SEMI-INFINITE HIGHEST WEIGHT CATEGORIES

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Dedicated to Jens Carsten Jantzen on the occasion of his 70th birthday.

Abstract. We develop the axiomatics of highest weight (and various more general stratified) categories, in order to incorporate two "semi-infinite" situations which are in Ringel duality with each other; the underlying posets are either *upper finite* or *lower finite*. We also discuss several well-known examples which fit into our setup.

Contents

1. Introduction

Highest weight categories were introduced by Cline, Parshall and Scott [CPS1] in order to provide an axiomatic framework encompassing a number of important examples which had previously arisen in representation theory. In this article, we give a detailed exposition of two semi-infinite generalizations of the notion of highest weight category, which we call *lower finite* and *upper finite highest weight categories*. Lower finite highest weight categories were already included in the original work of Cline, Parshall and Scott (although our terminology is different). Well-known examples include the category $\mathrm{Rep}(G)$ of finite-dimensional rational representations of a connected reductive algebraic group. Upper finite highest weight categories have also appeared in the literature in many examples, and an appropriate axiomatic framework was sketched out by Elias and Losev in [EL, §6.1.2]. However there are plenty of subtleties, so a full treatment seems desirable. We also explain how to extend Ringel duality to this setting:

$$\left\{\begin{array}{c}\text{lower finite}\\ \text{highest weight categories}\end{array}\right\} \xleftrightarrow{\text{Ringel duality}} \left\{\begin{array}{c}\text{upper finite}\\ \text{highest weight categories}\end{array}\right\}.$$

Other approaches to "semi-infinite Ringel duality" exist in the literature, but these typically require the existence of a $\mathbb{Z}$-grading; e.g., see [Soe] (in a Lie algebra setting) and also [Maz2]. We avoid this by working with finite-dimensional comodules over a coalgebra in the lower finite case, and with locally finite-dimensional modules over a locally unital algebra in the upper finite case.

We actually consider more generally what we call *ε-stratified categories* rather than highest weight categories. The idea of this definition is due to Ágoston, Dlab and Lukács: in [ADL, Definition 1.3] one finds the notion of a *stratified algebra of type ε*; the category

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of finite-dimensional left modules over such a finite-dimensional algebra is an example of a ε-stratified category in our sense. However, our precise setup does not seem to have appeared explicitly elsewhere in the literature, so we begin by explaining it in the finite-dimensional case. We work throughout over an algebraically closed field $\Bbbk$.

Consider a *finite Abelian category*, that is, a category $\mathcal{R}$ equivalent to the category $A\text{-mod}_{\text{fd}}$ of finite-dimensional left A-modules for some finite-dimensional $\Bbbk$-algebra A. Let $\mathbf{B}$ be a finite set indexing a full set of pairwise inequivalent irreducible objects $\{L(b) \mid b \in \mathbf{B}\}$. Let $P(b)$ (resp. $I(b)$) be a projective cover (resp. injective hull) of $L(b)$.

A *stratification* of $\mathcal{R}$ is the data of a function $\rho : \mathbf{B} \to \Lambda$ for some poset Λ. For $\lambda \in \Lambda$, let $\mathcal{R}_{\leqslant\lambda}$ (resp. $\mathcal{R}_{<\lambda}$) be the Serre subcategory of $\mathcal{R}$ generated by the irreducibles $L(b)$ for $b \in \mathbf{B}$ with $\rho(b) \leqslant \lambda$ (resp. $\rho(b) < \lambda$). Then define the *stratum* $\mathcal{R}_\lambda$ to be the Serre quotient $\mathcal{R}_{\leqslant\lambda}/\mathcal{R}_{<\lambda}$ with quotient functor $j^\lambda : \mathcal{R}_{\leqslant\lambda} \to \mathcal{R}_\lambda$. For $b \in \mathbf{B}_\lambda := \rho^{-1}(\lambda)$, let $L_\lambda(b) := j^\lambda L(b)$. These give a full set of pairwise inequivalent irreducible objects in $\mathcal{R}_\lambda$. Let $P_\lambda(b)$ (resp. $I_\lambda(b)$) be a projective cover (resp. an injective hull) of $L_\lambda(b)$ in $\mathcal{R}_\lambda$.

The functor j^λ has a left adjoint $j^\lambda_!$ and a right adjoint j^λ_*, which we refer to as the *standardization* and *costandardization functors*, respectively, following [LW, §2]. Then we introduce the *standard, proper standard, costandard* and *proper costandard* objects of $\mathcal{R}$ for $\lambda \in \Lambda$ and $b \in \mathbf{B}_\lambda$:

$$\Delta(b) := j^\lambda_! P_\lambda(b), \quad \bar\Delta(b) := j^\lambda_! L_\lambda(b), \quad \nabla(b) := j^\lambda_* I_\lambda(b), \quad \bar\nabla(b) := j^\lambda_* L_\lambda(b). \tag{1.1}$$

Equivalently, $\Delta(b)$ (resp. $\nabla(b)$) is the largest quotient of $P(b)$ (resp. the largest subobject of $I(b)$) that belongs to $\mathcal{R}_{\leqslant\lambda}$, and $\bar\Delta(b)$ (resp. $\bar\nabla(b)$) is the largest quotient of $\Delta(b)$ (resp. the largest subobject of $\nabla(b)$) such that all composition factors apart from its irreducible head (resp. its irreducible socle) belong to $\mathcal{R}_{<\lambda}$.

Fix a sign function $\varepsilon : \Lambda \to \{\pm\}$ and define the *ε-standard* and *ε-costandard objects*

$$\Delta_\varepsilon(b) := \begin{cases} \Delta(b) & \text{if } \varepsilon(\rho(b)) = + \\ \bar\Delta(b) & \text{if } \varepsilon(\rho(b)) = - \end{cases}, \qquad \nabla_\varepsilon(b) := \begin{cases} \bar\nabla(b) & \text{if } \varepsilon(\rho(b)) = + \\ \nabla(b) & \text{if } \varepsilon(\rho(b)) = - \end{cases}. \tag{1.2}$$

We call $\mathcal{R}$ a *finite ε-stratified category* if one of the following equivalent properties holds:

($P\Delta_\varepsilon$) For every $b \in \mathbf{B}$, the projective object $P(b)$ has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}$ with $\rho(c) \geqslant \rho(b)$.

($I\nabla_\varepsilon$) For every $b \in \mathbf{B}$, the injective object $I(b)$ has a ∇_ε-flag with sections $\nabla_\varepsilon(c)$ for $c \in \mathbf{B}$ with $\rho(c) \geqslant \rho(b)$.

The fact that these two properties are indeed equivalent was established originally in [ADL, Theorem 2.2] (under slightly more restrictive hypotheses than here), extending the earlier work of Dlab [Dla]. We give a self-contained proof in Theorem 3.6 below; see also §5.1 for some elementary examples. An equivalent statement is as follows.

Theorem 1.1 (Dlab,...). *Let $\mathcal{R}$ be a finite Abelian category equipped with a stratification as above. Then $\mathcal{R}$ is ε-stratified if and only if $\mathcal{R}^{\text{op}}$ is $(-\varepsilon)$-stratified.*

The notion of ε-stratified category generalizes the original notion of highest weight category from [CPS1]: a *highest weight category* is an ε-stratified category in which all strata $\mathcal{R}_\lambda$ are simple. In that case, the function ρ is a bijection, so we may identify $\mathbf{B}$ with the poset Λ, and moreover $\Delta(\lambda) = \bar\Delta(\lambda)$ and $\nabla(\lambda) = \bar\nabla(\lambda)$ for each $\lambda \in \Lambda$.

The next most important special cases arise when ε is the constant function $+$ or $-$. We refer to ε-stratified categories for these two choices of ε as *standardly stratified categories* and *costandardly stratified categories*, respectively. The notion of a finite standardly stratified category goes back to the work of Cline, Parshall and Scott [CPS2]:

it means the same as a category equivalent to the category of finite-dimensional modules over a *standardly stratified algebra* as defined in *loc. cit.*. By Theorem 1.1, $\mathcal{R}$ is costandardly stratified if and only if $\mathcal{R}^{\mathrm{op}}$ is standardly stratified.

Adopting the language suggested in the remark after [ADL, Definition 1.3], we say that $\mathcal{R}$ is *fully stratified* if it is both standardly and costandardly stratified; in that case, it is ε-stratified for all choices of the sign function ε. Fully stratified categories have appeared several times elsewhere in the literature but under different names: they are called "weakly properly stratified" in [Fri1], "exactly properly stratified" in [CZ], and "standardly stratified" in [LW]. The latter seems a particularly confusing choice since it clashes with the established notion from [CPS2] but we completely agree with the sentiment of [LW, Remark 2.2]: fully stratified categories have a particularly well-behaved structure theory. One reason for this is that all of the standardization and costandardization functors in a fully stratified category are exact. We note also that any ε-stratified category with a duality is automatically fully stratified; see Corollary 3.22.

Many examples of fully stratified categories arise in the context of categorification. This includes the pioneering examples of categorified tensor products of finite dimensional irreducible representations for the quantum group attached to $\mathfrak{sl}_k$ from [FKS] (in particular Remark 2.5 therein), and the categorified induced cell modules for Hecke algebras from [MS, 6.5]. Building on these examples and the subsequent work of Webster [Web1], [Web2], Losev and Webster [LW] formulated the important axiomatic definition of a *tensor product categorification*. On such categories, there is a fully stratified structure which gives a categorical interpretation of Lusztig's construction of tensor product of based modules for a quantum group.

The device of incorporating the sign function ε into the definition of stratified category is convenient as it streamlines many of the subsequent definitions and proofs. It also leads to some interesting new possibilities when it comes to the "tilting theory" which we discuss next, although it remains unclear to us if it the extra generality is really useful (beyond the special cases $\varepsilon = +$ or $-$).

Assuming $\mathcal{R}$ is a finite ε-stratified category, an *ε-tilting object* is an object of $\mathcal{R}$ which has both a Δ_ε-flag and a ∇_ε-flag. Isomorphism classes of *indecomposable* ε-tilting objects are parametrized in a canonical way by the set $\mathbf{B}$; see Theorem 4.2. We denote them by $T_\varepsilon(b)$, $b \in \mathbf{B}$. Now let T be a *full* ε-tilting object. The *Ringel dual* of $\mathcal{R}$ relative to T is the category

$$\widetilde{\mathcal{R}} := \mathrm{End}_{\mathcal{R}}(T)^{\mathrm{op}}\text{-mod}_{\mathrm{fd}}\,.$$

We note that the isomorphism classes of irreducible objects in $\widetilde{\mathcal{R}}$ are in bijection with the isomorphism classes of indecomposable summands of T, hence, they are also indexed naturally by the same set $\mathbf{B}$ that indexes the irreducibles in $\mathcal{R}$. Let

$$F := \mathrm{Hom}_{\mathcal{R}}(T, -) : \mathcal{R} \to \widetilde{\mathcal{R}}.$$

This is the *Ringel duality functor*. The following theorem is well known for highest weight categories (where it is due to Ringel [Rin] and Happel [Hap]) and for standardly stratified categories (see [AHLU]). We prove it for general ε in Theorem 4.11 below.

Theorem 1.2 (Ringel, Happel, ...). *Let $\widetilde{\mathcal{R}}$ be the Ringel dual of $\mathcal{R}$ relative to T.*

1. *Letting Λ^{op} be the opposite poset, the function $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$ defines a stratification of $\widetilde{\mathcal{R}}$ making it into a $(-\varepsilon)$-stratified category. Moreover, each stratum $\widetilde{\mathcal{R}}_\lambda$ of $\widetilde{\mathcal{R}}$ is equivalent to the corresponding stratum $\mathcal{R}_\lambda$ of $\mathcal{R}$.*
2. *The functor F defines an equivalence of categories between the categories of ∇_ε-filtered objects in $\mathcal{R}$ and $\Delta_{-\varepsilon}$-filtered objects in $\widetilde{\mathcal{R}}$. It sends ε-tilting objects (resp. injective objects) in $\mathcal{R}$ to projective objects (resp. $(-\varepsilon)$-tilting objects) in $\widetilde{\mathcal{R}}$.*

(3) If $\mathcal{R}_\lambda$ is of finite global dimension for each λ such that $\varepsilon(\lambda) = -$ then the total derived functor $\mathbb{R}F : D^b(\mathcal{R}) \to D^b(\widetilde{\mathcal{R}})$ is an equivalence between the bounded derived categories.

The original category $\mathcal{R}$ can be recovered from its Ringel dual $\widetilde{\mathcal{R}}$. Indeed, if we let I be an injective cogenerator in $\mathcal{R}$, then $\widetilde{T} := FI$ is a full $(-\varepsilon)$-tilting object in $\widetilde{\mathcal{R}}$ such that $\mathrm{End}_{\mathcal{R}}(I) \cong \mathrm{End}_{\widetilde{\mathcal{R}}}(\widetilde{T})$. Since $\mathcal{R}$ is equivalent to the category of finite-dimensional left $\mathrm{End}_{\mathcal{R}}(I)^{\mathrm{op}}$-modules, $\mathcal{R}$ is equivalent to the Ringel dual of $\widetilde{\mathcal{R}}$ relative to $\widetilde{T}$. The Ringel duality functor $\widetilde{F} := \mathrm{Hom}_{\widetilde{\mathcal{R}}}(\widetilde{T}, -)$ in the other direction induces an equivalence $\mathbb{R}\widetilde{F} : D^b(\widetilde{\mathcal{R}}) \to D^b(\mathcal{R})$ if $\mathcal{R}_\lambda$ is of finite global dimension for each λ with $\varepsilon(\lambda) = +$.

We do not consider here derived equivalences in the case of infinite global dimension, but instead refer to [PS], where this and involved t-structures are treated in detail by generalizing the classical theory of co(resolving) subcategories. This requires to use instead of ordinary derived categories certain coderived and contraderived categories.

Now we shift our attention to *the semi-infinite case*, which is really the main topic of the article. Following [EGNO], a *locally finite Abelian category* is a category that is equivalent to the category $\mathrm{comod}_{\mathrm{fd}}\text{-}C$ of finite-dimensional right comodules over some coalgebra C. Let $\mathcal{R}$ be such a category and $\{L(b) \mid b \in \mathbf{B}\}$ be a full set of pairwise inequivalent irreducible objects. Fix also a poset Λ that is *lower finite*, i.e., the intervals $(-\infty, \mu]$ are finite for all $\mu \in \Lambda$, a sign function $\varepsilon : \Lambda \to \{\pm\}$, and a function $\rho : \mathbf{B} \to \Lambda$ with finite fibers. For any lower set $\Lambda^\downarrow$ in Λ, we can consider the Serre subcategory $\mathcal{R}^\downarrow$ of $\mathcal{R}$ generated by the irreducible objects $\{L(b) \mid b \in \mathbf{B}^\downarrow\}$ where $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$. We say that $\mathcal{R}$ is a *lower finite ε-stratified category* if ρ defines a stratification making $\mathcal{R}^\downarrow$ into a finite ε-stratified category for every finite lower set $\mathbf{B}^\downarrow$. In particular, this means that each $\mathcal{R}^\downarrow$ must be finite Abelian.

In a lower finite ε-stratified category, there are ε-standard and ε-costandard objects $\Delta_\varepsilon(b)$ and $\nabla_\varepsilon(b)$; they are the same as the ε-standard and ε-costandard objects of the finite ε-stratified category $\mathcal{R}^\downarrow$ defined from any finite lower set $\Lambda^\downarrow$ containing $\rho(b)$. As well as (finite) Δ_ε- and ∇_ε-flags, one can consider certain infinite ∇_ε-flags in objects of the ind-competion $\mathrm{Ind}(\mathcal{R})$ (which is the category comod-C of all right C-comodules in the case that $\mathcal{R} = \mathrm{comod}_{\mathrm{fd}}\text{-}C$). We refer to these as *ascending ∇_ε-flags*; see Definition 3.51 for the precise formulation.

Theorem 3.59 establishes a cohomological criterion for an object to possess an ascending ∇_ε-flag, generalizing the well-known criterion for good filtrations in rational representations of reductive groups [Jan1, Proposition II.4.16]. From this, it follows that the injective hull $I(b)$ of $L(b)$ in $\mathrm{Ind}(\mathcal{R})$ has an ascending ∇_ε-flag. Moreover, the multiplicity of $\nabla_\varepsilon(c)$ as a section of such a flag satisfies

$$(I(b) : \nabla_\varepsilon(c)) = [\Delta_\varepsilon(c) : L(b)],$$

generalizing BGG reciprocity. This leads to an alternative "global" characterization of lower finite ε-stratified categories; see Definition 3.53.

In a lower finite ε-stratified category, there are also ε-tilting objects. Isomorphism classes of the indecomposable ones are labelled by $\mathbf{B}$ just like in the finite case. In fact, for $b \in \mathbf{B}$ the corresponding indecomposable ε-tilting object of $\mathcal{R}$ is the same as the object $T_\varepsilon(b)$ of the finite ε-stratified category $\mathcal{R}^\downarrow$ defined from any finite lower set $\Lambda^\downarrow$ containing $\rho(b)$. Let $(T_i)_{i \in I}$ be a *full ε-tilting family* in $\mathcal{R}$. Then we can define the *Ringel dual* $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ relative to $T := \bigoplus_{i \in I} T_i$ (an object in the ind-completion of $\mathcal{R}$): it is the category of locally finite-dimensional left modules over the locally unital algebra

$$A = \bigoplus_{i,j \in I} \mathrm{Hom}_{\mathcal{R}}(T_i, T_j),$$

with multiplication in A being the opposite of composition in $\mathcal{R}$. Saying that A is *locally unital* means that $A = \bigoplus_{i,j\in I} e_i A e_j$ where $\{e_i \mid i \in I\}$ are the mutually orthogonal idempotents defined by the identity endomorphisms of each T_i. A *locally finite-dimensional module* is an A-module $V = \bigoplus_{i\in I} e_i V$ with $\dim e_i V < \infty$ for each $i \in I$.

This brings us to the notion of an *upper finite ε-stratified category*, whose definition may be discovered by considering the nature of the categories $\widetilde{\mathcal{R}}$ that can arise as Ringel duals of lower finite ε-stratified categories as just defined. We refer to Definition 3.31 for the intrinsic formulation. In fact, starting from $\mathcal{R}$ that is a lower finite ε-stratified category, the Ringel dual $\widetilde{\mathcal{R}}$ is an upper finite $(-\varepsilon)$-stratified category with stratification defined by reversing the partial order on the poset Λ; see Theorem 4.20 which extends parts (1) and (2) of Theorem 1.2.

In general, in an upper finite ε-stratified category, the underlying poset is required to be upper finite, i.e., all of the intervals $[\lambda, \infty)$ are finite. There are ε-standard and ε-costandard objects, but now these can have infinite length (although composition multiplicities in such objects are finite). On the other hand, the indecomposable projectives and injectives do still have finite Δ_ε-flags and ∇_ε-flags, exactly like in $(P\Delta_\varepsilon)$ and $(I\nabla_\varepsilon)$. Perhaps the most interesting feature is that one can still make sense of ε-tilting objects. These are objects possessing certain infinite flags: both an *ascending Δ_ε-flag* and a *descending ∇_ε-flag*; see Definition 3.35. This allows to define the *Ringel dual* of an upper finite ε-stratified category; see Definition 4.21 and Theorem 4.22.

For $\widetilde{\mathcal{R}}$ arising as the Ringel dual of a lower finite ε-stratified category $\mathcal{R}$, the indecomposable $(-\varepsilon)$-tilting objects in $\widetilde{\mathcal{R}}$ are the images of the indecomposable injective objects of $\mathcal{R}$ under the Ringel duality functor

$$F := \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{R}}(T_i, -) : \mathcal{R} \to \widetilde{\mathcal{R}}.$$

Moreover, the Ringel dual of $\widetilde{\mathcal{R}}$ is equivalent to the original category $\mathcal{R}$. The proof of this relies on the following elementary observation: if U is any locally finite-dimensional module over a locally unital algebra A then the endomorphism algebra $\mathrm{End}_A(U)^{\mathrm{op}}$ is the linear dual C^* of a coalgebra C; see Lemma 2.8. Taking U to be a full $(-\varepsilon)$-tilting object in $\widetilde{\mathcal{R}}$, this produces a coalgebra C such that the original category $\mathcal{R}$ is equivalent to $\mathrm{comod}_{\mathrm{fd}}$-$C$.

A *lower finite* (resp. *upper finite*) *highest weight category* is a lower finite (resp. upper finite) ε-stratified category all of whose strata are simple. As we have already mentioned, the category $\mathrm{Rep}(G)$ for a reductive group G gives an example of a lower finite highest weight category. Its Ringel dual is an upper finite highest weight category. This case has been studied in particular by Donkin (e.g., see [Do2], [Do3]), but Donkin's approach involves truncating to a finite-dimensional algebra from the outset. Other important examples come from blocks of category $\mathcal{O}$ over an affine Lie algebra: in negative level one obtains lower finite highest weight categories, while positive level produces upper finite ones. These and several other prominent examples are outlined in §§5.2–5.5.

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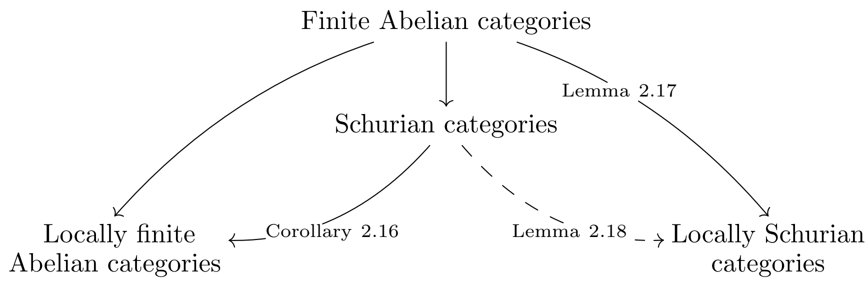
2. Some finiteness properties on Abelian categories

We fix an algebraically closed field $\Bbbk$. All algebras, categories, functors, etc. will be assumed to be linear over $\Bbbk$ without further mention. We write $\otimes$ for $\otimes_\Bbbk$. We begin by introducing some language for Abelian categories with various finiteness properties:



2.1. **Finite and locally finite Abelian categories.** According to [EGNO, Definition 1.8.5], a *finite Abelian category* is a category that is equivalent to the category $A\text{-mod}_{\text{fd}}$ of finite-dimensional (left) modules over some finite-dimensional algebra A. Equivalently, a finite Abelian category is a category equivalent to the category $\text{comod}_{\text{fd}}\text{-}C$ of finite-dimensional (right) comodules over some finite-dimensional coalgebra C. To explain this in more detail, recall that the dual $A := C^*$ of a finite-dimensional coalgebra C has a natural algebra structure with multiplication $A \otimes A \to A$ that is the dual of the comultiplication $C \to C \otimes C$; for this, one needs to use the canonical isomorphism

$$C^* \otimes C^* \to (C \otimes C)^*, \qquad f \otimes g \mapsto (v \otimes w \mapsto f(v)g(w)) \tag{2.1}$$

to identify $C^* \otimes C^*$ with $(C \otimes C)^*$. Then any right C-comodule can be viewed as a left A-module with action defined from $av := \sum_{i=1}^n a(c_i)v_i$, assuming here that the structure map $\eta : V \to V \otimes C$ sends $v \mapsto \sum_{i=1}^n v_i \otimes c_i$. This construction defines an *isomorphism* of categories $\text{comod}_{\text{fd}}\text{-}C \xrightarrow{\sim} A\text{-mod}_{\text{fd}}$.

A *locally finite Abelian category* is a category that is equivalent to $\text{comod}_{\text{fd}}\text{-}C$ for a (not necessarily finite-dimensional) coalgebra C. We refer to a choice of C as a *coalgebra realization* of $\mathcal{R}$. The following theorem of Takeuchi gives an intrinsic characterization of locally finite Abelian categories; see [Tak] and [EGNO, Theorem 1.9.15]. Note Takeuchi's original paper uses the language "locally finite Abelian" slightly differently (following [Gab]) but his formulation is equivalent to the one here (which follows [EGNO, Definition 1.8.1]). In *loc. cit.* it is shown moreover that the coalgebra realization can be chosen so that it is *pointed*, i.e., all of its irreducible comodules are one-dimensional; in that case, C is unique up to isomorphism.

Lemma 2.1. *An essentially small Abelian category $\mathcal{R}$ is locally finite if and only if all objects are of finite length and all morphism spaces are finite-dimensional.*

Now we summarize the main properties of the locally finite Abelian category

$$\mathcal{R} = \text{comod}_{\text{fd}}\text{-}C.$$

Fix a full set of pairwise inequivalent irreducible objects $\{L(b) \mid b \in \mathbf{B}\}$ in $\mathcal{R}$. By Schur's Lemma, we have that $\text{End}_{\mathcal{R}}(L(b)) = \Bbbk$ for each $b \in \mathbf{B}$.

The opposite category $\mathcal{R}^{\text{op}}$ is also a locally finite Abelian category. Moreover, a coalgebra realization for it is given by the opposite coalgebra C^{op}. To see this, note that there is a contravariant equivalence

$$* : \text{comod}_{\text{fd}}\text{-}C \to C\text{-comod}_{\text{fd}} \tag{2.2}$$

sending a finite-dimensional right comodule to the dual vector space viewed as a left comodule in the natural way: if $v_1, \dots, v_n$ is a basis for V, with dual basis $f_1, \dots, f_n$ for V^*, and the structure map $V \to V \otimes C$ sends $v_j \mapsto \sum_{i=1}^n v_i \otimes c_{i,j}$ then the dual's structure map $V^* \to C \otimes V^*$ sends $f_i \mapsto \sum_{i=1}^n c_{i,j} \otimes f_j$. Since we have that $C\text{-comod}_{\text{fd}} \cong \text{comod}_{\text{fd}}\text{-}C^{\text{op}}$, we deduce that $\mathcal{R}^{\text{op}}$ is equivalent to $\text{comod}_{\text{fd}}\text{-}C^{\text{op}}$.

In general, $\mathcal{R}$ need not have enough injectives or projectives. To get injectives, we pass to the *ind-completion* $\text{Ind}(\mathcal{R})$; see e.g. [KS, §6.1]. For $V, W \in \text{Ind}(\mathcal{R})$, we abuse notation by writing simply $\text{Ext}^n_{\mathcal{R}}(V, W)$ for $\text{Ext}^n_{\text{Ind}(\mathcal{R})}(V, W)$; it may be computed via an injective resolution of W in the ind-completion. More generally, we can consider the right derived functors $\mathbb{R}^n F$ of any left exact functor $F : \text{Ind}(\mathcal{R}) \to \mathcal{S}$.

Let comod-C be the category of all right C-comodules. Since every comodule is the union of its finite-dimensional subcomodules, any right C-comodule may be viewed as an object of $\text{Ind}(\mathcal{R})$. The resulting inclusion functor comod-$C \to \text{Ind}(\mathcal{R})$ is an equivalence of categories. This means that one can work simply with comod-C in place of $\text{Ind}(\mathcal{R})$, as we do in the next few paragraphs.

The category comod-C is a Grothendieck category: it is Abelian, it possesses all small coproducts, direct colimits of monomorphisms are monomorphisms, and there is a generator. A generating family may be obtained by choosing representatives for the isomorphism classes of finite-dimensional C-comodules. By the general theory of Grothendieck categories, every C-comodule has an injective hull. We use the notation $I(b)$ to denote an injective hull of $L(b)$. The right regular comodule decomposes as

$$C \cong \bigoplus_{b \in \mathbf{B}} I(b)^{\oplus \dim L(b)}. \tag{2.3}$$

By Baer's criterion for Grothendieck categories (e.g., see [KS, Proposition 8.4.7]), arbitrary direct sums of injectives are injective. It follows that an injective hull of $V \in$ comod-C comes from an injective hull of its socle: if $\text{soc}\, V \cong \bigoplus_{s \in S} L(b_s)$ then $\bigoplus_{s \in S} I(b_s)$ is an injective hull of V.

In any Abelian category, we write $[V : L]$ for the *composition multiplicity* of an irreducible object L in an object V. By definition, this is the supremum of the sizes of the sets $\{i = 1, \dots, n | V_i/V_{i-1} \cong L\}$ over all finite filtrations $0 = V_0 < V_1 < \cdots < V_n = V$; possibly, $[V : L] = \infty$. Composition multiplicity is additive on short exact sequences. For any right C-comodule V, we have by Schur's Lemma that

$$[V : L(b)] = \dim \text{Hom}_C(V, I(b)). \tag{2.4}$$

When C is infinite-dimensional, the map (2.1) is not an isomorphism, but one can still use it to make the dual vector space $B := C^*$ into a unital algebra. Since C is the union of its finite-dimensional subcoalgebras, the algebra B is isomorphic to the inverse limit of its finite-dimensional quotients, i.e., the canonical homomorphism $B \to \varprojlim (B/J)$ is an isomorphism where the limit is over all two-sided ideals J of B of finite codimension. These two-sided ideals J form a basis of neighborhoods of 0 making B into a *pseudocompact topological algebra*; see [Gab, Ch. IV] or [Sim, Definition 2.4]. We refer to the topology on B defined in this way as the *profinite topology.* The coalgebra C can be recovered from B as the *continuous dual*

$$B^* := \left\{ f \in B^* \,\middle|\, f \text{ vanishes on some two-sided ideal } J \text{ of finite codimension} \right\}. \tag{2.5}$$

It has a natural coalgebra structure dual to the algebra structure on B. This is discussed further in [Sim, §3]; see also [EGNO, §1.12] where B^* is called the *finite dual.* We note that any left ideal I of B of finite codimension contains a two-sided ideal J of finite codimension, namely, $J := \text{Ann}_B(B/I)$. So, in the definition (2.5) of continuous dual, "two-sided ideal J of finite codimension" can be replaced by "left ideal I of finite codimension." Similarly for right ideals.

Any right C-comodule V is naturally a left B-module by the same construction as in the finite-dimensional case. We deduce that the category comod-C of all right C-comodules is isomorphic to the full subcategory $B\text{-mod}_{\text{ds}}$ of B-mod consisting of all *discrete* left B-modules, that is, all B-modules which are the unions of their finite-dimensional submodules. In particular, $\text{comod}_{\text{fd}}\text{-}C$ and $B\text{-mod}_{\text{fd}}$ are identified under this construction. This means that any locally finite Abelian category may be realized as the category of finite-dimensional modules over an algebra which is pseudocompact with respect to the profinite topology; see also [Sim, §3].

The definition of the left C-comodule structure on the linear dual V^* of a right C-comodule V in (2.2) required V to be finite-dimensional in order for it to make sense. If V is an infinite-dimensional right C-comodule, it can be viewed equivalently as a discrete left module over the dual algebra $B := C^*$. Then its dual V^* is a *pseudocompact right B-module*, that is, a B-module isomorphic to the inverse limit of its finite-dimensional quotients. Viewing pseudocompact modules as topological B-modules with respect to the profinite topology (i.e., submodules of finite codimension form a basis of neighborhoods of 0), we obtain the category $\text{mod}_{\text{pc}}\text{-}B$ of all pseudocompact right B-modules and continuous B-module homomorphisms. The duality functor (2.2) extends to

$$* : B\text{-mod}_{\text{ds}} \to \text{mod}_{\text{pc}}\text{-}B. \tag{2.6}$$

This is contravariant equivalence with quasi-inverse given by the functor

$$\circledast : \text{mod}_{\text{pc}}\text{-}B \to B\text{-mod}_{\text{ds}} \tag{2.7}$$

taking $V \in \text{mod}_{\text{pc}}\text{-}B$ to its *continuous dual*

$$V^{\circledast} := \left\{ f \in V^* \;\middle|\; f \text{ vanishes on some submodule of } V \text{ finite codimension} \right\}. \tag{2.8}$$

Similarly, there are contravariant equivalences $*$ and $\circledast$ between $\text{mod}_{\text{ds}}\text{-}B = B^{\text{op}}\text{-mod}_{\text{pc}}$ and $B\text{-mod}_{\text{pc}} = \text{mod}_{\text{pc}}\text{-}B^{\text{op}}$. The category $B\text{-mod}_{\text{pc}}$ is equivalent to the *pro-completion*

$$\text{Pro}(\mathcal{R}) := (\text{Ind}(\mathcal{R}^{\text{op}}))^{\text{op}} \tag{2.9}$$

of $\mathcal{R} = \text{comod}_{\text{fd}}\text{-}C \cong B\text{-mod}_{\text{fd}}$. We will only occasionally need this point of view; for more details see [Gab, Ch. IV] or [Sim, §4].

Lemma 2.2. *Suppose that C is a coalgebra and $B := C^*$ is its dual algebra. For any right C-comodule V, composing with the counit $\varepsilon : C \to \Bbbk$ defines an isomorphism of left B-modules $\alpha_V : \text{Hom}_C(V, C) \xrightarrow{\sim} V^*$. When $V = C$, this map gives an algebra isomorphism $\text{End}_C(C)^{\text{op}} \cong B$.*

Proof. Let $\eta : V \to V \otimes C$ be the comodule structure map. To show that α_V is an isomorphism, one checks that the map $\beta_V : V^* \to \text{Hom}_C(V, C), f \mapsto (f \bar{\otimes} \text{id}) \circ \eta$ is its two-sided inverse; cf. [Sim, Lemma 4.9]. It remains to show that $\alpha_C : \text{End}_C(C)^{\text{op}} \xrightarrow{\sim} B$ is an algebra homomorphism: for $f, g \in B$ we have that

$$\begin{aligned}\alpha_C(\beta_C(g) \circ \beta_C(f)) &= \varepsilon \circ (g \bar{\otimes} \text{id}) \circ \eta \circ (f \bar{\otimes} \text{id}) \circ \eta \\ &= (g \bar{\otimes} \text{id}) \circ (\text{id} \otimes \varepsilon) \circ \eta \circ (f \bar{\otimes} \text{id}) \circ \eta = g \circ (f \bar{\otimes} \text{id}) \circ \eta = fg.\end{aligned}$$

□

2.2. **Locally unital algebras.** We are going to work with certain Abelian categories which are not locally finite, but which nevertheless have some well-behaved finiteness properties. We will define these in the next subsection. First we must review some basic notions about locally unital algebras. These ideas originate in the work of Mitchell [Mit].

A *locally unital algebra* is an associative (but not necessarily unital) algebra A equipped with a distinguished system $\{e_i \mid i \in I\}$ of mutually orthogonal idempotents such that

$$A = \bigoplus_{i,j \in I} e_i A e_j.$$

We say A is *locally* (resp., *essentially*) *finite-dimensional* if each subspace $e_i A e_j$ (resp., each right ideal $e_i A$ and each left ideal $A e_j$) is finite-dimensional.

Suppose that A is a locally unital algebra. An A*-module* means a left module V as usual such that $V = \bigoplus_{i \in I} e_i V$. An A-module V is

- *locally finite-dimensional* if $\dim e_i V < \infty$ for all $i \in I$;
- *finitely generated* if $V = Av_1 + \cdots + Av_n$ for $n \in \mathbb{N}$ and $v_1, \ldots, v_n \in V$.

Let A-mod (resp., A-mod$_{\text{lfd}}$, resp., A-mod$_{\text{fg}}$) be the category of all A-modules (resp., the locally finite-dimensional ones, resp., the finitely generated ones). Similarly, we define the categories mod-A, mod$_{\text{lfd}}$-A and mod$_{\text{fg}}$-A of right modules.

Lemma 2.3. *An essentially small Abelian category $\mathcal{R}$ is equivalent to* A-mod *for some locally unital algebra A if and only if $\mathcal{R}$ possesses all small coproducts and it has a projective generating family, i.e., there is a family $(P_i)_{i \in I}$ of compact (= finitely generated) projective objects such that $V \neq 0 \Rightarrow \operatorname{Hom}_{\mathcal{R}}(P_i, V) \neq 0$ for some $i \in I$.*

Proof. This is similar to [Frey, Exercise 5.F]. One shows that $\mathcal{R}$ is equivalent to A-mod for the locally unital algebra $A = \bigoplus_{i,j \in I} e_i A e_j$ defined by setting $e_i A e_j := \operatorname{Hom}_{\mathcal{R}}(P_i, P_j)$ with multiplication that is the opposite of composition in $\mathcal{R}$. The canonical equivalence $\mathcal{R} \to A$-mod is given by the functor $\bigoplus_{i \in I} \operatorname{Hom}_{\mathcal{R}}(P_i, -)$. □

Remark 2.4. The data of a locally unital algebra $A = \bigoplus_{i,j \in I} e_i A e_j$ is the same as the data of a category $\mathcal{A}$ with object set I and $\operatorname{Hom}_{\mathcal{A}}(j, i) = e_i A e_j$. In these terms, a locally finite-dimensional locally unital algebra is the same as a *finite-dimensional category*, that is, a small category whose morphism spaces are finite-dimensional. A left A-module (resp., a locally finite-dimensional A-module) is the same as a $\Bbbk$-linear functor from $\mathcal{A}$ to the category Vec (resp., Vec$_{\text{fd}}$) of vector spaces (resp., finite-dimensional vector spaces).

The following lemma is fundamental. It is the analog of "adjointness of tensor and hom" in the locally unital setting; see e.g. [BD1, §2.1] for a fuller discussion.

