \alpha_{t-i+1}) = (\varepsilon_{t-|\lambda|} - \varepsilon_t) + (\varepsilon_{\lambda_1 - 1} - \varepsilon_{-1}) + \cdots + (\varepsilon_{\lambda_k - k} - \varepsilon_{-k}) \tag{5.20}$$

for any $k \geqslant \ell(\lambda)$. Moreover, given another partition $\mu \in \mathcal{P}$, we have that $\mathrm{wt}_t(\lambda) = \mathrm{wt}_t(\mu)$ if and only if the infinite sequences $(t - |\lambda|, \lambda_1 - 1, \lambda_2 - 2, \dots)$ and $(t - |\mu|, \mu_1 - 1, \mu_2 - 2, \dots)$ are rearrangements of each other.

Proof. The first equality in (5.20) follows immediately from Lemma 5.7. To deduce the second equality, take $k \geqslant \ell(\lambda)$. For $1 \leqslant r \leqslant k$ the contents of the nodes in the rth row of the Young diagram of λ are $1-r, 2-r, \dots, \lambda_r - r$, and we have that $\alpha_{1-r} + \cdots + \alpha_{\lambda_r - r} = \varepsilon_{\lambda_r - r} - \varepsilon_{-r}$. Also $\alpha_t + \alpha_{t-1} + \cdots + \alpha_{t-n+1} = \varepsilon_t - \varepsilon_{t-n}$. Now the desired formula follows easily.

Rearranging the right hand side of (5.20) gives that $\varepsilon_{t-|\lambda|}+\varepsilon_{\lambda_1-1}+\varepsilon_{\lambda_2-2}+\cdots+\varepsilon_{\lambda_k-k}=\mathrm{wt}_t(\lambda)+\varepsilon_{-1}+\varepsilon_{-2}+\cdots+\varepsilon_{-k}+\varepsilon_t$ for any $k\geqslant \ell(\lambda)$. Hence, we have that $\mathrm{wt}_t(\lambda)=\mathrm{wt}_t(\mu)$ if and only if

$$\varepsilon_{t-|\lambda|}+\varepsilon_{\lambda_1-1}+\varepsilon_{\lambda_2-2}+\cdots+\varepsilon_{\lambda_k-k}=\varepsilon_{t-|\mu|}+\varepsilon_{\mu_1-1}+\varepsilon_{\mu_2-2}+\cdots+\varepsilon_{\mu_k-k}$$

for all $k\gg 0$. This is clearly equivalent to saying that the infinite sequences $(t-|\lambda|,\lambda_1-1,\lambda_2-2,\dots)$ and $(t-|\mu|,\mu_1-1,\mu_2-2,\dots)$ may be obtained from each other by permuting the entries. $\square$

The final assertion from Lemma 5.8 shows that $\sim_t$ is exactly the same as the equivalence relation on partitions defined in [CO, Def. 5.1]. The equivalence classes of this relation were investigated in detail in [CO, §5.3]. The following summarizes the results obtained there. For the statement, we say that $\lambda\in\mathcal{P}$ is *typical* if it is the only partition in its $\sim_t$-equivalence class; otherwise we say that λ is *atypical*. Of course, these notions depend on the fixed value of the parameter t.

Theorem 5.9 (Comes-Ostrik)**.** *If $t\notin\mathbb{N}$ then all partitions are typical. If $t\in\mathbb{N}$ then there is a bijection $\mathcal{P}_t\xrightarrow{\sim}\{$atypical $\sim_t$-equivalence classes$\}$ taking $\kappa\in\mathcal{P}_t$ to the $\sim_t$-equivalence class $\{\kappa^{(0)},\kappa^{(1)},\kappa^{(2)},\dots\}$ where*

$$\kappa^{(n)}:=(\kappa_1+1,\dots,\kappa_n+1,\kappa_{n+2},\kappa_{n+3},\dots)\in\mathcal{P}_{t+n-\kappa_{n+1}},\tag{5.21}$$

i.e., it is the partition obtained from κ by adding a node to the first n rows of its Young diagram then removing its $(n+1)$th row. Moreover, still assuming $t\in\mathbb{N}$, a partition $\lambda\in\mathcal{P}$ is typical if and only if $t-|\lambda|=\lambda_i-i$ for some $i\geqslant 1$.

Example 5.10. For any $t\in\mathbb{N}$, the $\sim_t$-equivalence class associated to $\kappa=(t)\in\mathcal{P}_t$ is

$$S=\left\{\varnothing,(t+1),(t+1,1),(t+1,1^2),\cdots\right\}.$$

For $t\in\mathbb{N}-\{0,1\}$, the $\sim_t$-equivalence class associated to $\kappa=(1^t)\in\mathcal{P}_t$ is

$$S=\left\{(1^{t-1}),(2,1^{t-2}),(2^2,1^{t-3}),\cdots,(2^{t-1}),(2^t),(2^t,1),(2^t,1^2),\cdots\right\}.$$

As noted in [CO, Cor. 5.23] (using a different argument for the forward implication), the first assertion of Theorem 5.9 allows us to recover the following well known result of Deligne [D, Th. 2.18]: $\underline{\mathrm{Rep}}(S_t)$ is semisimple if and only if $t\notin\mathbb{N}$. In terms of the path algebra Par_t, Deligne's result can be stated as follows.

Corollary 5.11 (Deligne)**.** *Par_t is semisimple if and only if $t\notin\mathbb{N}$.*

Proof. We already know that Par_t is not semisimple if $t\in\mathbb{N}$ by Corollary 4.2. Conversely, if $t\notin\mathbb{N}$, we apply the criterion from Lemma 5.3, noting that all $\approx_t$-equivalence classes are singletons thanks to (5.11) and the first part of Theorem 5.9. $\square$

Remark 5.12. When $t\notin\mathbb{N}$, the above arguments show for $\lambda,\mu\in\mathcal{P}$ with $\lambda\neq\mu$ that there is a central element in the subalgebra $Z_0(Par_t)$ of $Z(Par_t)$ which acts by different scalars on the irreducible modules $L(\lambda)$ and $L(\mu)$. It follows in these cases that $Z_0(Par_t)$ is a dense subalgebra of the pseudo-compact topological algebra $Z(Par_t)$[3]. We do not expect that this is the case when $t\in\mathbb{N}$, but nevertheless $Z_0(Par_t)$ is still sufficiently large to separate blocks. This will be established in Corollary 5.25 below, which shows for any value of t that the relations $\sim_t$ and $\approx_t$ coincide, so that "blocks" are blocks, and (5.12) is always the same decomposition as (5.9); see also [CO, Th. 5.3].

[3]These algebras are certainly not equal since $Z(Par_t)\cong\prod_{\lambda\in\mathcal{P}}\Bbbk_\lambda$ is of uncountable dimension.

5.3. **Special projective functors.** From now on, we will primarily be interested in parameter values $t \in \mathbb{N}$, so that Par_t is not semisimple. Consider the atypical block $\{\kappa^{(0)}, \kappa^{(1)}, \kappa^{(2)}, \dots\}$ associated to $\kappa \in \mathcal{P}_t$. From (5.21), it follows that $\kappa^{(n)}$ is obtained from $\kappa^{(n-1)}$ by adding $\kappa_n - \kappa_{n+1} + 1$ nodes to the nth row of its Young diagram, leaving all other rows unchanged. The partition $\kappa^{(0)}$ is the smallest of all of the $\kappa^{(n)}$, hence, it is maximal in the highest weight ordering from Theorem 3.3. It follows that

$$P(\kappa^{(0)}) \cong \Delta(\kappa^{(0)}). \tag{5.22}$$

The indecomposable projectives $\Delta(\kappa^{(0)})$ are exactly the ones of non-zero categorical dimension mentioned already in Remark 4.3, with the irreducible $\Bbbk S_t$-module associated to the image of $\Delta(\kappa^{(0)})$ under the equivalence $\overline{\psi}_t$ between the semisimplification of $\mathrm{Kar}(\mathcal{P}ar_t)$ and $\Bbbk S_t$-$\mathrm{Mod}_{\mathrm{fd}}$ being the Specht module $S(\kappa)$. It is also useful to note for $t \in \mathbb{N}$ and $\kappa \in \mathcal{P}_t$ that the associated block $\{\kappa^{(0)}, \kappa^{(1)}, \dots\}$ is the set $S(\gamma)$ from (5.18) for

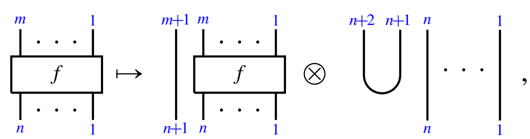
$$\gamma := (\varepsilon_{\kappa_1} - \varepsilon_t) + (\varepsilon_{\kappa_2 - 1} - \varepsilon_{-1}) + \dots + (\varepsilon_{\kappa_t - t + 1} - \varepsilon_{-t}) \in P. \tag{5.23}$$

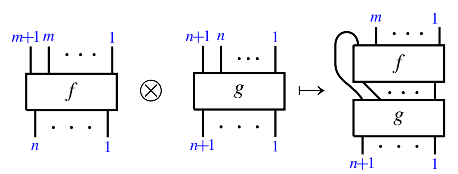
This is follows easily using (5.20) and (5.21).

