

QUASI-SPLIT iQUANTUM GROUP DEFINITIONS

1. Let Q be a **loop-free quiver**, vertex set I , $\#(i \rightarrow j)$ arrows from vertex i to vertex j .
2. The corresponding **Cartan matrix** $A = (a_{i,j})_{i,j \in I}$ is

$$a_{i,j} := \begin{cases} 2 & \text{if } i = j \\ -\#(i \rightarrow j) - \#(j \rightarrow i) & \text{if } i \neq j. \end{cases}$$

3. Fix a realization: free Abelian groups X and Y , a perfect pairing $Y \times X \rightarrow \mathbb{Z}$, linearly independent **simple coroots** $h_i \in Y$ and **simple roots** $\alpha_j \in X$ such that $h_i(\alpha_j) = a_{i,j}$.
4. Let $\tau : I \rightarrow I$ be an **involution** such that $\#(i \rightarrow j) = \#(\tau j \rightarrow \tau i)$. Let $\tau : X \rightarrow X$ be an involution such that $\tau(\alpha_i) = \alpha_{\tau i}$ and $\tau^*(h_i) = h_{\tau i}$.
5. Let $X^\tau := X / \text{im}(\text{id} + \tau)$ be the **iweight lattice**. For $\lambda \in X^\tau$ with pre-image $\hat{\lambda} \in X$ let

$$\lambda_i := (h_i - h_{\tau i})(\hat{\lambda}) \in \mathbb{Z},$$

$$\delta_i := \begin{cases} -1 & \text{if } i = \tau i \\ \#(i \rightarrow \tau i) & \text{if } i \neq \tau i. \end{cases}$$

6. The **(modified) iquantum group** $\dot{U}^\tau$ is the locally unital $\mathbb{Q}(q)$ -algebra, mutually orthogonal distinguished idempotents 1_λ ($\lambda \in X^\tau$), generators $b_i 1_\lambda = 1_{\lambda - \alpha_i} b_i$ ($i \in I, \lambda \in X^\tau$), relations

$$\sum_{n=0}^{1-a_{i,j}} (-1)^n b_i^{(n)} b_j b_i^{(1-a_{i,j}-n)} 1_\lambda = \delta_{i,\tau j} \prod_{r=1}^{-a_{i,j}} (q^r - q^{-r}) \cdot \frac{(-1)^{a_{i,j}} q^{\lambda_i - \delta_i - \binom{a_{i,j}}{2}} - q^{\binom{a_{i,j}}{2} + \delta_i - \lambda_i}}{q - q^{-1}} b_i^{(-a_{i,j})} 1_\lambda$$

for $i \neq j$ in I and $\lambda \in X^\tau$. Here, $b_i^{(n)} 1_\lambda$ is the **idivided power** defined by the recurrence relation

$$b_i b_i^{(n)} 1_\lambda = \begin{cases} [n+1] b_i^{(n+1)} 1_\lambda + [n] b_i^{(n-1)} 1_\lambda & \text{if } i = \tau i \text{ and } n \equiv h_i(\hat{\lambda}) \pmod{2} \\ [n+1] b_i^{(n+1)} 1_\lambda & \text{otherwise.} \end{cases}$$

Divided powers generate a $\mathbb{Z}[q, q^{-1}]$ -form $\dot{U}_{\mathbb{Z}}^\tau$ for $\dot{U}^\tau$.

7. Additional 2-quantum group parameters: A **ground field** $\mathbb{k}$ of characteristic $\neq 2$. A **normalization homomorphism** $c_i : X \rightarrow \mathbb{k}^\times$ such that

- $c_i(\alpha_j) = (-1)^{\#(j \rightarrow i)}$ for all $i, j \in I$;
- $c_{\tau i}(\tau(\lambda)) = (-1)^{h_i(\lambda)} c_i(\lambda)$ for all $\lambda \in X$ and $i \in I$ with $i \neq \tau i$.