Lemma 2.5. *Let $A = \bigoplus_{i,j \in I} e_i A e_j$ and $B = \bigoplus_{i,j \in J} f_i B f_j$ be locally unital algebras, and let $M = \bigoplus_{i \in I, j \in J} e_i M f_j$ be an (A, B)-bimodule.*

1. *The functor $M \otimes_B - :$ B-mod $\to$ A-mod is left adjoint to $\bigoplus_{j \in J} \operatorname{Hom}_A(M f_j, -)$.*
2. *The functor $- \otimes_A M :$ mod-A $\to$ mod-B is left adjoint to $\bigoplus_{i \in I} \operatorname{Hom}_B(e_i M, -)$.*

For any locally unital algebra A, there is a contravariant equivalence

$$\circledast : A\text{-mod}_{\text{lfd}} \to \text{mod}_{\text{lfd}}\text{-}A \tag{2.10}$$

sending a left module V to $V^\circledast := \bigoplus_{i \in I} (e_i V)^*$, viewed as a right module in the obvious way. The analogous duality functor $\circledast : \text{mod}_{\text{lfd}}\text{-}A \to A\text{-mod}_{\text{lfd}}$ gives a quasi-inverse. The contravariant functor (2.10) also makes sense on arbitrary left (or right) A-modules. It is no longer an equivalence, but we still have that

$$\operatorname{Hom}_A(V, W^\circledast) \cong \operatorname{Hom}_A(W, V^\circledast) \tag{2.11}$$

for any $V \in A$-mod and $W \in$ mod-A. To prove this, apply Lemma 2.5(1) to the $(\Bbbk, A)$-bimodule W to show that $\operatorname{Hom}_A(V, W^\circledast) \cong (W \otimes_A V)^*$, then apply Lemma 2.5(2) to the $(A, \Bbbk)$-bimodule V to show that $(W \otimes_A V)^* \cong \operatorname{Hom}_A(W, V^\circledast)$.

Lemma 2.6. *The dual $V^{\circledast}$ of a projective (left or right) A-module is an injective (right or left) A-module.*

Proof. Just like in the classic treatment of duality for vector spaces from [Mac, IV.2], (2.11) shows that the covariant functor $\circledast : A\text{-mod} \to (\text{mod-}A)^{\text{op}}$ is left adjoint to the exact covariant functor $\circledast : (\text{mod-}A)^{\text{op}} \to A\text{-mod}$. So it sends projective left A-modules to projectives in $(\text{mod-}A)^{\text{op}}$, which are injective right A-modules. □

Now we assume that A is a locally unital algebra and $U \in A\text{-mod}_{\text{lfd}}$. Let

$$B := \text{End}_A(U)^{\text{op}},$$

which is a unital algebra. Then U is an (A, B)-bimodule and the dual $U^{\circledast}$ is a (B, A)-bimodule. Let $U_i := e_i U$, so that $U = \bigoplus_{i \in I} U_i$ and $U^{\circledast} = \bigoplus_{i \in I} U_i^*$.

Lemma 2.7. *Suppose that $U = \bigoplus_{i \in I} U_i \in A\text{-mod}_{\text{lfd}}$ and $B := \text{End}_A(U)^{\text{op}}$ are as above. For any $V \in A\text{-mod}$, there is a natural isomorphism of right B-modules*

$$\text{Hom}_A(V, U) \xrightarrow{\sim} (U^{\circledast} \otimes_A V)^*, \qquad \theta \mapsto (f \otimes v \mapsto f(\theta(v))).$$

In particular, taking $V = U$, we get that $(U^{\circledast} \otimes_A U)^* \cong B$ as (B, B)-bimodules.

Proof. By Lemma 2.5(1) applied to the $(A, \Bbbk)$-bimodule $U^{\circledast}$, the functor $U^{\circledast} \otimes_A -$ is left adjoint to $\bigoplus_{i \in I} \text{Hom}_{\Bbbk}(U_i^*, -)$. Hence,

$$(U^{\circledast} \otimes_A V)^* = \text{Hom}_{\Bbbk}(U^{\circledast} \otimes_A V, \Bbbk) \cong \text{Hom}_A(V, \bigoplus_{i \in I} \text{Hom}_{\Bbbk}(U_i^*, \Bbbk)) \cong \text{Hom}_A(V, U).$$

This is the natural isomorphism in the statement of the lemma. We leave it to the reader to check that it is a B-module homomorphism. □

Continuing with this setup, let

$$C := U^{\circledast} \otimes_A U. \tag{2.12}$$

There is a unique way to make C into a coalgebra so that the bimodule isomorphism $B \xrightarrow{\sim} C^*$ from Lemma 2.7 is actually an algebra isomorphism (viewing the dual C^* of a coalgebra as an algebra as in the previous subsection). Explicitly, let $y_1^{(i)}, \dots, y_{n_i}^{(i)}$ be a basis for U_i and $x_1^{(i)}, \dots, x_{n_i}^{(i)}$ be the dual basis for U_i^*. Let $c_{s,r}^{(i)} := x_r^{(i)} \otimes y_s^{(i)} \in C$. Then the comultiplication $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to \Bbbk$ satisfy

$$\Delta(c_{r,s}^{(i)}) = \sum_{t=1}^{n_i} c_{r,t}^{(i)} \otimes c_{t,s}^{(i)}, \qquad \varepsilon(c_{r,s}^{(i)}) = \delta_{r,s} \tag{2.13}$$

for each $i \in I$ and $1 \leqslant r, s \leqslant n_i$. For the next lemma, we recall the definition of continuous dual of a pseudocompact topological algebra from (2.5).

Lemma 2.8. *The endomorphism algebra $B = \text{End}_A(U)^{\text{op}}$ of $U \in A\text{-mod}_{\text{lfd}}$ is a pseudocompact topological algebra with respect to the profinite topology, i.e., B is isomorphic to $\varprojlim B/J$ where the inverse limit is over all two-sided ideals J of finite codimension. Moreover, the coalgebra C from (2.12) may be identified with the continuous dual B^*.*

Proof. This follows because $B \cong C^*$ as algebras. □

Now consider the functor $U^{\circledast} \otimes_A - : A\text{-mod} \to B\text{-mod}$. Since U is locally finite-dimensional, it takes finitely generated A-modules to finite-dimensional B-modules. Any A-module V is the union of its finitely generated submodules, and $U^{\circledast} \otimes_A -$ commutes with direct limits, so we see that $U^{\circledast} \otimes_A V$ is actually a discrete B-module. This shows that we have a well-defined functor

$$U^{\circledast} \otimes_A - : A\text{-mod} \to B\text{-mod}_{\text{ds}}. \tag{2.14}$$

Since $B \cong C^*$, the category $B\text{-mod}_{\text{ds}}$ is isomorphic to comod-C. Consequently, for $V \in A\text{-mod}$, we can view $U^{\circledast} \otimes_A V$ instead as a right C-comodule. Its structure map $\eta : U^{\circledast} \otimes_A V \to U^{\circledast} \otimes_A V \otimes C$ is given explicitly by the formula

$$\eta(x_s^{(i)} \otimes v) = \sum_{r=1}^{n_i} x_r^{(i)} \otimes v \otimes c_{r,s}^{(i)}. \tag{2.15}$$

Recall the definition of the functor $*$ from (2.8).

Lemma 2.9. *Suppose that* $U = \bigoplus_{i \in I} U_i \in A\text{-mod}_{\text{lfd}}$ *and* $B := \operatorname{End}_A(U)^{\text{op}}$ *are as above. The functor* $U^{\circledast} \otimes_A -$ *just constructed is isomorphic to*

$$* \circ \operatorname{Hom}_A(-, U) : A\text{-mod} \to B\text{-mod}_{\text{ds}} \tag{2.16}$$

and left adjoint to the functor

$$\bigoplus_{i \in I} \operatorname{Hom}_B(U_i^*, -) : B\text{-mod}_{\text{ds}} \to A\text{-mod}\,. \tag{2.17}$$

Proof. The fact that (2.14) is left adjoint to (2.17) follows by Lemma 2.5. To see that it is isomorphic to (2.16), take $V \in A\text{-mod}$ and consider the natural isomorphism $\operatorname{Hom}_A(V, U) \cong (U^{\circledast} \otimes_A V)^*$ of right B-modules from Lemma 2.7. As $U^{\circledast} \otimes_A V$ is discrete, its dual is a pseudocompact left B-module, hence, $\operatorname{Hom}_A(V, U)$ is pseudocompact too. Then we apply $*$, using that it is quasi-inverse to $*$, to get that $\operatorname{Hom}_A(V, U)^* \in B\text{-mod}_{\text{ds}}$ is naturally isomorphic to $U^{\circledast} \otimes_A V$. □

Finally assume that A is locally finite-dimensional. Let $A\text{-mod}_{\text{lpc}}$ (resp., $\text{mod}_{\text{lpc}}\text{-}A$) be the subcategory of A-mod (resp., mod-A) consisting of all *locally pseudocompact modules*, that is, the modules which are isomorphic to the inverse limit of their locally finite-dimensional quotients. We put a topology on such a module V, the *locally profinite topology*, by declaring that the submodules W such that V/W is locally finite-dimensional form a basis of neighborhoods of 0. Then morphisms in $A\text{-mod}_{\text{lpc}}$ (resp., $\text{mod}_{\text{lpc}}\text{-}A$) are the continuous A-module homomorphisms.

As A is locally finite-dimensional, each Ae_i is a locally finite-dimensional A-module. Hence, finitely generated modules are locally finite-dimensional. Since any A-module is a union of finitely generated submodules, it follows that every A-module is the union of its locally finite-dimensional submodules. From this, it follows that the (faithful) duality functor $\circledast : A\text{-mod} \to \text{mod-}A$ defines a contravariant equivalence

$$\circledast : A\text{-mod} \to \text{mod}_{\text{lpc}}\text{-}A. \tag{2.18}$$

A quasi-inverse $\circledast : \text{mod}_{\text{lpc}}\text{-}A \to A\text{-mod}$ is given by taking the *continuous dual*, i.e.,

$$V^{\circledast} := \left\{ f \in V^{\circledast} \mid f(W) = 0 \text{ for some } W \leqslant V \text{ such that } V/W \in \text{mod}_{\text{lfd}}\text{-}A \right\}. \tag{2.19}$$

Similarly, we define quasi-inverse dualities $\circledast$ and $\circledast$ between mod-A and $A\text{-mod}_{\text{lpc}}$. All of this is similar to (2.6)–(2.7) except now the topology on A is the discrete topology.

Lemma 2.10. *Suppose that A is a locally finite-dimensional locally unital algebra. Let* $U = \bigoplus_{i \in I} U_i \in A\text{-mod}_{\text{lfd}}$ *and* $B := \operatorname{End}_A(U)^{\text{op}}$ *be as in Lemma 2.9. The functor*

$$\operatorname{Hom}_A(U, -) : A\text{-mod}_{\text{lpc}} \to B\text{-mod}_{\text{pc}} \tag{2.20}$$

has a left adjoint $\circledast \circ \bigoplus_{i \in I} \operatorname{Hom}_B(-, U_i^*) : B\text{-mod}_{\text{pc}} \to A\text{-mod}_{\text{lpc}}$. *Moreover, this left adjoint is isomorphic to the functor*

$$U \overline{\otimes}_B - : B\text{-mod}_{\text{pc}} \to A\text{-mod}_{\text{lpc}}, \tag{2.21}$$

taking $V \in B\text{-mod}_{\text{pc}}$ *to the largest quotient of* $U \otimes_B V$ *that lies in* $A\text{-mod}_{\text{lpc}}$.

Proof. We have that $\operatorname{End}_A(U^{\circledast})^{\mathrm{op}} \cong \operatorname{End}_A(U) = B^{\mathrm{op}}$. Hence, on applying Lemma 2.9 with U, A and B replaced by $U^{\circledast} = U^{\circledast}, A^{\mathrm{op}}$ and B^{op}, respectively, we get that

$$X := * \circ \operatorname{Hom}_A(-, U^{\circledast}) : \text{mod-}A \to \text{mod}_{\mathrm{ds}}\text{-}B$$

is left adjoint to $Y := \bigoplus_{i \in I} \operatorname{Hom}_B(U_i, -) : \text{mod}_{\mathrm{ds}}\text{-}B \to \text{mod-}A$. We deduce that $* \circ X \circ \circledast$ is right adjoint to $\circledast \circ Y \circ *$. These are isomorphic to the functors (2.20) and its left adjoint respectively. Note finally that $U \overline{\otimes}_B -$ is obviously also left adjoint to (2.20) as a consequence of the usual adjointness of tensor and hom. □

2.3. **Locally Schurian categories.** A *locally Schurian category* is a category $\mathcal{R}$ that is equivalent to $A\text{-mod}_{\mathrm{lfd}}$ for a locally finite-dimensional locally unital algebra A. This terminology was introduced in [BD1], but we warn the reader that it was used there to refer to a category equivalent to A-mod (rather than $A\text{-mod}_{\mathrm{lfd}}$) for such locally unital algebras A. We refer to A (together with the set I indexing its distinguished idempotents) as an *algebra realization* of $\mathcal{R}$. We say that A is *pointed* if it is basic, i.e., all of its irreducible modules are one-dimensional, and moreover each of its distinguished idempotents is primitive. Any locally Schurian category $\mathcal{R}$ has a unique (up to isomorphism) pointed algebra realization. We will discuss this further at the end of the subsection.

Let us summarize some of the basic properties of locally Schurian categories, referring to [BD1, §2] for a more detailed treatment. Assume for the remainder of the subsection that A is a locally finite-dimensional locally unital algebra and let

$$\mathcal{R} = A\text{-mod}_{\mathrm{lfd}}\,.$$

Let $\{L(b) \mid b \in \mathbf{B}\}$ be a full set of pairwise inequivalent irreducible objects of $\mathcal{R}$. Schur's Lemma holds: we have that $\operatorname{End}_{\mathcal{R}}(L(b)) = \Bbbk$ for each $b \in \mathbf{B}$.

The opposite category $\mathcal{R}^{\mathrm{op}}$ is also locally Schurian, and A^{op} gives an algebra realization for it. This follows because $\mathcal{R}^{\mathrm{op}} = (A\text{-mod}_{\mathrm{lfd}})^{\mathrm{op}}$ is equivalent to $\text{mod}_{\mathrm{lfd}}\text{-}A \cong (A^{\mathrm{op}})\text{-mod}_{\mathrm{lfd}}$ using the duality (2.10).

The *ind-completion* $\operatorname{Ind}(\mathcal{R})$ is the category A-mod of all A-modules. This follows because any A-module V is the union of its finitely generated submodules, which are all locally finite-dimensional. Although we avoid working in it as much as possible, one can also consider the *pro-completion*

$$\operatorname{Pro}(\mathcal{R}) = \operatorname{Ind}(\mathcal{R}^{\mathrm{op}})^{\mathrm{op}}. \tag{2.22}$$

The contravariant equivalence $\circledast : \text{mod-}A \to A\text{-mod}_{\mathrm{lpc}}$ identifies $\operatorname{Pro}(\mathcal{R})$ with the subcategory $A\text{-mod}_{\mathrm{lpc}}$ of A-mod.

Every A-module has an injective hull in A-mod, as follows by the general theory of Grothendieck categories. Since every A-module is a quotient of a direct sum of projective A-modules of the form Ae_i, the category A-mod also has enough projectives, It is *not* true that an arbitrary A-module has a projective cover, but we will see in Lemma 2.12 below that finitely generated A-modules do.

As we did in the case of locally finite Abelian categories, we write $\operatorname{Ext}^n_{\mathcal{R}}(V, W)$ in place of $\operatorname{Ext}^n_{\operatorname{Ind}(\mathcal{R})}(V, W)$ for any $V, W \in \operatorname{Ind}(\mathcal{R})$; it can be computed either from a projective resolution of V or from an injective resolution of W. We can also consider both right derived functors $\mathbb{R}^n F$ of a left exact functor $F : \operatorname{Ind}(\mathcal{R}) \to \mathcal{S}$ and left derived functors $\mathbb{L}_n G$ of a right exact functor $G : \operatorname{Ind}(\mathcal{R}) \to \mathcal{S}$. The following lemma is non-trivial since $\operatorname{Ind}(\mathcal{R})$ is not the same as $\operatorname{Pro}(\mathcal{R})$!

Lemma 2.11. *For $V, W \in \mathcal{R}$ and $n \geqslant 0$, there is a natural isomorphism*

$$\operatorname{Ext}^n_{\mathcal{R}}(V, W) \cong \operatorname{Ext}^n_{\mathcal{R}^{\mathrm{op}}}(W, V).$$

Proof. Using (2.10), we must show that $\mathrm{Ext}^n_A(V,W) \cong \mathrm{Ext}^n_A(W^{\circledast},V^{\circledast})$ for locally finite-dimensional A-modules V and W. To compute $\mathrm{Ext}^n_A(V,W)$, take a projective resolution

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow V \longrightarrow 0$$

of V in A-mod. By Lemma 2.6, on applying the exact functor $\circledast$, we get an injective resolution

$$0 \longrightarrow V^{\circledast} \longrightarrow P_0^{\circledast} \longrightarrow P_1^{\circledast} \longrightarrow \cdots$$

of $V^{\circledast}$ in mod-A. Since W is locally finite-dimensional, we can use (2.11) to see that $\mathrm{Hom}_A(P_i,W) \cong \mathrm{Hom}_A(W^{\circledast},P_i^{\circledast})$ for each i. So $\mathrm{Ext}^n_A(V,W) \cong \mathrm{Ext}^n_A(W^{\circledast},V^{\circledast})$. □

Let $I(b)$ be an injective hull of $L(b)$ in A-mod. The dual $(e_iA)^{\circledast}$ of the projective right A-module e_iA is injective in A-mod. Since $\mathrm{End}_A((e_iA)^{\circledast})^{\mathrm{op}} \cong \mathrm{End}_A(e_iA) \cong e_iAe_i$, which is finite-dimensional, the injective module $(e_iA)^{\circledast}$ can be written as a finite direct sum of indecomposable injectives. To determine which ones, we compute its socle: we have that $\mathrm{Hom}_A(L(b),(e_iA)^{\circledast}) \cong \mathrm{Hom}_A(e_iA,L(b)^{\circledast}) \cong (L(b)^{\circledast})e_i = (e_iL(b))^*$, hence,

$$(e_iA)^{\circledast} \cong \bigoplus_{b\in\mathbf{B}} I(b)^{\oplus \dim e_iL(b)}, \tag{2.23}$$

with all but finitely many summands on the right hand side being zero. In particular, this shows for fixed i that $\dim e_iL(b) = 0$ for all but finitely many $b \in \mathbf{B}$. Conversely, for fixed $b \in \mathbf{B}$, we can always choose $i \in I$ so that $e_iL(b) \neq 0$, and deduce that $I(b)$ is a summand of $(e_iA)^{\circledast}$. This means that each indecomposable injective $I(b)$ is a locally finite-dimensional left A-module.

Let $P(b)$ be the dual of the injective hull of the irreducible right A-module $L(b)^{\circledast}$. By dualizing the right module analog of the decomposition (2.23), we get also that

$$Ae_i \cong \bigoplus_{b\in\mathbf{B}} P(b)^{\oplus \dim e_iL(b)}, \tag{2.24}$$

with all but finitely many summands being zero. In particular, $P(b)$ is projective in A-mod, hence, it is a projective cover of $L(b)$ in A-mod.

The composition multiplicities of any A-module satisfy

$$[V:L(b)] = \dim \mathrm{Hom}_A(V,I(b)) = \dim \mathrm{Hom}_A(P(b),V). \tag{2.25}$$

Moreover, V is locally finite-dimensional if and only if $[V:L(b)] < \infty$ for all $b \in \mathbf{B}$. To see this, note that V is locally finite-dimensional if and only if $\dim \mathrm{Hom}_A(Ae_i,V) < \infty$ for each $i \in I$. Using the decompositon (2.24), this is if and only if $\dim \mathrm{Hom}_A(P(b),V) < \infty$ for each $b \in \mathbf{B}$, as claimed.

There is more to be said about finitely generated modules. Recall from the previous subsection that a module is finitely generated if $V = Av_1 + \cdots + Av_n$ for some $v_1,\dots,v_n \in V$. We say that V is *finitely cogenerated* if its dual is finitely generated. It is obvious from these definitions that $\mathrm{Hom}_A(V,W)$ is finite-dimensional either if V is finitely generated and W is locally finite-dimensional, or if V is locally finite-dimensional and W is finitely cogenerated. The following checks that our naive definitions are consistent with the usual notions of finitely generated and cogenerated objects of Grothendieck categories.

Lemma 2.12. *For $V \in A$-mod, the following properties are equivalent:*

(i) V is finitely generated;
(ii) the radical $\mathrm{rad}\, V$*, i.e., the sum of its maximal proper submodules, is a superfluous submodule and* $\mathrm{hd}\, V := V/\mathrm{rad}\, V$ *is of finite length;*
(iii) V is a quotient of a finite direct sum of the modules $P(b)$ for $b \in \mathbf{B}$;
(iv) V is a quotient of a finite direct sum of the modules Ae_i for $i \in I$.

Moreover, any finitely generated V has a projective cover.

Proof. We have already observed that $P(b)$ is a projective cover of $L(b)$. The lemma follows by standard arguments given this and the decomposition (2.24). □

Corollary 2.13. *For $V \in A\text{-mod}$, the following properties are equivalent:*

(i) V is finitely cogenerated;
(ii) $\operatorname{soc} V$ is an essential submodule of finite length;
(iii) V is isomorphic to a submodule of a finite direct sum of modules $I(b)$ for $b \in \mathbf{B}$.

Let us explain why any locally finite-dimensional locally unital algebra is Morita equivalent to a pointed locally finite-dimensional locally unital algebra, as asserted earlier. For $b \in \mathbf{B}$, pick $i(b) \in I$ such that $e_{i(b)}L(b) \neq 0$. In view of (2.24), we find a primitive idempotent $e_b \in e_{i(b)}Ae_{i(b)}$ such that $Ae_b \cong P(b)$. Then A is Morita equivalent to

$$B = \bigoplus_{a,b\in\mathbf{B}} e_a A e_b,$$

which is pointed. For an explicit equivalence $A\text{-mod} \rightarrow B\text{-mod}$, consider the functor sending an A-module V to $\bigoplus_{b\in\mathbf{B}} e_b V$.

In the case that A is pointed, its distinguished idempotents may be indexed by the same set $\mathbf{B}$ as is used to index the isomorphism classes of irreducible objects, so that $P(b) = Ae_b$ and $I(b) = (e_b A)^{\circledast}$. It is also easy to see that if A and B are pointed locally finite-dimensional locally unital algebras which are Morita equivalent, then they are actually isomorphic as locally unital algebras.

Remark 2.14. There is a common generalization of the notions of locally finite Abelian category and locally Schurian category, namely the category $\text{comod}_{\text{lfd}}\text{-}C$ of locally finite-dimensional right comodules over an I-graded coalgebra $C = \bigoplus_{i,j\in I} {}_iC_j$ (for some set I). When I is a singleton, we recover the notion of locally finite Abelian category, and when each ${}_iC_j$ is finite-dimensional we recover the notion of locally Schurian category.

2.4. **Schurian categories.** A *Schurian category* is a category equivalent to $A\text{-mod}_{\text{fd}}$ for an essentially finite-dimensional locally unital algebra $A = \bigoplus_{i,j\in I} e_i A e_j$. This terminology was introduced in [BLW, §2.1]. The following establishes the equivalence of the definition as just formulated and the definition adopted in *loc. cit.*.

Lemma 2.15. *An essentially small category $\mathcal{R}$ is equivalent to $A\text{-mod}_{\text{fd}}$ for a locally unital algebra $A = \bigoplus_{i,j\in I} e_i A e_j$ such that each left ideal Ae_j (resp., each right ideal e_iA) is finite-dimensional if and only if $\mathcal{R}$ is a locally finite Abelian category with enough projectives (resp., enough injectives).*

Proof. We just prove the result for left ideals and projectives; the parenthesized statement for right ideals and injectives follows by replacing $\mathcal{R}$ and A with $\mathcal{R}^{\text{op}}$ and A^{op}.

Suppose first that $A = \bigoplus_{i,j\in I} e_i A e_j$ is a locally unital algebra such that each left ideal Ae_j is finite-dimensional. Then $A\text{-mod}_{\text{fd}}$ is a locally finite Abelian category. It has enough projectives because the left ideals Ae_j are finite-dimensional.

Conversely, suppose $\mathcal{R}$ is a locally finite Abelian category with enough projectives. Let $\{L(b) \mid b \in \mathbf{B}\}$ be a full set of pairwise inequivalent irreducible objects, and $P(b) \in \mathcal{R}$ a projective cover of $L(b)$. Define A to be the locally unital algebra $A = \bigoplus_{a,b\in\mathbf{B}} e_a A e_b$ where $e_a A e_b := \operatorname{Hom}_{\mathcal{R}}(P(a), P(b))$ with multiplication that is the opposite of composition in $\mathcal{R}$. This is a pointed locally finite-dimensional locally unital algebra. As in the proof of Lemma 2.3, the functor $\bigoplus_{b\in\mathbf{B}} \operatorname{Hom}_{\mathcal{R}}(P(b), -)$ defines an equivalence $\mathcal{R} \rightarrow A\text{-mod}_{\text{fd}}$. It remains to note that the ideals Ae_b are finite-dimensional since they are the images under this functor of the projectives $P(b)$, which are of finite length. □

Corollary 2.16. *An essentially small category $\mathcal{R}$ is Schurian if and only if it is a locally finite Abelian category with enough injectives and projectives.*

Note that $\mathcal{R}$ is Schurian if and only if $\mathcal{R}^{\text{op}}$ is Schurian. Moreover, if A is an algebra realization for $\mathcal{R}$ then A^{op} is one for $\mathcal{R}^{\text{op}}$ by the obvious duality $* : A\text{-mod}_{\text{fd}} \to \text{mod}_{\text{fd}}\text{-}A$.

Schurian categories are almost as convenient to work with as finite Abelian categories since one can perform all of the usual constructions of homological algebra without needing to pass to the ind-completion.

Lemma 2.17. *For a category $\mathcal{R}$, the following are equivalent:*

(i) *$\mathcal{R}$ is a finite Abelian category;*
(ii) *$\mathcal{R}$ is a locally Schurian category with only finitely many isomorphism classes of irreducible objects;*
(iii) *$\mathcal{R}$ is a Schurian category with only finitely many isomorphism classes of irreducible objects;*
(iv) *$\mathcal{R}$ is a locally finite Abelian category with only finitely many isomorphism classes of irreducible objects and either enough projectives or enough injectives;*
(v) *$\mathcal{R}$ is both a locally finite Abelian category and a locally Schurian category.*

Proof. Clearly, (i) implies (ii) and (iii). The implication (ii)⇒(i) follows on considering a pointed algebra realization of $\mathcal{R}$. The implication (iii)⇒(iv) follows from Corollary 2.16. The implication (iv)⇒(i) follows from Lemma 2.15. Clearly (ii) and (iv) together imply (v). Finally, to see that (v) implies (ii), it suffices to note that a locally Schurian category with infinitely many isomorphism classes of irreducible objects cannot be locally finite Abelian: the direct sum of infinitely many non-isomorphic irreducibles is a well-defined object of $\mathcal{R}$ but it is not of finite length. □

Schurian categories with infinitely many isomorphism classes of irreducible objects are *not* locally Schurian categories. However they are closely related as we explain next.

- If $\mathcal{R}$ is Schurian, we define its *locally Schurian envelope* $\text{Loc}(\mathcal{R})$ to be the full subcategory of $\text{Ind}(\mathcal{R})$ consisting of all objects that have finite composition multiplicities.
- If $\mathcal{R}$ is locally Schurian, let $\text{Fin}(\mathcal{R})$ be the full subcategory of $\mathcal{R}$ consisting of all objects of finite length.

Lemma 2.18. *The operators* Loc *and* Fin *define mutually inverse equivalences*

$$\left\{\begin{array}{l}\textit{Schurian}\\ \textit{categories}\end{array}\right\} \underset{\text{Fin}}{\overset{\text{Loc}}{\rightleftarrows}} \left\{\begin{array}{l}\textit{Locally Schurian categories whose indecomposable}\\ \textit{injectives and projectives are all of finite length}\end{array}\right\}.$$

Proof. This is easy to see in terms of an algebra realization: if $\mathcal{R} = A\text{-mod}_{\text{fd}}$ for an essentially finite-dimensional locally unital algebra A then $\text{Loc}(\mathcal{R}) = A\text{-mod}_{\text{lfd}}$, so it is locally Schurian. Since the indecomposable injectives and projectives in $\text{Loc}(\mathcal{R})$ are the same as in $\mathcal{R}$, they have finite length. Conversely, suppose that $\mathcal{R} = A\text{-mod}_{\text{lfd}}$ for a *pointed* locally finite-dimensional locally unital algebra, such that all of the indecomposable injectives and projectives are of finite length. Since A is pointed, this means equivalently that all of the left ideals Ae_i and right ideals e_iA are finite-dimensional. Hence, A is essentially finite-dimensional, and $\text{Fin}(\mathcal{R}) = A\text{-mod}_{\text{fd}}$ is Schurian. □

2.5. **Recollement.** We conclude the section with some reminders about "recollement" in our algebraic setting; see [BBD, §1.4] or [CPS1, §2] for further background.

Let $\mathcal{R}$ be an Abelian category of one of the four types discussed in the previous subsections. Assume that we are given a full set $\{L(b) \mid b \in \mathbf{B}\}$ of pairwise inequivalent irreducible objects. Let $\mathbf{B}^{\downarrow}$ be a subset of $\mathbf{B}$ and $\mathcal{R}^{\downarrow}$ be the full subcategory of $\mathcal{R}$ consisting of all the objects V such that $[V : L(b)] \neq 0 \Rightarrow b \in \mathbf{B}^{\downarrow}$. Let $i : \mathcal{R}^{\downarrow} \to \mathcal{R}$ be the inclusion functor. The category $\mathcal{R}^{\downarrow}$ is a Serre subcategory of $\mathcal{R}$, with irreducible objects $\{L^{\downarrow}(b) \mid b \in \mathbf{B}^{\downarrow}\}$ defined simply by $L^{\downarrow}(b) := L(b)$. Moreover, the inclusion functor i has a left adjoint i^* and a right adjoint $i^!$ such that the counit of adjunction $i^* \circ i \to \mathrm{Id}_{\mathcal{R}^{\downarrow}}$ and the unit of adjunction $\mathrm{Id}_{\mathcal{R}^{\downarrow}} \to i^! \circ i$ are isomorphisms:

$$\begin{tikzcd} \mathcal{R}^{\downarrow} \arrow[r, "i"] & \mathcal{R}. \arrow[l, bend right, "i^!"'] \arrow[l, bend left, "i^*"] \end{tikzcd}$$

Explicitly, i^* (resp., $i^!$) sends an object of $\mathcal{R}$ to the largest quotient (resp., subobject) that belongs to $\mathcal{R}^{\downarrow}$. We use the same notation i, i^* and $i^!$ for the natural extensions of these functors to the ind-completions of $\mathcal{R}$ and $\mathcal{R}^{\downarrow}$.

We can also pass to the *Serre quotient* $\mathcal{R}^{\uparrow} := \mathcal{R}/\mathcal{R}^{\downarrow}$. This is an Abelian category equipped with an exact *quotient functor* $j : \mathcal{R} \to \mathcal{R}^{\uparrow}$ satisfying the following universal property: if $h : \mathcal{R} \to \mathcal{C}$ is any exact functor to an Abelian category $\mathcal{C}$ with $hL(b) = 0$ for all $b \in \mathbf{B}^{\downarrow}$, then there is a unique functor $\bar{h} : \mathcal{R}^{\uparrow} \to \mathcal{C}$ such that $h = \bar{h} \circ j$. The irreducible objects in $\mathcal{R}^{\uparrow}$ are $\{L^{\uparrow}(b) \mid b \in \mathbf{B}^{\uparrow}\}$ where $\mathbf{B}^{\uparrow} := \mathbf{B} \backslash \mathbf{B}^{\downarrow}$ and $L^{\uparrow}(b) := jL(b)$. For a fuller discussion of these statements, see e.g. [Gab].

Now $\mathcal{R}^{\downarrow}$ and $\mathcal{R}^{\uparrow}$ are of the same type (finite Abelian, locally finite Abelian, Schurian or locally Schurian) as the original category $\mathcal{R}$. This is clear in the locally finite Abelian case from Lemma 2.1. For the other three cases, fix a pointed algebra realization

$$A = \bigoplus_{a,b \in \mathbf{B}} e_a A e_b \tag{2.26}$$

of $\mathcal{R}$, so A is finite-dimensional, essentially finite-dimensional or locally finite-dimensional according to whether $\mathcal{R}$ is finite Abelian, Schurian or locally Schurian. Let

$$A^{\downarrow} = \bigoplus_{a,b \in \mathbf{B}^{\downarrow}} e_a A^{\downarrow} e_b := A/(e_c \mid c \in \mathbf{B}^{\uparrow}), \qquad A^{\uparrow} := \bigoplus_{a,b \in \mathbf{B}^{\uparrow}} e_a A e_b. \tag{2.27}$$

In the finite Abelian or Schurian cases, $\mathcal{R}^{\downarrow}$ is equivalent to $A^{\downarrow}\text{-mod}_{\mathrm{fd}}$ and $\mathcal{R}^{\uparrow}$ to $A^{\uparrow}\text{-mod}_{\mathrm{fd}}$. In the locally Schurian case, $\mathcal{R}^{\downarrow}$ is equivalent to $A^{\downarrow}\text{-mod}_{\mathrm{lfd}}$ and $\mathcal{R}^{\uparrow}$ to $A^{\uparrow}\text{-mod}_{\mathrm{lfd}}$. Since $A^{\downarrow}$ and $A^{\uparrow}$ satisfy the finiteness property of A, the claim follows.

For $\mathcal{R}$ finite Abelian, Schurian or locally Schurian, we claim moreover that the quotient functor $j : \mathcal{R} \to \mathcal{R}^{\uparrow}$ has a left adjoint $j_!$ and and a right adjoint j_* such that the unit $\mathrm{Id}_{\mathcal{R}^{\uparrow}} \to j \circ j_!$ and the counit $j \circ j_* \to \mathrm{Id}_{\mathcal{R}^{\uparrow}}$ of adjunction are isomorphisms:

$$\begin{tikzcd} \mathcal{R} \arrow[r, "j"] & \mathcal{R}^{\uparrow}. \arrow[l, bend right, "j_*"'] \arrow[l, bend left, "j_!"] \end{tikzcd}$$

We will use the same notation $j, j_!$ and j_* for the natural extensions of these functors to the ind-completions. To prove the claim, we again work in terms of the algebra realizations (2.26)–(2.27). Then j is the idempotent truncation functor

$$j : A\text{-mod} \to A^{\uparrow}\text{-mod}, \qquad V \mapsto \bigoplus_{a \in \mathbf{B}^{\uparrow}} e_a V. \tag{2.28}$$

This is isomorphic to the hom functor $\bigoplus_{b \in \mathbf{B}^{\uparrow}} \mathrm{Hom}_A(Ae_b, -)$, which has the left adjoint

$$j_! := \Big(\bigoplus_{b \in \mathbf{B}^{\uparrow}} Ae_b\Big) \otimes_{A^{\uparrow}} - : A^{\uparrow}\text{-mod} \to A\text{-mod} \tag{2.29}$$

by Lemma 2.5(1). From this and (2.27), it is clear that the unit of adjunction $\mathrm{Id}_{\mathcal{R}^\uparrow} \to j \circ j_!$ is an isomorphism. On the other hand, j is also isomorphic to the tensor functor $(\bigoplus_{b\in\mathbf{B}^\uparrow} e_b A)\otimes_A -$, so Lemma 2.5(1) also gives that j has the right adjoint

$$j_* := \bigoplus_{a\in\mathbf{B}} \mathrm{Hom}_{A^\uparrow}\Big(\bigoplus_{b\in\mathbf{B}^\uparrow} e_b A e_a, -\Big) : A^\uparrow\text{-mod} \to A\text{-mod}\,. \tag{2.30}$$

Again using this we see that the counit $j \circ j_* \to \mathrm{Id}_{\mathcal{R}^\uparrow}$ is an isomorphism.

Lemma 2.19. *Assuming that $\mathcal{R}$ is finite Abelian, Schurian or locally Schurian, let $j, j_!$ and j_* be as above. Let $P(b)$ (resp. $I(b)$) and $P^\uparrow(b)$ (resp. $I^\uparrow(b)$) be a projective cover (resp. an injective hull) of $L(b)$ in $\mathcal{R}$ and a projective cover (resp. an injective hull) of $L^\uparrow(b)$ in $\mathcal{R}^\uparrow$. For $b \in \mathbf{B}^\uparrow$, we have that*

$$jP(b) \cong P^\uparrow(b), \qquad jI(b) \cong I^\uparrow(b), \qquad j_!P^\uparrow(b) \cong P(b), \qquad j_*I^\uparrow(b) \cong I(b).$$

Moreover, $j_!$ sends finitely generated objects to finitely generated objects, and j_* sends finitely cogenerated objects to finitely cogenerated objects.