In order to understand the structure of the atypical blocks more fully, we are going to use the endofunctor $| \star : \mathcal{P}ar_t \to \mathcal{P}ar_t$. Let

$$D := \mathrm{res}_{|\star} = 1_{|\star} Par_t \otimes_{Par_t} : Par_t\text{-Mod} \to Par_t\text{-Mod} \tag{5.24}$$

be the corresponding restriction functor from (2.21). This obviously preserves locally finite-dimensional modules. The object $|$ is self-dual so, by Lemma 2.5, the restriction functor D is isomorphic to the induction functor $\mathrm{ind}_{|\star}$. By Corollary 2.6, D is a self-adjoint projective functor, so it preserves finitely generated projectives (and finitely cogenerated injectives). To make the canonical adjunction as explicit as possible, we note that its unit and counit are induced by the bimodule homomorphisms

$$\eta : Par_t \to 1_{|\star} Par_t \otimes_{Par_t} 1_{|\star} Par_t,$$  (5.25)

$$\varepsilon : 1_{|\star} Par_t \otimes_{Par_t} 1_{|\star} Par_t \to Par_t,$$  (5.26)

Using (2.13), it follows that D commutes with the duality $?^{\circledast}$ on Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$.

Lemma 5.13. *For $\lambda \in \mathcal{P}$, there is a filtration $0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 = D\Delta(\lambda)$ such that*

$$V_3/V_2 \cong \bigoplus_{a \in \mathrm{add}(\lambda)} \Delta\left(\lambda + \boxed{a}\right),$$

$$V_2/V_1 \cong \Delta(\lambda) \oplus \bigoplus_{b \in \mathrm{rem}(\lambda)} \bigoplus_{a \in \mathrm{add}(\lambda - \boxed{b})} \Delta\left(\left(\lambda - \boxed{b}\right) + \boxed{a}\right),$$

$$V_1/V_0 \cong \bigoplus_{b \in \mathrm{rem}(\lambda)} \Delta\left(\lambda - \boxed{b}\right).$$

Proof. By Lemma 2.4, D is isomorphic to the functor $Q(\square) \circledast ?$ defined by taking the induction product with the projective module $Q(\square)$. By Lemma 3.9(iii) and Lemma 3.13, $Q(\square)$ has a Δ-flag of length two with sections $\Delta(\square)$ at the top and $\Delta(\varnothing)$ at the bottom. Applying Theorem 3.11, we deduce that

$D\Delta(\lambda) \cong Q(\square)\circledast\Delta(\lambda)$ has a Δ-flag with one section $\Delta(\varnothing)\circledast\Delta(\lambda) \cong \Delta(\lambda)$ and other sections arising from the Δ-flag of $\Delta(\square)\circledast\Delta(\lambda)$ described in that theorem. For this, one just needs to know the values of the reduced Kronecker coefficients $\overline{G}^{\mu}_{\square,\lambda}$ which were worked out in Example 3.6. The Δ-flag can be ordered in the way described since $\operatorname{Ext}^1(\Delta(\mu), \Delta(\nu)) = 0$ if $|\mu| \leqslant |\nu|$. $\square$

Remark 5.14. Lemma 5.13 also follows from Lemma 5.29 below, which constructs the filtration explicitly. The proof of Lemma 5.29 is also valid over fields of positive characteristic.

Now we are going to use the affine partition category $\mathcal{APar}$ to decompose the endofunctor D as a direct sum of *special projective functors* $D_{b|a}$. The approach here is analogous to the way the affine symmetric category $\mathcal{ASym}$ was used to decompose E and F as direct sums of E_a and F_b in (2.39). As noted at the end of §4.3, $\mathcal{Par}_t$ is isomorphic to the quotient of $\mathcal{APar}$ by a left tensor ideal. Hence, $\mathcal{Par}_t$ is a strict $\mathcal{APar}$-module category. The self-adjoint functor D is also the restriction functor $\operatorname{res}_{|\star}$ arising from this categorical action of $\mathcal{APar}$ on $\mathcal{Par}_t$. Now the left and right dots give us natural transformations

Applying the general construction from (2.8) to these, we obtain commuting endomorphisms

$$x := \operatorname{res}_\alpha : D \Rightarrow D, \qquad\qquad y := \operatorname{res}_\beta : D \Rightarrow D. \tag{5.27}$$

Let $D_{b|a}$ be the summand of D that is the simultaneous generalized eigenspace of x and y of eigenvalues a and b, respectively. Explicitly, $D = \operatorname{res}_{|\star}$ is defined by tensoring with the bimodule $1_{|\star}\mathcal{Par}_t$, and the endomorphisms x and y of D are induced by the bimodule endomorphisms ρ and λ of $1_{|\star}\mathcal{Par}_t$ given by left multiplication by x^R_{m+1} and x^L_{m+1}, respectively, on the summand $1_{m+1}\mathcal{Par}_t$ of $1_{|\star}\mathcal{Par}_t$ for each $m \geqslant 0$. Then, $D_{b|a}$ is the functor defined by tensoring with the summand of $1_{|\star}\mathcal{Par}_t$ that is the simultaneous generalized eigenspaces of ρ and λ for the eigenvalues a and b, respectively. As $1_{m+1}\mathcal{Par}_t = \bigoplus_{n\geqslant 0} 1_{m+1}\mathcal{Par}_t 1_n$ with each $1_{m+1}\mathcal{Par}_t 1_n$ being finite-dimensional, these endomorphisms are locally finite, so we have that

$$D = \bigoplus_{a,b\in\Bbbk} D_{b|a}. \tag{5.28}$$

Lemma 5.15. *For $a, b \in \Bbbk$, the endofunctor $D_{b|a}$ commutes with the duality $?^{\circledast}$, i.e., $D_{b|a}\circ ?^{\circledast} \cong ?^{\circledast} \circ D_{b|a}$.*

Proof. This follows from the fact that D commutes with the duality $?^{\circledast}$, and σ fixes both the left dot and the right dot. $\square$

Lemma 5.16. *For $a, b \in \Bbbk$, the endofunctors $D_{b|a}$ and $D_{a|b}$ are biadjoint.*

Proof. The adjunction $(D_{a|b}, D_{b|a})$ is induced by the self-adjunction of D. The unit $\bar\eta$ of adjunction comes from the bimodule homomorphism that is the composition of the unit η from (5.25) with the projection onto the generalized a and b eigenspaces of ρ and λ on the left tensor factor and the generalized b and a eigenspaces of ρ and λ on the right tensor factor. The counit $\bar\varepsilon$ of adjunction comes from the composition of the counit ε from (5.26) with the inclusion of the generalized b and a eigenspaces of ρ and λ on the left tensor factor and the generalized a and b eigenspaces of ρ and λ on the right tensor factor. To check the zig-zag identities, one just needs to use the relations

i.e., the fact that the left and right dots are duals. $\square$

When $a \neq b$, Lemma 5.16 can also be proved a bit more easily using the description of $D_{b|a}$ given in the following lemma, since the projection functor pr_γ commutes with $?^{\circledast}$ thanks to (3.32).

Lemma 5.17. *Let* pr_γ *be the projection functor defined by* (5.19). *If* $a \neq b$ *then*

$$D_{b|a} \cong \bigoplus_{\gamma \in P} \mathrm{pr}_{\gamma+\alpha_a-\alpha_b} \circ D \circ \mathrm{pr}_\gamma .$$

Also $\bigoplus_{\gamma\in P} \mathrm{pr}_\gamma \circ D \circ \mathrm{pr}_\gamma \cong \bigoplus_{a \in \Bbbk} D_{a|a}$.

Proof. Take a module V in the "block" parametrized by $\gamma \in P$, so that $\mathrm{wt}_t(\lambda) = \gamma$ for all irreducible subquotients of V. We need to show that $D_{b|a}V$ is in the "block" parametrized by $\gamma + \alpha_a - \alpha_b$. Since $D_{b|a}$ is exact, we may assume that V is irreducible, so $V = L(\lambda)$ for $\lambda \in \mathcal{P}$ with $\mathrm{wt}_t(\lambda) = \gamma$. The module $DV = 1_{|\star} Par_t \otimes_{Par_t} V \cong 1_{|\star} V$ is generated by the finite-dimensional vector spaces $1_{m+1}V$ for all $m \geqslant 0$. Hence, $D_{b|a}V$ is generated by the simultaneous generalized eigenspaces of x^R_{m+1} and x^L_{m+1} on $1_{m+1}V$ of eigenvalues a and b, respectively. Consequently, if $L(\mu)$ is an irreducible subquotient of $D_{b|a}V$, then $c(u)$ must act on $L(\mu)$ in the same way as $| \star c_m(u)$ acts on a simultaneous eigenvector $v \in 1_{m+1}V$ for x^R_{m+1} and x^L_{m+1} of eigenvalues a and b. Also $c_{m+1}(u)$ acts on $v \in V$ as multiplication by $\chi_\lambda(c(u))$, the rational function displayed on the right hand side of (5.15). Using (4.65), we deduce that

$$\chi_\mu(c(u)) = \frac{\alpha_a(u)}{\alpha_b(u)} \times \chi_\lambda(c(u)).$$

Hence, $\mathrm{wt}_t(\mu) = \mathrm{wt}_t(\lambda) + \alpha_a - \alpha_b$. □

Our main combinatorial result about the functors $D_{b|a}$ is as follows.