Let $\xi_i := \pm 2^{\delta_i}$ choosing the signs of ξ_i and $\xi_{\tau i}$ so that exactly one is negative. Let

$$\gamma_i(\lambda) := \begin{cases} c_i(\hat{\lambda} - \tau(\hat{\lambda})) & \text{if } i \neq \tau i \\ (-1)^{h_i(\hat{\lambda})} & \text{if } i = \tau i, \end{cases}$$

Most important, let

$$Q_{i,j}(x, y) := \begin{cases} (x - y)^{\#(i \rightarrow j)} (y - x)^{\#(j \rightarrow i)} & \text{if } i \neq j \\ 0 & \text{if } i = j, \end{cases} \quad R_{i,j}(x, y) := \begin{cases} Q_{i,j}(x, y) & \text{if } i \neq j \\ 1/(x - y)^2 & \text{if } i = j, \end{cases}$$

$$Q_{i,j}^\tau(x, y) := (-1)^{\delta_{i,\tau j}} Q_{i,j}(x, y),$$

8. The **2-quantum group** $\mathfrak{U}^i$ is the graded $\mathbb{k}$ -linear 2-category with object set X^i , generating 1-morphisms $B_i \mathbb{1}_\lambda = \mathbb{1}_{\lambda - \alpha_i} B_i : \lambda \rightarrow \lambda - \alpha_i$ for $\lambda \in X^i$ and $i \in I$, identity 2-endomorphisms denoted by strings $\begin{array}{c} \lambda - \alpha_i \\ \downarrow \\ i \end{array}$, and generating 2-morphisms

Generator	Degree
$\begin{array}{c} i \\ \bullet \\ i \end{array} : B_i \mathbb{1}_\lambda \Rightarrow B_i \mathbb{1}_\lambda$	2
$\begin{array}{c} \lambda \\ \tau i \quad i \end{array} : B_{\tau i} B_i \mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$	$1 + \delta_i - \lambda_i$
$\begin{array}{c} \tau i \quad i \\ \lambda \end{array} : \mathbb{1}_\lambda \Rightarrow B_{\tau i} B_i \mathbb{1}_\lambda$	$1 + \delta_i - \lambda_i$
$\tau i \bigcirc \begin{array}{c} n \\ \lambda \end{array} : \mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$ for $0 \leq n \leq \delta_i - \lambda_i$	$2n$
$\begin{array}{c} j \quad i \\ \times \\ i \quad j \end{array} : B_i B_j \mathbb{1}_\lambda \Rightarrow B_j B_i \mathbb{1}_\lambda$	$-a_{i,j}$

Relations are expressed using **dot** and **bubble generating functions** (formal series in u^{-1}):

$$\begin{aligned} \bullet &:= \begin{array}{c} \bullet \\ \text{---} \end{array} \left(\frac{1}{u-x} \right) = \sum_{n \geq 0} n \begin{array}{c} \bullet \\ \text{---} \end{array} u^{-n-1}, \\ \tau i \bigcirc (u) \begin{array}{c} \lambda \end{array} &:= \begin{cases} -\frac{1}{2u} \text{id}_{\mathbb{1}_\lambda} + \sum_{n \geq 0} \tau i \bigcirc \begin{array}{c} n \\ \lambda \end{array} u^{-n-1} & \text{if } i = \tau i \\ \sum_{n=0}^{\delta_i - \lambda_i} \tau i \bigcirc \begin{array}{c} n \\ \lambda \end{array} u^{\delta_i - \lambda_i - n} + \sum_{n \geq 0} \tau i \bigcirc \begin{array}{c} n \\ \lambda \end{array} u^{-n-1} & \text{if } i \neq \tau i. \end{cases} \end{aligned}$$