Proof. The first statement can be checked directly using the explicit realizations (2.28)–(2.30) of these functors. The second part follows since $j_!$ is a left adjoint, hence, right exact, and j_* is a right adjoint, hence, left exact. □

Lemma 2.20. *Suppose that $\mathcal{R} = \mathrm{comod}_{\mathrm{fd}}\text{-}C$ for a coalgebra C and $\{L(b) \mid b \in \mathbf{B}\}$ is a full set of pairwise inequivalent irreducible right C-comodules. For $\mathbf{B}^\downarrow \subseteq \mathbf{B}$, let $\mathcal{R}^\downarrow$ be the Serre subcategory of $\mathcal{R}$ generated by $\{L(b) | b \in \mathbf{B}^\downarrow\}$. Let $C^\downarrow$ be the largest right coideal of C such that all of its irreducible subquotients belong to $\mathcal{R}^\downarrow$. Then $C^\downarrow$ is a subcoalgebra of C. Moreover, $\mathcal{R}^\downarrow$ consists of all $V \in \mathrm{comod}_{\mathrm{fd}}\text{-}C$ such that the image of the structure map $\eta : V \to V \otimes C$ is contained in $V \otimes C^\downarrow$, i.e., we have that $\mathcal{R}^\downarrow = \mathrm{comod}_{\mathrm{fd}}\text{-}C^\downarrow$.*

Proof. For a right comodule V with structure map $\eta : V \to V \otimes C$, we can consider $V \otimes C$ as a right comodule with structure map $\mathrm{id} \otimes \Delta$. The coassociative and counit axioms imply that η is an injective homomorphism of right comodules. We deduce that all irreducible subquotients of V belong to $\mathcal{R}^\downarrow$ if and only if $\eta(V) \subseteq V \otimes C^\downarrow$. Applying this with $V = C^\downarrow$ shows that $C^\downarrow$ is a subcoalgebra. Applying it to $V \in \mathcal{R}$ shows that $V \in \mathcal{R}^\downarrow$ if and only if $\eta(V) \subseteq V \otimes C^\downarrow$. □

Lemma 2.21. *Let $\mathcal{R}$ be a locally Schurian category and $\{L(b) \mid b \in \mathbf{B}\}$ be a full set of pairwise inequivalent irreducible objects. Suppose that $\mathbf{B}^\downarrow$ is a subset of $\mathbf{B}$ such that $\mathbf{B}^\uparrow := \mathbf{B}\backslash\mathbf{B}^\downarrow$ is finite. Let $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the inclusion of the corresponding Serre subcategory of $\mathcal{R}$. For any $V \in \mathcal{R}$ the object $V/i^!V$ is finitely cogenerated.*

Proof. Let $I(b)$ be an injective hull of $L(b)$ and $W := \bigoplus_{b\in\mathbf{B}^\uparrow} I(b)^{\oplus[V:L(b)]}$. Proceeding by induction on $\sum_{b\in\mathbf{B}^\uparrow}[V : L(b)] < \infty$, one constructs a morphism $f : V \to W$ such that $\ker f$ has no composition factors $L(b)$ for $b \in \mathbf{B}^\uparrow$. This means that $\ker f \subseteq i^!V$. The image of $i^!V$ under f is a subobject of the finitely cogenerated object W. As no constituent of the socle of W is a composition factor of $i^!V$, it must actually be zero. Thus, $i^!V \subseteq \ker f$ too. Hence, f induces a monomorphism $V/i^!V \hookrightarrow W$, and we are done since W is finitely cogenerated. □

3. Semi-infinite stratified and highest weight categories

In this section, we define the various sorts of "stratified category" in the four settings of finite Abelian, locally finite Abelian, Schurian and locally Schurian categories.

3.1. **Standard and costandard objects.** Let Λ be a poset. It is *interval finite* (resp., *upper finite*, resp., *lower finite*) if the interval $[\lambda,\mu] := \{\nu \in \Lambda \mid \lambda \leqslant \nu \leqslant \mu\}$ (resp., $[\lambda,\infty) := \{\nu \in \Lambda \mid \lambda \leqslant \nu\}$, resp., $(-\infty,\mu] := \{\nu \in \Lambda \mid \nu \leqslant \mu\}$) is finite for all $\lambda,\mu \in \Lambda$. A *lower set* (resp., *upper set*) means a subset $\Lambda^\downarrow$ (resp., $\Lambda^\uparrow$) such that $\mu \leqslant \lambda \in \Lambda^\downarrow \Rightarrow \mu \in \Lambda^\downarrow$ (resp., $\mu \geqslant \lambda \in \Lambda^\uparrow \Rightarrow \mu \in \Lambda^\uparrow$).

Definition 3.1. Let $\mathcal{R}$ be an Abelian category of one of the four types discussed in the previous section. A *stratification* $\rho : \mathbf{B} \to \Lambda$ of $\mathcal{R}$ is the data of

(S1) an interval finite poset $(\Lambda, \leqslant)$;
(S2) a set $\mathbf{B}$ indexing representatives $\{L(b) \mid b \in \mathbf{B}\}$ for the isomorphism classes of irreducible objects in $\mathcal{R}$;
(S3) a function $\rho : \mathbf{B} \to \Lambda$ with *finite* fibers $\mathbf{B}_\lambda := \rho^{-1}(\lambda)$.

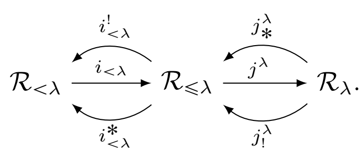
For each $\lambda \in \Lambda$, let $\mathcal{R}_{\leqslant\lambda}$ and $\mathcal{R}_{<\lambda}$ be the Serre subcategories of $\mathcal{R}$ associated to the subsets $\mathbf{B}_{\leqslant\lambda} := \{b \in \mathbf{B} \mid \rho(b) \leqslant \lambda\}$ and $\mathbf{B}_{<\lambda} := \{b \in \mathbf{B} \mid \rho(b) < \lambda\}$, respectively. Let $\mathcal{R}_\lambda$ be the quotient category $\mathcal{R}_{\leqslant\lambda}/\mathcal{R}_{<\lambda}$. We impose the following axiom:

(S4) each of the Abelian subcategories $\mathcal{R}_{\leqslant\lambda}$ has enough projectives and injectives.

In case ρ is a bijection, one can use it to identify $\mathbf{B}$ with Λ, and may simply write $L(\lambda)$ instead of $L(b)$ (similarly for all of the other families of objects introduced below).

Remark 3.2. If $\mathcal{R}$ is finite Abelian, Schurian or locally Schurian then the axiom (S4) holds automatically. However, it rules out many situations in which $\mathcal{R}$ is merely locally finite Abelian. For example, the category $\operatorname{Rep}(\mathbb{G}_a)$ of finite-dimensional rational representations of the additive group does not admit a stratification in the above sense.

Given a stratification of $\mathcal{R}$, we write $i_{\leqslant\lambda} : \mathcal{R}_{\leqslant\lambda} \to \mathcal{R}$ and $i_{<\lambda} : \mathcal{R}_{<\lambda} \to \mathcal{R}$ for the inclusion functors, and $j^\lambda : \mathcal{R}_{\leqslant\lambda} \to \mathcal{R}_\lambda$ for the quotient functor. Using Corollary 2.16 in the locally finite Abelian case, we see that $\mathcal{R}_{\leqslant\lambda}$ is always either Schurian or locally Schurian. Hence, $\mathcal{R}_\lambda$ is a finite Abelian category, and we are in a recollement situation:



As we said already in the introduction, we call $j^\lambda_!$ and j^λ_* the *standardization* and *costandardization* functors, respectively. Let $L_\lambda(b) := j^\lambda L(b)$, so that the objects $\{L_\lambda(b) | b \in \mathbf{B}\}$ give a full set of pairwise inequivalent irreducible objects in the finite Abelian category $\mathcal{R}_\lambda$. Let $P_\lambda(b)$ and $I_\lambda(b)$ be a projective cover and an injective hull of $L_\lambda(b)$ in $\mathcal{R}_\lambda$, respectively. Finally, define *standard, costandard, proper standard* and *proper costandard objects* $\Delta(b), \nabla(b), \bar{\Delta}(b)$ and $\bar{\nabla}(b)$ according to (1.1).

Lemma 3.3. *For $b \in \mathbf{B}_\lambda$, $\Delta(b)$ is a projective cover and $\nabla(b)$ is an injective hull of $L(b)$ in $\mathcal{R}_{\leqslant\lambda}$. Also, $\bar{\Delta}(b)$ is the largest quotient of $\Delta(b)$ such that $[\bar{\Delta}(b) : L(b)] = 1$ and all other composition factors are of the form $L(c)$ for $c \in \mathbf{B}_{<\lambda}$. Similarly, $\bar{\nabla}(b)$ is the largest subobject of $\nabla(b)$ such that $[\bar{\nabla}(b) : L(b)] = 1$ and all other composition factors are of the form $L(c)$ for $c \in \mathbf{B}_{<\lambda}$.*

Proof. The first assertion follows by Lemma 2.19. To prove the statement about $\bar{\Delta}(b)$, assume $[\bar{\Delta}(b) : L(c)] \neq 0$. Since $\bar{\Delta}(b) \in \mathcal{R}_{\leqslant\lambda}$, we have $\rho(c) \leqslant \rho(b)$. If $\rho(c) = \rho(b)$ then

$$[\bar{\Delta}(b) : L(c)] = [j^\lambda \bar{\Delta}(b) : L_\lambda(c)] = [L_\lambda(b) : L_\lambda(c)] = \delta_{b,c}.$$

Thus, $\bar{\Delta}(b)$ is such a quotient of $\Delta(b)$. To show that it is the largest such quotient, it suffices to show that the kernel K of $\Delta(b) \twoheadrightarrow \bar{\Delta}(b)$ is finitely generated with head that only involves irreducibles $L(c)$ with $\rho(c) = \rho(b)$. To see this, apply the right exact functor $j_!^\lambda$ to a short exact sequence $0 \to \widehat{K} \to P_\lambda(b) \to L_\lambda(b) \to 0$ to get an epimorphism $j_!^\lambda \widehat{K} \twoheadrightarrow K$. Then observe that $j_!^\lambda \widehat{K}$ is finitely generated by Lemma 2.19, and its head only involves irreducibles $L(c)$ with $\rho(c) = \rho(b)$. The latter assertion follows because $\operatorname{Hom}_{\mathcal{R}}(j_!^\lambda \widehat{K}, L(c)) \cong \operatorname{Hom}_{R_\lambda}(\widehat{K}, j^\lambda L(c))$ for $c \in \mathbf{B}_{\leqslant\lambda}$. The statement about $\bar{\nabla}(b)$ may be proved similarly. □

Definition 3.4. Suppose we are given a stratification $\rho : \mathbf{B} \to \Lambda$ of $\mathcal{R}$. For $\lambda \in \Lambda$, we say that the stratum $\mathcal{R}_\lambda$ is *simple* if it is equivalent to the category $\mathrm{Vec}_{\mathrm{fd}}$ of finite-dimensional vector spaces.

Lemma 3.5. *For a stratification $\rho : \mathbf{B} \to \Lambda$ of $\mathcal{R}$, the following are equivalent:*

(i) all of the strata are simple;
(ii) ρ is a bijection and $\Delta(\lambda) = \bar{\Delta}(\lambda)$ for all $\lambda \in \Lambda$;
(iii) ρ is a bijection and $\nabla(\lambda) = \bar{\nabla}(\lambda)$ for all $\lambda \in \Lambda$.

Proof. (i)⇒(ii): Take $\lambda \in \Lambda$. As the stratum $\mathcal{R}_\lambda$ is simple, $\mathbf{B}_\lambda = \{b_\lambda\}$ is a singleton and $P_\lambda(b_\lambda) = L_\lambda(b_\lambda)$. We deduce that ρ is a bijection and $\Delta(b_\lambda) = \bar{\Delta}(b_\lambda)$.

(ii)⇒(i): Take $\lambda \in \Lambda$. Then $\mathcal{R}_\lambda$ has just one irreducible object (up to isomorphism), namely, $j^\lambda \bar{\Delta}(\lambda)$. Since this equals $j^\lambda \Delta(\lambda)$, it is also projective. Hence, $\mathcal{R}_\lambda$ is simple.

(i)⇔(iii): Similar. □

Given a *sign function* $\varepsilon : \Lambda \to \{\pm\}$, we introduce the *$\varepsilon$-standard* and *$\varepsilon$-costandard objects* $\Delta_\varepsilon(b)$ and $\nabla_\varepsilon(b)$ as in (1.2). A *Δ_ε-flag* of $V \in \mathcal{R}$ means a finite filtration

$$0 = V_0 < V_1 < \cdots < V_n = V$$

with sections $V_m/V_{m-1} \cong \Delta_\varepsilon(b_m)$ for $b_m \in \mathbf{B}$. Similarly, we define ∇_ε-flags. We denote the exact subcategories of $\mathcal{R}$ consisting of all objects with a Δ_ε-flag or a ∇_ε-flag by $\Delta_\varepsilon(\mathcal{R})$ and $\nabla_\varepsilon(\mathcal{R})$, respectively.

3.2. **Finite and essentially finite stratified categories.** Throughout this subsection, $\mathcal{R}$ is a Schurian category equipped with a stratification $\rho : \mathbf{B} \to \Lambda$, and $\varepsilon : \Lambda \to \{\pm\}$ is a sign function. The most important case is when $\mathcal{R}$ is a finite Abelian category, i.e., the index set $\mathbf{B}$ is finite. Let $P(b)$ and $I(b)$ be a projective cover and an injective hull of $L(b)$, respectively. We need the objects from (1.1)–(1.2). For $b \in \mathbf{B}_\lambda$, $\Delta(b)$ (resp., $\nabla(b)$) is the largest quotient of $P(b)$ (resp., subobject of $I(b)$) that belongs to $\mathcal{R}_{\leqslant\lambda}$.

As we explained already in the introduction, the essence of the following fundamental theorem appeared originally in [ADL], extending the earlier work of Dlab [Dla].

Theorem 3.6. *In the above setup, the properties ($P\Delta_\varepsilon$) and ($I\nabla_\varepsilon$) from the introduction are equivalent. Given these two properties, the standardization functor $j_!^\lambda$ is exact whenever $\varepsilon(\lambda) = -$, and the costandardization functor j_*^λ is exact whenever $\varepsilon(\lambda) = +$.*

Proof. This follows from some lemmas to be proved in §3.4 below. Suppose that $\mathcal{R}$ satisfies ($P\Delta_\varepsilon$) (as assumed for all of those lemmas). Since $V = I(b)$ is a finitely cogenerated injective object, it satisfies the hypotheses of Lemma 3.48(ii). Hence, by that lemma, $I(b)$ has a ∇_ε-flag and the multiplicity $(I(b) : \nabla_\varepsilon(c))$ of $\nabla_\varepsilon(c)$ as a section of any such flag is given by

$$(I(b) : \nabla_\varepsilon(c)) = \dim \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(c), I(b)) = [\Delta_\varepsilon(c) : L(b)].$$

This is zero unless $\rho(b) \leqslant \rho(c)$. Thus, we have verified that $\mathcal{R}$ satisfies ($I\nabla_\varepsilon$). By the final assertion of Lemma 3.47 applied in the category $\mathcal{R}_{\leqslant\lambda}$, we get that j_*^λ is exact whenever $\varepsilon(\lambda) = +$. Exactly the same results but with $\mathcal{R}$ replaced by $\mathcal{R}^{\mathrm{op}}$ and ε replaced with $-\varepsilon$ show that ($I\nabla_\varepsilon$) implies ($P\Delta_\varepsilon$) and that $j_!^\lambda$ is exact whenever $\varepsilon(\lambda) = -$. □

This justifies the following, which is closely related to [ADL, Definition 1.3].

Definition 3.7. We say that $\mathcal{R}$ is a *finite* (resp., an *essentially finite*) *ε-stratified category* if it is a finite Abelian category (resp., a Schurian category) with a stratification as in Definition 3.1, such that one of the equivalent properties $(P\Delta_\varepsilon)$ or $(I\nabla_\varepsilon)$ holds.

Remark 3.8. The assumption in Definition 3.1 that the poset is interval finite, which one finds already in the original work of Cline, Parshall and Scott [CPS1], actually plays no role in any of the proofs in this subsection: the much stronger finiteness assumption that $\mathcal{R}$ is Schurian is all we use here. Nevertheless, we have kept it in place because it is often helpful to be able to study such categories by truncating to the finite case.

We can view $\{L(b) \mid b \in \mathbf{B}\}$ equivalently as a full set of pairwise inequivalent irreducible objects in $\mathcal{R}^{\mathrm{op}}$. The stratification of $\mathcal{R}$ is also one of $\mathcal{R}^{\mathrm{op}}$. The indecomposable projectives and injectives in $\mathcal{R}^{\mathrm{op}}$ are $I(b)$ and $P(b)$, while the $(-\varepsilon)$-standard and $(-\varepsilon)$-costandard objects in $\mathcal{R}^{\mathrm{op}}$ are $\nabla_\varepsilon(b)$ and $\Delta_\varepsilon(b)$, respectively. So we get the following from the equivalence of $(P\Delta_\varepsilon)$ and $(I\nabla_\varepsilon)$.

Lemma 3.9. *$\mathcal{R}$ is ε-stratified if and only if $\mathcal{R}^{\mathrm{op}}$ is $(-\varepsilon)$-stratified.*

The other basic observation which we use repeatedly is the following.

Lemma 3.10. *Assume $\mathcal{R}$ is ε-stratified. For $b, c \in \mathbf{B}$ with $\rho(b) \nleqslant \rho(c)$ we have that*

$$\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), \Delta_\varepsilon(c)) = \mathrm{Ext}^1_{\mathcal{R}}(\nabla_\varepsilon(c), \nabla_\varepsilon(b)) = 0.$$

Proof. This is Lemma 3.43 and its dual. □

We also have the following well-known *homological criterion* for good filtrations.

Theorem 3.11. *Assume that $\mathcal{R}$ is a finite or essentially finite ε-stratified category. Then the following properties are equivalent:*

(i) $V \in \nabla_\varepsilon(\mathcal{R})$;
(ii) $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$ *for all* $b \in \mathbf{B}$;
(iii) $\mathrm{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$ *for all* $b \in \mathbf{B}$ *and* $n \geqslant 1$.

The multiplicity $(V : \nabla_\varepsilon(b))$ of $\nabla_\varepsilon(b)$ as a section of a ∇_ε-flag of V is well-defined and independent of the choice of flag, and equals $\dim \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V)$.

Proof. This is established in §3.4 below: it is exactly Lemma 3.48. □

Corollary 3.12. $(I(b) : \nabla_\varepsilon(c)) = [\Delta_\varepsilon(c) : L(b)]$.

Corollary 3.13. *Let $0 \to U \to V \to W \to 0$ be a short exact sequence in a finite or essentially finite ε-stratified category. If U and V have ∇_ε-flags then so does W.*

Theorem 3.14. *Assume that $\mathcal{R}$ is a finite or essentially finite ε-stratified category. Then the following properties are equivalent:*

(i) $V \in \Delta_\varepsilon(\mathcal{R})$;
(ii) $\mathrm{Ext}^1_{\mathcal{R}}(V, \nabla_\varepsilon(b)) = 0$ *for all* $b \in \mathbf{B}$;
(iii) $\mathrm{Ext}^n_{\mathcal{R}}(V, \nabla_\varepsilon(b)) = 0$ *for all* $b \in \mathbf{B}$ *and* $n \geqslant 1$.

The multiplicity $(V : \Delta_\varepsilon(b))$ of $\Delta_\varepsilon(b)$ as a section of a Δ_ε-flag of V is well-defined and independent of the choice of flag, and equals $\dim \mathrm{Hom}_{\mathcal{R}}(V, \nabla_\varepsilon(b))$.

Proof. This is the equivalent dual statement to Theorem 3.11. In other words, one takes Theorem 3.11 with $\mathcal{R}$ replaced by $\mathcal{R}^{\mathrm{op}}$ and ε by $-\varepsilon$, then applies the contravariant isomorphism between $\mathcal{R}$ and $\mathcal{R}^{\mathrm{op}}$ that is the identity on objects and morphisms. □

Corollary 3.15. $(P(b) : \Delta_\varepsilon(c)) = [\nabla_\varepsilon(c) : L(b)]$.

Corollary 3.16. *Let $0 \to U \to V \to W \to 0$ be a short exact sequence in a finite or essentially finite ε-stratified category. If V and W have Δ_ε-flags then so does U.*

The following results about truncation to lower and upper sets are extremely useful.

Theorem 3.17. *Assume that $\mathcal{R}$ is a finite or essentially finite ε-stratified category. Suppose that $\Lambda^\downarrow$ is a lower set in Λ. Let $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$ and $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the corresponding Serre subcategory of $\mathcal{R}$ with the induced stratification. Then $\mathcal{R}^\downarrow$ is itself a finite or essentially finite ε-stratified category according to whether $\Lambda^\downarrow$ is finite or infinite. Moreover:*

1. *The distinguished objects in $\mathcal{R}^\downarrow$ satisfy $L^\downarrow(b) \cong L(b)$, $P^\downarrow(b) \cong i^*P(b)$, $I^\downarrow(b) \cong i^! I(b)$, $\Delta^\downarrow(b) \cong \Delta(b)$,$\bar\Delta^\downarrow(b) \cong \bar\Delta(b)$, $\nabla^\downarrow(b) \cong \nabla(b)$ and $\bar\nabla^\downarrow(b) \cong \bar\nabla(b)$ for $b \in \mathbf{B}^\downarrow$.*
2. *i^* is exact[1] on $\Delta_\varepsilon(\mathcal{R})$ with $i^*\Delta(b) \cong \Delta^\downarrow(b)$ and $i^*\bar\Delta(b) \cong \bar\Delta^\downarrow(b)$ for $b \in \mathbf{B}^\downarrow$; also $i^*\Delta(b) = i^*\bar\Delta(b) = 0$ for $b \notin \mathbf{B}^\downarrow$.*
3. *$\mathrm{Ext}^n_{\mathcal{R}}(V, iW) \cong \mathrm{Ext}^n_{\mathcal{R}^\downarrow}(i^*V, W)$ for $V \in \Delta_\varepsilon(\mathcal{R})$, $W \in \mathcal{R}^\downarrow$ and all $n \geqslant 0$.*
4. *$i^!$ is exact on $\nabla_\varepsilon(\mathcal{R})$ with $i^!\nabla(b) \cong \nabla^\downarrow(b)$ and $i^!\bar\nabla(b) \cong \bar\nabla^\downarrow(b)$ for $b \in \mathbf{B}^\downarrow$; also $i^!\nabla(b) = i^!\bar\nabla(b) = 0$ for $b \notin \mathbf{B}^\downarrow$.*
5. *$\mathrm{Ext}^n_{\mathcal{R}}(iV, W) \cong \mathrm{Ext}^n_{\mathcal{R}^\downarrow}(V, i^!W)$ for $V \in \mathcal{R}^\downarrow, W \in \nabla_\varepsilon(\mathcal{R})$ and all $n \geqslant 0$.*
6. *$\mathrm{Ext}^n_{\mathcal{R}}(iV, iW) \cong \mathrm{Ext}^n_{\mathcal{R}^\downarrow}(V, W)$ for $V, W \in \mathcal{R}^\downarrow$ and $n \geqslant 0$.*

Proof. Apart from (6), this follows from Lemma 3.44 proved in §3.4 below (applied both in $\mathcal{R}$ and in $\mathcal{R}^{\mathrm{op}}$). To prove (6), by the same argument as used to prove Lemma 3.44(4), it suffices to show that $(\mathbb{L}_n i^*)V = 0$ for $V \in \mathcal{R}^\downarrow$. Since any such V has finite length this follows if we can prove it for each irreducible object in $\mathcal{R}^\downarrow$, i.e., we must show that $(\mathbb{L}_n i^*)L(b) = 0$ for $b \in \mathbf{B}^\downarrow$. Take a short exact sequence $0 \to K \to \Delta_\varepsilon(b) \to L(b) \to 0$ and apply i^* and Lemma 3.44(3) to get

$$0 \longrightarrow (\mathbb{L}_1 i^*)L(b) \longrightarrow i^*K \longrightarrow i^*\Delta_\varepsilon(b) \longrightarrow i^*L(b) \longrightarrow 0.$$

But $K, \Delta_\varepsilon(b)$ and $L(b)$ all lie in $\mathcal{R}^\downarrow$ so i^* is the identity on them. We deduce that $(\mathbb{L}_1 i^*)L(b) = 0$. Degree shifting easily gives the result for $n > 1$. □

Theorem 3.18. *Assume that $\mathcal{R}$ is a finite or essentially finite ε-stratified category. Suppose that $\Lambda^\uparrow$ is an upper set in Λ. Let $\mathbf{B}^\uparrow := \rho^{-1}(\Lambda^\uparrow)$ and $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the corresponding Serre quotient category of $\mathcal{R}$ with the induced stratification. Then $\mathcal{R}^\uparrow$ is itself a finite or essentially finite ε-stratified category according to whether $\Lambda^\uparrow$ is finite or infinite. Moreover:*

1. *For $b \in \mathbf{B}^\uparrow$, the distinguished objects $L^\uparrow(b)$, $P^\uparrow(b)$, $I^\uparrow(b)$, $\Delta^\uparrow(b)$, $\bar\Delta^\uparrow(b)$, $\nabla^\uparrow(b)$ and $\bar\nabla^\uparrow(b)$ in $\mathcal{R}^\uparrow$ are isomorphic to the images under j of the corresponding objects of $\mathcal{R}$.*
2. *We have that $jL(b) = j\Delta(b) = j\bar\Delta(b) = j\nabla(b) = j\bar\nabla(b) = 0$ if $b \notin \mathbf{B}^\uparrow$.*
3. *$\mathrm{Ext}^n_{\mathcal{R}}(V, j_*W) \cong \mathrm{Ext}^n_{\mathcal{R}^\uparrow}(jV, W)$ for $V \in \mathcal{R}, W \in \nabla_\varepsilon(\mathcal{R}^\uparrow)$ and all $n \geqslant 0$.*
4. *j_* is exact on $\nabla_\varepsilon(\mathcal{R}^\uparrow)$ with $j_*\nabla^\uparrow(b) \cong \nabla(b)$, $j_*\bar\nabla^\uparrow(b) \cong \bar\nabla(b)$ and $j_*I^\uparrow(b) \cong I(b)$ for $b \in \mathbf{B}^\uparrow$.*
5. *$\mathrm{Ext}^n_{\mathcal{R}}(j_!V, W) \cong \mathrm{Ext}^n_{\mathcal{R}^\uparrow}(V, jW)$ for $V \in \Delta_\varepsilon(\mathcal{R}^\uparrow)$, $W \in \mathcal{R}$ and all $n \geqslant 0$.*
6. *$j_!$ is exact on $\Delta_\varepsilon(\mathcal{R}^\uparrow)$ with $j_!\Delta^\uparrow(b) \cong \Delta(b)$, $j_!\bar\Delta^\uparrow(b) \cong \bar\Delta(b)$ and $j_!P^\uparrow(b) = P(b)$ for $b \in \mathbf{B}^\uparrow$.*

Proof. Apart from (4) and (6), this follows from Lemma 3.49 proved in §3.4 below (applied both in $\mathcal{R}$ and in $\mathcal{R}^{\mathrm{op}}$). For (4) and (6), it suffices to prove (4), since (6) is the equivalent dual statement. The descriptions of $j_*\nabla^\uparrow(b)$, $j_*\bar\nabla^\uparrow(b)$ and $j_*I^\uparrow(b)$, follow from Lemma 3.49(2). It remains to prove the exactness. We can actually show

[1] We mean that it sends short exact sequences of objects with Δ_ε-flags to short exact sequences.

slightly more, namely, that $(\mathbb{R}^n j_*)V = 0$ for $V \in \nabla_\varepsilon(\mathcal{R}^\uparrow)$ and $n \geqslant 1$. Take $V \in \nabla_\varepsilon(\mathcal{R}^\uparrow)$. Consider a short exact sequence $0 \to V \to I \to Q \to 0$ in $\mathcal{R}^\uparrow$ with I injective. Apply the left exact functor j_* and consider the resulting long exact sequence:

$$0 \to j_*V \to j_*I \to j_*Q \to (\mathbb{R}^1 j_*)V \to 0.$$

As V has a ∇_ε-flag, we can use (3) to see that $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), j_*V) \cong \mathrm{Hom}_{\mathcal{R}^\uparrow}(j\Delta_\varepsilon(b), V)$ and $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), j_*V) \cong \mathrm{Ext}^1_{\mathcal{R}^\uparrow}(j\Delta_\varepsilon(b), V)$ for every $b \in \mathbf{B}$. Hence, by Lemma 3.48, j_*V has a ∇_ε-flag with

$$(j_*V : \nabla_\varepsilon(b)) = \dim \mathrm{Hom}_{\mathcal{R}}(j\Delta_\varepsilon(b), V) = \begin{cases} (V : \nabla^\uparrow_\varepsilon(b)) & \text{if } b \in \mathbf{B}^\uparrow, \\ 0 & \text{otherwise.} \end{cases}$$

Both I and Q have ∇_ε-flags too, so we get similar statements for j_*I and j_*Q. Since $(I : \nabla^\uparrow_\varepsilon(b)) = (V : \nabla^\uparrow_\varepsilon(b)) + (Q : \nabla^\uparrow_\varepsilon(b))$ by the exactness of the original sequence, we deduce that $0 \to j_*V \to j_*I \to j_*Q \to 0$ is exact. Hence, $(\mathbb{R}^1 j_*)V = 0$. This proves the result for $n = 1$. The result for $n > 1$ follows by a degree shifting argument. □

Corollary 3.19. *Let notation be as in Theorem 3.18 and set* $\mathbf{B}^\downarrow := \mathbf{B} \backslash \mathbf{B}^\uparrow$.

(1) For $V \in \nabla_\varepsilon(\mathcal{R})$, there is a short exact sequence $0 \to K \to V \xrightarrow{\gamma} j_*(jV) \to 0$ where γ comes from the unit of adjunction, $j_*(jV)$ has a ∇_ε-flag with sections $\nabla_\varepsilon(b)$ for $b \in \mathbf{B}^\uparrow$, and K has a ∇_ε-flag with sections $\nabla_\varepsilon(c)$ for $c \in \mathbf{B}^\downarrow$.

(2) For $V \in \Delta_\varepsilon(\mathcal{R})$, there is a short exact sequence $0 \to j_!(jV) \xrightarrow{\delta} V \to Q \to 0$ where δ comes from the counit of adjunction, $j_!(jV)$ has a Δ_ε-flag with sections $\Delta_\varepsilon(b)$ for $b \in \mathbf{B}^\uparrow$ and Q has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}^\downarrow$.

Proof. We prove only (1), since (2) is just the dual statement. Using Lemma 3.10, we can order the ∇_ε-flag of V to get a short exact sequence $0 \to K \to V \to Q \to 0$ such that K has a ∇_ε-flag with sections $\nabla_\varepsilon(b)$ for $b \in \mathbf{B}^\downarrow$ and Q has a ∇_ε-flag with sections $\nabla_\varepsilon(c)$ for $c \in \mathbf{B}^\uparrow$. For each $b \in \mathbf{B}^\uparrow$, the unit of adjunction $\nabla_\varepsilon(b) \to j_*(j\nabla_\varepsilon(b))$ is an isomorphism rhanks to Theorem 3.18(4) since it becomes an isomorphism on applying j. Since j_* is exact on $\nabla_\varepsilon(\mathcal{R}^\uparrow)$, we deduce that the the unit of adjunction $Q \to j_*(jQ)$ is an isomorphism too. It remains to note that $jV \cong jQ$, hence, $j_*(jV) \cong j_*(jQ)$. □

The definition of ε-stratified category has some important special cases. Again, the language here agrees with [ADL, Definition 1.3].

Definition 3.20. Let $\mathcal{R}$ be a finite Abelian category (resp., a Schurian category) with a stratification. We say that $\mathcal{R}$ is

- a *finite* (resp., an *essentially finite*) *standardly stratified category* if it is ε-stratified taking ε to be the constant function $+$;
- a *finite* (resp., an *essentially finite*) *costandardly stratified category* if it is ε-stratified taking ε to be the constant function $-$;
- a *finite* (resp., an *essentially finite*) *fully stratified category* if it is both standardly stratified and costandardly stratified.

By Lemma 3.9, $\mathcal{R}$ is standardly stratified if and only if $\mathcal{R}^{\mathrm{op}}$ is costandardly stratified. Hence, $\mathcal{R}$ is fully stratified if and only if $\mathcal{R}^{\mathrm{op}}$ is fully stratified. The following lemma shows that finite fully stratified categories are the same as the "standardly stratified categories" defined by Losev and Webster in [LW, §2].

Lemma 3.21. *The following are equivalent:*

(i) $\mathcal{R}$ is fully stratified;

(ii) $\mathcal{R}$ is ε-stratified for every choice of sign function $\varepsilon : \Lambda \to \{\pm\}$;

(iii) $\mathcal{R}$ is ε-stratified and $(-\varepsilon)$-stratified for some choice of sign function $\varepsilon : \Lambda \to \{\pm\}$;

(iv) $\mathcal{R}$ is ε-stratified for some $\varepsilon : \Lambda \to \{\pm\}$ and all of its standardization and costandardization functors are exact;
(v) $\mathcal{R}$ is standardly stratified and each $\Delta(b)$ has a $\bar{\Delta}$-flag with sections $\bar{\Delta}(c)$ for c with $\rho(c) = \rho(b)$;
(vi) $\mathcal{R}$ is costandardly stratified and each $\nabla(b)$ has a $\bar{\nabla}$-flag with sections $\bar{\nabla}(c)$ for c with $\rho(c) = \rho(b)$.

Proof. (i)⇒(ii)⇒(iii): Obvious.

(iii)⇒(iv): Take ε as in (iii) so that $\mathcal{R}$ is ε-stratified. The standardization functor $j_!^\lambda$ is exact when $\varepsilon(\lambda) = -$ by the last part of Theorem 3.6. Also $\mathcal{R}$ is $(-\varepsilon)$-stratified, so the same result gives that $j_!^\lambda$ is exact when $\varepsilon(\lambda) = +$. Similarly, all of the costandardization functors are exact too.

(iv)⇒(v): Applying the exact standardization functor $j_!^\lambda$ to a composition series of $P_\lambda(b)$, we deduce that $\Delta(b)$ has a $\bar{\Delta}$-flag with sections $\bar{\Delta}(c)$ for c with $\rho(c) = \rho(b)$. Similarly, applying j_*^λ, we get that $\nabla(b)$ has a $\bar{\nabla}$-flag with sections $\bar{\nabla}(c)$ for c with $\rho(c) = \rho(b)$.

To show that $\mathcal{R}$ is standardly stratified, we check that each $I(b)$ has a $\bar{\nabla}$-flag with sections $\bar{\nabla}(c)$ for c with $\rho(c) \geqslant \rho(b)$. This is immediate if $\varepsilon(b) = +$ since we are assuming that $\mathcal{R}$ is ε-stratified. If $\varepsilon(b) = -$ then $I(b)$ has a ∇-flag with sections $\nabla(c)$ for c with $\rho(c) \geqslant \rho(b)$. Hence, by the previous paragraph, it also has the required sort of $\bar{\nabla}$-flag.

(v)⇒(i): We just need to show that $\mathcal{R}$ is costandardly stratified. We know that each $P(b)$ has a Δ-flag with sections $\Delta(c)$ for c with $\rho(c) \geqslant \rho(b)$. Now use the given $\bar{\Delta}$-flags of each $\Delta(c)$ to see that each $P(b)$ also has the appropriate sort of $\bar{\Delta}$-flag.

(v)⇔(vi): This follows from the above using the observation made earlier that $\mathcal{R}$ is fully statified if and only if $\mathcal{R}^{\mathrm{op}}$ is fully stratified. □

Corollary 3.22. *Suppose that $\mathcal{R}$ is a finite or essentially finite ε-stratified category admitting a duality compatible with the stratification, i.e., there is a contravariant equivalence $\# : \mathcal{R} \to \mathcal{R}$ such that $L(b)^\# \cong L(c)$ implies $\rho(b) = \rho(c)$ for $b, c \in \mathbf{B}$. Then $\mathcal{R}$ is fully stratified.*

Proof. Since $\mathcal{R}$ is ε-stratified, $\mathcal{R}^{\mathrm{op}}$ is $(-\varepsilon)$-stratified. Using the duality, we deduce that $\mathcal{R}$ is also $(-\varepsilon)$-stratified. This verifies the property from Lemma 3.21(3). □

One reason fully stratified categories are particularly natural is due to the following.

Lemma 3.23. *Suppose that $\mathcal{R}$ is a finite or essentially finite fully stratified category. For $b, c \in \mathcal{R}$ and $n \geqslant 0$, we have that*

$$\mathrm{Ext}^n_{\mathcal{R}}(\bar{\Delta}(b), \bar{\nabla}(c)) \cong \begin{cases} \mathrm{Ext}^n_{\mathcal{R}_\lambda}(L(b), L(c)) & \text{if } \lambda = \mu \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda := \rho(b)$ and $\mu := \rho(c)$.

Proof. Choose ε so that $\varepsilon(\lambda) = -$, hence, $\bar{\Delta}(b) = \Delta_\varepsilon(b)$. By Lemma 3.21, $\mathcal{R}$ is ε-stratified, so we can apply Theorem 3.17(4) with $\mathcal{R}^\downarrow = \mathcal{R}_{\leqslant\mu}$ to deduce that

$$\mathrm{Ext}^n_{\mathcal{R}}(\bar{\Delta}(b), \bar{\nabla}(c)) \cong \mathrm{Ext}^n_{\mathcal{R}_{\leqslant\mu}}(i^*_{\leqslant\mu}\bar{\Delta}(b), \bar{\nabla}(c)).$$

This is zero unless $\lambda \leqslant \mu$. If $\lambda \leqslant \mu$, it is simply $\mathrm{Ext}^n_{\mathcal{R}_{\leqslant\mu}}(\bar{\Delta}(b), \bar{\nabla}(c))$. Now we change ε so that $\varepsilon(\mu) = +$, hence, $\bar{\nabla}(c) = \nabla_\varepsilon(c)$. Then by Theorem 3.18(3) with $\mathcal{R} = \mathcal{R}_{\leqslant\mu}$ and $\mathcal{R}^\uparrow = \mathcal{R}_\mu$ we get that $\mathrm{Ext}^n_{\mathcal{R}_{\leqslant\mu}}(\bar{\Delta}(b), \bar{\nabla}(c)) \cong \mathrm{Ext}^n_{\mathcal{R}_\mu}(j^\mu\bar{\Delta}(b), L(c))$. This is zero unless $\lambda = \mu$, when $j^\mu\bar{\Delta}(b) = L(b)$ and we are done. □

The next results are concerned with global dimension.