Theorem 5.18. *For* $\lambda \in \mathcal{P}$ *and* $a, b \in \Bbbk$, *there is a filtration* $0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 = D_{b|a}\Delta(\lambda)$ *such that*

$$V_3/V_2 \cong \begin{cases} \Delta(\lambda + \boxed{a}) & \text{if } a \in \mathrm{add}(\lambda) \text{ and } b = t - |\lambda| \\ 0 & \text{otherwise,} \end{cases}$$

$$V_2/V_1 \cong \begin{cases} \Delta(\lambda) \oplus \Delta(\lambda) & \text{if } t - |\lambda| = a = b \in \mathrm{rem}(\lambda) \\ \Delta(\lambda) & \text{if } t - |\lambda| \neq a = b \in \mathrm{rem}(\lambda) \text{ or } t - |\lambda| = a = b \notin \mathrm{rem}(\lambda) \\ \Delta((\lambda - \boxed{b}) + \boxed{a}) & \text{if } a \neq b \in \mathrm{rem}(\lambda) \text{ and } a \in \mathrm{add}(\lambda - \boxed{b}) \\ 0 & \text{otherwise,} \end{cases}$$

$$V_1/V_0 \cong \begin{cases} \Delta(\lambda - \boxed{b}) & \text{if } a = t - |\lambda| + 1 \text{ and } b \in \mathrm{rem}(\lambda) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, when $t \in \mathbb{Z}$, *the functor* $D_{b|a}$ *is zero unless both* a *and* b *are integers.*

Proof. See §5.5 below. □

The following corollary is an immediate consequence of the theorem, but actually it has a much easier proof which we include below.

Corollary 5.19. *For* $\lambda \in \mathcal{P}$ *and* $a, b \in \Bbbk$ *with* $a \neq b$, *there is a filtration* $0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 = D_{b|a}\Delta(\lambda)$ *such that*

$$V_3/V_2 \cong \begin{cases} \Delta(\lambda + \boxed{a}) & \text{if } a \in \mathrm{add}(\lambda) \text{ and } b = t - |\lambda| \\ 0 & \text{otherwise,} \end{cases}$$

$$V_2/V_1 \cong \begin{cases} \Delta((\lambda - \boxed{b}) + \boxed{a}) & \text{if } b \in \mathrm{rem}(\lambda) \text{ and } a \in \mathrm{add}(\lambda - \boxed{b}) \\ 0 & \text{otherwise,} \end{cases}$$

$$V_1/V_0 \cong \begin{cases} \Delta(\lambda - \boxed{b}) & \text{if } a = t - |\lambda| + 1 \text{ and } b \in \mathrm{rem}(\lambda) \\ 0 & \text{otherwise.} \end{cases}$$

Direct proof avoiding Theorem 5.18. Let $\gamma := \mathrm{wt}_t(\lambda)$. By Lemma 5.17, we can compute $D_{b|a}\Delta(\lambda)$ by applying $\mathrm{pr}_{\gamma+\alpha_a-\alpha_b}$ to the Δ-flag for $D\Delta(\lambda)$ from Lemma 5.13. This produces a module with a Δ-flag consisting of all $\Delta(\mu)$ in the original Δ-flag such that $\mathrm{wt}_t(\mu) - \mathrm{wt}_t(\lambda) = \alpha_a - \alpha_b$. It just remains to compute $\mathrm{wt}_t(\mu) - \mathrm{wt}_t(\lambda)$ for the various possible μ. If $\mu = \lambda + \boxed{c}$ for $c \in \mathrm{add}(\lambda)$ then, by a computation using the first equality from (5.20), we have that $\mathrm{wt}_t(\mu) - \mathrm{wt}_t(\lambda) = \alpha_c - \alpha_{t-|\lambda|}$; for this to equal $\alpha_a - \alpha_b$ we must have $b = t - |\lambda|$ and $c = a$. If $\mu = \lambda - \boxed{d}$ for $d \in \mathrm{rem}(\lambda)$ then, by a similar computation, $\mathrm{wt}_t(\mu) - \mathrm{wt}_t(\lambda) = \alpha_{t-|\lambda|+1} - \alpha_d$; for this to equal $\alpha_a - \alpha_b$ we must have $d = b$ and $a = t - |\lambda| + 1$. Finally if $\mu = (\lambda - \boxed{d}) + \boxed{c}$ for $d \in \mathrm{rem}(\lambda)$ and $c \in \mathrm{add}(\lambda - \boxed{d})$ then $\mathrm{wt}_t(\mu) - \mathrm{wt}_t(\lambda) = \alpha_c - \alpha_d$; for this to equal $\alpha_a - \alpha_b$ we must have $c = a$ and $d = b$. $\square$

5.4. **Blocks.** We assume throughout the subsection that $t \in \mathbb{N}$. We are going to describe the structure of the atypical "blocks," revealing in particular that they are indecomposable, hence, they are actually blocks. Recall from Theorem 5.9 that the atypical "blocks" are parametrized by partitions $\kappa \in \mathcal{P}_t$, with the irreducible modules in the "block" being the ones labelled by the partitions $\{\kappa^{(0)}, \kappa^{(1)}, \dots\}$. This is the set $S(\gamma)$ from (5.18) where $\gamma \in P$ is obtained from κ according to (5.23).

The first step is to show that all of the atypical "blocks" are equivalent to each other. The proof of this uses the special projective functors $D_{b|a}$ with $a \neq b$. These are the ones which can be defined just using information about central characters rather than requiring the Jucys-Murphy elements; cf. Lemma 5.17 and Corollary 5.19. In view of Remark 4.27, this sort of information was already available to Comes and Ostrik in an equivalent form, and indeed they were also able to prove a similar result by an analogous argument; see [CO, Lem. 5.18(2)] and [CO, Prop. 6.6].

Lemma 5.20. *Let κ and $\tilde{\kappa}$ be partitions of t such that $\tilde{\kappa}$ is obtained from κ by moving a node from the first row of its Young diagram to its $(r+1)$th row for some $r \geqslant 1$. Let $a := \kappa_{r+1} - r + 1$ and $b := \kappa_1$. Then for all $n \geqslant 0$ we have that $D_{b|a}\Delta(\kappa^{(n)}) \cong \Delta(\tilde{\kappa}^{(n)})$ and $D_{a|b}\Delta(\tilde{\kappa}^{(n)}) \cong \Delta(\kappa^{(n)})$.*

Proof. Let $\gamma, \tilde{\gamma} \in P$ be defined from κ and $\tilde{\kappa}$ according to (5.23). From this formula it follows that $\tilde{\gamma} = \gamma + \alpha_a - \alpha_b$ where $a = \kappa_{r+1} - r + 1$ and $b = \kappa_1$ as in the statement of the lemma. Note that $a \neq b$. So we can apply Lemma 5.17 to see that $D_{b|a}\Delta(\kappa^{(n)}) = \mathrm{pr}_{\gamma+\alpha_a-\alpha_b}(D\Delta(\kappa^{(n)}))$ and $D_{a|b}\Delta(\tilde{\kappa}^{(n)}) = \mathrm{pr}_{\gamma-\alpha_a+\alpha_b}(D\Delta(\tilde{\kappa}^{(n)}))$. Now we use this description to show that $D_{b|a}\Delta(\kappa^{(n)}) \cong \Delta(\tilde{\kappa}^{(n)})$. The proof that $D_{a|b}\Delta(\tilde{\kappa}^{(n)}) \cong \Delta(\kappa^{(n)})$ is similar and we leave this to the reader.

Fix $n \geqslant 0$ and let B_n be the set of $\mu \in \mathcal{P}$ which are obtained from $\kappa^{(n)}$ by removing a node, removing a node then adding a different node, or adding a node. Bearing in mind that $a \neq b$, the standard modules $\Delta(\mu)$ for $\mu \in B_n$ include all of the ones which are sections of the Δ-flag from Lemma 5.13 which could possibly be in the same block as $\Delta(\tilde{\kappa}^{(n)})$. Now it suffices to show for $m \geqslant 0$ that $\tilde{\kappa}^{(m)} \in B_n$ if and only if $m = n$. There are four cases to consider.

Case one: $n = 0$. We have that $\kappa^{(0)} = (\kappa_2, \kappa_3, \dots, \kappa_{r+1}, \dots)$ and $\tilde{\kappa}^{(0)} = (\kappa_2, \kappa_3, \dots, \kappa_{r+1} + 1, \dots)$, which is $\kappa^{(0)}$ with one node added to the rth row of its Young diagram. We definitely have that $\tilde{\kappa}^{(0)} \in B_0$. All other $\mu \in B_0$ satisfy $|\mu| \leqslant |\tilde{\kappa}^{(0)}|$. Since all $\tilde{\kappa}^{(m)}$ with $m > 0$ have $|\tilde{\kappa}^{(m)}| > |\tilde{\kappa}^{(0)}|$, none of these belong to B_0.

Case two: $1 \leqslant n < r$. We have that $\kappa^{(n)} = (\kappa_1 + 1, \kappa_2 + 1, \dots, \kappa_n + 1, \dots, \kappa_{r+1}, \dots)$ and $\tilde{\kappa}^{(n)} = (\kappa_1, \kappa_2 + 1, \dots, \kappa_n + 1, \dots, \kappa_{r+1} + 1, \dots)$, which is $\kappa^{(n)}$ with a node removed from the first row and a node added to the rth row of its Young diagram. We definitely have that $\tilde{\kappa}^{(n)} \in B_n$. For $m < n$, $\tilde{\kappa}^{(m)}$ is of smaller size than $\kappa^{(n)}$ and its rth row is of length $\kappa_{r+1} + 1$. This cannot be obtained from $\kappa^{(n)}$ by removing a node since $\kappa^{(n)}$ has rth row of length κ_{r+1}. So it does not belong to B_n. For $m > n$, $\tilde{\kappa}^{(m)}$ is of greater size than $\kappa^{(n)}$ and its first row is of length κ_1. This cannot be obtained from $\kappa^{(n)}$ by adding a node since $\kappa^{(n)}$ has first row of length $\kappa_1 + 1$. So again it does not belong to B_n.