Also x, y, z denote dots on strings in order from left to right. Defining relations:

$$\begin{aligned} \left[\tau i \bigcirc (u) \begin{array}{c} \lambda \end{array} \right]_{u: \geq \delta_i - \lambda_i} &= \xi_i \gamma_i(\lambda) u^{\delta_i - \lambda_i} \text{id}_{\mathbb{1}_\lambda}, & \left[\tau i \bigcirc (u) \begin{array}{c} i \end{array} \bigcirc (-u) \right]_{u: < -a_{i, \tau i}} &= 0, \\ \tau i \bigcirc (u) \begin{array}{c} j \end{array} \left(R_{i,j}(u, x) \right) &= \left(R_{\tau i, j}(-u, x) \right) \begin{array}{c} j \end{array} \tau i \bigcirc (u), & \begin{array}{c} i \\ \cup \end{array} &= - \begin{array}{c} i \\ \cup \end{array}, & \begin{array}{c} i \\ \cap \end{array} &= - \begin{array}{c} i \\ \cap \end{array}, \\ \begin{array}{c} \cup \\ i \end{array} &= \begin{array}{c} \cup \\ i \end{array} = \begin{array}{c} \cup \\ i \end{array}, & \begin{array}{c} \cap \\ i \end{array} &= \left[\begin{array}{c} \bullet \\ i \end{array} \bigcirc (-u) \right]_{u: -1}, & \begin{array}{c} i \quad j \\ \cup \end{array} &= \begin{array}{c} i \quad j \\ \cup \end{array}, & \begin{array}{c} i \quad j \\ \cap \end{array} &= \begin{array}{c} i \quad j \\ \cap \end{array}, \\ \begin{array}{c} \times \\ i \quad j \end{array} - \begin{array}{c} \times \\ i \quad j \end{array} &= \delta_{i,j} \begin{array}{c} | \\ i \quad j \end{array} - \delta_{i, \tau j} \begin{array}{c} j \\ \cup \end{array} = \begin{array}{c} \times \\ i \quad j \end{array} - \begin{array}{c} \times \\ i \quad j \end{array}, \\ \begin{array}{c} \times \\ i \quad j \end{array} &= \left(\frac{Q_{i,j}^i(x, y)}{\text{---}} \right) \begin{array}{c} | \\ i \quad j \end{array} + \delta_{i, \tau j} \left[\begin{array}{c} i \\ \bullet \end{array} \begin{array}{c} j \\ \bullet \end{array} \bigcirc (u) \right]_{u: -1}, \\ \begin{array}{c} \times \\ i \quad j \quad k \end{array} - \begin{array}{c} \times \\ i \quad j \quad k \end{array} &= \delta_{i,k} \begin{array}{c} | \\ i \quad j \quad k \end{array} \left(\frac{Q_{i,j}^i(x, y) - Q_{i,j}^i(z, y)}{x-z} \right) + \delta_{i, \tau j} \delta_{j, \tau k} \left[\begin{array}{c} i \\ \bullet \end{array} \begin{array}{c} j \\ \bullet \end{array} \bigcirc (u) \begin{array}{c} k \\ \bullet \end{array} - \begin{array}{c} i \\ \bullet \end{array} \begin{array}{c} j \\ \bullet \end{array} \bigcirc (-u) \begin{array}{c} k \\ \bullet \end{array} \right]_{u: -1} \\ &\quad - \delta_{i, \tau j} \begin{array}{c} j \\ \bullet \end{array} \begin{array}{c} k \\ \bullet \end{array} \begin{array}{c} \times \\ i \quad j \quad k \end{array} \left(\frac{Q_{j,k}^i(x, y) - Q_{j,k}^i(z, y)}{x-z} \right) - \delta_{j, \tau k} \begin{array}{c} k \\ \bullet \end{array} \begin{array}{c} i \\ \bullet \end{array} \begin{array}{c} \times \\ i \quad j \quad k \end{array} \left(\frac{Q_{j,i}^i(x, y) - Q_{j,i}^i(z, y)}{x-z} \right). \end{aligned}$$