Lemma 3.24. *Let $\mathcal{R}$ be a finite ε-stratified category.*

(1) *All $V \in \Delta_\varepsilon(\mathcal{R})$ are of finite projective dimension if and only if all negative strata*[2] *have finite global dimension.*
(2) *All $V \in \nabla_\varepsilon(\mathcal{R})$ are of finite injective dimension if and only if all positive strata have finite global dimension.*

Proof. It suffices to prove (1). Assume that all negative strata have finite global dimension. By [Wei, Exercise 4.1.2], it suffices to show that $\operatorname{pd}\Delta_\varepsilon(b) < \infty$ for each $b \in \mathbf{B}$. We proceed by downwards induction on the partial order. For the induction step, consider $\Delta_\varepsilon(b)$ for $b \in \mathbf{B}_\lambda$, assuming that $\operatorname{pd}\Delta_\varepsilon(c) < \infty$ for each c with $\rho(c) > \lambda$. We show first that $\operatorname{pd}\Delta(b) < \infty$. Using Lemma 3.10 with $\varepsilon(\lambda) = -$, we see that there is a short exact sequence $0 \to Q \to P(b) \to \Delta(b) \to 0$ such that Q has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for c with $\rho(c) > \lambda$. By the induction hypothesis, Q has finite projective dimension, hence, so does $\Delta(b)$. This verifies the induction step in the case that $\varepsilon(\lambda) = +$. Instead, suppose that $\varepsilon(\lambda) = -$, i.e., $\Delta_\varepsilon(b) = \bar{\Delta}(b)$. Let $0 \to P_n \to \cdots \to P_0 \to L_\lambda(b) \to 0$ be a finite projective resolution of $L_\lambda(b)$ in the stratum $\mathcal{R}_\lambda$. Applying $j_!^\lambda$, which is exact thanks to Theorem 3.6, we obtain an exact sequence $0 \to V_n \to \cdots \to V_0 \to \bar{\Delta}(b) \to 0$ such that each V_m is a direct sum of standard objects $\Delta(c)$ for $c \in \mathbf{B}_\lambda$. The result already established plus [Wei, Exercise 4.1.3] implies that $\operatorname{pd} V_m < \infty$ for each m. Arguing like in the proof of [Wei, Theorem 4.3.1], we deduce that $\operatorname{pd}\bar{\Delta}(b) < \infty$ too.

Conversely, suppose that $\operatorname{pd}\Delta_\varepsilon(b) < \infty$ for all $b \in \mathbf{B}$. Take $\lambda \in \Lambda$ with $\varepsilon(\lambda) = -$. Suppose first that λ is maximal. Applying the exact functor j^λ to finite projective resolutions of $\bar{\Delta}(b)$ for each $b \in \mathbf{B}_\lambda$, we obtain finite projective resolutions of $L_\lambda(b)$ in $\mathcal{R}_\lambda$, showing that the stratum $\mathcal{R}_\lambda$ is of finite global dimension. Finally, when λ is not maximal, we let $\Lambda^\downarrow := \Lambda \backslash (\lambda, \infty)$ and $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the corresponding ε-stratified Serre subcategory of $\mathcal{R}$. Given $b \in \mathbf{B}_\lambda$, take a finite projective resolution $0 \to P_n \to \cdots \to P_0 \to \Delta_\varepsilon(b) \to 0$ of $\Delta_\varepsilon(b)$ in $\mathcal{R}$. Applying i^*, we obtain an exact sequence $0 \to i^*P_n \to \cdots \to i^*P_0 \to \Delta_\varepsilon(b) \to 0$ in $\mathcal{R}^\downarrow$. This sequence is exact due to Theorem 3.17(2); to see this one also needs to use Corollary 3.16 to break the sequence into short exact sequences in $\Delta_\varepsilon(\mathcal{R})$. Since λ is maximal in $\Lambda^\downarrow$, we are reduced to the case already discussed. □

Corollary 3.25. *Suppose that $\mathcal{R}$ is a finite ε-stratified category. If $\mathcal{R}$ is of finite global dimension then all of its strata are of finite global dimension too.*

Corollary 3.26. *Suppose that $\mathcal{R}$ is a finite standardly stratified category, a finite costandardly stratified category, or a finite fully stratified category. If all of the strata are of finite global dimension then $\mathcal{R}$ is of finite global dimension.*

Proof. We just explain this in the case that $\mathcal{R}$ is costandardly stratified, i.e., it is ε-stratified for $\varepsilon = -$. Lemma 3.24(1) implies that $\bar{\Delta}(b)$ is of finite projective dimension for each $b \in \mathbf{B}$. Moreover, there is a short exact sequence $0 \to K \to \bar{\Delta}(b) \to L(b) \to 0$ where all composition factors of K are of the form $L(c)$ for c with $\rho(c) < \rho(b)$. Ascending induction on the partial order implies that each $L(b)$ has finite projective dimension. □

Remark 3.27. In the fully stratified case, Lemma 3.23 can be used to give a precise bound on the global dimension of $\mathcal{R}$ in Corollary 3.26. Let

$$|\lambda| := \sup\left\{ \frac{\max(\operatorname{gl.dim}\mathcal{R}_{\lambda_0}, \dots, \operatorname{gl.dim}\mathcal{R}_{\lambda_n})}{2} + n \;\middle|\; \begin{array}{l} n \geqslant 0 \text{ and } \lambda_0, \lambda_1, \dots, \lambda_n \in \Lambda \\ \text{with } \lambda_0 < \lambda_1 < \cdots < \lambda_n = \lambda \end{array} \right\}.$$

By mimicking the proof of [Do4, Proposition A2.3], one shows that $\operatorname{Ext}^n_{\mathcal{R}}(L(b), L(c)) = 0$ for $b, c \in \mathbf{B}$ and any $n > |\rho(b)| + |\rho(c)|$. Hence, $\operatorname{gl.dim}\mathcal{R} \leqslant 2\max\{|\lambda| \mid \lambda \in \Lambda\}$.

[2] We mean the strata $\mathcal{R}_\lambda$ for $\lambda \in \Lambda$ such that $\varepsilon(\lambda) = -$.

Finally, we record the following well-known definition, which goes back to [CPS1]; for detailed historical remarks we refer to [Do4, §A5].

Definition 3.28. A *finite* (resp., an *essentially finite*) *highest weight category* is a finite Abelian (resp., a Schurian) category $\mathcal{R}$ with a stratification such that all of the strata are simple (cf. Lemma 3.5) and one of the following equivalent properties holds:

($P\Delta$) Each $P(\lambda)$ has a Δ-flag with sections $\Delta(\mu)$ for $\mu \geqslant \lambda$.
($I\nabla$) Each $I(\lambda)$ has a ∇-flag with sections $\nabla(\mu)$ for $\mu \geqslant \lambda$.

In particular, Corollary 3.26 recovers the following well-known result, see e.g. [CPS1]:

Corollary 3.29. *Finite highest weight categories are of finite global dimension.*

3.3. **Upper finite stratified categories.** In this subsection we assume that $\mathcal{R}$ is locally Schurian with a stratification $\rho : \mathbf{B} \to \Lambda$ such that the poset Λ is upper finite. Also ε is a sign function. Let $I(b)$ and $P(b)$ be an injective hull and a projective cover of $L(b)$ in $\mathcal{R}$, respectively, and recall (1.1)–(1.2).

Theorem 3.30. *Theorem 3.6 holds in the present setup too.*

Proof. This is exactly the same as the proof of Theorem 3.6 above: the proof before depended on various lemmas in §3.4 below, which are proved there both for the Schurian case needed before and the new locally Schurian case needed now. □

Consequently, we can make essentially the same definition as before.

Definition 3.31. We say that $\mathcal{R}$ is an *upper finite ε-stratified category* if it is a locally Schurian category equipped with a stratification as in Definition 3.1 such that

- the poset Λ is upper finite;
- one of the equivalent properties ($P\Delta_\varepsilon$) or ($I\nabla_\varepsilon$) holds.

Lemmas 3.9 and 3.10 still hold in the same way as before.

The next goal is to adapt the other results from §3.2 to the upper finite case. First, we state the following, which is the analog of Theorem 3.17; we have dropped part (6) since the proof of that required objects of $\mathcal{R}^\downarrow$ to have finite length.

Theorem 3.32. *Assume that $\mathcal{R}$ is an upper finite ε-stratified category. Suppose that $\Lambda^\downarrow$ is a lower set in Λ. Let $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$ and $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the corresponding Serre subcategory of $\mathcal{R}$ with the induced stratification. Then $\mathcal{R}^\downarrow$ is an upper finite ε-stratified category. Moreover:*

(1) The distinguished objects in $\mathcal{R}^\downarrow$ satisfy $L^\downarrow(b) \cong L(b)$, $P^\downarrow(b) \cong i^*P(b)$, $I^\downarrow(b) \cong i^!I(b)$, $\Delta^\downarrow(b) \cong \Delta(b)$,$\bar{\Delta}^\downarrow(b) \cong \bar{\Delta}(b)$, $\nabla^\downarrow(b) \cong \nabla(b)$ and $\bar{\nabla}^\downarrow(b) \cong \bar{\nabla}(b)$ for $b \in \mathbf{B}^\downarrow$.
(2) i^* is exact on $\Delta_\varepsilon(\mathcal{R})$ with $i^*\Delta(b) \cong \Delta^\downarrow(b)$ and $i^*\bar{\Delta}(b) \cong \bar{\Delta}^\downarrow(b)$ for $b \in \mathbf{B}^\downarrow$; also $i^*\Delta(b) = i^*\bar{\Delta}(b) = 0$ for $b \notin \mathbf{B}^\downarrow$.
(3) $\operatorname{Ext}^n_{\mathcal{R}}(V, iW) \cong \operatorname{Ext}^n_{\mathcal{R}^\downarrow}(i^*V, W)$ for $V \in \Delta_\varepsilon(\mathcal{R})$, $W \in \mathcal{R}^\downarrow$ and all $n \geqslant 0$.
(4) $i^!$ is exact on $\nabla_\varepsilon(\mathcal{R})$ with $i^!\nabla(b) \cong \nabla^\downarrow(b)$ and $i^!\bar{\nabla}(b) \cong \bar{\nabla}^\downarrow(b)$ for $b \in \mathbf{B}^\downarrow$; also $i^!\nabla(b) = i^!\bar{\nabla}(b) = 0$ for $b \notin \mathbf{B}^\downarrow$.
(5) $\operatorname{Ext}^n_{\mathcal{R}}(iV, W) \cong \operatorname{Ext}^n_{\mathcal{R}^\downarrow}(V, i^!W)$ for $V \in \mathcal{R}^\downarrow, W \in \nabla_\varepsilon(\mathcal{R})$ and all $n \geqslant 0$.

Proof. This follows from Lemma 3.44 proved in §3.4 (applied both in $\mathcal{R}$ and in $\mathcal{R}^{\mathrm{op}}$). For (5), one also needs Lemma 2.11 in order to establish its equivalence with the dual statement (or (5) could be proved directly by dualizing the proof in Lemma 3.44). □

The other essential Ext-vanishing result is the following. Like in the previous subsection, it implies for $V \in \nabla_\varepsilon(\mathcal{R})$) (resp., $V \in \Delta_\varepsilon(\mathcal{R})$) that the multiplicity $(V : \nabla_\varepsilon(b))$ (resp., $(V : \Delta_\varepsilon(b))$) is well-defined and independent of the choice of flag, as it is equal to the dimension of $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V)$ (resp., $\operatorname{Hom}_{\mathcal{R}}(V, \nabla_\varepsilon(b))$).

Lemma 3.33. *In any upper finite ε-stratified category, we have that*

$$\dim \operatorname{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), \nabla_\varepsilon(c)) = \delta_{b,c}\delta_{n,0}$$

for any $b, c \in \mathbf{B}$ and $n \geqslant 0$.

Proof. This is Lemma 3.46 below. □

Corollary 3.34. $(P(b) : \Delta_\varepsilon(c)) = [\nabla_\varepsilon(c) : L(b)]$ *and* $(I(b) : \nabla_\varepsilon(c)) = [\Delta_\varepsilon(c) : L(b)]$.

Next we introduce the two (in fact dual) notions of ascending Δ_ε- and descending ∇_ε-flags, generalizing the finite flags discussed already. One might be tempted to say that an ascending Δ_ε-flag in V is an ascending chain $0 = V_0 < V_1 < V_2 < \cdots$ of subobjects of V with $V = \sum_{n\in\mathbb{N}} V_n$ such that $V_m/V_{m-1} \cong \Delta_\varepsilon(b_m)$, and a descending ∇_ε-flag is a descending chain $V = V_0 > V_1 > V_2 > \cdots$ of subobjects of V such that $\bigcap_{n\in\mathbb{N}} V_n = 0$ and $V_{m-1}/V_m \cong \Delta_\varepsilon(b_m)$, for $b_m \in \mathbf{B}$. These would be serviceable definitions when Λ is countable. In order to avoid this unnecessary restriction, we will work instead with the following more general formulations.

Definition 3.35. Suppose that $\mathcal{R}$ is an upper finite ε-stratified category and $V \in \mathcal{R}$.

- An *ascending Δ_ε-flag* in V is the data of a directed set Ω with smallest element 0 and a direct system $(V_\omega)_{\omega\in\Omega}$ of subobjects of V such that $V_0 = 0$, $\sum_{\omega\in\Omega} V_\omega = V$, and $V_\upsilon/V_\omega \in \Delta_\varepsilon(\mathcal{R})$ for each $\omega < \upsilon$. Let $\Delta^{\text{asc}}_\varepsilon(\mathcal{R})$ be the exact subcategory of $\mathcal{R}$ consisting of all objects V possessing such a flag. For any choice $(V_\omega)_{\omega\in\Omega}$ of ascending Δ_ε-flag in $V \in \Delta^{\text{asc}}_\varepsilon(\mathcal{R})$, we have that
$$\operatorname{Hom}_{\mathcal{R}}(V, \nabla_\varepsilon(b)) \cong \operatorname{Hom}_{\mathcal{R}}(\varinjlim V_\omega, \nabla_\varepsilon(b)) \cong \varprojlim \operatorname{Hom}_{\mathcal{R}}(V_\omega, \nabla_\varepsilon(b)).$$
Hence, $(V : \Delta_\varepsilon(b)) := \dim \operatorname{Hom}_{\mathcal{R}}(V, \nabla_\varepsilon(b)) = \sup\left\{(V_\omega : \Delta_\varepsilon(b)) \,\middle|\, \omega \in \Omega\right\} \in \mathbb{N}$.
- A *descending ∇_ε-flag* in V is the data of a directed set Ω with smallest element 0 and an inverse system $(V/V_\omega)_{\omega\in\Omega}$ of quotients of V such that $V_0 = V$, $\bigcap_{\omega\in\Omega} V_\omega = 0$, and $V_\upsilon/V_\omega \in \nabla_\varepsilon(\mathcal{R})$ for each $\upsilon < \omega$. Let $\nabla^{\text{dsc}}_\varepsilon(\mathcal{R})$ be the exact subcategory of $\mathcal{R}$ consisting of all objects V possessing such a flag. By the dual arguments to the above, we see for any choice $(V/V_\omega)_{\omega\in\Omega}$ of descending ∇_ε-flag in $V \in \nabla^{\text{dsc}}_\varepsilon(\mathcal{R})$ that $(V : \nabla_\varepsilon(b)) := \dim \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) = \sup\left\{(V/V_\omega : \nabla_\varepsilon(b)) \,\middle|\, \omega \in \Omega\right\} \in \mathbb{N}$.

Lemma 3.36. *Assume that $\mathcal{R}$ is an upper finite ε-stratified category. For $V \in \Delta^{\text{asc}}_\varepsilon(\mathcal{R})$, $W \in \nabla^{\text{dsc}}_\varepsilon(\mathcal{R})$ and $n \geqslant 1$, we have that* $\operatorname{Ext}^n_{\mathcal{R}}(V, W) = 0$.

Proof. We first prove this in the special case that $W = \nabla_\varepsilon(b)$. Let $(V_\omega)_{\omega\in\Omega}$ be an ascending Δ_ε-flag in V, so that $V \cong \varinjlim V_\omega$. Since $\operatorname{Ext}^n_{\mathcal{R}}(V_\omega, W) = 0$ by Lemma 3.33, it suffices to show that
$$\operatorname{Ext}^n_{\mathcal{R}}(V, W) \cong \varprojlim \operatorname{Ext}^n_{\mathcal{R}}(V_\omega, W).$$
To see this, like in [Wei, Application 3.5.10], we need to check a Mittag-Leffler condition. We show simply that the natural map $\operatorname{Ext}^{n-1}_{\mathcal{R}}(V_\upsilon, W) \to \operatorname{Ext}^{n-1}_{\mathcal{R}}(V_\omega, W)$ is surjective for each $\omega < \upsilon$ in Ω. Applying $\operatorname{Hom}_{\mathcal{R}}(-, W)$ to the short exact sequence $0 \to V_\omega \to V_\upsilon \to V_\upsilon/V_\omega \to 0$ gives an exact sequence
$$\operatorname{Ext}^{n-1}_{\mathcal{R}}(V_\upsilon, W) \longrightarrow \operatorname{Ext}^{n-1}_{\mathcal{R}}(V_\omega, W) \longrightarrow \operatorname{Ext}^{n}_{\mathcal{R}}(V_\upsilon/V_\omega, W).$$
It remains to observe that $\operatorname{Ext}^n_{\mathcal{R}}(V_\upsilon/V_\omega, W) = 0$ by Lemma 3.33 again, since we know from the definition of ascending Δ_ε-flag that $V_\upsilon/V_\omega \in \Delta_\varepsilon(\mathcal{R})$.

The result just proved with $\mathcal{R}$ repaced by $\mathcal{R}^{\text{op}}$ shows that $\operatorname{Ext}^n_{\mathcal{R}}(V, W) = 0$ for $n \geqslant 1$ in the case that $V = \Delta_\varepsilon(b)$ and $W \in \nabla^{\text{dsc}}_\varepsilon(\mathcal{R})$. Repeating the argument of the previous paragraph and using this assertion in place of Lemma 3.33 gives the result for general $V \in \Delta^{\text{asc}}_\varepsilon(\mathcal{R})$ and $W \in \nabla^{\text{dsc}}_\varepsilon(\mathcal{R})$. □

Theorem 3.37. *Assume that $\mathcal{R}$ is an upper finite ε-stratified category. Suppose that $\Lambda^\uparrow$ is an upper set in Λ. Let $\mathbf{B}^\uparrow := \rho^{-1}(\Lambda^\uparrow)$ and $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the corresponding Serre quotient category of $\mathcal{R}$ with the induced stratification. Then $\mathcal{R}^\uparrow$ is itself a finite or upper finite ε-stratified category according to whether $\Lambda^\uparrow$ is finite or infinite. Moreover:*

(1) For $b \in \mathbf{B}^\uparrow$, the distinguished objects $L^\uparrow(b)$, $P^\uparrow(b)$, $I^\uparrow(b)$, $\Delta^\uparrow(b)$, $\bar\Delta^\uparrow(b)$, $\nabla^\uparrow(b)$ and $\bar\nabla^\uparrow(b)$ in $\mathcal{R}^\uparrow$ are isomorphic to the images under j of the corresponding objects of $\mathcal{R}$.
(2) We have that $jL(b) = j\Delta(b) = j\bar\Delta(b) = j\nabla(b) = j\bar\nabla(b) = 0$ if $b \notin \mathbf{B}^\uparrow$.
(3) $\operatorname{Ext}^n_{\mathcal{R}}(V, j_*W) \cong \operatorname{Ext}^n_{\mathcal{R}^\uparrow}(jV, W)$ for $V \in \mathcal{R}, W \in \nabla^{\mathrm{dsc}}_\varepsilon(\mathcal{R}^\uparrow)$ and all $n \geqslant 0$.
(4) j_* is exact on $\nabla_\varepsilon(\mathcal{R}^\uparrow)$ with $j_*\nabla^\uparrow(b) \cong \nabla(b)$, $j_*\bar\nabla^\uparrow(b) \cong \bar\nabla(b)$ and $j_*I^\uparrow(b) \cong I(b)$ for $b \in \mathbf{B}^\uparrow$.
(5) $\operatorname{Ext}^n_{\mathcal{R}}(j_!V, W) \cong \operatorname{Ext}^n_{\mathcal{R}^\uparrow}(V, jW)$ for $V \in \Delta^{\mathrm{asc}}_\varepsilon(\mathcal{R}^\uparrow)$, $W \in \mathcal{R}$ and all $n \geqslant 0$.
(6) $j_!$ is exact on $\Delta_\varepsilon(\mathcal{R}^\uparrow)$ with $j_!\Delta^\uparrow(b) \cong \Delta(b)$, $j_!\bar\Delta^\uparrow(b) \cong \bar\Delta(b)$ and $j_!P^\uparrow(b) = P(b)$ for $b \in \mathbf{B}^\uparrow$.

Proof. If $\Lambda^\uparrow$ is finite, this is proved in just the same way as Theorem 3.18, noting that the lemmas from §3.4 referenced in the argument also hold in the locally Schurian case.

Assume instead that $\Lambda^\uparrow$ is infinite. Then the same arguments prove everything up to and including (2), but the proofs of the remaining parts need some slight modifications. It suffices to prove (3) and (4), since (5) and (6) are the same results for $\mathcal{R}^{\mathrm{op}}$ (bearing in mind also Lemma 2.11). For (3), the original argument from the proof of Lemma 3.49(4) reduces to checking that j sends projectives to objects that are acyclic for $\operatorname{Hom}_{\mathcal{R}^\uparrow}(-, W)$. To see this, it suffices to show that $\operatorname{Ext}^n_{\mathcal{R}^\uparrow}(jP(b), W) = 0$ for $n \geqslant 1$ and $b \in \mathbf{B}$, which follows from Lemma 3.36.

Finally, for (4), the argument from the proof of Theorem 3.18(4) cannot be used since it depends on $\mathcal{R}^\uparrow$ being Schurian. So we provide an alternate argument. Take a short exact sequence $0 \to U \to V \to W \to 0$ in $\nabla_\varepsilon(\mathcal{R}^\uparrow)$. Applying j_*, we get $0 \to j_*U \to j_*V \to j_*W$, and just need to show that the final morphism here is an epimorphism. This follows because, by (3) and Lemma 3.48, j_*U, j_*V and j_*W all have ∇_ε-flags such that $(j_*V : \nabla_\varepsilon(b)) = (j_*U : \nabla_\varepsilon(b)) + (j_*W : \nabla_\varepsilon(b))$ for all $b \in \mathbf{B}$. □

Theorem 3.38. *Assume that $\mathcal{R}$ is an upper finite ε-stratified category. For $V \in \mathcal{R}$, the following are equivalent:*

(i) $V \in \Delta^{\mathrm{asc}}_\varepsilon(\mathcal{R})$;
(ii) $\operatorname{Ext}^1_{\mathcal{R}}(V, \nabla_\varepsilon(b)) = 0$ for all $b \in \mathbf{B}$;
(iii) $\operatorname{Ext}^n_{\mathcal{R}}(V, \nabla_\varepsilon(b)) = 0$ for all $b \in \mathbf{B}$ and $n \geqslant 1$.

Assuming these properties, we have that $V \in \Delta_\varepsilon(\mathcal{R})$ if and only if it is finitely generated.

Proof. (i)⇒(iii) follows from Lemma 3.36, and (iii)⇒(ii) is trivial. It remains to show (ii)⇒(i). Let Ω be the directed set of finite upper sets in Λ. Take $\omega \in \Omega$; it is some finite upper set $\Lambda^\uparrow$. In the notation from Theorem 3.37, let $V_\omega := j_!(jV)$. By Theorem 3.37(3) and (4), we have that $\operatorname{Ext}^1_{\mathcal{R}^\uparrow}(jV, \nabla^\uparrow_\varepsilon(b)) \cong \operatorname{Ext}^1_{\mathcal{R}}(V, \nabla_\varepsilon(b)) = 0$ for all $b \in \mathbf{B}^\uparrow$. Hence, by Theorem 3.14, we have that $jV \in \Delta_\varepsilon(\mathcal{R}^\uparrow)$. Applying Theorem 3.37(6), it follows that $V_\omega := j_!(jV) \in \Delta_\varepsilon(\mathcal{R})$. Let $f_\omega : V_\omega \to V$ be the morphism induced by the counit of adjunction. We claim for any $b \in \mathbf{B}^\uparrow$ that the map

$$f_\omega(b) : \operatorname{Hom}_{\mathcal{R}}(P(b), V_\omega) \to \operatorname{Hom}_{\mathcal{R}}(P(b), V),\ \theta \mapsto f_\omega \circ \theta$$

is an isomorphism. To see this, we assume that $\mathcal{R} = A\text{-mod}_{\mathrm{lfd}}$ for a pointed locally finite-dimensional locally unital algebra $A = \bigoplus_{a,b\in\mathbf{B}} e_aAe_b$. Then $\mathcal{R}^\uparrow = eAe\text{-mod}_{\mathrm{lfd}}$ where $e = \sum_{a\in\mathbf{B}^\uparrow} e_a$, and $V_\omega = Ae \otimes_{eAe} eV$. From this, it is obvious that, for $b \in \mathbf{B}^\uparrow$, the natural multiplication gives an isomorphism $e_bV_\omega \xrightarrow{\sim} e_bV$. This proves the claim.

Now take $\upsilon > \omega$, i.e., another finite upper set $\Lambda^{\Uparrow} \supset \Lambda^{\uparrow}$, and let $k : \mathcal{R} \to \mathcal{R}^{\Uparrow}$ be the associated quotient. The quotient functor $j : \mathcal{R} \to \mathcal{R}^{\uparrow}$ factors as $j = \bar{\jmath} \circ k$ for another quotient functor $\bar{\jmath} : \mathcal{R}^{\Uparrow} \to \mathcal{R}^{\uparrow}$, and we have by definition that

$$V_\omega = (\bar{\jmath} \circ k)_!((\bar{\jmath} \circ k)V) = k_!(\bar{\jmath}_!(\bar{\jmath}(kV))), \qquad V_\upsilon = k_!(kV).$$

By Corollary 3.19(2), there is a short exact sequence $0 \to \bar{\jmath}_!(\bar{\jmath}(kV)) \to kV \to Q \to 0$ such that both $\bar{\jmath}_!(\bar{\jmath}(kV))$ and Q belong to $\Delta_\varepsilon(\mathcal{R}^{\uparrow})$. Applying $k_!$ and using the exactness from Theorem 3.37(6), we get an embedding $f^\upsilon_\omega : V_\omega \hookrightarrow V_\upsilon$ such that $V_\upsilon / V_\omega \cong k_! Q \in \Delta_\varepsilon(\mathcal{R})$. Since the morphisms all came from counits of adjunction, we have that $f_\upsilon \circ f^\upsilon_\omega = f_\omega$.

Now we can show that each f_ω is a monomorphism. It suffices to show that $f_\omega(b) : \operatorname{Hom}_{\mathcal{R}}(P(b), V_\omega) \to \operatorname{Hom}_{\mathcal{R}}(P(b), V)$ is injective for all $b \in \mathbf{B}$. Choose υ in the previous paragraph to be sufficiently large so as to ensure that $b \in \mathbf{B}^{\Uparrow}$. Then we know that $f_\upsilon(b)$ is injective by the claim in the opening paragraph. Since $f_\omega = f_\upsilon \circ f^\upsilon_\omega$ and f^υ_ω is a momorphism, it follows that $f_\omega(b)$ is injective too. Thus, identifying V_ω with its image under f_ω, we have defined a direct system $(V_\omega)_{\omega \in \Omega}$ of subobjects of V such that $V_\upsilon / V_\omega \in \Delta_\varepsilon(\mathcal{R})$ for each $\omega < \upsilon$. It remains to observe that $V_\varnothing = 0$ and $\sum_{\omega \in \Omega} V_\omega = V$. The latter equality follows because we have also shown that for each $b \in \mathbf{B}$ that $f_\omega(b)$ is surjective for sufficiently large ω.

Final part: If $V \in \Delta_\varepsilon(\mathcal{R})$, it is obvious that it is finitely generated since each $\Delta_\varepsilon(b)$ is. Conversely, suppose that V is finitely generated and has an ascending Δ_ε-flag. To see that it is actually a finite flag, it suffices to show that $\operatorname{Hom}_{\mathcal{R}}(V, \nabla_\varepsilon(b)) = 0$ for all but finitely many $b \in \mathbf{B}$. Say $\operatorname{hd} V \cong L(b_1) \oplus \cdots \oplus L(b_n)$. If $V \to \nabla_\varepsilon(b)$ is a non-zero homomorphism, we must have that $\rho(b_i) \leqslant \rho(b)$ for some $i = 1, \dots, n$. Hence, there are only finitely many choices for b as the poset is upper finite. □

Corollary 3.39. *Let $0 \to U \to V \to W \to 0$ be a short exact sequence in an upper finite ε-stratified category. If V and W belong to $\Delta_\varepsilon(\mathcal{R})$ (resp., to $\Delta^{\mathrm{asc}}_\varepsilon(\mathcal{R})$) so does U.*

Theorem 3.40. *Assume that $\mathcal{R}$ is an upper finite ε-stratified category. For $V \in \mathcal{R}$, the following are equivalent:*

(i) $V \in \nabla^{\mathrm{dsc}}_\varepsilon(\mathcal{R})$;
(ii) $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$ *for all* $b \in \mathbf{B}$;
(iii) $\operatorname{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$ *for all* $b \in \mathbf{B}$ *and* $n \geqslant 1$.

With these properties we have that $V \in \nabla_\varepsilon(\mathcal{R})$ if and only if it is finitely cogenerated.

Proof. By Lemma 2.11, this is the equivalent dual statement to Theorem 3.38. □

Corollary 3.41. *Let $0 \to U \to V \to W \to 0$ be a short exact sequence in an upper finite ε-stratified category. If U and V belong to $\nabla_\varepsilon(\mathcal{R})$ (resp., to $\nabla^{\mathrm{dsc}}_\varepsilon(\mathcal{R})$) so does W.*

Like in the previous subsection, there are various special cases.

Definition 3.42. Let $\mathcal{R}$ be a locally Schurian category equipped with a stratification $\rho : \mathbf{B} \to \Lambda$ such that the poset Λ is upper finite. Adopt the usual notation for its various distinguished objects. We say that $\mathcal{R}$ is

- an *upper finite standardly stratified category* if it is $+$-stratified;
- an *upper finite costandardly stratified category* if it is $-$-stratified;
- an *upper finite fully stratified category* if it is both $+$- and $-$-stratified;
- an *upper finite highest weight category* if all strata are simple (cf. Lemma 3.5) and one of the equivalent properties $(P\Delta)$ or $(I\nabla)$ from Definition 3.28 holds.

The reader should have no difficulty in transporting Lemma 3.21 and Corollary 3.22 to the upper finite setting.

3.4. **Lemmas for §§3.2–3.3.** In this subsection, we prove a series of lemmas needed in both §3.2 and in §3.3. Let $\mathcal{R}$ be a category which is *either* Schurian (§3.2) *or* locally Schurian (§3.3). Also let $\rho : \mathbf{B} \to \Lambda$ be a stratification, and $\varepsilon : \Lambda \to \{\pm\}$ be a sign function. In the locally Schurian case, we assume that the poset Λ is upper finite. In both cases, we assume that property $(P\Delta_\varepsilon)$ holds.

Lemma 3.43. *We have that* $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), \Delta_\varepsilon(c)) = 0$ *for* $b, c \in \mathbf{B}$ *such that* $\rho(b) \nless \rho(c)$.

Proof. Using the assumed property $(P\Delta_\varepsilon)$, it follows that the first two terms of a minimal projective resolution of $\Delta_\varepsilon(b)$ are of the form $\bigoplus_{\rho(a) \geqslant \rho(b)} P(a)^{\oplus n_a} \to P(b) \to \Delta_\varepsilon(b) \to 0$ for some $n_a \geqslant 0$. Now apply $\operatorname{Hom}_{\mathcal{R}}(-, \Delta_\varepsilon(c))$. □

Lemma 3.44. *Let* $\Lambda^\downarrow$ *be a lower set in* Λ *and* $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$. *Let* $i : \mathcal{R}^\downarrow \to \mathcal{R}$ *be the corresponding Serre subcategory of* $\mathcal{R}$ *equipped with the induced stratification.*

1. *The standard, proper standard and indecomposable projective objects of* $\mathcal{R}^\downarrow$ *are the objects* $\Delta(b)$, $\bar{\Delta}(b)$ *and* $i^*P(b)$ *for* $b \in \mathbf{B}^\downarrow$.
2. $i^*P(b)$ *is zero unless* $b \in \mathbf{B}^\downarrow$, *in which case it has a* Δ_ε*-flag with sections* $\Delta_\varepsilon(c)$ *for* $c \in \mathbf{B}^\downarrow$ *with* $\rho(c) \geqslant \rho(b)$. *Hence,* $(P\Delta_\varepsilon)$ *holds in* $\mathcal{R}^\downarrow$.
3. $(\mathbb{L}_n i^*)V = 0$ *for* $V \in \Delta_\varepsilon(\mathcal{R})$ *and* $n \geqslant 1$.
4. $\operatorname{Ext}^n_{\mathcal{R}}(V, iW) \cong \operatorname{Ext}^n_{\mathcal{R}^\downarrow}(i^*V, W)$ *for* $V \in \Delta_\varepsilon(\mathcal{R})$, $W \in \mathcal{R}^\downarrow$ *and* $n \geqslant 0$.

Proof. (1) For projectives, this follows from the usual adjunction properties. For standard and proper standard objects, just note that the standardization functors for $\mathcal{R}^\downarrow$ are some of the ones for $\mathcal{R}$.

(2) Consider a Δ_ε-flag of $P(b)$. Using Lemma 3.43, we can rearrange this filtration if necessary so that all of the sections $\Delta_\varepsilon(c)$ with $c \in \mathbf{B}^\downarrow$ appear above the sections $\Delta_\varepsilon(d)$ with $d \in \mathbf{B}\backslash\mathbf{B}^\downarrow$. So there exists a short exact sequence $0 \to K \to P(b) \to Q \to 0$ in which Q has a finite filtration with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}^\downarrow$ with $\rho(c) \geqslant \rho(b)$, and K has a finite filtration with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}\backslash\mathbf{B}^\downarrow$. It follows easily that $i^*P(b)$ is isomorphic to Q, so it has the appropriate filtration.

(3) It suffices to show that $(\mathbb{L}_n i^*)\Delta_\varepsilon(b) = 0$ for all $b \in \mathbf{B}$ and $n > 0$. Take a short exact sequence $0 \to K \to P(b) \to \Delta_\varepsilon(b) \to 0$ and apply i^* to obtain a long exact sequence

$$0 \longrightarrow (\mathbb{L}_1 i^*)\Delta_\varepsilon(b) \longrightarrow i^*K \longrightarrow i^*P(b) \longrightarrow i^*\Delta_\varepsilon(b) \longrightarrow 0$$

and isomorphisms $(\mathbb{L}_{n+1} i^*)\Delta_\varepsilon(b) \cong (\mathbb{L}_n i^*)K$ for $n > 0$. We claim that $(\mathbb{L}_1 i^*)\Delta_\varepsilon(b) = 0$. We know that K has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for c with $\rho(c) \geqslant \rho(b)$. If $b \in \mathbf{B}\backslash\mathbf{B}^\downarrow$ we deduce that $i^*K = 0$, hence, $(\mathbb{L}_1 i^*)\Delta_\varepsilon(b) = 0$. If $b \in \mathbf{B}^\downarrow$, we use Lemma 3.43 to order the Δ_ε-flag of K so that it yields a short exact sequence $0 \to L \to K \to Q \to 0$ in which Q has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}^\downarrow$, and L has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}\backslash\mathbf{B}^\downarrow$. It follows that $i^*K = Q$ and there is a short exact sequence $0 \to \iota^*K \to \iota^*P(b) \to \Delta_\varepsilon(b) \to 0$. Comparing with the long exact sequence, we deduce that $(\mathbb{L}_1 i^*)\Delta_\varepsilon(b) = 0$. Finally some degree shifting using the isomorphisms $(\mathbb{L}_{n+1} i^*)\Delta_\varepsilon(b) \cong (\mathbb{L}_n i^*)K$ gives that $(\mathbb{L}_n i^*)\Delta_\varepsilon(b) = 0$ for $n > 1$ too.

(4) By the adjunction, we have that $\operatorname{Hom}_{\mathcal{R}}(-, iW) \cong \operatorname{Hom}_{\mathcal{R}^\downarrow}(-, W) \circ i^*$, i.e., the result holds when $n = 0$. Also i^* sends projectives to projectives as it is left adjoint to an exact functor. Now the result for $n > 0$ follows by a standard Grothendieck spectral sequence argument; the spectral sequence degenerates due to (3). □

Lemma 3.45. *Assume that* $\lambda \in \Lambda$ *is maximal and* $\varepsilon(\lambda) = +$. *For any* $V \in \mathcal{R}_\lambda$ *and* $b \in \mathbf{B}$, *we have that* $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), j^\lambda_* V) = 0$.

Proof. If $b \in \mathbf{B}_\lambda$ then $\Delta_\varepsilon(b) = \Delta(b)$, which is projective in $\mathcal{R}$ by the maximality of λ. So we get the Ext^1-vanishing in this case. For the remainder of the proof, suppose that

$b \notin \mathbf{B}_\lambda$. Let I be an injective hull of V in $\mathcal{R}_\lambda$. Applying j_*^λ to a short exact sequence $0 \to V \to I \to Q \to 0$, we get an exact sequence $0 \to j_*^\lambda V \to j_*^\lambda I \to j_*^\lambda Q$. By properties of adjunctions, $j_*^\lambda Q$ is finitely cogenerated and all constituents of its socle are $L(c)$ for $c \in \mathbf{B}_\lambda$. The same is true for $j_*^\lambda I / j_*^\lambda V$ since it embeds into $j_*^\lambda Q$. We deduce that $\mathrm{Hom}_\mathcal{R}(\Delta_\varepsilon(b), j_*^\lambda I / j_*^\lambda V) = 0$.

Now take an extension $0 \to j_*^\lambda V \to E \to \Delta_\varepsilon(b) \to 0$. Since $j_*^\lambda I$ is injective, we can find morphisms f and g making the following diagram with exact rows commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_*^\lambda V & \longrightarrow & E & \longrightarrow & \Delta_\varepsilon(b) & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \downarrow g & & \\ 0 & \longrightarrow & j_*^\lambda V & \longrightarrow & j_*^\lambda I & \longrightarrow & j_*^\lambda I / j_*^\lambda V & \longrightarrow & 0. \end{array}$$

The previous paragraph implies that $g = 0$. Hence, $\mathrm{im}\, f \subseteq j_*^\lambda V$. Thus, f splits the top short exact sequence, and we have shown that $\mathrm{Ext}^1_\mathcal{R}(\Delta_\varepsilon(b), j_*^\lambda V) = 0$. □

Lemma 3.46. *For $b, c \in \mathbf{B}$ and $n \geqslant 0$, we have that* $\dim \mathrm{Ext}^n_\mathcal{R}(\Delta_\varepsilon(b), \nabla_\varepsilon(c)) = \delta_{b,c}\delta_{n,0}$.