Case three: $n = r$. We have that $\kappa^{(n)} = (\kappa_1 + 1, \kappa_2 + 1, \dots, \kappa_r + 1, \kappa_{r+2}, \dots)$ and $\tilde{\kappa}^{(n)} = (\kappa_1, \kappa_2 + 1 \dots, \kappa_r + 1, \kappa_{r+2}, \dots)$, which is $\kappa^{(n)}$ with a node removed from the first row of its Young diagram. We definitely have that $\tilde{\kappa}^{(n)} \in B_n$. The $\tilde{\kappa}^{(m)}$ with $m < n$ have $|\tilde{\kappa}^{(m)}| \leqslant |\tilde{\kappa}^{(n)}| - 1 = |\kappa^{(n)}| - 2$ so are not elements of B_n. The $\tilde{\kappa}^{(m)}$ with $m > n$ have $(r + 1)$th row of length $\kappa_{r+1} + 2$, so these are not elements of B_n either since this is at least two more than the length of the $(r + 1)$th row of $\kappa^{(n)}$.

Case four: $n > r$. We have that $\kappa^{(n)} = (\kappa_1 + 1, \kappa_2 + 1, \dots, \kappa_{r+1} + 1, \dots)$ and $\tilde{\kappa}^{(n)} = (\kappa_1, \kappa_2 + 1, \dots, \kappa_{r+1} + 2, \dots)$, which is $\kappa^{(n)}$ with a node removed from its first row and a node added to its $(r + 1)$th row. We definitely have that $\tilde{\kappa}^{(n)} \in B_n$. The $\tilde{\kappa}^{(m)}$ with $m > n$ are of greater size than $\kappa^{(n)}$ and have first row of length κ_1; these cannot be obtained by adding a node to $\kappa^{(n)}$. The $\tilde{\kappa}^{(m)}$ with $r + 1 \leqslant m < n$ are of smaller size than $\kappa^{(n)}$ and have $(r + 1)$th row of length $\kappa_{r+1} + 2$; these cannot be obtained by removing a node from $\kappa^{(n)}$. The $\tilde{\kappa}^{(m)}$ with $m \leqslant r$ have first row of length $\leqslant \kappa_1$ and $(r + 1)$th row of length κ_{r+2}, whereas these two rows of $\kappa^{(n)}$ are of lengths $\kappa_1 + 1$ and $\kappa_{r+1} + 1 > \kappa_{r+2}$, so these are not elements of B_n. □

Theorem 5.21 (Comes-Ostrik). *Let κ and $\tilde{\kappa}$ be partitions of t, denoting the associated $\sim_t$-equivalence classes by $S := \{\kappa^{(0)}, \kappa^{(1)}, \dots\}$ and $\widetilde{S} := \{\tilde{\kappa}^{(0)}, \tilde{\kappa}^{(1)}, \dots\}$. There is an equivalence of categories*

$$\Sigma : 1_S \mathit{Par}_t\text{-Mod} \to 1_{\widetilde{S}} \mathit{Par}_t\text{-Mod}$$

between the corresponding "blocks" such that $\Sigma L(\kappa^{(n)}) \cong L(\tilde{\kappa}^{(n)})$ for all $n \geqslant 0$. The functor Σ is a composition of the special projective functors $D_{b|a}$ $(a \neq b)$, hence, it is a projective functor.

Proof. We may assume that $\tilde{\kappa}$ is obtained from κ by moving a node from the first row of its Young diagram to its $(r + 1)$th row for some $r \geqslant 1$. Thus, we are in the situation of Lemma 5.20. The lemma gives us functors $D_{b|a} : 1_S \mathit{Par}_t\text{-Mod} \to 1_{\widetilde{S}} \mathit{Par}_t\text{-Mod}$ and $D_{a|b} : 1_{\widetilde{S}} \mathit{Par}_t\text{-Mod} \to 1_S \mathit{Par}_t\text{-Mod}$ such that $D_{b|a}\Delta(\kappa^{(n)}) \cong \Delta(\tilde{\kappa}^{(n)})$ and $D_{a|b}\Delta(\tilde{\kappa}^{(n)}) \cong \Delta(\kappa^{(n)})$. These functors are also biadjoint thanks to Lemma 5.16. It follows easily that they are quasi-inverse equivalences of categories as claimed in the theorem. In more detail, the unit and counit of one of the adjunctions gives natural transformations $D_{a|b} \circ D_{b|a} \Rightarrow \mathrm{Id}$ and $\mathrm{Id} \Rightarrow D_{b|a} \circ D_{a|b}$. We claim that these natural transformations are isomorphisms. They are non-zero, hence, they are isomorphisms on all standard modules. The functors are exact and indecomposable projectives have finite Δ-flags, so it follows that the natural transformations are isomorphisms on all indecomposable projectives. Then we get that they are isomorphisms on an arbitrary module by considering a two step projective resolution and applying the Five Lemma. □

The next lemma does use the functors $D_{b|a}$ in the case $a = b$, i.e., it definitely requires the full strength of Theorem 5.18 rather than merely Corollary 5.19.

Lemma 5.22. *Let $\kappa \in \mathcal{P}_t$ and $S := \{\kappa^{(0)}, \kappa^{(1)}, \dots\}$ be the corresponding $\sim_t$-equivalence class. For each $n \geqslant 0$, there is an endofunctor $\Pi_n : \mathit{Par}_t\text{-Mod} \to \mathit{Par}_t\text{-Mod}$ such that $\Pi_n\Delta(\kappa^{(m)}) = 0$ for $m \neq n, n + 1$, and moreover there exist short exact sequences $0 \to \Delta(\kappa^{(n)}) \to \Pi_n\Delta(\kappa^{(n)}) \to \Delta(\kappa^{(n+1)}) \to 0$ and $0 \to \Delta(\kappa^{(n)}) \to \Pi_n\Delta(\kappa^{(n+1)}) \to \Delta(\kappa^{(n+1)}) \to 0$. The functor Π_n is a composition of the special projective functors $D_{b|a}$ $(a, b \in \mathbb{Z})$, hence, it is a projective functor.*

Proof. In view of Theorem 5.21, it suffices to prove the lemma in the special case that $\kappa = (t)$, when $S = \{\varnothing, (t+1), (t+1, 1), (t+1, 1^2), \dots\}$ as in Example 5.10. Then we take $\Pi_0 := D_{0|t} \circ \cdots \circ D_{t-1|1} \circ D_{t|0}$ and $\Pi_n := D_{-n|-n}$ for $n > 0$. Now it is just a matter of applying Theorem 5.18 to see that these functors have the stated properties.

The situation for Π_0 is the most interesting. To understand this, let $u := \lceil \frac{t}{2} \rceil$ and $v := \lfloor \frac{t}{2} \rfloor$. Then one checks that $D_{v+1|u-1} \circ \cdots \circ D_{t-1|1} \circ D_{t|0}(\Delta(\varnothing)) \cong \Delta((u))$; each of these functors adds a single node to the first row of the Young diagram. After that we apply $D_{v|u}$ to get a module with a two step Δ-flag, with a copy of $\Delta((u+1))$ at the top and a copy of $\Delta((v))$ at the bottom. Note this is obtained from

Theorem 5.18 in a slightly different way according to whether $u = v$ (i.e., t is even) or $u = v+1$ (i.e., t is odd). Also, this is now a module in an atypical block. Finally we apply $D_{0|t} \circ D_{1|t-1} \circ \cdots D_{v-1|u+1}$ to end up with the desired two step Δ-flag with a copy of $\Delta(\kappa^{(1)}) = \Delta((t+1))$ at the top and $\Delta(\kappa^{(0)}) = \Delta(\varnothing)$ at the bottom; each of these functors adds a single node to the first row of the Young diagram labelling the module at the top and removes a node from the Young diagram labelling the module at the bottom. This is what Π_0 is meant to do to $\Delta(\varnothing)$. A similar argument shows that $\Pi_0\Delta((t+1))$ has a Δ-flag with the same two sections. It is also easy to check that $\Pi_0\Delta(\kappa^{(m)}) = 0$ for $m > 1$, indeed, $D_{t|0}$ already annihilates these standard modules.

The functors $\Pi_n = D_{-n|-n}$ for $n > 0$ are easier to analyze. Noting that $\kappa^{(n)} = \Delta((t+1, 1^{n-1}))$, the module $\Pi_n\Delta(\kappa^{(n)})$ has a two step Δ-flag with $\Delta(\kappa^{(n+1)}) = \Delta((t+1, 1^n))$ at the top and $\Delta(\kappa^{(n)})$ at the bottom; this uses the $t - |\lambda| = a = b \notin \mathrm{rem}(\lambda)$ case from Theorem 5.18. Similarly, $\Pi_n\Delta(\kappa^{(n+1)})$ has a Δ-flag with the same two sections. Finally, one checks that $\Pi_n\Delta(\kappa^{(m)}) = 0$ for $m \neq n, n+1$. □

Remark 5.23. In the proof of the next theorem, we will show that the functor Π_n from Lemma 5.22 satisfies $\Pi_n\Delta(\kappa^{(n)}) \cong \Pi_n\Delta(\kappa^{(n+1)}) \cong \Pi_n L(\kappa^{(n+1)}) \cong P(\kappa^{(n+1)})$ for all $n \geqslant 0$.

Now we can prove the main result about blocks. This can also be deduced from [CO, Th. 6.10], but the proof of that appealed to results of Martin [M2] in order to obtain the precise submodule structure of the indecomposable projectives, whereas we are able to establish this by exploiting the highest weight structure and the Chevalley duality $?^{\circledast}$.

Theorem 5.24. *Let $\kappa \in \mathcal{P}ar_t$ and $S := \{\kappa^{(0)}, \kappa^{(1)}, \dots\}$ be the corresponding $\sim_t$-equivalence class.*

(i) For each $n \geqslant 0$, the standard module $\Delta(\kappa^{(n)})$ is of length two with head $L(\kappa^{(n)})$ and socle $L(\kappa^{(n+1)})$.

(ii) The indecomposable projective module $P(\kappa^{(0)})$ is isomorphic to $\Delta(\kappa^{(0)})$, while for $n \geqslant 1$ the module $P(\kappa^{(n)})$ has a two step Δ-flag with top section $\Delta(\kappa^{(n)})$ and bottom section $\Delta(\kappa^{(n-1)})$.

(iii) For each $n \geqslant 1$, $P(\kappa^{(n)})$ is self-dual with irreducible head and socle isomorphic to $L(\kappa^{(n)})$ and completely reducible heart $\mathrm{rad}\, P(\kappa^{(n)}) / \mathrm{soc}\, P(\kappa^{(n)}) \cong L(\kappa^{(n-1)}) \oplus L(\kappa^{(n+1)})$.

Proof. To improve the readability, we write simply $P(n), \Delta(n)$ and $L(n)$ in place of $P(\kappa^{(n)}), \Delta(\kappa^{(n)})$ and $L(\kappa^{(n)})$. For $n \geqslant 0$, Lemma 5.22 shows that the module $P_n := \Pi_{n-1} \circ \cdots \Pi_1 \circ \Pi_0(\Delta(0))$ has a two step Δ-flag with top section $\Delta(n)$ and bottom section $\Delta(n-1)$. Since $\Delta(0)$ is projective by the minimality observed in (5.22) and each Π_i is a projective functor, P_n is projective. Since P_n has $L(n)$ in its head, it must contain the indecomposable projective $P(n)$ as a summand, so we either have that $P(n) \cong P_n$ if P_n is indecomposable, or $P(n) \cong \Delta(n)$ otherwise. In the former case, $(P(n) : \Delta(m)) = \delta_{m,n} + \delta_{m,n-1}$, while $(P(n) : \Delta(m)) = \delta_{m,n}$ in the latter situation. Now we apply BGG reciprocity to deduce for any $m \geqslant 0$ that $[\Delta(m) : L(n)] = \delta_{n,m} + \delta_{n,m+1}$ if P_n is indecomposable and $[\Delta(m) : L(n)] = \delta_{n,m}$ otherwise. Hence, for each $m \geqslant 0$, we either have that $\Delta(m) \cong L(m)$, or $\Delta(m)$ is of composition length two with composition factors $L(m)$ and $L(m+1)$.

We claim for any $n \geqslant 0$ that $\Delta(n) \cong L(n)$ if and only if $\Delta(n+1) \cong L(n+1)$. Suppose first that $\Delta(n) \cong L(n)$. Since Π_n commutes with duality by Lemma 5.15, this implies that $\Pi_n\Delta(n)$ is self-dual. But this module has a two step Δ-flag with top section $\Delta(n+1)$ and bottom section $\Delta(n) \cong L(n)$. The only way such a module can be self-dual is if $\Delta(n+1) \cong L(n+1)$ (and the module must be completely reducible). Conversely, suppose for a contradiction that $\Delta(n+1) \cong L(n+1)$ but $\Delta(n) \ncong L(n)$. Then $\Delta(n)$ is of length two with composition factors $L(n)$ and $L(n+1)$, so that $P(n+1)$ has a two step Δ-flag with top section $\Delta(n+1) \cong L(n+1)$ and bottom section $\Delta(n)$. Since $\Pi_{n+1}\Delta(n) = 0$ according to Lemma 5.22 and Π_{n+1} is exact, we must have that $\Pi_{n+1}L(n+1) = 0$. Since $\Delta(n+1) \cong L(n+1)$, this implies that $\Pi_{n+1}\Delta(n+1) = 0$, which contradicts Lemma 5.22.

From the claim, we see that if $\Delta(n)$ is irreducible for any one $n \geqslant 0$, then it is irreducible for all $n \geqslant 0$. Since all atypical "blocks" are equivalent by Theorem 5.21, it follows in that case that the standard modules $\Delta(\lambda)$ for *all* $\lambda \in \mathcal{P}$ are irreducible. This implies that the minimal ordering $\succeq_t$ from Remark 5.5 is trivial, hence, the blocks are trivial and Par_t is semisimple, which contradicts Corollary 5.11. Thus, we have proved that $\Delta(n)$ must be of length two for every $n \geqslant 0$, and (i) is proved.

Property (ii) follows immediately from (i) and BGG reciprocity as noted earlier.

It remains to prove (iii). Take $n \geqslant 1$. By Lemma 5.22, we have that $\Pi_{n-1}\Delta(n+1) = 0$. Since Π_{n-1} is exact and $L(n+1)$ is a composition factor of $\Delta(n+1)$, it follows that $\Pi_{n-1}L(n+1) = 0$ too. From this, we deduce that $\Pi_{n-1}\Delta(n) \cong \Pi_{n-1}L(n)$. By Lemma 5.22 again, $\Pi_{n-1}\Delta(n-1)$ has the same composition length as $\Pi_{n-1}\Delta(n) \cong \Pi_{n-1}L(n)$. Also $\Delta(n-1)$ has $L(n)$ as a constituent. Using the exactness of Π_{n-1} again, we must therefore have that $\Pi_{n-1}\Delta(n-1) \cong \Pi_{n-1}L(n)$. As observed earlier in the proof, this module is isomorphic to $P(n)$, so using that $L(n)$ is self-dual and Π_{n-1} commutes with duality, we now see that $P(n)$ is self-dual. We also know that it has length four with irreducible head $L(n)$, $[P(n) : L(n)] = 2$ and $[P(n) : L(n-1)] = [P(n) : L(n+1)] = 1$. The only possible structure is the one claimed. ◻

Corollary 5.25 (Comes-Ostrik)**.** *All "blocks" of Par_t-*Mod *are indecomposable, hence, they coincide with the blocks.*

Corollary 5.26. *The minimal ordering $\succeq_t$ from Remark 5.5 is the partial order such that $\kappa^{(m)} \succeq_t \kappa^{(n)}$ for each $\kappa \in \mathcal{P}_t$ and $m \leqslant n$, with all other pairs of partitions being incomparable.*

In general, in an upper finite highest weight category, the standard objects can have infinite length. Our final corollary, which is also noted in [SS2, Rem. 6.4], shows that this is not the case in Par_t-$\mathrm{Mod}_{\mathrm{lfd}}$. Consequently, the full subcategory consisting of all modules of finite length has enough projectives and injectives, indeed, this subcategory is an essentially finite highest weight category in the sense of [BS, Def. 3.7].

Corollary 5.27. *The locally unital algebra Par_t is locally Artinian, i.e., the left ideals $Par_t 1_n$ and the right ideals $1_n Par_t$ are of finite length for all $n \geqslant 0$.*

Proof. Theorem 5.24 shows that all indecomposable projective left Par_t-modules are of finite length, hence, all finitely generated projectives are of finite length too. This includes all of the left ideals $Par_t 1_n$. Since there is a duality $?^{\circledast}$, it also follows that all fintely cogenerated injective left Par_t-modules are of finite length. This includes all of the duals $(1_n Par_t)^{\circledast}$, hence, each $1_n Par_t$ is of finite length as a right module. ◻

5.5. **Proof of Theorem 5.18.** It just remains to prove Theorem 5.18. In fact, we will prove the following slightly stronger result, from which Theorem 5.18 follows easily on applying the functors involved to the Specht module $S(\lambda)$. To state this stronger result, let $j_! : Sym\text{-}\mathrm{Mod}_{\mathrm{fd}} \to Par_t\text{-}\mathrm{Mod}_{\mathrm{lfd}}$ be the standardization functor from (3.23), E_a and F_b be the refined induction and restriction functors from (2.39), $D_{b|a}$ be the special projective functor from (5.28), and $\mathrm{pr}_c : Sym\text{-}\mathrm{Mod}_{\mathrm{fd}} \to Sym\text{-}\mathrm{Mod}_{\mathrm{fd}}$ be the functor defined by multiplication by the identity element of the symmetric group S_c if $c \in \mathbb{N}$, i.e., it is the projection onto $\Bbbk S_c$-$\mathrm{Mod}_{\mathrm{fd}}$ followed followed by the inclusion of $\Bbbk S_c$-$\mathrm{Mod}_{\mathrm{fd}}$ into Sym-$\mathrm{Mod}_{\mathrm{fd}}$, or the zero functor if $c \in \Bbbk - \mathbb{N}$.