Proof. This is clear in the case $n = 0$, so assume that $n > 0$. Suppose that $b \in \mathbf{B}_\lambda$ and $c \in \mathbf{B}_\mu$. By Lemma 3.44(4), we have that

$$\mathrm{Ext}^n_\mathcal{R}(\Delta_\varepsilon(b), \nabla_\varepsilon(c)) \cong \mathrm{Ext}^n_{\mathcal{R}_{\leqslant \mu}}(i_\mu^* \Delta_\varepsilon(b), \nabla_\varepsilon(c)).$$

If $\lambda \nleqslant \mu$ then $i_\mu^* \Delta_\varepsilon(b) = 0$ and we get the desired vanishing. Now assume that $\lambda \leqslant \mu$. If $\varepsilon(\mu) = -$ then $\nabla_\varepsilon(c) = \nabla(c)$, and the result follows since $\nabla(c)$ is injective in $\mathcal{R}_{\leqslant \mu}$. So we may assume also that $\varepsilon(\mu) = +$. If $\lambda = \mu$ then $i_\mu^* \Delta_\varepsilon(b) = \Delta(b)$ which is projective in $\mathcal{R}_{\leqslant \mu}$, so again we are done. Finally, we are reduced to $\lambda < \mu$ and $\varepsilon(\mu) = +$, and need to show that $\mathrm{Ext}^n_{\mathcal{R}_{\leqslant \mu}}(\Delta_\varepsilon(b), \bar\nabla(c)) = 0$ for $n > 0$. If $n = 1$, we get the desired conclusion from Lemma 3.45. Then for $n \geqslant 2$ we use a standard degree shifting argument: let $P := i_{\leqslant \mu}^* P(b)$ be the projective cover of $L(b)$ in $\mathcal{R}_{\leqslant \mu}$. By Lemma 3.44(2), there is a short exact sequence $0 \to Q \to P \to \Delta_\varepsilon(b) \to 0$ such that Q has a Δ_ε-flag with sections $\Delta_\varepsilon(a)$ for a with $\lambda \leqslant \rho(a) \leqslant \mu$. Applying $\mathrm{Hom}_{\mathcal{R}_{\leqslant \mu}}(-, \bar\nabla(c))$ we obtain $\mathrm{Ext}^n_{\mathcal{R}_{\leqslant \mu}}(\Delta_\varepsilon(b), \bar\nabla(c)) \cong \mathrm{Ext}^{n-1}_{\mathcal{R}_{\leqslant \mu}}(Q, \bar\nabla(c))$, which is zero by induction. □

Lemma 3.47. *Suppose that $\lambda \in \Lambda$ is maximal. Assume that $V \in \mathcal{R}$ satisfies the following properties:*

(1) $\mathrm{Ext}^1_\mathcal{R}(\Delta_\varepsilon(b), V) = 0$ *for all* $b \in \mathbf{B}$*;*

(2) V is finitely cogenerated with $\mathrm{soc}\, V \cong L(b_1) \oplus \cdots \oplus L(b_n)$ *for* $b_1, \dots, b_n \in \rho^{-1}(\lambda)$.

Then we have that

$$\begin{cases} V \cong j_*^\lambda(j^\lambda V) \in \bar\nabla(\mathcal{R}) & \text{if } \varepsilon(\lambda) = +, \\ V \cong \nabla(b_1) \oplus \cdots \oplus \nabla(b_n) \in \nabla(\mathcal{R}) & \text{if } \varepsilon(\lambda) = -. \end{cases}$$

Also, the functor j_*^λ is exact in the case $\varepsilon(\lambda) = +$.

Proof. We first treat the case that $\varepsilon(\lambda) = -$. Let $W := \nabla(b_1) \oplus \cdots \oplus \nabla(b_n) \in \nabla(\mathcal{R})$. Since λ is maximal, this is an injective hull of $\mathrm{soc}\, V$, so there is a short exact sequence $0 \to V \to W \to W/V \to 0$. Apply $\mathrm{Hom}_\mathcal{R}(\Delta_\varepsilon(b), -)$ and use (1) to get a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_\mathcal{R}(\Delta_\varepsilon(b), V) \xrightarrow{f} \mathrm{Hom}_\mathcal{R}(\Delta_\varepsilon(b), W) \longrightarrow \mathrm{Hom}_\mathcal{R}(\Delta_\varepsilon(b), W/V) \longrightarrow 0.$$

If $\rho(b) < \lambda$ then $\mathrm{Hom}_\mathcal{R}(\Delta_\varepsilon(b), W) = 0$ as none of the composition factors of $\Delta_\varepsilon(b)$ are constituents of $\mathrm{soc}\, W$. If $\rho(b) = \lambda$ then $\Delta_\varepsilon(b) = \bar\Delta(b)$ and any homomorphism $\bar\Delta(b) \to W$ must factor through the unique irreducible quotient $L(b)$ of $\bar\Delta(b)$. So its image is contained in $\mathrm{soc}\, W \subseteq V$, showing that f is an isomorphism. These arguments show that $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), W/V) = 0$ for all $b \in \mathbf{B}$. In view of $(P\Delta_\varepsilon)$, it follows that $\mathrm{Hom}_{\mathcal{R}}(P(b), W/V) = 0$ for all $b \in \mathbf{B}$ too. Hence, $W/V = 0$, i.e., $V = W$ as required.

Now suppose that $\varepsilon(b) = +$. We first claim for any V satisfying properties (1)–(2) that $V \cong j_*^\lambda(j^\lambda V)$. The unit of adjunction gives us a homomorphism $g : V \to j_*^\lambda(j^\lambda V)$. Since g becomes an isomorphism when we apply j^λ, its kernel belongs to $\mathcal{R}_{<\lambda}$. In view of (2), we deduce that $\ker g = 0$, so g is a monomorphism. To show that g is an epimorphism as well, let $W := j_*^\lambda(j^\lambda V)$. Like in the previous paragraph, we apply $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), -)$ to $0 \to V \to W \to W/V \to 0$ to get a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) \xrightarrow{f} \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), W) \longrightarrow \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), W/V) \longrightarrow 0.$$

The middle morphism space is isomorphic to $\mathrm{Hom}_{\mathcal{R}_\lambda}(j^\lambda \Delta_\varepsilon(b), j^\lambda V)$, which is zero if $\rho(b) < \lambda$. If $\rho(b) = \lambda$ then $\Delta_\varepsilon(b) = \Delta(b)$ is the projective cover of $L(b)$ in $\mathcal{R}$ and $j^\lambda \Delta_\varepsilon(b)$ is the projective cover of $L_\lambda(b)$ in $\mathcal{R}_\lambda$. We deduce that both the first and second morphism spaces are of the same dimension $[V : L(b)] = [j^\lambda V : L_\lambda(b)]$, so f must be an isomorphism. These arguments show that $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), W/V) = 0$, hence, $V = W$. This proves the claim.

To complete the proof, we must show that j_*^λ is exact when $\varepsilon(\lambda) = +$. Given this result, the remaining assertion of the lemma, namely, that $j_*^\lambda(j^\lambda V) \in \bar{\nabla}(\mathcal{R})$ follows easily. To prove it, we use induction on composition length to show that j_*^λ is exact on any short exact sequence $0 \to U \to V \to W \to 0$ in $\mathcal{R}_\lambda$. For the induction step, suppose we are given such an exact sequence with $U, W \neq 0$. By induction, $j_*^\lambda U$ and $j_*^\lambda W$ both have filtrations with sections $\bar{\nabla}(b)$ for $b \in \mathbf{B}_\lambda$. Hence, by Lemma 3.46, we have that $\mathrm{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), j_*^\lambda U) = \mathrm{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), j_*^\lambda W) = 0$ for all $n \geqslant 1$ and $b \in \mathbf{B}$. As it is a right adjoint, j_*^λ is left exact, so there is an exact sequence

$$0 \longrightarrow j_*^\lambda U \longrightarrow j_*^\lambda V \longrightarrow j_*^\lambda W. \tag{3.1}$$

Let $X := j_*^\lambda V / j_*^\lambda U$, so that there is a short exact sequence

$$0 \longrightarrow j_*^\lambda U \longrightarrow j_*^\lambda V \longrightarrow X \longrightarrow 0. \tag{3.2}$$

To complete the argument, it suffices to show that $X \cong j_*^\lambda W$. To establish this, we will show in the next paragraph that X satisfies both of the properties (1)–(2). Assuming this has been established, the previous paragraph and exactness of j^λ imply that $X \cong j_*^\lambda(j^\lambda X) \cong j_*^\lambda(V/U) \cong j_*^\lambda W$, and we are done.

To see that X satisfies (1), we apply $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), -)$ to (3.2) to get an exact sequence

$$\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), j_*^\lambda V) \longrightarrow \mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), X) \longrightarrow \mathrm{Ext}^2_{\mathcal{R}}(\Delta_\varepsilon(b), j_*^\lambda U).$$

The first Ext^1 is zero by Lemma 3.45. Since we already know that the Ext^2 term is zero, $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), X) = 0$ as required. To see that X satisfies (2), note comparing (3.1)–(3.2) that $X \hookrightarrow j_*^\lambda W$. By adjunction properties, $j_*^\lambda W$ is finitely cogenerated with socle of the desired form due to what we know about its $\bar{\nabla}$-flag, thus the same holds for X. □

Lemma 3.48. *For a finitely cogenerated object $V \in \mathcal{R}$, the following are equivalent:*

- *(i)* $V \in \nabla_\varepsilon(\mathcal{R})$*;*
- *(ii)* $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$ *for all* $b \in \mathbf{B}$*;*
- *(iii)* $\mathrm{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$ *for all* $b \in \mathbf{B}$ *and* $n \geqslant 1$*.*

Moreover, the multiplicity $(V : \nabla_\varepsilon(b))$ of $\nabla_\varepsilon(b)$ as a section of a ∇_ε-flag of $V \in \nabla_\varepsilon(\mathcal{R})$ is well-defined and satisfies $(V : \nabla_\varepsilon(b)) = \dim \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V)$.

Proof. (iii)⇒(ii): Trivial.

(i)⇒(iii) and the final assertion of the lemma: These follow directly from Lemma 3.46.

(ii)⇒(i): Assume V is finitely cogenerated and satisfies (ii). We claim that the set

$$\Lambda(V) := \{\rho(b) \mid b \in \mathbf{B} \text{ such that } \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) \neq 0\}$$

is finite. This is clear in the Schurian case since V is of finite length. To see it in the locally Schurian case, suppose that soc $V \cong L(b_1) \oplus \cdots \oplus L(b_n)$. If $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) \neq 0$ then $\rho(b) \geqslant \rho(b_i)$ for some i. The claim follows since we are assuming that the poset is upper finite. We are going to show that (i) holds by induction on

$$d(V) := \sum_{b \in \mathbf{B}} \dim \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V),$$

which is finite thanks to the claim just proved. For the induction step, let $\lambda \in \Lambda(V)$ be minimal. Then it is a maximal element of $\Lambda^\downarrow := \Lambda \backslash \Lambda^\uparrow$ where

$$\Lambda^\uparrow := (\lambda, \infty) \cup \bigcup_{\lambda \neq \mu \in \Lambda(V)} [\mu, \infty).$$

Let $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$ and $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the inclusion of the corresponding Serre subcategory of $\mathcal{R}$. This is a Schurian or locally Schurian category which also satisfies $(P\Delta_\varepsilon)$ thanks to Lemma 3.44(2). Let $W := i^! V$. This is a subobject of V, so it is finitely cogenerated. Moreover, we have that $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), W) \neq 0$ only if $b \in \mathbf{B}_\lambda$, hence, soc $W \cong L(b_1) \oplus \cdots \oplus L(b_n)$ for $b_1, \ldots, b_n \in \rho^{-1}(\lambda)$. Thus, $W \in \mathcal{R}^\downarrow$ satisfies hypothesis (2) from Lemma 3.47. To see that it satisfies hypothesis (1), we apply $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), -)$ to a short exact sequence $0 \to W \to V \to Q \to 0$ to get exact sequences

$$0 \to \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), W) \to \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) \to \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), Q) \to \mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), W) \to 0,$$
$$0 \to \mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), Q) \to \mathrm{Ext}^2_{\mathcal{R}}(\Delta_\varepsilon(b), W).$$

By Lemma 2.21, Q is finitely cogenerated, and its socle has no constituent $L(b)$ for $b \in \mathbf{B}^\downarrow$. So for $b \in \mathbf{B}^\downarrow$ the space $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), Q)$ is zero, and from the first exact sequence we get that $\mathrm{Ext}^1_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), W) = 0$ for all $b \in \mathbf{B}^\downarrow$. Now we can apply Lemma 3.47 to deduce that $W \in \nabla_\varepsilon(\mathcal{R}^\downarrow)$. Hence, $W \in \nabla_\varepsilon(\mathcal{R})$.

In view of Lemma 3.46, we get that $\mathrm{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), W) = 0$ for all $n \geqslant 1$ and $b \in \mathbf{B}$. So, by the above exact sequences, we have that $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), Q) = 0$ and $d(Q) = d(V) - d(W) < d(V)$. Finally we appeal to the induction hypothesis to deduce that $Q \in \Delta_\varepsilon(\mathcal{R})$. Since we already know that $W \in \Delta_\varepsilon(\mathcal{R})$, this shows that $V \in \Delta_\varepsilon(\mathcal{R})$ as required. □

Lemma 3.49. *Let $\Lambda^\uparrow$ be an upper set in Λ and $\mathbf{B}^\uparrow := \rho^{-1}(\Lambda^\uparrow)$. Let $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the corresponding Serre quotient category of $\mathcal{R}$ equipped with the induced stratification.*

1. *The standard, costandard and indecomposable projective of $\mathcal{R}^\uparrow$ are $\Delta^\uparrow(b) := j\Delta(b)$, $\bar{\Delta}^\uparrow(b) := j\bar{\Delta}(b)$, and $P^\uparrow(b) := jP(b)$ for $b \in \mathbf{B}^\uparrow$.*
2. *For $b \in \mathbf{B}^\uparrow$, we have that $j_! \Delta^\uparrow(b) \cong \Delta(b)$, $j_! \bar{\Delta}^\uparrow(b) \cong \bar{\Delta}(b)$ and $j_! P^\uparrow(b) \cong P(b)$.*
3. *For any $b \in \mathbf{B}$, the object $jP(b)$ has a finite filtration with sections $\Delta^\uparrow_\varepsilon(c)$ for $c \in \mathbf{B}^\uparrow$ with $\rho(c) \geqslant \rho(b)$. In particular, $(P\Delta_\varepsilon)$ holds in $\mathcal{R}^\uparrow$.*
4. *$\mathrm{Ext}^n_{\mathcal{R}}(V, j_* W) \cong \mathrm{Ext}^n_{\mathcal{R}^\uparrow}(jV, W)$ for $V \in \mathcal{R}$, $W \in \nabla_\varepsilon(\mathcal{R}^\uparrow)$ and $n \geqslant 0$.*

Proof. (1)–(2) By a general property of Serre quotient functors, $P^\uparrow(b) = jP(b)$ for each $b \in \mathbf{B}^\uparrow$. Now take $b \in \mathbf{B}_\lambda$ for $\lambda \in \Lambda^\uparrow$. Let $j^\lambda : \mathcal{R}_{\leqslant \lambda} \to \mathcal{R}_\lambda$ be the quotient functor as usual, and denote the analogous functor for $\mathcal{R}^\uparrow$ by $k^\lambda : \mathcal{R}^\uparrow_{\leqslant \lambda} \to \mathcal{R}^\uparrow_\lambda$. The universal property of quotient category gives us an exact functor $\bar{\jmath} : \mathcal{R}_\lambda \to \mathcal{R}^\uparrow_\lambda$ making the diagram

$$\begin{array}{ccc} \mathcal{R}_{\leqslant\lambda} & \xrightarrow{j} & \mathcal{R}^\uparrow_{\leqslant\lambda} \\ {\scriptstyle j^\lambda}\downarrow & & \downarrow{\scriptstyle k^\lambda} \\ \mathcal{R}_\lambda & \xrightarrow[\bar{\jmath}]{} & \mathcal{R}^\uparrow_\lambda \end{array}$$

commute. In fact, $\bar{\jmath}$ is an equivalence of categories because it sends the indecomposable projective $j^\lambda P(b)$ in $\mathcal{R}_\lambda$ to the indecomposable projective $k^\lambda P^\uparrow(b)$ in $\mathcal{R}^\uparrow_\lambda$ for each $b \in \mathbf{B}_\lambda$. We deduce that there is an isomorphism of functors $j_! \circ k^\lambda_! \circ \bar{\jmath} \cong j^\lambda_!$. Applying this to $P_\lambda(b)$ and to $L_\lambda(b)$ gives that $j_!\Delta^\uparrow(b) \cong \Delta(b)$ and $j_!\bar{\Delta}^\uparrow(b) \cong \bar{\Delta}(b)$. Also by adjunction properties we have that $j_!P^\uparrow(b) \cong P(b)$. Then applying j to the isomorphisms in (2) and using $j \circ j_! \cong \mathrm{Id}_{\mathcal{R}^\uparrow}$ completes the proof of (1) too.

(3) This follows from (1) and the exactness of j, using also that $j\Delta_\varepsilon(b) = 0$ if $b \notin \mathbf{B}^\uparrow$.

(4) The adjunction gives an isomorphism $\mathrm{Hom}_{\mathcal{R}}(-, j_*W) \cong \mathrm{Hom}_{\mathcal{R}^\uparrow}(-, W) \circ j$. This proves the result when $n = 0$. For $n > 0$, the functor j is exact, so all that remains is to check that j sends projectives to objects that are acyclic for $\mathrm{Hom}_{\mathcal{R}^\uparrow}(-, W)$. This follows from (3), plus the analog of Lemma 3.46 for $\mathcal{R}^\uparrow$. □

3.5. **Lower finite stratified categories.** In this subsection, Λ is a lower finite poset, $\mathcal{R}$ is a locally finite Abelian category equipped with a stratification $\rho : \mathbf{B} \to \Lambda$, and we fix a sign function $\varepsilon : \Lambda \to \{\pm\}$. For $b \in \mathbf{B}$ define $\Delta_\varepsilon(b)$ and $\nabla_\varepsilon(b)$ as usual following (1.1)–(1.2) and let $I(b)$ be an injective hull of $L(b)$ in $\mathrm{Ind}(\mathcal{R})$ Note for $b \in \mathbf{B}_\lambda$ that $\nabla(b)$ could be defined equivalently as $i^!_{\leqslant\lambda} I(b)$. The essential axiom (S4) from Definition 3.1 ensures that this object is of finite length, i.e., it belongs to $\mathcal{R}$ rather than $\mathrm{Ind}(\mathcal{R})$. More generally,

Lemma 3.50. *Assume that Λ is a lower finite poset, $\mathcal{R}$ is a locally finite Abelian category, and $\rho : \mathbf{B} \to \Lambda$ satisfies properties (S1)–(S3) from Definition 3.1. The property (S4) holds too if and only if $i^!_{\leqslant\lambda} I(b)$ is of finite length for each $b \in \mathbf{B}$ and $\lambda \in \Lambda$.*

Proof. By properties of adjunctions, $i^!_{\leqslant\lambda} I(b)$ is an injective hull of $L(b)$ in $\mathrm{Ind}(\mathcal{R}_{\leqslant\lambda})$ for each $b \in \mathbf{B}_{\leqslant\lambda}$. If (S4) holds then this injective hull lies in $\mathcal{R}_{\leqslant\lambda}$, so is of finite length. Conversely, if all $i^!_{\leqslant\lambda} I(b)$ are of finite length then $\mathcal{R}_{\leqslant\lambda}$ has enough injectives. As Λ is lower finite, $\mathcal{R}_{\leqslant\lambda}$ is a finite Abelian category by the discussion after Corollary 2.16. This means that it has enough projectives. □

We need to consider another sort of infinite good filtration in objects of $\mathrm{Ind}(\mathcal{R})$, which we call *ascending ∇_ε-flag*. Usually (e.g., if Λ is countable), it is sufficient to restrict attention to ascending ∇_ε-flags that are given simply by a chain of subobjects $0 = V_0 < V_1 < V_2 < \cdots$ such that $V = \sum_{n \in \mathbb{N}} V_n$ and $V_m/V_{m-1} \cong \nabla_\varepsilon(b_m)$ for some $b_m \in \mathbf{B}$. Here is the general definition which avoids this restriction.

Definition 3.51. An *ascending ∇_ε-flag* in $V \in \mathrm{Ind}(\mathcal{R})$ is the data of a direct system $(V_\omega)_{\omega\in\Omega}$ of subobjects of V such that $V = \sum_{\omega\in\Omega} V_\omega$ and each V_ω has a ∇_ε-flag. *Ascending $\bar{\nabla}$-flags* (resp., *ascending ∇-flags*) mean ascending ∇_ε-flags in the special case that $\varepsilon = +$ (resp., $\varepsilon = -$).

Remark 3.52. One could also introduce the notion of a *descending Δ_ε-flag* in an object of the pro-completion $\mathrm{Pro}(\mathcal{R})$. We will not discuss this here.

Definition 3.53. We say that $\mathcal{R}$ is a *lower finite ε-stratified category* if it is a locally finite Abelian category equipped with a stratification as in Definition 3.1 such that

- the poset Λ is lower finite;
- each $I(b)$ has an ascending ∇_ε-flag $(V_\omega)_{\omega\in\Omega}$ such that for every $\omega \in \Omega$ there is a ∇_ε-flag in V_ω which only involves sections $\nabla_\varepsilon(c)$ for c with $\rho(c) \geqslant \rho(b)$.

We are going to adapt Theorem 3.17 to such categories.

Lemma 3.54. *Suppose $\mathcal{R}$ is a lower finite ε-stratified category and take $b, c \in \mathbf{B}$ with $\rho(b) \nleqslant \rho(c)$. Then we have that Then* $\operatorname{Ext}^1_{\mathcal{R}}(\nabla_\varepsilon(c), \nabla_\varepsilon(b)) = 0$.

Proof. Consider a minimal injective resolution $0 \to \nabla_\varepsilon(b) \to I_0 \to I_1 \to \cdots$ in $\operatorname{Ind}(\mathcal{R})$. We have that $I_0 = I(b)$ since $\nabla_\varepsilon(b)$ has irreducible socle $L(b)$. Let $(V_\omega)_{\omega\in\Omega}$ be an ascending ∇_ε-flag in $I(b)$ such that each V_ω is non-zero and has a ∇_ε-flag which only involves sections $\nabla_\varepsilon(a)$ for a with $\rho(a) \geqslant \rho(b)$. Then $\nabla_\varepsilon(b) \hookrightarrow V_\omega$ for every $\omega \in \Omega$, so $\nabla_\varepsilon(b) \hookrightarrow I(b) = \sum_{\omega\in\Omega} V_\omega$. Moreover, $I(b)/\nabla_\varepsilon(b) = \sum_{\omega\in\Omega}(V_\omega/\nabla_\varepsilon(b))$, so its socle only involves constituents $L(a)$ with $\rho(a) \geqslant \rho(b)$. So I_1 is a direct sum of $I(a)$ with $\rho(a) \geqslant \rho(b)$. The Ext^1-vanishing now follows on applying $\operatorname{Hom}_{\mathcal{R}}(\nabla_\varepsilon(c), -)$ to the resolution and taking homology. □

Here is the first half of the analog of Theorem 3.17.

Theorem 3.55. *Suppose $\mathcal{R}$ is a lower finite ε-stratified category. Let $\Lambda^\downarrow$ be a finite lower set, $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$, and $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the corresponding Serre subcategory of $\mathcal{R}$ with the induced stratification. Then $\mathcal{R}^\downarrow$ is a finite ε-stratified category with distinguished objects $L^\downarrow(b) \cong L(b)$, $I^\downarrow(b) \cong i^! I(b)$, $\Delta^\downarrow(b) \cong \Delta(b)$, $\bar\Delta^\downarrow(b) \cong \bar\Delta(b)$, $\nabla^\downarrow(b) \cong \nabla(b)$ and $\bar\nabla^\uparrow(b) \cong \bar\nabla(b)$ for $b \in \mathbf{B}^\downarrow$.*

Proof. The identification of the distinguished objects of $\mathcal{R}^\downarrow$ is straightforward. In particular, the objects $\nabla_\varepsilon(b)$ in $\mathcal{R}^\downarrow$ are just the same as the ones in $\mathcal{R}$ indexed by $b \in \mathbf{B}^\downarrow$, while the indecomposable injectives in $\operatorname{Ind}(\mathcal{R}^\downarrow)$ are the objects $i^! I(b)$ for $b \in \mathbf{B}^\downarrow$. To complete the proof, we need to prove the following for each $b \in \mathbf{B}^\downarrow$:

(1) $i^! I(b)$ has finite length, i.e., it actually lies in $\mathcal{R}^\downarrow$;
(2) $i^! I(b)$ satisfies the property $(I\nabla_\varepsilon)$.

The first of these implies that $\mathcal{R}^\downarrow$ is a locally finite Abelian category with finitely many isomorphism classes of irreducible objects and with enough injectives. Hence, $\mathcal{R}^\downarrow$ is a finite Abelian category by the discussion after Corollary 2.16. Then (2) checks that it is ε-stratified as in Definition 3.7.

To prove (1)–(2), take $b \in \mathbf{B}^\downarrow$. Let $(V_\omega)_{\omega\in\Omega}$ be an ascending ∇_ε-flag in $I(b)$ as in Definition 3.53, and fix also a ∇_ε-flag in each V_ω which only involves sections $\nabla_\varepsilon(c)$ for c with $\rho(c) \geqslant \rho(b)$. For $\lambda \in \Lambda^\downarrow$, let $m(\lambda, \omega)$ be the sum of the multiplicities of the objects $\nabla_\varepsilon(c)$ $(c \in \mathbf{B}_\lambda)$ as sections of the ∇_ε-flag of V_ω. Let $m(\lambda) := \sup\{m(\lambda, \omega) \mid \omega \in \Omega\}$. We claim that $m(\lambda) < \infty$. To see this, suppose for a contradiction that it is not the case. Choose λ minimal so that $m(\lambda) = \infty$. Then for any $n \in \mathbb{N}$, we can find $\omega \in \Omega$ such that the sum of the multiplicities of the objects $\nabla_\varepsilon(c)$ $(c \in \mathbf{B}_\lambda)$ as sections of the ∇_ε-flag of V_ω is greater than n. Using Lemma 3.50 and the minimality of λ, we can rearrange this flag if necessary so that the only other sections appearing *below* these ones are of the form $\nabla_\varepsilon(d)$ for $d \in \mathbf{B}_{<\lambda}$. Then we deduce that $\sum_{c\in\mathbf{B}_\lambda}[i^!_{\leqslant\lambda} V_\omega : L(c)] > n$ too. This contradicts the property from Lemma 3.50 that $i^!_{\leqslant\lambda} I(b)$ has finite length.

Using Lemma 3.54 again, we can rearrange the ∇_ε-flag in each V_ω if necessary to deduce that there are short exact sequences $0 \to V'_\omega \to V_\omega \to V''_\omega \to 0$ such that V'_ω has a ∇_ε-flag with sections $\nabla_\varepsilon(c)$ for c with $\rho(b) \leqslant \rho(c) \in \Lambda^\downarrow$ and V''_ω has a ∇_ε-flag with sections $\nabla_\varepsilon(d)$ for d with $\rho(d) \notin \Lambda^\downarrow$. Moreover, the finiteness property established in the previous paragraph means that the length of the ∇_ε-flag of V'_ω is bounded by

$\sum_{\lambda \in \Lambda^\downarrow} m(\lambda)$ independent of $\omega \in \Omega$. Consequently, we can find some sufficiently large ω in the directed set Ω so that $V'_\upsilon = V'_\omega$ for all $\upsilon > \omega$. Then $i^! I(b) = V'_\omega$ for this ω. □

Corollary 3.56. *In a lower finite ε-stratified category $\mathcal{R}$, we have for each $b, c \in \mathbf{B}$,*

$$\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), \nabla_\varepsilon(c)) = 0.$$

Proof. Given b, c, let $\Lambda^\downarrow$ be the finite lower set of Λ that they generate. Let $\mathcal{R}^\downarrow$ be the corresponding finite ε-stratified subcategory of $\mathcal{R}$. Since $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), \nabla_\varepsilon(c)) = \mathrm{Ext}^1_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), \nabla_\varepsilon(c))$, it is zero thanks to Theorem 3.11. □

Suppose that $V \in \mathrm{Ind}(\mathcal{R})$ has an ascending ∇_ε-flag $(V_\omega)_{\omega \in \Omega}$. Corollary 3.56 implies as usual that the multiplicity $(V_\omega : \nabla_\varepsilon(b))$ of $\nabla_\varepsilon(b)$ as a section of an ∇_ε-flag of V_ω is well-defined for every ω. Since $\Delta_\varepsilon(b)$ is finitely generated, we have that

$$\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), \varinjlim V_\omega) \cong \varinjlim \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V_\omega).$$

We deduce that

$$(V : \nabla_\varepsilon(b)) := \sup\left\{(V_\omega : \nabla_\varepsilon(b)) \,\middle|\, \omega \in \Omega\right\} = \dim \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) \in \mathbb{N} \cup \{\infty\}, \qquad (3.3)$$

which is independent of the particular choice of ∇_ε-flag. Having made sense of these multiplicities, we let $\nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ be the exact subcategory consisting of all objects V that possess an ascending ∇_ε-flag such that $(V : \nabla_\varepsilon(b)) < \infty$ for all $b \in \mathbf{B}$. We write $\nabla^{\mathrm{asc}}(\mathcal{R})$ and $\bar{\nabla}^{\mathrm{asc}}(\mathcal{R})$ for $\nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ in the special cases $\varepsilon = -$ and $\varepsilon = +$, respectively.

Corollary 3.57. *Assume that $\mathcal{R}$ is a lower finite ε-stratified category. For $V \in \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ and $b \in \mathbf{B}$, we have that $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$.*

Proof. Let $(V_\omega)_{\omega \in \Omega}$ be an ascending ∇_ε-flag in V. Take an extension $V \hookrightarrow E \twoheadrightarrow \Delta_\varepsilon(b)$. We can find a subobject E_1 of E of finite length such that $V + E_1 = V + E$; this follows easily by induction on the length of $\Delta_\varepsilon(b)$ as explained at the start of the proof of [CPS1, Lemma 3.8(a)]. Since $V \cap E_1$ is of finite length, there exists $\omega \in \Omega$ with $V \cap E_1 \subseteq V_\omega$. Then we have that $V \cap E_1 = V_\omega \cap E_1$ and

$$(V_\omega + E_1)/V_\omega \cong E_1/V_\omega \cap E_1 = E_1/V \cap E_1 \cong (V + E_1)/V = (V + E)/V \cong \Delta_\varepsilon(b).$$

Thus, there is a short exact sequence $0 \to V_\omega \to V_\omega + E_1 \to \Delta_\varepsilon(b) \to 0$. By Corollary 3.56, this splits, so we can find a subobject $E_2 \cong \Delta_\varepsilon(b)$ of $V_\omega + E_1$ such that $V_\omega + E_1 = V_\omega \oplus E_2$. Then $V + E = V + E_1 = V + V_\omega + E_1 = V + V_\omega + E_2 = V + E_2 = V \oplus E_2$, and our original short exact sequence splits. □

Corollary 3.58. *In the notation of Theorem 3.55, if $V \in \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ then $i^! V \in \nabla_\varepsilon(\mathcal{R}^\downarrow)$.*

Proof. Take a short exact sequence $0 \to i^! V \to V \to Q \to 0$. Note that

$$\mathrm{Hom}_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), i^! V) \cong \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V)$$

is finite-dimensional for each $b \in \mathbf{B}^\downarrow$. Since $\mathcal{R}^\downarrow$ is finite Abelian, it follows that $i^! V \in \mathcal{R}^\downarrow$ (rather than $\mathrm{Ind}(\mathcal{R}^\downarrow)$). Moreover, $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), Q) = 0$ for $b \in \mathbf{B}^\downarrow$. So, on applying $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), -)$ and considering the long exact sequence using Corollary 3.57, we get that $\mathrm{Ext}^1_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), i^! V) = \mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), i^! V) = 0$ for all $b \in \mathbf{B}^\downarrow$. Thus, by Theorem 3.11, we have that $i^! V \in \nabla_\varepsilon(\mathcal{R}^\downarrow)$. □

The following homological criterion for ascending ∇_ε-flags is similar to [Jan1, Proposition II.4.16]. It generalizes Theorem 3.11.

Theorem 3.59. *Assume that $\mathcal{R}$ is a lower finite ε-stratified category. For $V \in \mathrm{Ind}(\mathcal{R})$ such that $\dim \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) < \infty$ for all $b \in \mathbf{B}$, the following are equivalent:*

(i) V has an ascending ∇_ε-flag;

(ii) $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0)$ *for all* $b \in \mathbf{B}$*;*
(iii) $\mathrm{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0)$ *for all* $b \in \mathbf{B}$ *and* $n \geqslant 1$*.*

Proof. (ii)⇒(i): Let Ω be the directed set consisting of all finite lower sets in Λ. Take $\omega \in \Omega$. It is a finite lower set $\Lambda^\downarrow \subseteq \Lambda$, so we can associate a corresponding finite ε-stratified subcategory $\mathcal{R}^\downarrow$ as in Theorem 3.55. Letting $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the inclusion, we set $V_\omega := i^! V$. By Corollary 3.58, we have that $V_\omega \in \nabla_\varepsilon(\mathcal{R})$. So we have the required data $(V_\omega)_{\omega \in \Omega}$ of an ascending ∇_ε-flag in V. Finally, we let $V' := \sum_{\omega \in \Omega} V_\omega$ and complete the proof by showing that $V = V'$. To see this, apply $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), -)$ to the short exact sequence $0 \to V' \to V \to V/V' \to 0$ using Corollary 3.57, to deduce that there is a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V') \longrightarrow \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) \longrightarrow \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V/V') \longrightarrow 0$$

for every $b \in \mathbf{B}$. But any homomorphism $\Delta_\varepsilon(b) \to V$ has image contained in V_ω for sufficiently large ω, hence, also in V'. Thus the first morphism in this short exact sequence is an isomorphism, and $\mathrm{Hom}_{\mathcal{R}}(\Delta(b), V/V') = 0$ for all $b \in \mathbf{B}$. This implies that $V/V' = 0$ as required.

(i)⇒(ii): This is Corollary 3.57.

(iii)⇒(ii): Trivial.

(i)⇒(iii): This follows from Theorem 3.61(2). The forward reference causes no issues since we will only appeal to the equivalence of (i) and (ii) prior to that point. □

Corollary 3.60. *Let* $0 \to U \to V \to W \to 0$ *be a short exact sequence in a lower finite* ε*-stratified category. If* $U, V \in \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ *then* $W \in \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ *too. Moreover*

$$(V : \nabla_\varepsilon(b)) = (U : \nabla_\varepsilon(b)) + (W, \nabla_\varepsilon(b)).$$

The second half of our analog of Theorem 3.17 is as follows.

Theorem 3.61. *Suppose* $\mathcal{R}$ *is a lower finite* ε*-stratified category. Let* $\Lambda^\downarrow$ *be a finite lower set,* $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda)$*, and* $i : \mathcal{R}^\downarrow \to \mathcal{R}$ *be the inclusion of the corresponding finite* ε*-stratified subcategory of* $\mathcal{R}$ *as in Theorem 3.55.*

(1) $(\mathbb{R}^n i^!)V = 0$ *for* $n \geqslant 1$ *and either* $V \in \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ *or* $V \in \mathcal{R}^\downarrow$*.*
(2) $\mathrm{Ext}^n_{\mathcal{R}}(iV, W) \cong \mathrm{Ext}^n_{\mathcal{R}^\downarrow}(V, i^! W)$ *for* $V \in \mathcal{R}^\downarrow, W \in \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ *and all* $n \geqslant 0$*.*
(3) $\mathrm{Ext}^n_{\mathcal{R}}(iV, iW) \cong \mathrm{Ext}^n_{\mathcal{R}^\downarrow}(V, W)$ *for* $V, W \in \mathcal{R}^\downarrow$ *and all* $n \geqslant 0$*.*

Proof. (1) Assume first that $V \in \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$. Let I be an injective hull of soc V. Note that I is of the form $\bigoplus_{a \in \mathbf{B}} I(a)^{\oplus n_a}$ for $0 \leqslant n_a \leqslant (V : \nabla_\varepsilon(a)) < \infty$. It has an ascending ∇_ε-flag. Moreover, $\dim \mathrm{Hom}_{\mathcal{R}}(\Delta(b), I) = \sum_{a \in \mathbf{B}} n_a [\Delta_\varepsilon(b) : L(a)] < \infty$, hence, $I \in \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$.

Now consider the short exact sequence $0 \to V \to I \to Q \to 0$. By Corollary 3.60, we have that $Q \in \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ too. Applying $i^!$ and considering the long exact sequence, we see that to prove that $(\mathbb{R}^1 i^!)V = 0$ it suffices to show that the last morphism in the exact sequence $0 \to i^! V \to i^! I \to i^! Q$ is an epimorphism. Once that has been proved we can use degree shifting to establish the desired vanishing for all higher n; it is important for the induction step that we have already established that $Q \in \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ just like V.