Theorem 5.28. *For $a, b \in \Bbbk$, there is a filtration of the functor $D_{b|a} \circ j_! : Sym\text{-}\mathrm{Mod}_{\mathrm{fd}} \to Par_t\text{-}\mathrm{Mod}_{\mathrm{lfd}}$ by subfunctors $0 = S_0 \subseteq S_1 \subseteq S_2 \subseteq S_3 \subseteq S_4 = D_{b|a} \circ j_!$ such that*

$$S_4/S_3 \cong j_! \circ E_a \circ \mathrm{pr}_{t-b},$$
$$S_3/S_2 \cong j_! \circ \mathrm{pr}_{t-a} \circ \mathrm{pr}_{t-b},$$

$$S_2/S_1 \cong j_! \circ E_a \circ F_b,$$
$$S_1/S_0 \cong j_! \circ \mathrm{pr}_{t-a} \circ F_b.$$

(Recall that a subfunctor S *of a functor* $T : \mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}} \to \mathit{Par}_t\text{-}\mathrm{Mod}_{\mathrm{lfd}}$ *is a functor* $S : \mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}} \to \mathit{Par}_t\text{-}\mathrm{Mod}_{\mathrm{lfd}}$ *such that* SV *is a submodule of* TV *for all* $V \in \mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}}$ *and* $Sf = Tf|_{SV}$ *for all* $f \in \mathrm{Hom}_{\mathit{Sym}}(V, V')$*; then the quotient* T/S *is the obvious functor with* $(T/S)(V) := TV/SV$*.)*

The proof will take up the rest of the subsection. We begin by constructing a filtration of the functor $D \circ j_! : \mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}} \to \mathit{Par}_t\text{-}\mathrm{Mod}_{\mathrm{lfd}}$. Note that $D \circ j_! \cong M \otimes_{\mathit{Sym}}$ where M is the $(\mathit{Par}_t, \mathit{Sym})$-bimodule

$$M := 1_{|\star} \mathit{Par}_t \otimes_{\mathit{Par}^\sharp} \mathrm{infl}^\sharp\, \mathit{Sym}. \tag{5.29}$$

We also have the $(\mathit{Par}_t, \mathit{Sym})$-bimodules

$$N_4 = \mathit{Par}_t \otimes_{\mathit{Par}^\sharp} \mathrm{infl}^\sharp(\mathit{Sym} 1_{|\star}), \tag{5.30}$$
$$N_3 := \mathit{Par}_t \otimes_{\mathit{Par}^\sharp} \mathrm{infl}^\sharp\, \mathit{Sym}, \tag{5.31}$$
$$N_2 := \mathit{Par}_t \otimes_{\mathit{Par}^\sharp} \mathrm{infl}^\sharp(\mathit{Sym} 1_{|\star} \otimes_{\mathit{Sym}} 1_{|\star} \mathit{Sym}) \tag{5.32}$$
$$N_1 := \mathit{Par}_t \otimes_{\mathit{Par}^\sharp} \mathrm{infl}^\sharp(1_{|\star} \mathit{Sym}). \tag{5.33}$$

The functors $\mathit{Sym}\text{-}\mathrm{Mod}_{\mathrm{fd}} \to \mathit{Par}_t\text{-}\mathrm{Mod}_{\mathrm{lfd}}$ defined by tensoring with N_4, N_3, N_2 and N_1 are isomorphic to $j_! \circ E$, $j_!$, $j_! \circ E \circ F$ and $j_! \circ F$, respectively.

For $m \geqslant n \geqslant 0$, let $B_{m,n}$ be the basis for $1_m \mathit{Par}^- 1_n$ defined by representatives for the equivalence classes of normally ordered upward partition diagrams. By Theorem 3.2, the vector space M is isomorphic to $1_{|\star} \mathit{Par}^- \otimes_{\mathbb{K}} \mathit{Sym}$, hence, it has basis

$$\left\{ f \otimes g \;\middle|\; m \geqslant 0, n \geqslant 0, m+1 \geqslant n, f \in B_{m+1,n}, g \in S_n \right\}. \tag{5.34}$$

For any $f \in B_{m+1,n}$, let $c(f)$ be the connected component of the diagram containing the top left vertex. In the language from §3.2, this component could be a trunk, an upward tree, an upward leaf, or an upward branch. Then we introduce the following subspaces of M:

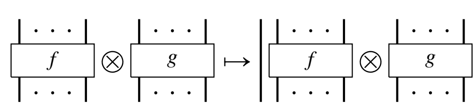
- Let M_1 be the subspace of M spanned by all $f \otimes g$ in this basis such that $c(f)$ is a trunk.
- Let M_2 be the subspace spanned by all $f \otimes g$ such that $c(f)$ is either a trunk or an upward tree.
- Let M_3 be the subspace spanned by all $f \otimes g$ such that $c(f)$ is either a trunk, an upward tree, or an upward leaf.
- Let $M_0 := 0$ and $M_4 := M$.

The following is a generalization of Lemma 5.13.

Lemma 5.29. *The subspaces* $0 = M_0 \subset M_1 \subset M_2 \subset M_3 \subset M_4 = M$ *are sub-bimodules of the* $(\mathit{Par}_t, \mathit{Sym})$*-bimodule* M*. Moreover, there are bimodule isomorphisms* $\theta_i : N_i \xrightarrow{\sim} M_i/M_{i-1}$ *for each* $i = 1, \dots, 4$*.*

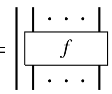
Proof. The fact that each M_i is a sub-bimodule of M is easily checked by vertically composing a basis vector $f \otimes g$ with an arbitrary partition diagram on the top and with any permutation diagram on the bottom. One just needs to note that the action on top involves $\mathrm{res}_{|\star}$, so that the top left vertex is untouched. This implies that the type $c(f)$ does not change if it is a trunk or an upward leaf, while if it is an upward tree it can only be changed to another upward tree or to a trunk.

We show in this paragraph that there is a bimodule isomorphism

$\theta_1 : N_1 \to M_1,$  (5.35) for any $m \geqslant 0, n > 0$, $f \in 1_m Par_t 1_{n-1}$ and $g \in S_n$. This is a well-defined bimodule homomorphism. By Theorem 3.2, N_1 is isomorphic as a vector space to $Par^- \otimes_{\mathbb{K}} 1_{|\star}Sym$, hence, it has basis

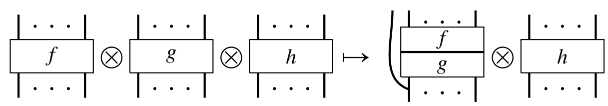
$$\left\{ f \otimes g \,\middle|\, m \geqslant n-1 \geqslant 0, f \in B_{m,n-1}, g \in S_n \right\}. \tag{5.36}$$

The vector space M_1 has basis given by all $f_1 \otimes g$ for $m+1 \geqslant n > 0$, $f_1 \in B_{m+1,n}$ and $g \in S_n$ such that $c(f_1)$ is a trunk. As it is normally ordered, any such f_1 is of the form

$f_1 =$ 

for a unique $f \in B_{m,n-1}$. Moreover, $f_1 \otimes g = \theta_1(f \otimes g)$ for every $g \in S_n$. It follows that θ_1 takes a basis for N_1 to a basis for M_1, so it is an isomorphism.

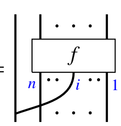
Next we show that there is a bimodule isomorphism

$\theta_2 : N_2 \to M_2/M_1,$  $+ M_1$ (5.37)

for $m \geqslant 0, n > 0$, $f \in 1_m Par_t 1_n$ and $g, h \in S_n$. Again, this is a well-defined bimodule homomorphism. By Theorem 3.2, N_2 is isomorphic to $Par^- \otimes_{\mathbb{K}} Sym1_{|\star} \otimes_{Sym} 1_{|\star}Sym$. Also $\Bbbk S_n$ is free as a right $\Bbbk S_{n-1}$-module with basis given by $\{(i\ i{+}1\ \cdots\ n) \mid 1 \leqslant i \leqslant n\}$, which is a set of S_n/S_{n-1}-cosets. It follows that N_2 has basis

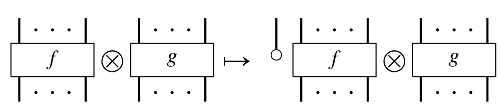
$$\left\{ f \otimes (i\ i{+}1\ \cdots\ n) \otimes g \,\middle|\, m \geqslant n > 0, f \in B_{m,n}, 1 \leqslant i \leqslant n, g \in S_n \right\}. \tag{5.38}$$

The vector space M_2/M_1 has a basis given by all $f_2 \otimes g + M_1$ for $m+1 \geqslant n > 0$, $f_2 \in B_{m+1,n}$ and $g \in S_n$ such that $c(f_2)$ is an upward tree. Any such f_2 is equal to

$f_2 =$ 

for a unique $f \in B_{m,n}$ and a unique $1 \leqslant i \leqslant n$ (the index of the string at which the component $c(f_2)$ meets the bottom of f). Moreover, $f_2 \otimes g = \theta_2(f \otimes (i\ i{+}1\ \cdots\ n) \otimes g)$ for each $g \in S_n$. It follows that θ_2 takes a basis for N_2 to a basis for M_2/M_1, so it is an isomorphism.