To prove the surjectivity, look at $0 \to i^! I / i^! V \to i^! Q \to C \to 0$. Both $i^! I$ and $i^! V$ have ∇_ε-flags by Corollary 3.58. Hence, so does $i^! I / i^! V$, and on applying $\mathrm{Hom}_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), -)$ for $b \in \mathbf{B}^\downarrow$, we get a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), i^! I / i^! V) \longrightarrow \mathrm{Hom}_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), i^! Q) \longrightarrow \mathrm{Hom}_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), C) \longrightarrow 0.$$

The first space here has dimension

$$(i^! I : \nabla_\varepsilon(b)) - (i^! V : \nabla_\varepsilon(b)) = (I : \nabla_\varepsilon(b)) - (V : \nabla_\varepsilon(b)) = (Q : \nabla_\varepsilon(b)) = (i^! Q : \nabla_\varepsilon(b)),$$

which is the dimension of the second space. This shows that the first morphism is an isomorphism. Hence, $\mathrm{Hom}_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), C) = 0$. This implies that $C = 0$ as required.

Finally let $V \in \mathcal{R}^\downarrow$. Then V is of finite length, so it suffices just to consider the case that $V = L(b)$ for $b \in \mathbf{B}^\downarrow$. Then we consider the short exact sequence $0 \to L(b) \to \nabla_\varepsilon(b) \to Q \to 0$. Applying $i^!$ and using the vanishing established so far gives $0 \to i^! L(b) \to i^! \nabla_\varepsilon(b) \to i^! Q \to (\mathbb{R}^1 i^!) L(b) \to 0$ and isomorphisms $(\mathbb{R}^n i^!) Q \cong (\mathbb{R}^{n+1} i^!) L(b)$ for $n \geqslant 1$. But $i^!$ is the identity on $L(b), \nabla_\varepsilon(b)$ and Q, so this immediately yields $(\mathbb{R}^1 i^!) L(b) = 0$, and then $(\mathbb{R}^n i^!) L(b) = 0$ for higher n by degree shifting.

(2), (3) These follow by the usual Grothendieck spectral sequence argument starting from the adjunction isomorphism $\mathrm{Hom}_{\mathcal{R}^\downarrow}(iV, -) \cong \mathrm{Hom}_{\mathcal{R}}(V, -) \circ i^!$. One just needs (1) and the observation that $i^!$ sends injectives to injectives. □

The following is an alternate characterization of "lower finite ε-stratified category."

Lemma 3.62. *Let $\mathcal{R}$ be a locally finite Abelian category with a stratification and a sign function as usual. Assume the poset Λ is lower finite. Then $\mathcal{R}$ is a lower finite ε-stratified category if and only if the Serre subcategory $\mathcal{R}^\downarrow$ of $\mathcal{R}$ associated to $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$ is a finite ε-stratified category with the induced stratification for each finite lower set $\Lambda^\downarrow$.*

Proof. ($\Rightarrow$): This is the content of Theorem 3.55.

($\Leftarrow$): Assume each $\mathcal{R}^\downarrow$ is a finite ε-stratified category. Then we can repeat the proof of the implication (ii)$\Rightarrow$(i) of Theorem 3.59 in the given category $\mathcal{R}$; the arguments given above only actually used the conclusions of Theorem 3.55 (which we are assuming) rather than Definition 3.53. Since $I(b)$ satisfies the homological criterion of Theorem 3.59(ii), we deduce that $I(b) \in \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$. Moreover, $(I(b) : \nabla_\varepsilon(c)) = [\Delta_\varepsilon(c) : L(b)]$ which is zero unless $\rho(c) \geqslant \rho(b)$. Hence, $\mathcal{R}$ is a lower finite ε-stratified category. □

Corollary 3.63. *$\mathcal{R}$ is a lower finite ε-stratified category if and only if $\mathcal{R}^{\mathrm{op}}$ is a lower finite $(-\varepsilon)$-stratified category.*

Proof. This follows from Lemma 3.62 plus Lemma 3.9 which treats the finite case. □

Finally, we record the definitions of the various special cases. All of the results above specialize in an obvious way to these cases.

Definition 3.64. Let $\mathcal{R}$ be a locally Abelian category equipped with a stratification $\rho : \mathbf{B} \to \Lambda$ such that the poset Λ is lower finite. Adopt the usual notation for its various distinguished objects. We say that $\mathcal{R}$ is

- a *lower finite standardly stratified category* if each $I(b)$ has an ascending $\bar\nabla$-flag involving sections $\bar\nabla(c)$ for c with $\rho(c) \geqslant \rho(b)$;
- a *lower finite costandardly stratified category* if each $I(b)$ has an ascending ∇-flag involving sections $\nabla(c)$ for c with $\rho(c) \geqslant \rho(b)$;
- a *lower finite fully stratified category* if each $I(b)$ has an ascending ∇-flag involving sections $\nabla(c)$ for c with $\rho(c) \geqslant \rho(b)$ and each $\nabla(b)$ has a $\bar\nabla$-flag with sections $\bar\nabla(c)$ for $\rho(c) = \rho(b)$;
- a *lower finite highest weight category* if all strata are simple (cf. Lemma 3.5) and each $I(\lambda)$ has an ascending ∇-flag with sections $\nabla(\mu)$ for $\mu \geqslant \lambda$.

Note that $\mathcal{R}$ is a lower finite highest weight category in the sense of Definition 3.64 if and only if $\mathrm{Ind}(\mathcal{R})$ is a highest weight category with weight poset that is lower finite in the original sense of [CPS1].

4. Tilting modules and semi-infinite Ringel duality

We now develop the theory of tilting objects and Ringel duality. Even in the finite case, we are not aware of a complete exposition of these results in the existing literature in the general ε-stratified setting.

4.1. **Tilting objects in the finite and lower finite cases.** In this subsection, $\mathcal{R}$ will be a finite or lower finite ε-stratified category with stratification $\rho : \mathbf{B} \to \Lambda$ and sign function $\sigma : \Lambda \to \{\pm\}$; see Definitions 3.7 and 3.53. By an *ε-tilting object*, we mean an object of the following full subcategory of $\mathcal{R}$:

$$\operatorname{Tilt}_\varepsilon(\mathcal{R}) := \Delta_\varepsilon(\mathcal{R}) \cap \nabla_\varepsilon(\mathcal{R}). \tag{4.1}$$

When $\varepsilon = +$ (resp., $\varepsilon = -$), we may simply refer to ε-tilting objects as *tilting objects* (resp., *cotilting objects*). The following shows that $\operatorname{Tilt}_\varepsilon(\mathcal{R})$ is a Karoubian subcategory of $\mathcal{R}$.

Lemma 4.1. *Direct summands of ε-tilting objects are ε-tilting objects.*

Proof. This follows easily from Corollaries 3.13 and 3.16. In the lower finite case, one needs to pass first to a finite ε-stratified subcategory containing the object in question using Theorem 3.55. □

The next goal is to construct and classify ε-tilting objects. Our exposition is based roughly on [Do4, Appendix], which in turn goes back to the work of Ringel [Rin]. There are some additional complications in the ε-stratified setting.

Theorem 4.2. *Assume that $\mathcal{R}$ is a finite or lower finite ε-stratified category. For $b \in \mathbf{B}$ with $\rho(b) = \lambda$, there is an indecomposable object $T_\varepsilon(b) \in \operatorname{Tilt}_\varepsilon(\mathcal{R})$ satisfying the following properties:*

(i) $T_\varepsilon(b)$ has a Δ_ε-flag with bottom section isomorphic to $\Delta_\varepsilon(b)$;
(ii) $T_\varepsilon(b)$ has a ∇_ε-flag with top section isomorphic to $\nabla_\varepsilon(b)$;
(iii) $T_\varepsilon(b) \in \mathcal{R}_{\leqslant\lambda}$ and $j^\lambda T_\varepsilon(b) \cong \begin{cases} P_\lambda(b) & \text{if } \varepsilon(\lambda) = + \\ I_\lambda(b) & \text{if } \varepsilon(\lambda) = - \end{cases}$.

These properties determine $T_\varepsilon(b)$ uniquely up to isomorphism: if U is any indecomposable object of $\operatorname{Tilt}_\varepsilon(\mathcal{R})$ satisfying any one of the properties (i)–(iii) then $U \cong T_\varepsilon(b)$; hence, it satisfies the other two properties as well.

Proof. By replacing $\mathcal{R}$ by the Serre subcategory associated to a sufficiently large but finite lower set $\Lambda^\downarrow$ in Λ, chosen so as to contain λ and (for the uniqueness statement) all $\rho(b)$ for b such that $[T : L(b)] \neq 0$, one reduces to the case that $\mathcal{R}$ is a finite ε-stratified category. This reduction depends only on Theorem 3.55. Thus, we may assume henceforth that Λ is finite.

Existence: The main step is to construct an indecomposable object $T_\varepsilon(b) \in \operatorname{Tilt}_\varepsilon(\mathcal{R})$ such that (iii) holds. The argument for this proceeds by induction on $|\Lambda|$. If $\lambda \in \Lambda$ is minimal, we set $T_\varepsilon(b) := \Delta(b)$ if $\varepsilon(\lambda) = +$ or $\nabla(b)$ if $\varepsilon(\lambda) = -$. Since $\bar\Delta(b) = L(b) = \bar\nabla(b)$ by the minimality of λ, this has both a Δ_ε- and a ∇_ε-flag. It is indecomposable, and we get (iii) from Lemma 2.19.

For the induction step, suppose that λ is not minimal and pick $\mu < \lambda$ that is minimal. Let $\Lambda^\uparrow := \Lambda \backslash \{\mu\}, \mathbf{B}^\uparrow := \rho^{-1}(\Lambda^\uparrow)$, and $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the corresponding Serre quotient. By induction, there is an indecomposable object $T_\varepsilon^\uparrow(b) \in \operatorname{Tilt}_\varepsilon(\mathcal{R}^\uparrow)$ satisfying the analogue of (iii). Now there are two cases according to whether $\varepsilon(\mu) = +$ or $-$.

Case $\varepsilon(\mu) = +$: For any $V \in \mathcal{R}$, let $d_+(V) := \sum_{c \in \mathbf{B}_\mu} \dim \operatorname{Ext}^1_{\mathcal{R}}(\Delta(c), V)$. We recursively construct $n \geqslant 0$ and $T_0, T_1, \dots, T_n$ so that $d_+(T_0) > d_+(T_1) > \cdots > d_+(T_n) = 0$ and the following properties hold for all m:

(1) $T_m \in \Delta_\varepsilon(\mathcal{R})$;
(2) $j^\lambda T_m \cong P_\lambda(b)$ if $\varepsilon(\lambda) = +$ or $I_\lambda(b)$ if $\varepsilon(\lambda) = -$;
(3) $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), T_m) = 0$ for all $a \in \mathbf{B} \backslash \mathbf{B}_\mu$.

To start with, set $T_0 := j_! T^\uparrow_\varepsilon(b)$. This satisfies all of the above properties: (1) follows from Theorem 3.18(6); (2) follows because j^λ factors through j and we know that $T^\uparrow_\varepsilon(b)$ satisfies the analogous property; (3) follows by Theorem 3.18(5). For the recursive step, assume that we are given T_m satisfying (1)–(3) and $d_+(T_m) > 0$. We can find $c \in \mathbf{B}_\mu$ and a non-split extension

$$0 \longrightarrow T_m \longrightarrow T_{m+1} \longrightarrow \Delta(c) \longrightarrow 0. \tag{4.2}$$

This constructs T_{m+1}. We claim that $d_+(T_{m+1}) < d_+(T_m)$ and that T_{m+1} satisfies (1)–(3). Part (1) is clear from the definition. For (2), we just apply the exact functor j^λ to the exact sequence (4.2), noting that $j^\lambda \Delta(c) = 0$. For (3), take $a \in \mathbf{B} \backslash \mathbf{B}_\mu$ and apply the functor $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), -)$ to the short exact sequence (4.2) to get

$$\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), T_m) \longrightarrow \mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), T_{m+1}) \longrightarrow \mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), \Delta(c)).$$

The first and last term are zero by hypothesis and by Lemma 3.10 which implies $\mathrm{Ext}^1_{\mathcal{R}}(T_{m+1}, \nabla_\varepsilon(a)) = 0$. It remains to show $d_+(T_{m+1}) < d_+(T_m)$. For $a \in \mathbf{B}_\mu$, we have $\mathrm{Ext}^1_{\mathcal{R}}(\Delta(a), \Delta(c)) = 0$ by Lemma 3.10, so again we have an exact sequence

$$\mathrm{Hom}_{\mathcal{R}}(\Delta(a), \Delta(c)) \xrightarrow{f} \mathrm{Ext}^1_{\mathcal{R}}(\Delta(a), T_m) \longrightarrow \mathrm{Ext}^1_{\mathcal{R}}(\Delta(a), T_{m+1}) \longrightarrow 0.$$

This shows that $\dim \mathrm{Ext}^1_{\mathcal{R}}(\Delta(a), T_{m+1}) \leqslant \dim \mathrm{Ext}^1_{\mathcal{R}}(\Delta(a), T_m)$, and we just need to observe that the inequality is actually a strict one in the case $a = c$. To see this, note that the first morphism f is non-zero in the case $a = c$ as $f(\mathrm{id}_{\Delta(c)}) \neq 0$ due to the fact that the original short exact sequence was not split. This completes the claim. We have now defined an object $T_n \in \Delta_\varepsilon(\mathcal{R})$ such that $j^\lambda T_n \cong P_\lambda(b)$ if $\varepsilon(\lambda) = +$ or $I_\lambda(b)$ if $\varepsilon(\lambda) = -$, and moreover $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), T_n) = 0$ for all $a \in \mathbf{B}$. By Theorem 3.11, we deduce that $T_n \in \nabla_\varepsilon(\mathcal{R}_{\leqslant\lambda})$ too, hence, it is an ε-tilting object. Decompose T_n into indecomposables $T_n = T_{n,1} \oplus \cdots \oplus T_{n,r}$. Then each $T_{n,i}$ is also an ε-tilting object. Since $j^\lambda T_n$ is indecomposable, we must have that $j^\lambda T_n = j^\lambda T_{n,i}$ for some unique i. Then we set $T_\varepsilon(b) := T_{n,i}$ for this i. This gives us the desired indecomposable ε-tilting object.

Case $\varepsilon(\mu) = -$: Consider $d_-(V) := \sum_{c \in \mathbf{B}_\mu} \dim \mathrm{Ext}^1_{\mathcal{R}}(V, \nabla(c))$. We recursively construct $T_0, T_1, \dots, T_n$ so that $d_-(T_0) > \cdots > d_-(T_n) = 0$ and

(1′) $T_m \in \nabla_\varepsilon(\mathcal{R})$;
(2′) $j^\lambda T_m \cong P_\lambda(b)$ if $\varepsilon(\lambda) = +$ or $I_\lambda(b)$ if $\varepsilon(\lambda) = -$;
(3′) $\mathrm{Ext}^1_{\mathcal{R}}(T_m, \nabla_\varepsilon(a)) = 0$ for all $a \in \mathbf{B} \backslash \mathbf{B}_\mu$.

We start from $T_0 := j_* T^\uparrow_\varepsilon(b)$. For the recursive step, assume that we are given T_m satisfying (1′)–(3′) and with $d_-(T_m) > 0$. We can find $c \in \mathbf{B}_\mu$ and a non-split extension

$$0 \longrightarrow \nabla(c) \longrightarrow T_{m+1} \longrightarrow T_m \longrightarrow 0.$$

This constructs T_{m+1}, and it remains to check that $d_-(T_{m+1}) < d_-(T_m)$ and that T_{m+1} satisfies (1′)–(3′). This is similar to before. For (3′), take $a \in \mathbf{B} \backslash \mathbf{B}_\mu$ and apply the functor $\mathrm{Hom}_{\mathcal{R}}(-, \nabla_\varepsilon(a))$ to get

$$\mathrm{Ext}^1_{\mathcal{R}}(T_m, \nabla_\varepsilon(a)) \longrightarrow \mathrm{Ext}^1_{\mathcal{R}}(T_{m+1}, \nabla_\varepsilon(a)) \longrightarrow \mathrm{Ext}^1_{\mathcal{R}}(\nabla(c), \nabla_\varepsilon(a)).$$

We deduce that $\mathrm{Ext}^1_{\mathcal{R}}(T_{m+1}, \nabla_\varepsilon(a)) = 0$, and it remains to show that $d_-(T_{m+1}) < d_-(T_m)$. For $a \in \mathbf{B}_\mu$, we have an exact sequence

$$\mathrm{Hom}_{\mathcal{R}}(\nabla(c), \nabla(a)) \xrightarrow{f} \mathrm{Ext}^1_{\mathcal{R}}(T_m, \nabla(a)) \longrightarrow \mathrm{Ext}^1_{\mathcal{R}}(T_{m+1}, \nabla(a)) \longrightarrow 0.$$

This shows that $\dim \mathrm{Ext}^1_{\mathcal{R}}(T_{m+1}, \nabla(a)) \leqslant \dim \mathrm{Ext}^1_{\mathcal{R}}(T_m, \nabla(a))$, and like before the inequality is actually a strict one in the case $a = c$. This completes the recursion. Then, we decompose T_n into indecomposables $T_n = T_{n,1} \oplus \cdots \oplus T_{n,r}$. By Theorem 3.14 each $T_{n,i}$ is an ε-tilting object. Since $j^\lambda T_n$ is indecomposable, we must have that $j^\lambda T_n = j^\lambda T_{n,i}$ for some unique i, and finally set $T_\varepsilon(b) := T_{n,i}$ for this i.

This completes the construction of $T_\varepsilon(b)$ in general. We have shown it satisfies (iii). Let us show that it also satisfies (i) and (ii). For (i), we know by (iii) that $T_\varepsilon(b)$ belongs to $\mathcal{R}_{\leqslant\lambda}$, and it has a Δ_ε-flag. By Lemma 3.10, we may order this flag so that the sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}_\lambda$ appear at the bottom. Thus, there is a short exact sequence $0 \to K \to T_\varepsilon(b) \to Q \to 0$ such that K has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}_\lambda$ and $j^\lambda Q = 0$. Then $j^\lambda K \cong j^\lambda T_\varepsilon(b)$. If $\varepsilon(\lambda) = +$, this is $P_\lambda(b)$. Since j^λ is exact and $j^\lambda \Delta(c) = P_\lambda(c)$ for each $c \in \mathbf{B}_\lambda$, we must have that $K \cong \Delta(b)$, and (1) follows. Instead, if $\varepsilon(\lambda) = -$, the bottom section of the $\bar{\nabla}$-flag of K must be $\bar{\nabla}(b)$ since $j^\lambda K \cong I_\lambda(b)$ has irreducible socle $L_\lambda(b)$, giving (i) in this case too. The proof of (ii) is similar.

Uniqueness: Let $T := T_\varepsilon(b)$ and U be some other indecomposable object of $\mathrm{Tilt}_\varepsilon(\mathcal{R})$ satisfying one of the properties (i)–(iii). We must prove that $T \cong U$. By the argument from the previous paragaph, we may assume actually that U satisfies either (i) or (ii). We just explain how to see this in the case that U satisfies (i); the dual argument treats the case that U satisfies (ii). So there are short exact sequences $0 \to \Delta_\varepsilon(b) \xrightarrow{f} U \to Q_1 \to 0$ and $0 \to \Delta_\varepsilon(b) \xrightarrow{g} T \to Q_2 \to 0$ such that Q_1, Q_2 have Δ_ε-flags. Applying $\mathrm{Hom}_{\mathcal{R}}(-, T)$ to the first and using $\mathrm{Ext}^1_{\mathcal{R}}(Q_1, T) = 0$, we get that $\mathrm{Hom}_{\mathcal{R}}(U, T) \twoheadrightarrow \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), T)$. Hence, g extends to a homomorphism $\bar{g} : U \to T$. Similarly, f extends to $\bar{f} : T \to U$. We have constructed morphisms making the triangles in the following diagram commute:

$$\begin{array}{ccc} & \overset{f}{\nearrow} & U \\ \Delta_\varepsilon(b) & & \bar{g}\downarrow\uparrow\bar{f} \\ & \underset{g}{\searrow} & T \end{array}$$

Since $\bar{f} \circ \bar{g} \circ f = f$, we deduce that $\bar{f} \circ \bar{g}$ is not nilpotent. Since U is indecomposable, Fitting's Lemma implies $\bar{f} \circ \bar{g}$ is an isomorphism. Similarly, so is $\bar{g} \circ \bar{f}$. Hence, $U \cong T$. $\square$

Remark 4.3. Let $b \in \mathbf{B}_\lambda$. When $\varepsilon(\lambda) = +$, Theorem 4.2 implies that $(T_\varepsilon(b) : \Delta_\varepsilon(b)) = 1$ and $(T_\varepsilon(b) : \Delta_\varepsilon(c)) = 0$ for all other $c \in \mathbf{B}_\lambda$. Similarly, when $\varepsilon(\lambda) = -$, we have that $(T_\varepsilon(b) : \nabla_\varepsilon(b)) = 1$ and $(T_\varepsilon(b) : \nabla_\varepsilon(c)) = 0$ for all other $c \in \mathbf{B}_\lambda$.

The following corollaries show that ε-tilting objects behave well with respect to the inclusion and quotient constructions from Theorems 3.17, 3.55 and 3.18. This follows easily from those theorems plus the characterization of tilting objects in Theorem 4.2; the situation is just like [Do4, Lemma A4.5].

Corollary 4.4. *Let $\mathcal{R}$ be a finite or lower finite ε-stratified category and $\mathcal{R}^\downarrow$ the finite ε-stratified subcategory associated to a finite lower set $\Lambda^\downarrow$ of Λ. For $b \in \mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$, the corresponding indecomposable ε-tilting object of $\mathcal{R}^\downarrow$ is $T_\varepsilon(b)$ (the same as in $\mathcal{R}$).*

Corollary 4.5. *Assume $\mathcal{R}$ is a finite ε-stratified category and let $\Lambda^\uparrow$ be an upper set in Λ with quotient $j : \mathcal{R} \to \mathcal{R}^\uparrow$. Let $b \in \mathbf{B}^\uparrow := \rho^{-1}(\Lambda^\uparrow)$. The corresponding indecomposable ε-tilting object $T^\uparrow_\varepsilon(b)$ of $\mathcal{R}^\uparrow$ satisfies $T^\uparrow_\varepsilon(b) \cong jT_\varepsilon(b)$. Also $jT_\varepsilon(b) = 0$ if $b \notin \mathbf{B}^\uparrow$.*

The next result is concerned with tilting resolutions.

Definition 4.6. Assume that $\mathcal{R}$ is a finite or lower finite ε-stratified category. An *ε-tilting resolution* $d : T_\bullet \to V$ of $V \in \mathcal{R}$ is the data of an exact sequence

$$\cdots \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} V \longrightarrow 0$$

such that

(TR1) $T_m \in \operatorname{Tilt}_\varepsilon(\mathcal{R})$ for each $m = 0, 1, \dots$;

(TR2) $\operatorname{im} d_m \in \nabla_\varepsilon(\mathcal{R})$ for $m \gg 0$.

Similarly, an *ε-tilting coresolution* $d : V \to T^\bullet$ of $V \in \mathcal{R}$ is the data of an exact sequence

$$0 \longrightarrow V \xrightarrow{d^0} T^0 \xrightarrow{d^1} T^1 \xrightarrow{d^2} \cdots$$

such that

(TC1) $T^m \in \operatorname{Tilt}_\varepsilon(\mathcal{R})$ for $m = 0, 1, \dots$;

(TC2) $\operatorname{coim} d^m \in \Delta_\varepsilon(\mathcal{R})$ for $m \gg 0$.

We say it is a *finite* resolution (resp., coresolution) if there is some n such that $T_m = 0$ (resp., $T^m = 0$) for $m > n$. Note in the finite case that the second axiom is redundant.

Lemma 4.7. *If $d : T_\bullet \to V$ is an ε-tilting resolution of $V \in \mathcal{R}$ then* $\operatorname{im} d_m \in \nabla_\varepsilon(\mathcal{R})$ *for* all $m \geqslant 0$. *In particular, $V \in \nabla_\varepsilon(\mathcal{R})$.*

Proof. It suffices to show that for any exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in a finite or lower finite ε-stratified category $B \in \nabla_\varepsilon(\mathcal{R})$ and $\operatorname{im} f \in \nabla_\varepsilon(\mathcal{R})$ implies $\operatorname{im} g \in \nabla_\varepsilon(\mathcal{R})$. Since $\operatorname{im} f = \ker g$, there is a short exact sequence $0 \to \operatorname{im} f \to B \to \operatorname{im} g \to 0$. Now apply Corollary 3.13 (or Corollary 3.60). □

Corollary 4.8. *If $d : V \to T^\bullet$ is an ε-tilting coresolution of $V \in \mathcal{R}$ then* $\operatorname{im} d^m \in \Delta_\varepsilon(\mathcal{R})$ *for* all $m \geqslant 0$. *In particular, $V \in \Delta_\varepsilon(\mathcal{R})$.*

Proof. An ε-tilting coresolution of V in $\mathcal{R}$ is the same thing as a $(-\varepsilon)$-tilting resolution of V in $\mathcal{R}^{\mathrm{op}}$. Hence, the corollary is the equivalent dual statement to Lemma 4.7. □

Theorem 4.9. *Let $\mathcal{R}$ be a finite or lower finite ε-stratified category and take $V \in \mathcal{R}$.*

(1) V has an ε-tilting resolution if and only if $V \in \nabla_\varepsilon(\mathcal{R})$.

(2) V has an ε-tilting coresolution if and only if $V \in \Delta_\varepsilon(\mathcal{R})$.

Proof. We just prove (1), since (2) is the equivalent dual statement. If V has an ε-tilting resolution, then we must have that $V \in \nabla_\varepsilon(\mathcal{R})$ thanks to Lemma 4.7. For the converse, we claim for $V \in \nabla_\varepsilon(\mathcal{R})$ that there is a short exact sequence $0 \to S_V \to T_V \to V \to 0$ with $S_V \in \nabla_\varepsilon(\mathcal{R})$ and $T_V \in \operatorname{Tilt}_\varepsilon(\mathcal{R})$. Given the claim, one can construct an ε-tilting resolution of V by "Splicing" (e.g., see [Wei, Figure 2.1]), to complete the proof.

To prove the claim, we argue by induction on the length $\sum_{b \in \mathbf{B}} (V : \nabla_\varepsilon(b))$ of a ∇_ε-flag of V. If this number is one, then $V \cong \nabla_\varepsilon(b)$ for some $b \in \mathbf{B}$, and there is a short exact sequence $0 \to S_V \to T_V \to V \to 0$ with $S_V \in \nabla_\varepsilon(b)$ and $T_V := T_\varepsilon(b)$ due to Theorem 4.2(ii). If it is greater than one, then there is a short exact sequence $0 \to U \to V \to W \to 0$ such that U and W have strictly shorter ∇_ε-flags. By induction, there are short exact sequences $0 \to S_U \to T_U \to U \to 0$ and $0 \to S_W \to T_W \to W \to 0$ with $S_U, S_W \in \nabla_\varepsilon(\mathcal{R})$ and $T_U, T_W \in \operatorname{Tilt}_\varepsilon(\mathcal{R})$. It remains to show that these short exact sequences can be assembled to produce the desired short exact sequence for V. The argument is like in the proof of the "Horseshoe Lemma" in [Wei, Lemma 2.2.8].

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_U & \longrightarrow & T_U & \xrightarrow{i} & U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & S_V & \longrightarrow & T_V & \xrightarrow{j} & V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & \overset{\hat{k}}{\nearrow} & \downarrow g & & \\ 0 & \longrightarrow & S_W & \longrightarrow & T_W & \xrightarrow{k} & W & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{4.3}$$

Since $\operatorname{Ext}^1_{\mathcal{R}}(T_W, U) = 0$, we can lift $k : T_W \to W$ to $\hat{k} : T_W \to V$ so that $k = g \circ \hat{k}$. Let $T_V := T_U \oplus T_W$ and $j : T_V \to V$ be $\operatorname{diag}(fi, \hat{k})$. This gives us a split short exact sequence in the middle column in (4.3), such that the right hand squares commute. Then we let $S_V := \ker j$, and see that there are induced maps making the left hand column and middle row into short exact sequences such that the left hand squares commute too. □

4.2. **Finite Ringel duality.** Throughout, we assume that $\mathcal{R}$ is a finite ε-stratified category with stratification defined by $\rho : \mathbf{B} \to \Lambda$. Always, Λ^{op} denotes Λ viewed as a poset with the opposite partial order. We are going to review the theory of Ringel duality for finite ε-stratified categories. The exposition here is based in part on [Do4, Appendix], which gives a self-contained treatment in the highest weight setting, and [AHLU], where the standardly stratified case is considered assuming $\mathbf{B} = \Lambda = \{1 < \cdots < n\}$.

Definition 4.10. By a *full ε-tilting object* in $\mathcal{R}$, we mean $T \in \operatorname{Tilt}_\varepsilon(\mathcal{R})$ such that T has a summand isomorphic to $T_\varepsilon(b)$ for each $b \in \mathbf{B}$. Given such an object, we define the *Ringel dual* of $\mathcal{R}$ relative to T to be the finite Abelian category $\widetilde{\mathcal{R}} := A\text{-mod}_{\mathrm{fd}}$ where $A := \operatorname{End}_{\mathcal{R}}(T)^{\mathrm{op}}$. We also define the two (covariant) *Ringel duality functors*

$$F := \operatorname{Hom}_{\mathcal{R}}(T, -) : \mathcal{R} \to \widetilde{\mathcal{R}}, \tag{4.4}$$

$$G := * \circ \operatorname{Hom}_{\mathcal{R}}(-, T) : \mathcal{R} \to \widetilde{\mathcal{R}}. \tag{4.5}$$

Theorem 4.11. *The Ringel dual $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ relative to T is a finite $(-\varepsilon)$-stratified category with stratification defined from $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$ and distinguished objects satisfying*

$$\widetilde{P}(b) \cong FT_\varepsilon(b), \qquad \widetilde{I}(b) \cong GT_\varepsilon(b), \qquad \widetilde{L}(b) \cong \operatorname{hd} \widetilde{P}(b) \cong \operatorname{soc} \widetilde{I}(b),$$

$$\widetilde{\Delta}_{-\varepsilon}(b) \cong F\nabla_\varepsilon(b), \qquad \widetilde{\nabla}_{-\varepsilon}(b) \cong G\Delta_\varepsilon(b), \qquad \widetilde{T}_{-\varepsilon}(b) \cong FI(b) \cong GP(b).$$

The restrictions $F : \nabla_\varepsilon(\mathcal{R}) \to \Delta_{-\varepsilon}(\widetilde{\mathcal{R}})$ and $G : \Delta_\varepsilon(\mathcal{R}) \to \nabla_{-\varepsilon}(\widetilde{\mathcal{R}})$ are equivalences.

Before the proof, we give some applications. The first is a *double centralizer property.* It implies that our situation fits into the setup from [Wak, (A1), (A2)], and T is an example of a *Wakamatsu module.*

Corollary 4.12. *Suppose that the ε-stratified category $\mathcal{R}$ in Theorem 4.11 is B-mod$_{\mathrm{fd}}$ for a finite-dimensional algebra B, so that T is a (B, A)-bimodule. Let $\widetilde{T} := T^*$ be the dual (A, B)-bimodule. Then the following holds.*

1. *$\widetilde{T}$ is a full $(-\varepsilon)$-tilting object in $\widetilde{\mathcal{R}}$ such that $B = \operatorname{End}_A(\widetilde{T})^{\mathrm{op}}$. Hence, the Ringel dual of $\widetilde{\mathcal{R}}$ relative to $\widetilde{T}$ is the original category $\mathcal{R}$.*
2. *Denoting the analogs of the Ringel duality functors F and G for $\widetilde{R}$ by*

$$\widetilde{G} := \operatorname{Hom}_{\widetilde{\mathcal{R}}}(\widetilde{T}, -) : \widetilde{\mathcal{R}} \to \mathcal{R}, \qquad \widetilde{F} := * \circ \operatorname{Hom}_{\widetilde{\mathcal{R}}}(-, \widetilde{T}) : \widetilde{\mathcal{R}} \to \mathcal{R}, \tag{4.6}$$

respectively, we have that

$$\widetilde{F} \cong T \otimes_A -, \qquad\qquad G \cong \widetilde{T} \otimes_B -. \tag{4.7}$$

Hence, $(\widetilde{F}, F)$ *and* $(G, \widetilde{G})$ *are adjoint pairs.*

(3) *The restrictions* $\widetilde{F} : \Delta_{-\varepsilon}(\widetilde{\mathcal{R}}) \to \nabla_\varepsilon(\mathcal{R})$ *and* $\widetilde{G} : \nabla_{-\varepsilon}(\widetilde{\mathcal{R}}) \to \Delta_\varepsilon(\mathcal{R})$ *are quasi-inverses of the equivalences defined by* F *and* G*, respectively.*

Proof. (1) Note that GB is a full $(-\varepsilon)$-tilting object since $GP(b) \cong \widetilde{T}_{-\varepsilon}(b)$ for $b \in \mathbf{B}$. Actually, $GB = \operatorname{Hom}_B(B, T)^* \cong T^* = \widetilde{T}$. Thus, $\widetilde{T}$ is a full $(-\varepsilon)$-tilting object in $\widetilde{\mathcal{R}}$. Its opposite endomorphism algebra is B since G defines an algebra isomorphism

$$B \cong \operatorname{End}_B(B)^{\mathrm{op}} \xrightarrow{\sim} \operatorname{End}_A(GB)^{\mathrm{op}} \cong \operatorname{End}_A(\widetilde{T})^{\mathrm{op}}.$$

(2) As $\widetilde{F}$ is right exact and commutes with direct sums, it is isomorphic to $(\widetilde{F}(A) \otimes_A - \cong T \otimes_A -$. Thus, $\widetilde{F}$ is left adjoint to F. Similarly, $G \cong \widetilde{T} \otimes_B -$ is left adjoint to $\widetilde{G}$.

(3) By Theorem 4.11, $F : \nabla_\varepsilon(\mathcal{R}) \to \Delta_{-\varepsilon}(\widetilde{R})$ and $\widetilde{F} : \Delta_{-\varepsilon}(\widetilde{R}) \to \nabla_\varepsilon(\mathcal{R})$ are equivalences. They must be quasi-inverse to each other as they are adjoints. Similarly for G and $\widetilde{G}$. □

The next corollary deals with the strata $\widetilde{\mathcal{R}}_\lambda$ of the Ringel dual category. Let $\tilde{\jmath}^\lambda_! : \widetilde{\mathcal{R}}_\lambda \to \widetilde{\mathcal{R}}_{\geqslant\lambda}$ and $\tilde{\jmath}^\lambda_* : \widetilde{\mathcal{R}}_\lambda \to \widetilde{\mathcal{R}}_{\geqslant\lambda}$ be its standardization and costandardization functors.

Corollary 4.13. *For* $\lambda \in \Lambda$*, the strata* $\mathcal{R}_\lambda$ *and* $\widetilde{\mathcal{R}}_\lambda$ *are equivalent. More precisely:*

(1) *If* $\varepsilon(\lambda) = +$ *the functor* $F_\lambda := \tilde{\jmath}^\lambda \circ F \circ j^\lambda_* : \mathcal{R}_\lambda \to \widetilde{\mathcal{R}}_\lambda$ *is a well-defined equivalence.*
(2) *If* $\varepsilon(\lambda) = -$ *the functor* $G_\lambda := \tilde{\jmath}^\lambda \circ G \circ j^\lambda_! : \mathcal{R}_\lambda \to \widetilde{\mathcal{R}}_\lambda$ *is a well-defined equivalence.*

Proof. We just prove (1), since (2) is similar. So assume that $\varepsilon(\lambda) = +$. Note that the definition of F_λ makes sense: j^λ_* is exact by Theorem 3.6 so it sends objects of $\mathcal{R}_\lambda$ to objects of $\mathcal{R}$ which have filtrations with sections $\nabla_\varepsilon(b)$ for $b \in \mathbf{B}_\lambda$; F sends such objects into $\Delta_{-\varepsilon}(\widetilde{\mathcal{R}}_{\geqslant\lambda})$ on which $\tilde{\jmath}^\lambda$ is defined. This shows moreover that F_λ is exact. Adopting the setup of Corollary 4.12, we can also define

$$\widetilde{F}_\lambda := j^\lambda \circ \widetilde{F} \circ \tilde{\jmath}^\lambda_! : \widetilde{\mathcal{R}}_\lambda \to \mathcal{R}_\lambda,$$

and get that $\widetilde{F}_\lambda$ is well-defined by similar arguments. We complete the proof by showing that F_λ and $\widetilde{F}_\lambda$ are quasi-inverse equivalences. Note that $\widetilde{F}_\lambda$ is left adjoint to F_λ. The counit of adjunction gives us a natural transformation $\widetilde{F}_\lambda \circ F_\lambda \to \mathrm{Id}_{\mathcal{R}_\lambda}$. We claim this is an isomorphism. Since both functors are exact, it suffices to prove this on irreducible objects: we have $\widetilde{F}_\lambda(F_\lambda L_\lambda(b)) \cong \widetilde{F}_\lambda \widetilde{L}_\lambda(b) \cong L_\lambda(b)$. Similar argument shows that the unit of adjunction is an isomorphism in the other direction. □

Corollary 4.14. *Let* $\mathcal{R}$ *be a finite* ε*-stratified category.*

(1) *All* $V \in \nabla_\varepsilon(\mathcal{R})$ *have finite* ε*-tilting resolutions if and only if all positive strata are of finite global dimension.*
(2) *All* $V \in \Delta_\varepsilon(\mathcal{R})$ *have finite* ε*-tilting coresolutions if and only if all negative strata are of finite global dimension.*

Proof. We just explain the proof of (1). By Theorem 4.11, all $V \in \nabla_\varepsilon(\mathcal{R})$ have finite ε-tilting resolutions if and only if all $\widetilde{V} \in \Delta_{-\varepsilon}(\widetilde{\mathcal{R}})$ have finite projective resolutions. By Lemma 3.24(1), this is equivalent to all negative strata of the $(-\varepsilon)$-stratified category $\widetilde{\mathcal{R}}$ are of finite global dimension. Equivalently, by Corollary 4.13, all positive strata of the ε-stratified category $\mathcal{R}$ are of finite global dimension. □

For the final application, let $\mathbb{R}F$ and $\mathbb{L}G$ be the total derived functors of F respectively G. These give triangulated functors between the bounded derived categories $D^b(\mathcal{R})$ and $D^b(\widetilde{\mathcal{R}})$. The following theorem is a consequence of Happel's general tilting theory for finite dimensional algebras from [Hap].

Theorem 4.15. *Let $\widetilde{\mathcal{R}}$ be the Ringel dual of a finite ε-stratified category $\mathcal{R}$. Assume that all negative strata (resp., all positive strata) of $\mathcal{R}$ are of finite global dimension. Then $\mathbb{R}F : D^b(\mathcal{R}) \to D^b(\widetilde{\mathcal{R}})$ (resp., $\mathbb{L}G : D^b(\mathcal{R}) \to D^b(\widetilde{\mathcal{R}})$) is an equivalence of triangulated categories. Moreover, if $\mathcal{R}$ is of finite global dimension, then so is $\widetilde{\mathcal{R}}$.*

Proof. Assuming $\mathcal{R}$ has finite global dimension, this all follows by [Hap, Lemma 2.9, Theorem 2.10]; the hypotheses there hold thanks to Corollary 4.14. To get the derived equivalence without assuming $\mathcal{R}$ has finite global dimension, we cite instead Keller's exposition of Happel's result in [Kel, Theorem 4.1], since it assumes slightly less; the hypotheses (a) and (c) there hold due to Corollary 4.14(2) and Lemma 3.24(1). □

Corollary 4.16. *If $\mathcal{R}$ is standardly stratified (resp., costandardly stratified) and $\mathcal{R}'$ is the Ringel dual relative to a full tilting object (resp., a full cotilting object), then $\mathbb{R}F : D^b(\mathcal{R}) \to D^b(\widetilde{\mathcal{R}})$ (resp., $\mathbb{L}G : D^b(\mathcal{R}) \to D^b(\widetilde{\mathcal{R}})$) is an equivalence.*

Proof of Theorem 4.11. This follows the same steps as in [Do4, pp.158–160]. Assume without loss of generality that $\mathcal{R} = B\text{-mod}_{\text{fd}}$ for a finite-dimensional algebra B. For each $b \in \mathbf{B}$, let $e_b \in A = \operatorname{End}_A(T)^{\text{op}}$ be an idempotent such that $Te_b \cong T_\varepsilon(b)$. Then $\widetilde{P}(b) := Ae_b$ is an indecomposable projective A-module and the modules

$$\left\{\widetilde{L}(b) := \text{hd } \widetilde{P}(b) \,\middle|\, b \in \mathbf{B}\right\}$$

give a full set of pairwise inequivalent irreducible A-modules. Since $\widetilde{\mathcal{R}}$ is a finite Abelian category, it is immediate that $\rho : \mathbf{B} \to \Lambda^{\text{op}}$ defines a stratification of it. Let $\widetilde{\Delta}_{-\varepsilon}(b)$ and $\widetilde{\nabla}_{-\varepsilon}(b)$ be the $(-\varepsilon)$-standard and $(-\varepsilon)$-costandard objects of $\widetilde{\mathcal{R}}$ defined from this stratification. Set $V(b) := F\nabla_\varepsilon(b)$.

(1) *For $b \in \mathbf{B}$ we have that $\widetilde{P}(b) = FT_\varepsilon(b)$.* This follows immediately from the equality $\operatorname{Hom}_B(T_\varepsilon, T_\varepsilon)e_b = \operatorname{Hom}_B(T_\varepsilon, T_\varepsilon e_b)$.

(2) *The functor F is exact on $\nabla_\varepsilon(\mathcal{R})$.* This is the usual Ext^1-vanishing between Δ_ε- and ∇_ε-filtered objects.

(3) *For $a, b \in \mathbf{B}$, we have that $[V(b) : \widetilde{L}(a)] = (T_\varepsilon(a) : \Delta_\varepsilon(b))$.* The left hand side is $\dim e_a V(b) = \dim e_a \operatorname{Hom}_B(T_\varepsilon, \nabla_\varepsilon(b)) = \dim \operatorname{Hom}_B(T_\varepsilon(a), \nabla_\varepsilon(b))$, which equals the right hand side.

(4) *$V(b)$ is a non-zero quotient of $\widetilde{P}(b)$, thus,* hd $V(b) = \widetilde{L}(b)$. By Theorem 4.2(i), $\nabla_\varepsilon(b)$ is a quotient of $T_\varepsilon(b)$, hence $V(b)$ is quotient of $\widetilde{P}(b)$ by (2). It is non-zero by (3).

(5) *We have that $V(b) \cong \widetilde{\Delta}_{-\varepsilon}(b)$.* Let $\lambda := \rho(b)$. We treat the cases $\varepsilon(\lambda) = +$ and $\varepsilon(\lambda) = -$ separately.

If $\varepsilon(\lambda) = +$ we must show that $V(b)$ is the largest quotient of $\widetilde{P}(b)$ such that $[V(b) : \widetilde{L}(a)] \neq 0 \Rightarrow \rho(a) \geqslant \rho(b)$. We have already observed in (4) that $V(b)$ is a quotient of $\widetilde{P}(b)$. Also $(T_\varepsilon(a) : \Delta_\varepsilon(b)) \neq 0 \Rightarrow \rho(b) \leqslant \rho(a)$ by Theorem 4.2(iii). Using (3), this imples that $V(b)$ has the property $[V(b) : \widetilde{L}(a)] \neq 0 \Rightarrow \rho(a) \geqslant \rho(b)$. It remains to show that any strictly larger quotient of $\widetilde{P}(b)$ fails this condition. To see this, since $\varepsilon(\lambda) = +$, a ∇_ε-flag in $T_\varepsilon(b)$ has $\nabla_\varepsilon(b)$ at the top and other sections $\nabla_\varepsilon(c)$ for c with $\rho(c) < \rho(b)$. In view of (4), any strictly larger quotient of $\widetilde{P}(b)$ than $V(b)$ therefore has an additional composition factor $\widetilde{L}(c)$ arising from the head of $V(c)$ for some c with $\rho(c) < \rho(b)$.

Instead, if $\varepsilon(\lambda) = -$, we use the characterization of $\widetilde{\Delta}_{-\varepsilon}(b)$ from Lemma 3.3: we must show that $V(b)$ is the largest quotient of $\widetilde{P}(b)$ such that $[V(b) : \widetilde{L}(b)] = 1$ and $[V(b) : \widetilde{L}(a)] \neq 0 \Rightarrow \rho(a) > \rho(b)$ for $a \neq b$. Since $\varepsilon(\lambda) = -$, we have that $(T_\varepsilon(b) : \nabla_\varepsilon(b)) = 1$ and $(T_\varepsilon(b) : \nabla_\varepsilon(a)) \neq 0 \Rightarrow \rho(a) < \rho(b)$ for $a \neq b$. Hence, using (3) again, the quotient $V(b)$ of $\widetilde{P}(b)$ has the required properties. A ∇_ε-flag in $T_\varepsilon(b)$ has $\nabla_\varepsilon(b)$ at the top and other sections $\nabla_\varepsilon(c)$ for c with $\rho(c) \leqslant \rho(b)$. So any strictly larger quotient of $\widetilde{P}(b)$ than $V(b)$ has a composition factor $\widetilde{L}(c)$ arising from the head of $V(c)$ for c with $\rho(c) \leqslant \rho(b)$. In case $c = b$, this violates the requirement that the quotient has $\widetilde{L}(b)$ appearing with multiplicity one; otherwise, it violates the requirement that all other composition factors of the quotient are of the form $\widetilde{L}(a)$ with $\rho(a) > \rho(b)$.

(6) *$\widetilde{\mathcal{R}}$ is $(-\varepsilon)$-stratified.* In view of (5), it suffices to show that $\widetilde{P}(b)$ has a filtration with sectons $V(c)$ for c with $\rho(c) \leqslant \rho(b)$. Since $T_\varepsilon(b)$ has a ∇_ε-flag with sections $\nabla_\varepsilon(c)$ for c with $\rho(c) \leqslant \rho(b)$, this follows using (1) and (2).

(7) *For any $U \in \mathrm{Tilt}_\varepsilon(\mathcal{R})$ and $V \in \mathcal{R}$, the map $f : \mathrm{Hom}_B(U, V) \to \mathrm{Hom}_A(FU, FV)$ induced by F is an isomorphism.* It suffices to prove this when $U = T$, so that the right hand space is $\mathrm{Hom}_A(A, FV)$ and $FV = \mathrm{Hom}_B(T, V)$. Then it is straightforward to check that f is the inverse of the isomorphism $\mathrm{Hom}_A(A, FV) \to FV, \theta \mapsto \theta(1)$.

(8) *For any $V, W \in \nabla_\varepsilon(\mathcal{R})$ and $n \geqslant 0$, the functor F induces a linear isomorphism $\mathrm{Ext}^n_{\mathcal{R}}(V, W) \xrightarrow{\sim} \mathrm{Ext}^n_{\widetilde{\mathcal{R}}}(FV, FW)$.* Take an ε-tilting resolution $d : T_\bullet \to V$ in the sense of Definition 4.6, which exists thanks to Theorem 4.9. The functor F takes $\cdots \to T_1 \to T_0 \to V \to 0$ to a complex $\cdots \to FT_1 \to FT_0 \to FV \to 0$. In fact, this complex is exact. To see this, take $m \geqslant 0$ and consider the short exact sequence $0 \to \ker d_m \to T_m \to \operatorname{im} d_m \to 0$. All of $\ker d_m$, T_m and $\operatorname{im} d_m$ have ∇_ε-flags due to Lemma 4.7. Hence, thanks to (2), we get a short exact sequence

$$0 \longrightarrow F(\ker d_m) \xrightarrow{i} FT_m \xrightarrow{p} F(\operatorname{im} d_m) \longrightarrow 0$$

on applying F. Since F is left exact, the canonical map $F(\operatorname{im} d_m) \to FT_{m-1}$ is a monomorphism. Its image is all $\theta : T \to T_{m-1}$ with image contained in $\operatorname{im} d_m$. As p is an epimorphism, any such θ can be written as $d_m \circ \phi$ for $\phi : T \to T_m$, i.e., $\theta \in \operatorname{im}(F d_m)$. Thus, $F(\operatorname{im} d_m) \cong \operatorname{im}(F d_m)$, and $0 \to \ker(F d_m) \longrightarrow FT_m \to \operatorname{im}(F d_m) \to 0$ is exact, as required. In view of (1), we have thus constructed a projective resolution of FV in $\widetilde{\mathcal{R}}$,

$$\cdots \longrightarrow FT_1 \longrightarrow FT_0 \longrightarrow FV \longrightarrow 0.$$

In this paragraph, we use the projective resolution just constructed to compute $\mathrm{Ext}^n_{\widetilde{\mathcal{R}}}(FV, FI)$ for any injective $I \in \mathcal{R}$. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{R}}(V, I) & \longrightarrow & \mathrm{Hom}_{\mathcal{R}}(T_0, I) & \longrightarrow & \mathrm{Hom}_{\mathcal{R}}(T_1, I) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \\
0 & \longrightarrow & \mathrm{Hom}_{\widetilde{\mathcal{R}}}(FV, FI) & \longrightarrow & \mathrm{Hom}_{\widetilde{\mathcal{R}}}(FT_0, FI) & \longrightarrow & \mathrm{Hom}_{\widetilde{\mathcal{R}}}(FT_1, FI) & \longrightarrow & \cdots
\end{array}$$

with vertical maps induced by F. The maps $f_0, f_1, \dots$ are isomorphisms due to (7). Also the top row is exact as I is injective. We deduce that the bottom row is exact at the positions $\mathrm{Hom}_{\widetilde{\mathcal{R}}}(FT_m, FI)$ for all $m \geqslant 1$. It is exact at positions $\mathrm{Hom}_{\widetilde{\mathcal{R}}}(FV, FI)$ and $\mathrm{Hom}_{\widetilde{\mathcal{R}}}(FT_0, FI)$ as $\mathrm{Hom}_{\widetilde{\mathcal{R}}}(-, FI)$ is left exact. Thus, the bottom row is exact everywhere. So the map f is an isomorphism too and $\mathrm{Ext}^n_{\widetilde{\mathcal{R}}}(FV, FI) = 0$ for $n > 0$.

Finally, take a short exact sequence $0 \to W \to I \to Q \to 0$ in $\mathcal{R}$ with I injective. We have that $Q \in \nabla_\varepsilon(\mathcal{R})$ by Corollary 3.13. Hence, using (2) and the previous paragraph, there is a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathcal{R}}(V,W) & \hookrightarrow & \mathrm{Hom}_{\mathcal{R}}(V,I) & \longrightarrow & \mathrm{Hom}_{\mathcal{R}}(V,Q) & \twoheadrightarrow & \mathrm{Ext}^1_{\mathcal{R}}(V,W) \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} \\
\mathrm{Hom}_{\widetilde{\mathcal{R}}}(FV,FW) & \hookrightarrow & \mathrm{Hom}_{\widetilde{\mathcal{R}}}(FV,FI) & \longrightarrow & \mathrm{Hom}_{\widetilde{\mathcal{R}}}(FV,FQ) & \twoheadrightarrow & \mathrm{Ext}^1_{\widetilde{\mathcal{R}}}(FV,FW)
\end{array}$$

with exact rows. As f_2 is an isomorphism, we get that f_1 is injective. Since this is proved for all V, this means that f_3 is injective too. Then a diagram chase gives that f_1 is surjective, hence, f_3 is surjective and f_4 is an isomorphism. Finally, degree shifting gives the isomorphisms $\mathrm{Ext}^n_{\mathcal{R}}(V,W) \xrightarrow{\sim} \mathrm{Ext}^n_{\widetilde{\mathcal{R}}}(FV,FW)$ for $n \geqslant 2$ as well.

(9) *We have that* $\widetilde{T}_{-\varepsilon}(b) \cong FI(b)$. By (5) and (8), we get that $\mathrm{Ext}^1_{\widetilde{\mathcal{R}}}(\widetilde{\Delta}_{-\varepsilon}(a), FI(b)) \cong \mathrm{Ext}^1_{\mathcal{R}}(\nabla_\varepsilon(a), I(b)) = 0$ for all $a \in \mathbf{B}$. Hence, by the homological criterion, $FI(b)$ has a $\nabla_{-\varepsilon}$-flag. It also has a $\Delta_{-\varepsilon}$-flag with bottom section isomorphism to $\widetilde{\Delta}_{-\varepsilon}(b)$ due to (2) and (5). So $FI(b) \in \mathrm{Tilt}_{-\varepsilon}(\widetilde{\mathcal{R}})$. It is indecomposable as $\mathrm{End}_{\widetilde{\mathcal{R}}}(FI(b)) \cong \mathrm{End}_{\mathcal{R}}(I(b))$ by (8), which is local. Therefore $FI(b) \cong \widetilde{T}_{-\varepsilon}(b)$ due to Theorem 4.2(i).

(10) *The restriction* $F : \nabla_\varepsilon(\mathcal{R}) \to \Delta_{-\varepsilon}(\mathcal{R}')$ *is an equivalence of categories.* It is full and faithful by (8). It remains to show that it is dense, i.e., for any $V \in \Delta_{-\varepsilon}(\mathcal{R}')$ there exists $\widetilde{V} \in \nabla_\varepsilon(\mathcal{R})$ with $F\widetilde{V} \cong V$. The proof of this goes by induction on the length of a $\Delta_{-\varepsilon}$-flag of V. If this length is one, we are done by (5). For the induction step, consider V fitting into a short exact sequence $0 \to U \to V \to W \to 0$ for shorter $U, W \in \Delta_{-\varepsilon}(\mathcal{R}')$. By induction there are $\widetilde{U}, \widetilde{W} \in \nabla_\varepsilon(\mathcal{R})$ such that $F\widetilde{U} \cong U, F\widetilde{W} \cong W$. Then we use the isomorphism $\mathrm{Ext}^1_{\mathcal{R}'}(F\widetilde{W}, F\widetilde{U}) \cong \mathrm{Ext}^1_{\mathcal{R}}(\widetilde{W}, \widetilde{U})$ from (8) to see that there is an extension $\widetilde{V}$ of $\widetilde{U}$ and $\widetilde{W}$ in $\mathcal{R}$ such that $F\widetilde{V} \cong V$.

(11) *The right B-module T^* is a full $(-\varepsilon)$-tilting object in the $(-\varepsilon)$-stratified category* $\mathrm{mod}_{\mathrm{fd}}$-$B$ *such that* $\mathrm{End}_B(T^*)^{\mathrm{op}} = A^{\mathrm{op}}$. *Let* $F^{\mathrm{op}} := \mathrm{Hom}_B(T^*, -) : \mathrm{mod}_{\mathrm{fd}}\text{-}B \to \mathrm{mod}_{\mathrm{fd}}\text{-}A$ *be the corresponding Ringel duality functor. Then we have that* $G \cong * \circ F^{\mathrm{op}} \circ *$. The first statement is clear from Lemma 3.9, observing that $\mathrm{End}_B(T^*)^{\mathrm{op}} \cong \mathrm{End}_B(T)$ since since $* : B\text{-}\mathrm{mod}_{\mathrm{fd}} \to \mathrm{mod}_{\mathrm{fd}}\text{-}B$ is a contravariant equivalence. It remains to observe that $* \circ F^{\mathrm{op}} \circ * \cong * \circ \mathrm{Hom}_B(T^*, (-)^*) \cong * \circ \mathrm{Hom}_B(-, T) = G$.

(12) *The restriction* $G : \Delta_\varepsilon(\mathcal{R}) \to \nabla_{-\varepsilon}(\widetilde{\mathcal{R}})$ *is an equivalence such that* $GT_\varepsilon(b) \cong \widetilde{I}(b)$, $G\Delta_\varepsilon(b) \cong \widetilde{\nabla}_{-\varepsilon}(b)$ *and* $GP(b) \cong \widetilde{T}_{-\varepsilon}(b)$. This follows from (11) together with the analogs of (1), (5), (9) and (10) with $\mathrm{mod}_{\mathrm{fd}}$-$B$, $\mathrm{mod}_{\mathrm{fd}}$-$A$ and F^{op} replacing $\mathcal{R} = B\text{-}\mathrm{mod}_{\mathrm{fd}}, \widetilde{\mathcal{R}} = A\text{-}\mathrm{mod}_{\mathrm{fd}}$ and F, respectively. □

4.3. **Tilting objects in the upper finite case.** Throughout the subsection, $\mathcal{R}$ will be an upper finite ε-stratified category.

We are going to extend the definition of tilting objects to this situation. Using the notion of ascending Δ_ε-flags and descending ∇_ε-flags from Definition 3.35, we set

$$\mathrm{Tilt}_\varepsilon(\mathcal{R}) := \Delta^{\mathrm{asc}}_\varepsilon(\mathcal{R}) \cap \nabla^{\mathrm{dsc}}_\varepsilon(\mathcal{R}). \tag{4.8}$$

We emphasize that objects of $\mathrm{Tilt}_\varepsilon(\mathcal{R})$ are in particular objects of $\mathcal{R}$, so all of their composition multiplicities are finite. Like in Lemma 4.1, $\mathrm{Tilt}_\varepsilon(\mathcal{R})$ is a Karoubian subcategory of $\mathcal{R}$.

Theorem 4.17. *Assume that $\mathcal{R}$ is an upper finite ε-stratified category. For $b \in \mathbf{B}$ with $\rho(b) = \lambda$, there is an indecomposable object $T_\varepsilon(b) \in \mathrm{Tilt}_\varepsilon(\mathcal{R})$ satisfying the following properties:*

(i) $T_\varepsilon(b)$ *has an ascending* Δ_ε*-flag with bottom section*[3] *isomorphic to* $\Delta_\varepsilon(b)$*;*
(ii) $T_\varepsilon(b)$ *has a descending* ∇_ε*-flag with top section*[4] *isomorphic to* $\nabla_\varepsilon(b)$*;*
(iii) $T_\varepsilon(b) \in \mathcal{R}_{\leqslant\lambda}$ *and* $j^\lambda T_\varepsilon(b) \cong \begin{cases} P_\lambda(b) & \text{if } \varepsilon(\lambda) = + \\ I_\lambda(b) & \text{if } \varepsilon(\lambda) = - \end{cases}$.

These properties determine $T_\varepsilon(b)$ *uniquely up to isomorphism: if* T *is any indecomposable object of* $\mathrm{Tilt}_\varepsilon(\mathcal{R})$ *satisfying any one of the properties (i)–(iii) then* $T \cong T_\varepsilon(b)$*; hence, it satisfies the other two properties as well.*

Proof. Existence: Replacing $\mathcal{R}$ by $\mathcal{R}_{\leqslant\lambda}$ if necessary and using Theorem 3.32, we reduce to the special case that λ is the largest element of the poset Λ. Assuming this, the first step in the construction of $T_\varepsilon(b)$ is to define a direct system $(V_\omega)_{\omega\in\Omega}$ of objects of $\mathcal{R}$. This is indexed by the directed set Ω of all finite upper sets in Λ. Let $V_\varnothing := 0$. Then take $\varnothing \neq \omega \in \Omega$ and denote it instead by $\Lambda^\uparrow$. Letting $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the corresponding finite ε-stratified quotient of $\mathcal{R}$, we set $V_\omega := j_! T_\varepsilon^\uparrow(b)$. By Theorem 3.37(6), this has a Δ_ε-flag. Given also $\omega < \upsilon \in \Omega$, i.e., another upper set $\Lambda^{\uparrow\uparrow}$ containing $\Lambda^\uparrow$, let $k : \mathcal{R} \to \mathcal{R}^{\uparrow\uparrow}$ be the corresponding quotient. Then j factors as $j = \bar{\jmath} \circ k$ for an induced quotient functor $\bar{\jmath} : \mathcal{R}^{\uparrow\uparrow} \to \mathcal{R}^\uparrow$. Since $\bar{\jmath} T_\varepsilon^{\uparrow\uparrow}(b) \cong T_\varepsilon^\uparrow(b)$ by Corollary 4.5, we deduce from Corollary 3.19(2) that there is a short exact sequence

$$0 \longrightarrow \bar{\jmath}_! T_\varepsilon^\uparrow(b) \longrightarrow T_\varepsilon^{\uparrow\uparrow}(b) \longrightarrow Q \longrightarrow 0$$

such that Q has a Δ_ε-flag with sections $\Delta_\varepsilon^{\uparrow\uparrow}(c)$ for c with $\rho(c) \in \Lambda^{\uparrow\uparrow} \backslash \Lambda^\uparrow$. Applying $k_!$ and using the exactness from Theorem 3.37(6) again, we deduce that there is an embedding $f_\omega^\upsilon : V_\omega \hookrightarrow V_\upsilon$ with $\operatorname{coker} f_\omega^\upsilon \in \Delta_\varepsilon(\mathcal{R})$. Thus, we have a direct system $(V_\omega)_{\omega\in\Omega}$. Now let $T_\varepsilon(b) := \varinjlim V_\omega \in \mathrm{Ind}(\mathcal{R})$. Using the induced embeddings $f_\omega : V_\omega \hookrightarrow T_\varepsilon(b)$, we identify each V_ω with a subobject of $T_\varepsilon(b)$. We have shown for $\omega < \upsilon$ that $V_\upsilon / V_\omega \in \Delta_\varepsilon(\mathcal{R})$ and, moreover, $jV_\upsilon = jV_\omega$ where $j : \mathcal{R} \to \mathcal{R}^\uparrow$ is the quotient associated to ω.

In this paragraph, we show that $T_\varepsilon(b)$ actually lies in $\mathcal{R}$ rather than $\mathrm{Ind}(\mathcal{R})$, i.e., all of the composition multiplicities $[T_\varepsilon(b) : L(c)]$ are finite. To see this, take $c \in \mathbf{B}$. Let $\omega = \Lambda^\uparrow \in \Omega$ be some fixed finite upper set such that $\rho(c) \in \Lambda^\uparrow$, and $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the quotient functor as usual. Then for any $\upsilon \geqslant \omega$ we have that

$$[V_\upsilon : L(c)] = [jV_\upsilon : L^\uparrow(c)] = [jV_\omega : L^\uparrow(c)] = [V_\omega : L(c)].$$

Hence, $[T_\varepsilon(b) : L(c)] = [V_\omega : L(c)] < \infty$.

So now we have defined $T_\varepsilon(b) \in \mathcal{R}$ together with an ascending Δ_ε-flag $(V_\omega)_{\omega\in\Omega}$. The smallest non-empty element of Ω is $\omega := \{\lambda\}$, and $V_\omega = j_!^\lambda P_\lambda(b) = \Delta_\varepsilon(b)$ if $\varepsilon(\lambda) = +$, or $j_!^\lambda I_\lambda(b)$ if $\varepsilon(\lambda) = -$. Since $j^\lambda T_\varepsilon(b) = j^\lambda V_\omega$, we deduce that (iii) holds. Also by construction $T_\varepsilon(b)$ has an ascending Δ_ε-flag. To see that it has a descending ∇_ε-flag, take any $a \in \mathbf{B}$. Let $\omega = \Lambda^\uparrow \in \Omega$ be such that $\rho(a) \in \Lambda^\uparrow$. Then $\Delta_\varepsilon(a) = j_! \Delta_\varepsilon^\uparrow(a)$ and $jT_\varepsilon(b) = jV_\omega = T_\varepsilon^\uparrow(b)$, so by Theorem 3.37(5) we get that

$$\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), T_\varepsilon(b)) \cong \mathrm{Ext}^1_{\mathcal{R}^\uparrow}(\Delta_\varepsilon^\uparrow(a), T_\varepsilon^\uparrow(b)) = 0.$$

By Theorem 3.40, this shows that $T_\varepsilon(b) \in \nabla_\varepsilon^{\mathrm{dsc}}(\mathcal{R})$.

Note finally that $T_\varepsilon(b)$ is indecomposable. This follows because $jT_\varepsilon(b)$ is indecomposable for every $j : \mathcal{R} \to \mathcal{R}^\uparrow$ (adopting the usual notation). Indeed, by the construction we have that $jT_\varepsilon(b) \cong T_\varepsilon^\uparrow(b)$ This completes the construction of the indecomposable object $T_\varepsilon(b) \in \mathrm{Tilt}_\varepsilon(\mathcal{R})$. We have shown that it satisfies (iii), and it follows easily that it also satisfies (i) and (ii).

[3]We mean that there is an ascending Δ_ε-flag $(V_\omega)_{\omega\in\Omega}$ in which Ω has a smallest non-zero element 1 such that $V_1 \cong \Delta_\varepsilon(b)$.

[4]Similarly, we mean that $V/V_1 \cong \nabla_\varepsilon(b)$.

Uniqueness: Since (iii) implies (i) and (ii), it suffices to show that any indecomposable $U \in \operatorname{Tilt}_\varepsilon(\mathcal{R})$ satisfying either (i) or (ii) is isomorphic to the object $T := T_\varepsilon(b)$ just constructed. We explain this just in the case of (i), since the argument for (ii) is similar. We take a short exact sequence $0 \to \Delta_\varepsilon(b) \to T \to Q \to 0$ with $Q \in \Delta_\varepsilon^{\mathrm{asc}}(\mathcal{R})$. Using the Ext-vanishing from Lemma 3.36, we deduce like in the proof of Theorem 4.2 that the inclusion $f : \Delta_\varepsilon(b) \hookrightarrow T$ extends to $\bar{f} : U \to T$. In fact, $\bar{f}$ is an isomorphism. To see this, take a finite upper set $\Lambda^\uparrow$ containing λ and consider the quotient $j : \mathcal{R} \to \mathcal{R}^\uparrow$ as usual. Both jU and jT are isomorphic to $T_\varepsilon^\uparrow(b)$ by the uniqueness in Theorem 4.2. The proof there implies that any homomorphism $jT \to jU$ which restricts to an isomorphism on the subobject $\Delta_\varepsilon^\uparrow(b)$ is an isomorphism. We deduce that $j\bar{f}$ is an isomorphism. Since holds for all choices of $\Lambda^\uparrow$, it follows that $\bar{f}$ itself is an isomorphism. □

Corollary 4.18. *Any object of* $\operatorname{Tilt}_\varepsilon(\mathcal{R})$ *is isomorphic to* $\bigoplus_{b \in \mathbf{B}} T_\varepsilon(b)^{\oplus n_b}$ *for unique multiplicities* $n_b \in \mathbb{N}$*. Conversely, any such direct sum belongs to* $\operatorname{Tilt}_\varepsilon(\mathcal{R})$*.*

Proof. Let us first show that any direct sum $U := \bigoplus_{b \in \mathbf{B}} T_\varepsilon(b)^{\oplus n_b}$ belongs to $\operatorname{Tilt}_\varepsilon(\mathcal{R})$. The only issue is to see that U actually belongs to $\mathcal{R}$ rather than $\operatorname{Ind}(\mathcal{R})$, i.e., it has finite composition multiplicities. But for a given $c \in \mathbf{B}$, the multiplicity $[T_\varepsilon(b) : L(c)]$ is zero unless $\rho(c) \leqslant \rho(b)$. There are only finitely many such $b \in \mathbf{B}$, so $[U : L(c)] = \sum_{b \in \mathbf{B}} n_b [T_\varepsilon(b) : L(c)] < \infty$.

Now take any $U \in \operatorname{Tilt}_\varepsilon(\mathcal{R})$. Let Ω be the directed set of all finite upper sets in Λ. Take $\omega \in \Omega$, say it is the finite upper set $\Lambda^\uparrow$. Let $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the quotient functor as usual. Then we have that $jU \in \operatorname{Tilt}_\varepsilon(\mathcal{R}^\uparrow)$, so it decomposes as a finite direct sum as $jU \cong \bigoplus_{b \in \mathbf{B}^\uparrow} T_\varepsilon^\uparrow(b)^{\oplus n_b(\omega)}$ for $n_b(\omega) \in \mathbb{N}$. There is a corresponding direct summand $T_\omega \cong \bigoplus_{b \in \mathbf{B}^\uparrow} T_\varepsilon(b)^{\oplus n_b(\omega)}$ of U. Then $T = \varinjlim T_\omega$. Moreover, for $b \in \mathbf{B}^\uparrow$, the multiplicities $n_b(\omega)$ are stable in the sense that $n_b(\upsilon) = n_b(\omega)$ for all $\upsilon > \omega$. We deduce that $U \cong \bigoplus_{b \in \mathbf{B}} T_\varepsilon(b)^{\oplus n_b}$ where $n_b := n_b(\omega)$ for any sufficiently large ω. □

There are also obvious analogs of Corollaries 4.4 and 4.5 in the upper finite setting.

4.4. **Semi-infinite Ringel duality.** Throughout the subsection, Λ will be a lower finite poset and $\varepsilon : \Lambda \to \{\pm\}$ is a sign function. The opposite poset Λ^{op} is upper finite. The goal is to extend Ringel duality to include stratifications indexed by Λ or Λ^{op}. The situation is not as symmetric as in the finite case so we need two different formulations, one in each direction.

If we start with a lower finite ε-stratified category, the Ringel dual is an upper finite $(-\varepsilon)$-stratified category:

Definition 4.19. Let $\mathcal{R}$ be a lower finite ε-stratified category with stratification defined by $\rho : \mathbf{B} \to \Lambda$. A *full ε-tilting family* is a family $(T_i)_{i \in I}$ of ε-tilting objects in $\mathcal{R}$ such that every $T_\varepsilon(b)$ is isomorphic to a summand of T_i for some $i \in I$. Define the *Ringel dual* of $\mathcal{R}$ relative to $T := \bigoplus_{i \in I} T_i \in \operatorname{Ind}(\mathcal{R})$ to be $\widetilde{\mathcal{R}} := A\text{-mod}_{\mathrm{lfd}}$ where $A = \bigoplus_{i,j \in I} e_i A e_j$ is the locally finite-dimensional locally unital algebra with $e_i A e_j := \operatorname{Hom}_{\mathcal{R}}(T_i, T_j)$ and multiplication that is the opposite of composition in $\mathcal{R}$. Identifying $\operatorname{Ind}(\widetilde{\mathcal{R}})$ with A-mod as usual, we have the (covariant) *Ringel duality functors*

$$F := \bigoplus_{i \in I} \operatorname{Hom}_{\mathcal{R}}(T_i, -) : \operatorname{Ind}(\mathcal{R}) \to \operatorname{Ind}(\widetilde{\mathcal{R}}), \tag{4.9}$$

$$G := \circledast \circ \bigoplus_{i \in I} \operatorname{Hom}_{\mathcal{R}}(-, T_i) : \operatorname{Pro}(\mathcal{R}) \to \operatorname{Pro}(\widetilde{\mathcal{R}}). \tag{4.10}$$

These are instances of the functor (2.17) and 2.20), respectively. Both F and G take objects of $\mathcal{R}$ to objects of $\widetilde{\mathcal{R}}$.

Theorem 4.20. *In the setup of Definition 4.19, $\widetilde{\mathcal{R}}$ is an upper finite $(-\varepsilon)$-stratified category with stratification defined from $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$. Its distinguished objects satisfy*

$$\widetilde{P}(b) \cong FT_\varepsilon(b), \qquad \widetilde{I}(b) \cong GT_\varepsilon(b), \qquad \widetilde{L}(b) \cong \mathrm{hd}\ \widetilde{P}(b) \cong \mathrm{soc}\ \widetilde{I}(b),$$
$$\widetilde{\Delta}_{-\varepsilon}(b) \cong F\nabla_\varepsilon(b), \qquad \widetilde{\nabla}_{-\varepsilon}(b) \cong G\Delta_\varepsilon(b), \qquad \widetilde{T}_{-\varepsilon}(b) \cong FI(b).$$

The restrictions $F : \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R}) \to \Delta^{\mathrm{asc}}_{-\varepsilon}(\widetilde{\mathcal{R}})$ and $G : \Delta_\varepsilon(\mathcal{R}) \to \nabla_{-\varepsilon}(\widetilde{\mathcal{R}})$ are equivalences.

The proof will be explained later. In the other direction, if we start from an upper finite $(-\varepsilon)$-stratified category, the Ringel dual is a lower finite ε-stratified category:

Definition 4.21. Let $\widetilde{\mathcal{R}}$ be an upper finite $(-\varepsilon)$-stratified category with stratification defined by $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$. A *full $(-\varepsilon)$-tilting object* is an object $\widetilde{T} \in \mathrm{Tilt}_{-\varepsilon}(\widetilde{\mathcal{R}})$ such that $\widetilde{T}_{-\varepsilon}(b)$ is a summand of $\widetilde{T}$ for every $b \in \mathbf{B}$. By Lemma 2.8, the algebra $B := \mathrm{End}_{\widetilde{\mathcal{R}}}(\widetilde{T})^{\mathrm{op}}$ is a pseudocompact topological algebra with respect to the profinite topology; let C be the coalgebra that is its continuous dual. Then the *Ringel dual* of $\mathcal{R}$ relative to $\widetilde{T}$ is the category $\mathcal{R} := B\text{-mod}_{\mathrm{fd}} \cong \mathrm{comod}_{\mathrm{fd}}\text{-}C$. Recalling the continuous duality functor $*$ from (2.7), there are *Ringel duality functors*

$$\widetilde{F} := * \circ \mathrm{Hom}_{\widetilde{\mathcal{R}}}(-, \widetilde{T}) : \mathrm{Ind}(\widetilde{\mathcal{R}}) \to \mathrm{Ind}(\mathcal{R}), \tag{4.11}$$
$$\widetilde{G} := \mathrm{Hom}_{\widetilde{\mathcal{R}}}(\widetilde{T}, -) : \mathrm{Pro}(\widetilde{\mathcal{R}}) \to \mathrm{Pro}(\mathcal{R}). \tag{4.12}$$

These are instances of the functors (2.16) and (2.21), respectively. The functor $\widetilde{F}$ sends finitely generated objects of $\widetilde{\mathcal{R}}$ to objects of $\mathcal{R}$, while $\widetilde{G}$ sends finitely cogenerated objects of $\widetilde{\mathcal{R}}$ to objects of $\mathcal{R}$.

Theorem 4.22. *In the setup of Definition 4.21, $\mathcal{R}$ is a lower finite ε-stratified category with stratification defined from $\rho : \mathbf{B} \to \Lambda$. Its distinguished objects satisfy*

$$I(b) \cong \widetilde{F}\widetilde{T}_{-\varepsilon}(b), \qquad L(b) \cong \mathrm{soc}\ I(b),$$
$$\Delta_\varepsilon(b) \cong \widetilde{G}\widetilde{\nabla}_{-\varepsilon}(b), \qquad \nabla_\varepsilon(b) \cong \widetilde{F}\widetilde{\Delta}_{-\varepsilon}(b), \qquad T_\varepsilon(b) \cong \widetilde{F}\widetilde{P}(b) \cong \widetilde{G}\widetilde{I}(b).$$

The restrictions $\widetilde{F} : \Delta^{\mathrm{asc}}_{-\varepsilon}(\widetilde{\mathcal{R}}) \to \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ and $\widetilde{G} : \nabla_{-\varepsilon}(\widetilde{\mathcal{R}}) \to \Delta_\varepsilon(\mathcal{R})$ are equivalences.

The following corollaries are the analogs of the double centralizer property from Corollary 4.12 in the semi-infinite setting.

Corollary 4.23. *Let notation be as in Definition 4.19. Assume in addition that $\mathcal{R} =$ $\mathrm{comod}_{\mathrm{fd}}$-$C$ for a coalgebra C. Let $B := C^*$ be the dual algebra, so that T is a (B, A)-bimodule. Let $\widetilde{T} := T^{\circledast}$ be the dual (A, B)-bimodule.*

(1) $\widetilde{T}$ is a full $(-\varepsilon)$-tilting object in $\widetilde{\mathcal{R}}$ such that $B = \mathrm{End}_{\widetilde{\mathcal{R}}}(\widetilde{T})^{\mathrm{op}}$. Thus, the Ringel dual of $\widetilde{\mathcal{R}}$ relative to $\widetilde{T}$ is the original category $\mathcal{R}$.