The isomorphism θ_3 is defined by

$\theta_3 : N_3 \to M_3/M_2,$  $+ M_2$ (5.39)

for $m \geqslant 0, n \geqslant 0$, $f \in 1_m Par_t 1_n$ and $g \in S_n$. This is obviously a well-defined bimodule homomorphism. It is an isomorphism because it takes the basis

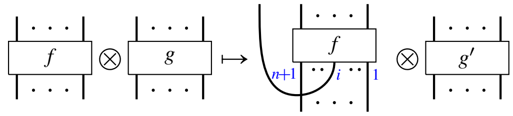
$$\{ f \otimes g \mid m \geqslant n \geqslant 0, f \in B_{m,n}, g \in S_n \} \tag{5.40}$$

for N_3 to the basis for M_3/M_2 consisting of all $f_3 \otimes g + M_2$ for $m+1 > n \geqslant 0$, $f_3 \in B_{m+1,n}$ and $g \in S_n$ such that $c(f_3)$ is an upward leaf.

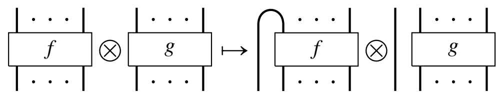
Finally, we construct the isomorphism θ_4. The vector space N_4 has basis

$$\{ f \otimes g \mid m-1 \geqslant n \geqslant 0, f \in B_{m,n+1}, g \in S_{n+1} \}. \tag{5.41}$$

We define the linear map

$\theta_4 : N_4 \to M_4/M_3,$  $+ M_3,$ (5.42)

where $f \otimes g$ is a vector from the basis for N_4 just displayed, and $g' \in S_n$ and $1 \leqslant i \leqslant n+1$ are defined from the equation $g = (i\ i{+}1\ \cdots\ n{+}1)g'$. To see that this linear map is actually a bimodule isomorphism, we construct a bimodule homorphism in the other direction and show that it is a two-sided inverse of θ_4. Consider the map

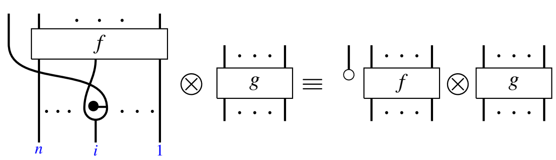
$\phi : M \to N_4,$  (5.43)

for $m \geqslant 0, n \geqslant 0, f \in 1_{m+1} \mathcal{P}ar_t 1_n$ and $g \in S_n$. It is easy to show that this is a well-defined bimodule homomorphism. Moreover, $M_3 \subseteq \ker \phi$ since applying ϕ to any basis vector $f \otimes g \in M_3$ produces a downward leaf, a cap or a downward tree which can be pushed across the tensor to act as zero on $\mathrm{infl}^\sharp\, \mathcal{S}ym$. Hence, ϕ induces a homomorphism $\bar{\phi} : M_4/M_3 \to N_4$. It remains to check that $\bar{\phi} \circ \theta_4$ and $\theta_4 \circ \bar{\phi}$ are both identity morphisms, which is straightforward. □

In the next two lemmas, we finally need to make some explicit calculations with the relations involving the left and right dots in the affine partition category. However, we are working now with $\mathcal{P}ar_t$, not with $\mathcal{AP}ar$, so all string diagrams from now on should be interpreted as the canonical images of these morphisms in $\mathcal{AP}ar$ under the functor $p_t : \mathcal{AP}ar \to \mathcal{P}ar_t$ from (4.43). We will also use the notation from (2.37) for an open dot on the interior of a string, meaning the canonical image of this morphism in $\mathcal{AS}ym$ under the functor $p : \mathcal{AS}ym \to \mathcal{S}ym$ from (2.35). This is quite different from an open dot at the end of a string!

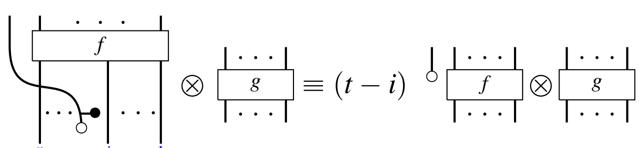
Lemma 5.30. *Suppose that $m \geqslant 0, n \geqslant 0, f \in 1_m \mathcal{P}ar_t 1_n$ and $g \in S_n$.*

(i) The following holds in the bimodule $M = 1_{|\star} \mathcal{P}ar_t \otimes_{\mathcal{P}ar^\sharp} \mathrm{infl}^\sharp\, \mathcal{S}ym$ for $i = 1, \dots, n$:



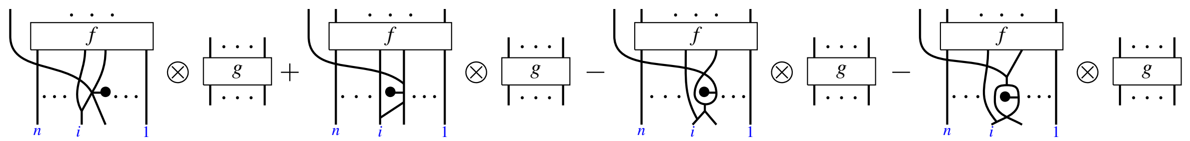
(mod M_2).

(ii) The following holds in the bimodule M for $i = 0, 1, \dots, n$ (the case $i = 0$ is when there are no strings to the right of the dangling dots):

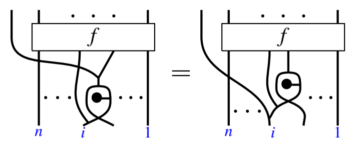


(mod M_2).

Proof. (i) We proceed by induction on $i = 1, \dots, n$. The base case $i = 1$ follows from (4.44). For the induction step, we take $i > 1$ and assume the result has been proved for $i-1$. Then we apply (4.27) to commute the left dot past the string to its right. This produces a sum of five terms. Ordering these terms in the same way as they appear on the right hand side of (4.27), the induction hypothesis can be applied to the first term, to produce the right hand side that we are after. It remains to show that the other four terms lies in M_2. These terms are as follows:

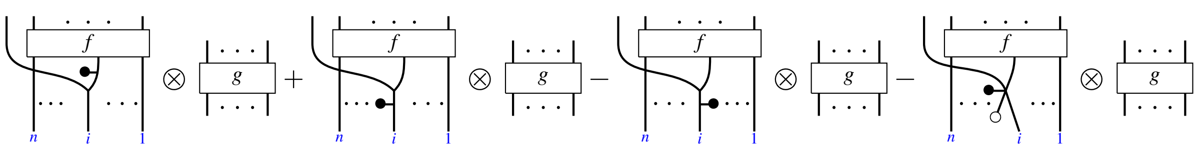


The second and third terms here are zero already in M because, in both of them, the diagram to the left of the tensor is equivalent to a diagram with a downward tree at the bottom. It remains to show that the first and the fourth terms lie in M_2. For the fourth term, we note that

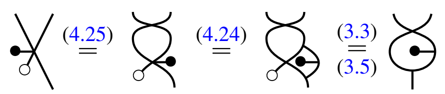


The left dot can now be absorbed into the morphism f, changing it to some other morphism f'. The result is a linear combination of morphisms in all of which the top left vertex is connected to the bottom edge, so that the connected component containing this vertex is either a tree or a trunk, and it belongs to the sub-bimodule M_2. The reason the first term lies in M_2 is very similar, one just needs to rewrite the right crossing using (4.24), and then it is easy to see that the top left vertex is again connected to the bottom edge.

(ii) Again we proceed by induction. The base case $i = 0$ follows from (4.44) using that $T = t1_{\mathbb{1}}$. For the induction step, we consider some $i > 0$. Then we apply (4.28) to commute the right dot past the string to its right. This produces a sum of five terms. This time, the induction hypothesis can be applied to the first term, to produce the vector that we are after but scaled by $(t - i + 1)$ rather than the desired $(t - i)$. The remaining four terms are as follows:



In the first term here, the left dot is some morphism in Par_t, which has the effect of changing f to some other morphism f'. After doing that, it is clear that the top left vertex is still connected to the bottom edge, so the first term lies in M_2. For the second and third terms, the left and right dots can be commuted across the tensor using (5.13), then again we see that these morphisms lie in M_2 since the top left vertex is connected to the bottom edge again. For the final term, we note that



Making this substitution in the middle of the picture reveals that the final term is exactly the expression studied in (i). On applying the conclusion of (i), we deduce that it contributes exactly the needed correction to complete the proof. □

Lemma 5.31. *Consider the bimodule endomorphisms ρ and λ of M defined on $1_{m+1}Par_t 1_n \otimes \Bbbk S_n$ by the left action of $x^R_{m+1} \otimes 1_n$ and $x^L_{m+1} \otimes 1_n$, respectively, for each $m, n \geqslant 0$. These endomorphisms preserve each of the sub-bimodules M_i $(i = 1, 2, 3, 4)$, hence, ρ and λ induce endomorphisms also denoted ρ*

and λ of each of the subquotients M_i/M_{i-1}. Moreover, for each i, the isomorphism θ_i from Lemma 5.29 satisfies

$$\theta_i \circ \rho_i = \rho \circ \theta_i, \qquad\qquad \theta_i \circ \lambda_i = \lambda \circ \theta_i, \tag{5.44}$$

where $\rho_i, \lambda_i : N_i \to N_i$ are defined as follows:

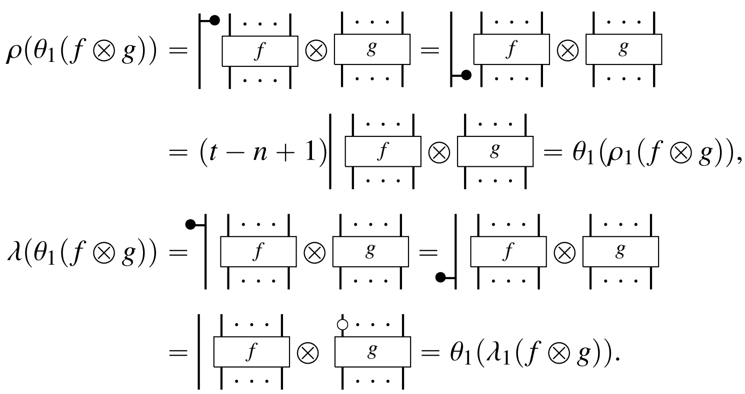
(i) ρ_1 and λ_1 are the bimodule endomorphisms of N_1 defined on the subspace $1_m Par_t 1_{n-1} \otimes \Bbbk S_n$ by the left actions of $(t-n+1)1_m \otimes 1_n$ and $1_m \otimes x_n$, respectively, for each $m \geqslant 0, n > 0$.

(ii) ρ_2 and λ_2 are the bimodule endomorphisms of N_2 defined on $1_m Par_t 1_n \otimes \Bbbk S_n \otimes \Bbbk S_n$ by the right action of $1_n \otimes x_n \otimes 1_n$ and the left action of $1_m \otimes 1_n \otimes x_n$, respectively.

(iii) ρ_3 and λ_3 are both equal to the bimodule endomorphism of N_3 defined on $1_m Par_t 1_n \otimes \Bbbk S_n$ by multiplication by $(t-n)$.

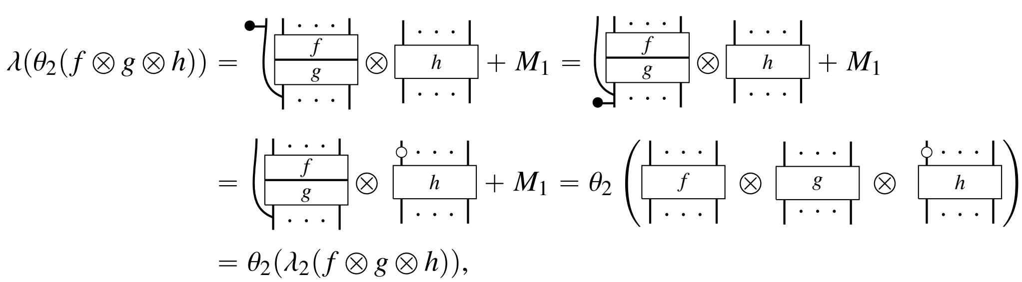
(iv) ρ_4 and λ_4 are the bimodule endomorphisms of N_4 defined on $1_m Par_t 1_{n+1} \otimes \Bbbk S_{n+1}$ by the right actions of $1_{n+1} \otimes x_{n+1}$ and $(t-n)1_{n+1} \otimes 1_{n+1}$, respectively.

Proof. (i) Recall the definition of θ_1 from (5.35). Take a vector $f \otimes g$ in the basis for N_1 from (5.36). By (5.13), we have that $x_n^L \equiv x_n \pmod{K_n}$ and $x_n^R \equiv (t-n+1)1_n \pmod{K_n}$ where K_n is the two sided ideal of $1_n Par_t 1_n$ from (5.1). Since the strictly downward partition diagrams which generate K^+ are zero on $\mathrm{infl}^\sharp\, Sym$, it follows that

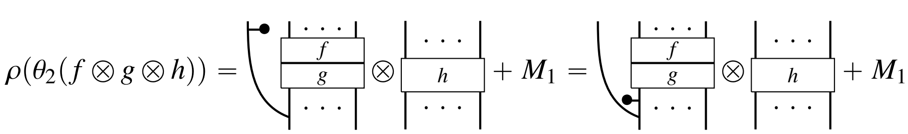


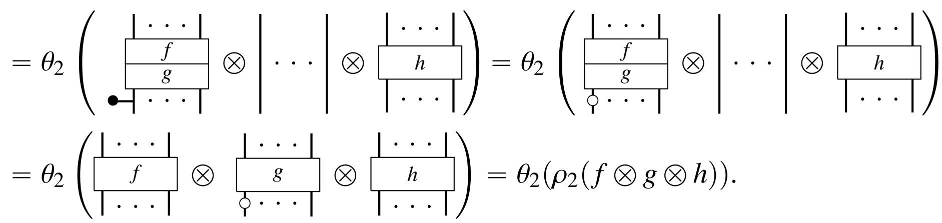
This shows as the same time that ρ and λ both leave M_1 invariant.

(ii) Recall the definition of θ_2 from (5.37). The argument for λ is similar to in (i). It follows from the calculation



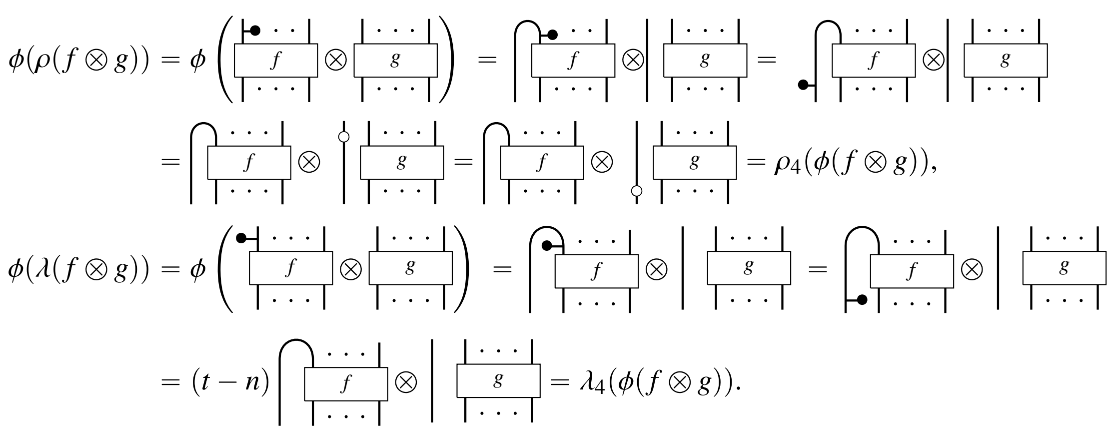
where $f \otimes g \otimes h$ is one of the basis vectors for N_2 from (5.38). For ρ, we instead have that





(iii) Recall the definition of θ_3 from (5.39). Note that $\rho = \lambda$ by the third relation from (4.22). For ρ, we need to show that $\rho(\theta_3(f \otimes g)) = (t-n)\theta_3(f \otimes g)$ for any basis vector $f \otimes g \in 1_m Par_t 1_n \otimes \Bbbk S_n \subset N_3$ from (5.40). This follows from Lemma 5.30(ii) taking $i = n$.

(iv) Instead of working with θ_4 from (5.42), it is easier to use the inverse map $\bar{\phi}$ induced by the homomorphism $\phi : M \to N_4$ from (5.43). We need to show that $\phi \circ \rho = \rho_4 \circ \phi$. This follows from the following calculations for $f \otimes g \in 1_{m+1} Par_t 1_n \otimes \Bbbk S_n$ and $m, n \geqslant 0$:



$\square$

Proof of Theorem 5.28. The functor $D_{b|a} \circ j_!$ is defined by tensoring with the bimodule $\overline{M}$ that is the simultaneous generalized a eigenspace of the endomorphism ρ and generalized b eigenspace of the endomorphism λ defined in Lemma 5.31. Lemma 5.29 defines a filtration of M with sections $M_i/M_{i-1} \cong N_i$ for $i = 1, \dots, 4$. Then Lemma 5.31 shows that the endomorphisms ρ and λ preserve this filtration, hence, the filtration of M induces a filtration of the summand $\overline{M}$. Moreover, for each i, $\overline{M}_i/\overline{M}_{i-1}$ is isomorphic to the summand $\overline{N}_i$ of N defined by the simultaneous generalized a-eigenspace of the endomorphism ρ_i and generalized b eigenspace of the endomorphism λ_i. By the descriptions of ρ_i and λ_i, it follows that $\overline{N}_i \otimes_{Sym}$ is isomorphic to the functor $j_! \circ E_a \circ \mathrm{pr}_{t-b}$, $j_! \circ \mathrm{pr}_{t-a} \circ \mathrm{pr}_{t-b}$, $j_! \circ E_a \circ F_b$ or $j_! \circ \mathrm{pr}_{t-a} \circ F_b$ for $i = 4, 3, 2, 1$, respectively. It remains to observe that Sym is semisimple, so every Sym-module is flat. This means that the filtration of $\overline{M}$ induces a filtration $0 = S_0 \subseteq S_1 \subseteq S_2 \subseteq S_3 \subseteq S_4 = D_{b|a} \circ j_!$ such that $S_i \cong \overline{M}_i/\overline{M}_{i-1} \otimes_{Sym} \cong \overline{N}_i \otimes_{Sym}$. $\square$

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