(2) Recalling (2.21), the functors (4.11)–(4.12) satisfy

$$\widetilde{F} \cong T \otimes_A -, \qquad G \cong \widetilde{T}\overline{\otimes}_B - . \tag{4.13}$$

Hence, $(\widetilde{F}, F)$ and $(G, \widetilde{G})$ are adjoint pairs.

(3) The restrictions $\widetilde{F} : \Delta^{\mathrm{asc}}_{-\varepsilon}(\widetilde{\mathcal{R}}) \to \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ and $\widetilde{G} : \nabla_{-\varepsilon}(\widetilde{\mathcal{R}}) \to \Delta_\varepsilon(\mathcal{R})$ are quasi-inverses of the equivalences defined by F and G, respectively.

Proof. (1) By Lemma 2.2, we have that $\mathrm{Hom}_C(T_i, C) \cong T_i^*$ as right B-modules, hence, $FC \cong \widetilde{T}$ as an (A, B)-bimodule. Since every $I(b)$ appears as a summand of the regular comodule, and $FI(b) \cong \widetilde{T}_{-\varepsilon}(b)$ by Theorem 4.20, we deduce that $\widetilde{T}$ is a full $(-\varepsilon)$-tilting module in $\widetilde{\mathcal{R}}$. To see that $B = \operatorname{End}_A(\widetilde{T})^{\mathrm{op}}$, we use the fact that F is an equivalence on ∇-filtered objects to deduce that

$$\operatorname{End}_A(\widetilde{T})^{\mathrm{op}} \cong \operatorname{End}_A(FC)^{\mathrm{op}} \cong \operatorname{End}_C(C)^{\mathrm{op}} \cong B,$$

using Lemma 2.2 again for the final algebra isomorphism.

(2) We get that $\widetilde{F} = * \circ \operatorname{Hom}_A(-, \widetilde{T}) \cong T \otimes_A -$ and $\widetilde{F}$ is left adjoint to F by Lemma 2.9. Also $G \cong \widetilde{T} \otimes_B -$ and G is left adjoint to $\widetilde{G}$ by Lemma 2.10.

(3) Theorems 4.20 and 4.22 together show that F and $\widetilde{F}$ restrict to equivalences between $\Delta^{\mathrm{asc}}_{-\varepsilon}(\widetilde{\mathcal{R}})$ and $\nabla^{\mathrm{asc}}_{\varepsilon}(\mathcal{R})$. They are quasi-inverse to each other since $\widetilde{F}$ is left adjoint to F. Similarly for G and $\widetilde{G}$. □

Corollary 4.24. *Let notation be as in Definition 4.21, and assume in addition that $\widetilde{R} = A\text{-mod}_{\mathrm{lfd}}$ for a locally finite-dimensional locally unital algebra $A = \bigoplus_{i,j \in I} e_i A e_j$. Let $T := \widetilde{T}^{\circledast}$, which is a (B, A)-bimodule. Set $T_i := T e_i \in B\text{-mod}_{\mathrm{fd}}$.*

(1) $(T_i)_{i \in I}$ is a full ε-tilting family in $\mathcal{R}$ such that $A = \bigoplus_{i,j \in I} \operatorname{Hom}_{\mathcal{R}}(T_i, T_j)$ with multiplication coming from the opposite of composition in $\mathcal{R}$. Thus, the Ringel dual of $\mathcal{R}$ relative to T is the category $\widetilde{\mathcal{R}}$.

(2) Defining F and G from (4.9)–(4.10), we again have the isomorphisms (4.13).

(3) The restrictions $F : \nabla^{\mathrm{asc}}_{\varepsilon}(\mathcal{R}) \to \Delta^{\mathrm{asc}}_{-\varepsilon}(\widetilde{\mathcal{R}})$ and $G : \Delta_{\varepsilon}(\mathcal{R}) \to \nabla_{-\varepsilon}(\widetilde{\mathcal{R}})$ are quasi-inverses of the equivalences defined by $\widetilde{F}$ and $\widetilde{G}$, respectively.

Proof. Note that $T_i = \widetilde{F}(Ae_i)$. So Theorem 4.22 implies that $(T_i)_{i \in I}$ is a full ε-tilting family in $\mathcal{R}$. Moreover, $\operatorname{Hom}_{\mathcal{R}}(T_i, T_j) = \operatorname{Hom}_{\mathcal{R}}(\widetilde{F}(Ae_i), \widetilde{F}(Ae_j)) \cong \operatorname{Hom}_A(Ae_i, Ae_j)$. Part (1) follows easily. This puts us in exactly the same situation as Corollary 4.23. The remaining parts (2) and (3) follow from the corresponding parts of that corollary. □

Corollary 4.13 carries over to the semi-infinite case. We leave this to the reader. We have not investigated derived equivalences or any analog of Theorem 4.15 in this setting.

Remark 4.25. In the context of Corollaries 4.23–4.24, let $x^{(i)}_1, \dots, x^{(i)}_{n_i}$ be a basis for T_i, $y^{(i)}_1, \dots, y^{(i)}_{n_i}$ be the dual basis for T_i^*, and set $c^{(i)}_{s,r} := x^{(i)}_r \otimes y^{(i)}_s \in T \otimes_A \widetilde{T}$. Then the coalgebra C may be identified with $T \otimes_A \widetilde{T}$ viewed as a coalgebra via (2.13) so that the elements $c^{(i)}_{r,s}$ are the coefficient functions of the right C-comodule T with respect to the chosen basis, i.e., the comultiplication $\eta : T \to T \otimes C$ satisies $\eta(x^{(i)}_s) = \sum_{r=1}^{n_i} x^{(i)}_r \otimes c^{(i)}_{r,s}$. This follows from Lemma 2.8 and the discussion just before it.

Proof of Theorem 4.20. We may assume that $\mathcal{R} = \text{comod}_{\mathrm{fd}}\text{-}C$ for a coalgebra C. Let $B := C^*$ be the dual algebra, so that $\mathcal{R}$ is identified also with $B\text{-mod}_{\mathrm{fd}}$. As in steps (11)–(12) of the proof of Theorem 4.11, all of the statements involving the functor G follow from the ones involving F, or rather, its right module analog F^{op}. This depends on the isomorphism $G \cong \circledast \circ F^{\mathrm{op}} \circ *$ which was noted already in the proof of Lemma 2.10. For example, the fact that G restricts to a well-defined equivalence between $\Delta_{\varepsilon}(\mathcal{R})$ and $\nabla_{-\varepsilon}(\widetilde{\mathcal{R}})$ follows because F^{op} restricts to an equivalence $\nabla_{-\varepsilon}(\mathcal{R}^{\mathrm{op}})$ and $\Delta_{-\varepsilon}(\widetilde{\mathcal{R}}^{\mathrm{op}})$. We will not mention the functor G again in the remainder of the proof.

Observe next that we can replace the full ε-tilting family $(T_i)_{i \in I}$ with any other such family. This just has the effect of transforming A into a Morita equivalent locally unital algebra. Consequently, without loss of generality, we may assume that $I = \mathbf{B}$ and $(T_i)_{i \in I} = (T_{\varepsilon}(b))_{b \in \mathbf{B}}$. Then

$$A = \bigoplus_{a,b \in \mathbf{B}} \operatorname{Hom}_B(T_{\varepsilon}(a), T_{\varepsilon}(b))$$

is a pointed locally finite-dimensional locally unital algebra, with primitive idempotents denoted $\{e_b \mid b \in \mathbf{B}\}$. Let $\widetilde{P}(b) := Ae_b$ and $\widetilde{L}(b) := \operatorname{hd} \widetilde{P}(b)$. Then $\widetilde{\mathcal{R}} = A\text{-mod}_{\mathrm{lfd}}$ is a locally Schurian category, the A-modules $\{\widetilde{L}(b) \mid b \in \mathbf{B}\}$ give a full set of pairwise inequivalent irreducible objects, and $\widetilde{P}(b)$ is a projective cover of $\widetilde{L}(b)$. It is immediate that $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$ gives a stratification of $\widetilde{\mathcal{R}}$. Let $\widetilde{\Delta}_{-\varepsilon}(b)$ and $\widetilde{\nabla}_{-\varepsilon}(b)$ be its $(-\varepsilon)$-standard and $(-\varepsilon)$-costandard objects. Also let $V(b) := F\nabla_\varepsilon(b)$.

Now one checks that steps (1)–(6) from the proof of Theorem 4.11 carry over to the present situation with very minor modifications. We will not rewrite these steps here, but cite them freely below. In particular, (6) establishes that $\widetilde{\mathcal{R}}$ is an upper finite $(-\varepsilon)$-stratified category. Also, $F\nabla_\varepsilon(b) \cong \widetilde{\Delta}_{-\varepsilon}(b)$ by (5). It just remains to show that

- F restricts to an equivalence of categories between $\nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ and $\Delta_{-\varepsilon}^{\mathrm{asc}}(\widetilde{\mathcal{R}})$;
- $FI(b) \cong \widetilde{T}_{-\varepsilon}(b)$.

This requires some different arguments compared to the ones from (7)–(10) in the proof of Theorem 4.11.

Let Ω be the directed poset consisting of all finite lower sets in Λ. Take $\omega \in \Omega$, say it is the lower set $\Lambda^\downarrow$. Let $\nabla_\varepsilon(\mathcal{R}, \omega)$ be the full subcategory of $\nabla_\varepsilon(\mathcal{R})$ consisting of the ∇_ε-filtered objects with sections $\nabla_\varepsilon(b)$ for $b \in \mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$. Similarly, we define the subcategory $\Delta_{-\varepsilon}(\widetilde{\mathcal{R}}, \omega)$ of $\Delta_{-\varepsilon}(\widetilde{\mathcal{R}})$. By (2) and (5), F restricts to a well-defined functor

$$F : \nabla_\varepsilon(\mathcal{R}, \omega) \to \Delta_{-\varepsilon}(\widetilde{\mathcal{R}}, \omega). \tag{4.14}$$

We claim that this is actually an equivalence of categories. To prove it, let $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the finite ε-stratified subcategory of $\mathcal{R}$ associated to $\Lambda^\downarrow$. Let $e := \sum_{b \in \mathbf{B}^\downarrow} e_b \in A$. Then $T^\downarrow := \bigoplus_{b \in \mathbf{B}^\downarrow} T_\varepsilon(b)$ is a full ε-tilting object in $\mathcal{R}^\downarrow$. As $\operatorname{End}_{\mathcal{R}^\downarrow}(T^\downarrow)^{\mathrm{op}} = eAe$, the Ringel dual of $\mathcal{R}^\downarrow$ relative to $T^\downarrow$ is the category $\widetilde{\mathcal{R}}^\downarrow := eAe\text{-mod}_{\mathrm{fd}}$; let $F^\downarrow := \operatorname{Hom}_C(T^\downarrow, -)$ be the corresponding Ringel duality functor. We know that $\widetilde{\mathcal{R}}^\downarrow$ is the finite $(-\varepsilon)$-stratified quotient of $\widetilde{\mathcal{R}}$ associated to $\Lambda^\downarrow$ (which is a finite upper set in Λ^{op}). Let $\tilde{\jmath} : \widetilde{\mathcal{R}} \to \widetilde{\mathcal{R}}^\downarrow$ be the quotient functor, i.e., the functor defined by multiplication by the idempotent e. For a right C-comodule V, we have that

$$F^\downarrow(i^*V) \cong \bigoplus_{b \in \mathbf{B}^\downarrow} \operatorname{Hom}_C(T_\varepsilon(b), i^*V) \cong e \bigoplus_{b \in \mathbf{B}} \operatorname{Hom}_C(T_\varepsilon(b), V) \cong \tilde{\jmath}(FV).$$

This shows that

$$F^\downarrow \circ i^* \cong \tilde{\jmath} \circ F. \tag{4.15}$$

By Theorem 4.11, $F^\downarrow$ gives an equivalence $\nabla_\varepsilon(\mathcal{R}^\downarrow) \to \Delta_{-\varepsilon}(\widetilde{\mathcal{R}}^\downarrow)$. Also $i^* : \nabla_\varepsilon(\mathcal{R}, \omega) \to \nabla_\varepsilon(\mathcal{R}^\downarrow)$ and $\tilde{\jmath} : \Delta_{-\varepsilon}(\widetilde{\mathcal{R}}, \omega) \to \Delta_{-\varepsilon}(\widetilde{\mathcal{R}}^\downarrow)$ are equivalences. This is clear for i^*. To see it for $\tilde{\jmath}$, one shows using Theorem 3.37 that the left adjoint $\tilde{\jmath}_!$ gives a quasi-inverse equivalence. Putting these things together, we deduce that (4.14) is an equivalence as claimed.

Now we can show that F restricts to an equivalence $F : \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R}) \to \Delta_{-\varepsilon}^{\mathrm{asc}}(\widetilde{\mathcal{R}})$. Take $V \in \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$. Then V has a distinguished ascending ∇_ε-flag $(V_\omega)_{\omega \in \Omega}$ indexed by the set Ω of finite lower sets in Λ. This is defined by setting $V_\omega := i^! V$ in the notation of the previous paragraph; see the proof of Theorem 3.59. As each comodule $T_\varepsilon(b)$ is finite-dimensional, hence, compact, the functor F commutes with direct limits. Hence, $FV \cong \varinjlim(FV_\omega)$. In fact, $(FV_\omega)_{\omega \in \Omega}$ is the data of an ascending $\Delta_{-\varepsilon}$-flag in $FV \in \widetilde{\mathcal{R}}$. To see this, we have that $FV_\omega \in \Delta_{-\varepsilon}(\widetilde{\mathcal{R}})$ by the previous paragraph. For $\omega < \upsilon$ the quotient V_υ / V_ω has a ∇_ε-flag thanks to Corollary 3.60, so $FV_\upsilon / FV_\omega \cong F(V_\upsilon / V_\omega)$ has a $\Delta_{-\varepsilon}$-flag. Finally we must show that $FV \in \widetilde{\mathcal{R}}$ (rather than $\operatorname{Ind}(\widetilde{\mathcal{R}})$). It suffices to show that $\dim \operatorname{Hom}_A(FV, \widetilde{I}(b)) < \infty$ for each $b \in \mathbf{B}$. Since $\widetilde{I}(b)$ has a finite $\nabla_{-\varepsilon}$-flag, this reduces to checking that $\dim \operatorname{Hom}_A(FV, \widetilde{\nabla}_{-\varepsilon}(b)) < \infty$ for each b, which holds because the multiplicities $(V_\omega : \nabla_\varepsilon(b))$ are bounded by the definition of the category $\nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$. This shows in particular that F restricts to a well-defined functor

$$F : \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R}) \to \Delta^{\mathrm{asc}}_{-\varepsilon}(\widetilde{\mathcal{R}}).$$

We prove that this is an equivalence by showing that the left adjoint $\widetilde{F} := T \otimes_A -$ to F gives a quasi-inverse. The left mate of (4.15) gives an isomorphism

$$i \circ \widetilde{(F^\downarrow)} \cong \widetilde{F} \circ \tilde{\jmath}_! . \tag{4.16}$$

Combining this with Corollary 4.12, we deduce that $\widetilde{F}$ restricts to a quasi-inverse of the equivalence (4.14) for each $\omega \in \Omega$. Also, $\widetilde{F}$ commutes with direct limits, and again any $\widetilde{V} \in \Delta^{\mathrm{asc}}_{-\varepsilon}(\widetilde{\mathcal{R}})$ has a distinguished ascending $\Delta_{-\varepsilon}$-flag $(\widetilde{V}_\omega)_{\omega\in\Omega}$ as we saw in the proof of Theorem 3.38. These facts are enough to show that $\widetilde{F}$ restricts to a well-defined functor $\widetilde{F} : \Delta^{\mathrm{asc}}_{-\varepsilon}(\widetilde{\mathcal{R}}) \to \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ which is quasi-inverse to F.

Finally, we check that $FI(b) \cong \widetilde{T}_{-\varepsilon}(b)$. Let $V := I(b)$ and $(V_\omega)_{\omega\in\Omega}$ be its distinguished ascending ∇_ε-flag indexed by the set Ω of finite lower sets in Λ. Using the same notation as above, for ω that is a lower set $\Lambda^\downarrow$ satisfying $\rho(b) \in \Lambda^\downarrow$, we know that V_ω is an injective hull of $L(b)$ in $\mathcal{R}^\downarrow$. Hence, by Theorem 4.11, $F^\downarrow V_\omega \cong \widetilde{T}^\downarrow_{-\varepsilon}(b) \in \widetilde{\mathcal{R}}^\downarrow$. From this, we see that the ascending $\Delta_{-\varepsilon}$-flag (FV_ω) in $FI(b)$ coincides with the distinsguished ascending $\Delta_{-\varepsilon}$-flag in $\widetilde{T}_{-\varepsilon}(b)$ from the construction from the proof of Theorem 4.17. □

Proof of Theorem 4.22. We may assume that $\widetilde{\mathcal{R}} = A\text{-mod}_{\mathrm{lfd}}$ for a pointed locally unital algebra $A = \bigoplus_{a,b\in\mathbf{B}} e_a A e_b$, so that $\widetilde{T}$ is a locally finite-dimensional left A-module. Let $T := \widetilde{T}^{\circledast}$ and $C := T \otimes_A T^{\circledast}$, which we view as a coalgebra according to (2.13). By Lemma 2.8, this coalgebra is the continuous dual of $B = \mathrm{End}_A(\widetilde{T})^{\mathrm{op}}$. We may identify $\mathcal{R}$ with $\mathrm{comod}_{\mathrm{fd}}\text{-}C$, which is a locally finite Abelian category. Applying Lemma 2.9, we can also identify the Ringel duality functor $\widetilde{F}$ with the functor $T \otimes_A - : A\text{-mod} \to \text{comod-}C$, the comodule structure map of $T \otimes_A V$ being defined as in (2.15). Let

$$I(b) := \widetilde{F}\widetilde{T}_{-\varepsilon}(b), \qquad \nabla_\varepsilon(b) := \widetilde{F}\widetilde{\Delta}_{-\varepsilon}(b), \qquad L(b) := \mathrm{soc}\; I(b). \tag{4.17}$$

Each $I(b)$ is an indecomposable injective right C-comodule, and $\{L(b) \,|\, b \in \mathbf{B}\}$ is a full set of pairwise inequivalent irreducible comodules. Since $\widetilde{\Delta}_{-\varepsilon}(b) \hookrightarrow \widetilde{T}_{-\varepsilon}(b)$, and $\widetilde{F}$ is exact on $\Delta^{\mathrm{asc}}_{-\varepsilon}(\widetilde{\mathcal{R}})$ by the original definition of $\widetilde{F}$ and the Ext^1-vanishing from Lemma 3.36, we see that $\nabla_\varepsilon(b) \hookrightarrow I(b)$. Thus, we also have that $L(b) = \mathrm{soc}\; \nabla_\varepsilon(b)$.

Now let $\Lambda^\downarrow$ be a finite lower set in Λ. Set $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$, and let $\tilde{\jmath} : \widetilde{\mathcal{R}} \to \widetilde{\mathcal{R}}^\downarrow$ be the corresponding Serre quotient of $\widetilde{\mathcal{R}}$. Since $\Lambda^\downarrow$ is a finite upper set in Λ^{op}, this is a finite $(-\varepsilon)$-stratified category thanks to Theorem 3.37. In fact, $\widetilde{\mathcal{R}}^\downarrow = eAe\text{-mod}_{\mathrm{lfd}}$ where $e := \sum_{b\in\mathbf{B}^\downarrow} e_b$, and $\tilde{\jmath}$ is the functor defined by multiplying by e. By the upper finite analog of Corollary 4.5, $e\widetilde{T}$ is a full $(-\varepsilon)$-tilting object in $\widetilde{\mathcal{R}}^\downarrow$. Let $B^\downarrow := \mathrm{End}_{eAe}(e\widetilde{T})^{\mathrm{op}}$ be its (finite-dimensional) endomorphism algebra. Then $\mathcal{R}^\downarrow := B^\downarrow\text{-mod}_{\mathrm{fd}}$ is the Ringel dual of $\widetilde{\mathcal{R}}^\downarrow$ relative to $e\widetilde{T}$. By the finite Ringel duality from Theorem 4.11, $\mathcal{R}^\downarrow$ is a finite ε-stratified category. Let $\widetilde{F}^\downarrow : \widetilde{\mathcal{R}}^\downarrow \to \mathcal{R}^\downarrow$ be its Ringel duality functor.

The functor $\tilde{\jmath}$ defines an algebra homomorphism

$$\pi : B \to B^\downarrow. \tag{4.18}$$

We claim that π is surjective. To prove this, consider the short exact sequence

$$0 \longrightarrow Ae \otimes_{eAe} e\widetilde{T} \longrightarrow \widetilde{T} \longrightarrow Q \longrightarrow 0 \tag{4.19}$$

which comes from the upper-finite counterpart of Lemma 3.19(2); thus, $Q \in \Delta^{\mathrm{asc}}_{-\varepsilon}(\mathcal{R})$ and all of its sections are of the form $\widetilde{\Delta}_{-\varepsilon}(b)$ for $b \notin \mathbf{B}^\downarrow$. Applying the functor

$\mathrm{Hom}_A(-,\widetilde{T})$ and using that $\mathrm{Ext}^1_A(Q,\widetilde{T}) = 0$, we deduce that the natural restriction map $\mathrm{Hom}_A(\widetilde{T},\widetilde{T}) \to \mathrm{Hom}_A(Ae\otimes_{eAe} e\widetilde{T},\widetilde{T})$ is surjective. Since $\mathrm{Hom}_A(Ae\otimes_{eAe} e\widetilde{T},\widetilde{T}) \cong \mathrm{Hom}_{eAe}(e\widetilde{T}, e\widetilde{T})$, this proves the claim.

From (4.18), we see that the natural inflation functor $i : \mathcal{R}^{\downarrow} \to \mathcal{R}$ includes $\mathcal{R}^{\downarrow}$ as an Abelian subcategory of $\mathcal{R}$. We claim moreover that that

$$i \circ \widetilde{F}^{\downarrow} \cong \widetilde{F} \circ \tilde{\jmath}_!. \tag{4.20}$$

This can be proved in the same way as (4.16) above, but the following alternative argument is more convenient in the present setting: consider the dual coalgebra

$$C^{\downarrow} := (B^{\downarrow})^*.$$

The dual map π^* to (4.18) defines a coalgebra homomorphism $C^{\downarrow} \to C$. Moreover, if we identify $C^{\downarrow}$ with $Te \otimes_{eAe} e\widetilde{T}$ like in Lemma 2.7 then π^* corresponds to the obvious coalgebra homomorphism $\pi^* : Te\otimes_{eAe} e\widetilde{T} \to T\otimes_A \widetilde{T}$ induced by the inclusion $Te\otimes e\widetilde{T} \hookrightarrow T \otimes \widetilde{T}$. Since π is surjective, the dual map π^* is injective, so it identifies $C^{\downarrow}$ with a subcoalgebra of C. Now the functor $\widetilde{F}^{\downarrow}$ is $Te \otimes_{eAe} - : eAe\text{-mod} \to \text{comod-}C^{\downarrow}$, and we get (4.20) since $T \otimes_A Ae \otimes_{eAe} V \cong Te \otimes_{eAe} V$ for any eAe-module V.

From (4.20) and Theorem 3.37(6), we see that ε-costandard objects of $\mathcal{R}^{\downarrow}$ are the comodules $\{\nabla_\varepsilon(b) | b \in \mathbf{B}^{\downarrow}\}$ defined by (4.17). Representatives for the isomorphism classes of irreducible objects in $\mathcal{R}^{\downarrow}$ are given by the socles $\{L(b) \mid b \in \mathbf{B}^{\downarrow}\}$ of these costandard objects. In fact, $\mathcal{R}^{\downarrow}$ is the Serre subcategory of $\mathcal{R}$ generated by $\{L(b) \mid b \in \mathbf{B}^{\downarrow}\}$. To prove this, by Lemma 2.20, it suffices to show that $C^{\downarrow}$ is the largest right coideal of C such that all of its irreducible subquotients are of the form $\{L(b) \mid b \in \mathbf{B}^{\downarrow}\}$. Apply $\widetilde{F}$ to (4.19), using the exactness noted before, to get a short exact sequence

$$0 \longrightarrow Te \otimes_{eAe} e\widetilde{T} \longrightarrow T \otimes_A \widetilde{T} \longrightarrow \widetilde{F}Q \longrightarrow 0.$$

Since $C^{\downarrow} = Te \otimes_{eAe} e\widetilde{T}$ and $C = T \otimes_A \widetilde{T}$, this shows that $C/C^{\downarrow} \cong \widetilde{F}Q$. To finish the argument we show that all irreducible constituents of $\mathrm{soc}\,(\widetilde{F}Q)$ are of the form $L(b)$ for $b \notin \mathbf{B}^{\downarrow}$. Fix an ascending $\Delta_{-\varepsilon}$-flag $(V_\omega)_{\omega\in\Omega}$ in Q. As $\widetilde{F}$ commutes with direct limits and is exact on $\Delta_{-\varepsilon}$-flags, we deduce that $\widetilde{F}Q$ is the union of subobjects of the form $\widetilde{F}V_\omega$. Now the sections in a Δ_ε-flag in V_ω are $\widetilde{\Delta}_{-\varepsilon}(b)$ for $b \notin \mathbf{B}^{\downarrow}$, hence, $\widetilde{F}V_\omega$ has a ∇_ε-flag with sections $\nabla_\varepsilon(b)$ for $b \notin \mathbf{B}^{\downarrow}$. It follows that $\mathrm{soc}\,(\widetilde{F}V_\omega)$ is of the desired form for each ω, hence, the socle of $\widetilde{F}Q$ is too.

We can now complete the proof of the theorem. Taking the finite lower set $\Lambda^{\downarrow}$ in the above to be $(-\infty, \lambda]$, we deduce that $\mathcal{R}_{\leqslant\lambda}$ is a finite Abelian category, verifying the property (S4) from Definition 3.1. This means that $\rho : \mathbf{B} \to \Lambda$ defines a stratification of $\mathcal{R}$. Then we get that $\mathcal{R}$ is a lower finite ε-stratified category by applying Lemma 3.62. Theorem 4.11 once again shows for any choice of $\Lambda^{\downarrow}$ that the ε-tilting object of $\mathcal{R}^{\downarrow}$ indexed by $b \in \mathbf{B}^{\downarrow}$ is

$$T^{\downarrow}_\varepsilon(b) := \widetilde{F}^{\downarrow}(\tilde{\jmath}\widetilde{P}(b)) \cong \widetilde{F}(\tilde{\jmath}_!(\tilde{\jmath}\widetilde{P}(b))) \cong \widetilde{F}\widetilde{P}(b).$$

This is also the ε-tilting object $T_\varepsilon(b)$ of $\mathcal{R}$ due to Corollary 4.4. Also, for $a, b \in \mathbf{B}^{\downarrow}$, we have that

$$\mathrm{Hom}_C(T_\varepsilon(a), T_\varepsilon(b)) \cong \mathrm{Hom}_{C^{\downarrow}}(T^{\downarrow}_\varepsilon(a), T^{\downarrow}_\varepsilon(b)) \cong \mathrm{Hom}_{eAe}(e\widetilde{P}(a), e\widetilde{P}(b)) \cong e_a A e_b.$$

These things are true for all choices of $\Lambda^{\downarrow}$, so we see that the Ringel dual of $\mathcal{R}$ relative to $\bigoplus_{b\in\mathbf{B}} T_\varepsilon(b)$ is the original category $A\text{-mod}_{\mathrm{lfd}}$. This puts us in the situation of Corollary 4.23, and finally we invoke that corollary (whose proof did not depend on Theorem 4.22) to establish that $\widetilde{F} : \Delta^{\mathrm{asc}}_{-\varepsilon}(\widetilde{\mathcal{R}}) \to \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R})$ is an equivalence. $\square$

4.5. **The essentially finite case.** In this subsection, we let $\mathcal{R}$ be an essentially finite ε-stratified category with stratification defined from $\rho : \mathbf{B} \to \Lambda$. As usual, Λ^{op} denotes the opposite poset. Since Λ is interval finite, unions of lower sets of the form $\Lambda_{\leqslant\lambda}$ are upper finite. If $\mathcal{R}^{\downarrow}$ is the Serre subcategory of $\mathcal{R}$ associated to such an upper finite lower set then its locally Schurian envelope $\mathrm{Loc}(\mathcal{R}^{\downarrow})$ is an upper finite ε-stratified category. This follows from Theorem 3.17.

For $b \in \mathbf{B}$, we define the corresponding ε-tilting object $T_\varepsilon(b) \in \mathrm{Loc}(\mathcal{R})$ as follows: pick any upper finite lower set $\Lambda^{\downarrow}$ such that $\rho(b) \in \Lambda^{\downarrow}$, let $\mathcal{R}^{\downarrow}$ be the corresponding Serre subcategory of $\mathcal{R}$, then let $T_\varepsilon(b)$ be the ε-tilting object in $\mathrm{Loc}(\mathcal{R}^{\downarrow})$ from Theorem 4.17. This is well-defined independent of the choice of $\Lambda^{\downarrow}$ by the uniqueness part of Theorem 4.17. We will only consider Ringel duality in the essentially finite case under the hypothesis that $\mathcal{R}$ is *ε-tilting-bounded*, meaning that the matrix

$$\big(\dim \mathrm{Hom}_{\mathcal{R}}(T_\varepsilon(a), T_\varepsilon(b))\big)_{a,b\in\mathbf{B}} \tag{4.21}$$

has only finitely many non-zero entries in every row and in every column. This condition implies in particular that each $T_\varepsilon(b)$ is of finite length, i.e., it belongs to $\mathcal{R}$ rather than $\mathrm{Loc}(\mathcal{R})$. From the recursive construction of Theorem 4.17, there is no apparent reason why this should hold, but it does in all of the examples that we are aware of "in nature."

Assuming $\mathcal{R}$ is ε-tilting-bounded, we define

$$\mathrm{Tilt}_\varepsilon(\mathcal{R}) := \Delta_\varepsilon(\mathcal{R}) \cap \nabla_\varepsilon(\mathcal{R}) \tag{4.22}$$

just like in (4.1). Theorem 4.2 carries over easily, to show that $\{T_\varepsilon(b)\,|\, b \in \mathbf{B}\}$ gives a full set of the indecomposable objects in the Karoubian category $\mathrm{Tilt}_\varepsilon(\mathcal{R})$. The construction of Theorem 4.9 also carries over unchanged. So all objects of $\nabla_\varepsilon(\mathcal{R})$ have ε-tilting resolutions and all objects of $\Delta_\varepsilon(\mathcal{R})$ have ε-cotilting resolutions.

Definition 4.26. A *full ε-tilting family* in $\mathcal{R}$ means a family $(T_i)_{i\in I}$ of objects $T_i \in \mathcal{R}$ such that each T_i is a direct sum of the objects $T_\varepsilon(b)$ and every $T_\varepsilon(b)$ appears as a summand of at least one and at most finitely many different T_i. Given such a family, we define the *Ringel dual* of $\mathcal{R}$ relative to $(T_i)_{i\in I}$ to be the category $\widetilde{\mathcal{R}} := A\text{-mod}_{\mathrm{fd}}$ where $A := \bigoplus_{i,j\in I} \mathrm{Hom}_{\mathcal{R}}(T_i, T_j)$ with multiplication that is the opposite of composition in $\mathcal{R}$. Also define the two Ringel duality functors

$$F := \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{R}}(T_i, -) : \mathcal{R} \to \widetilde{\mathcal{R}}, \tag{4.23}$$

$$G := * \circ \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{R}}(-, T_i) : \mathcal{R} \to \widetilde{\mathcal{R}}. \tag{4.24}$$

Theorem 4.27. *Assume that $\mathcal{R}$ is an ε-tilting-bounded essentially finite ε-stratified category. Let $\widetilde{\mathcal{R}}$ be the Ringel dual of $\mathcal{R}$ relative to $(T_i)_{i\in I}$ as in Definition 4.26. Then $\widetilde{\mathcal{R}}$ is a $(-\varepsilon)$-tilting-bounded essentially finite $(-\varepsilon)$-stratified category with stratification defined from $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$. Its distinguished objects satisfy*

$$\widetilde{P}(b) \cong FT_\varepsilon(b), \qquad \widetilde{I}(b) \cong GT_\varepsilon(b), \qquad \widetilde{L}(b) \cong \mathrm{hd}\, \widetilde{P}(b) \cong \mathrm{soc}\, \widetilde{I}(b),$$
$$\widetilde{\Delta}_{-\varepsilon}(b) \cong F\nabla_\varepsilon(b), \qquad \widetilde{\nabla}_{-\varepsilon}(b) \cong G\Delta_\varepsilon(b), \qquad \widetilde{T}_{-\varepsilon}(b) \cong FI(b) \cong GP(b).$$

The restrictions $F : \nabla_\varepsilon(\mathcal{R}) \to \Delta_{-\varepsilon}(\widetilde{\mathcal{R}})$ and $G : \Delta_\varepsilon(\mathcal{R}) \to \nabla_{-\varepsilon}(\widetilde{\mathcal{R}})$ are equivalences.

Proof. We may assume that $\mathcal{R} = B\text{-mod}_{\mathrm{fd}}$ for an essentially finite-dimensional locally unital algebra $B = \bigoplus_{i,j\in J} f_i B f_j$. The assumption that $\mathcal{R}$ is ε-tilting-bounded implies that $\sum_{i\in I} \dim \mathrm{Hom}_{\mathcal{R}}(T_i, T_j) < \infty$ and $\sum_{j\in I} \dim \mathrm{Hom}_{\mathcal{R}}(T_i, T_j) < \infty$ for each $i, j \in I$. Thus, the locally unital algebra A is also essentially finite-dimensional, i.e., $\widetilde{\mathcal{R}}$ is Schurian.

For $b \in \mathbf{B}$, pick $i(b) \in I$ and a primitive idempotent $e_b \in e_{i(b)} A e_{i(b)}$ such that $T_{i(b)} e_b \cong T_\varepsilon(b)$. Then $\widetilde{P}(b) := Ae_b$ is an indecomposable projective A-module, and

$$\left\{ \widetilde{L}(b) := \operatorname{hd} \widetilde{P}(b) \;\middle|\; b \in \mathbf{B} \right\}$$

is a full set of pairwise inequivalent irreducibles. It is immediate that $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$ defines a stratification of $\widetilde{\mathcal{R}}$. One checks that steps (1)–(12) from the proof of Theorem 4.11 all go through essentially unchanged in the present setting. This completes the proof but for one point: we must observe finally that $\widetilde{\mathcal{R}}$ is $(-\varepsilon)$-tilting-bounded. This follows because the analog of the matrix (4.21) for $\widetilde{\mathcal{R}}$ is the Cartan matrix

$$\left(\dim \operatorname{Hom}_B(P(a), P(b)) \right)_{a,b \in \mathbf{B}}$$

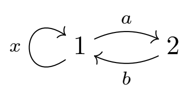
of $\mathcal{R}$. Its rows and columns have only finitely many non-zero entries as B is essentially finite-dimensional. □

We leave it to the reader to adapt Corollaries 4.12 and 4.13 to the present setting.

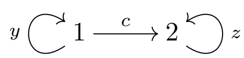
5. Examples

In this section, we explain several examples. The ones in §5.1 are new (but entirely elementary). The others are well known. For the ones in §§5.3–5.4 we give very few details, but have tried to outline carefully the relevant ingredients from the existing literature.

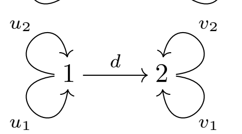
5.1. **Some examples via quiver and relations.** The following examples are of a similar nature to the one in [AHLU, Example 2.3]. Let A, B and C be the basic finite-dimensional algebras defined as the path algebra of the following quivers:

$A\ (1 < 2):$ 

with relations $x^2 = 0, ab = 0, baxb = 0,$ (5.1)

$B\ (1 > 2):$ 

with relations $y^2 = 0, z^2 = 0, zc = 0,$ (5.2)

$C\ (1 > 2):$ 

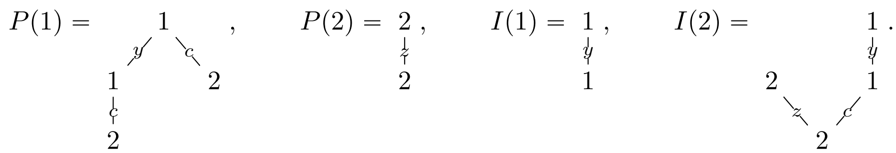
with relations $u_i u_j = 0, v_i v_j = 0, v_i d = 0.$ (5.3)

The algebra A has basis $\{e_1, x, ba, bax, xba, xbax; e_2, axb; b, xb; a, ax, axba, axbax\}$, B has basis $\{e_1, y; e_2, z; c, cy\}$, and C has basis $\{e_1, u_1, u_2; d, du_1, du_2; e_2, v_1, v_2\}$. The centers $Z(A)$ and $Z(B)$ are isomorphic, with bases $\{e_1 + e_2, bax + xba + axb\}$ and $\{e_1 + e_2, z\}$, respectively. The center $Z(C)$ has basis $\{e_1 + e_2, v_1, v_2\}$.

The irreducible A-, B- and C-modules are all indexed by the set $\{1, 2\}$. We are going to consider $A\text{-mod}_{\mathrm{fd}}$, $B\text{-mod}_{\mathrm{fd}}$ and $C\text{-mod}_{\mathrm{fd}}$ with the stratifications defined by the orders $1 < 2$, $1 > 2$ and $1 > 2$, respectively.

We first look at $B\text{-mod}_{\mathrm{fd}}$. As usual, we denote its irreducibles by $L(i)$, indecomposable projectives by $P(i)$, standards by $\Delta(i)$, etc...

The indecomposable projectives and injectives look as follows (where we abbreviate the simple module $L(i)$ just by i):



From this, one computes the structure of the standards and costandards:

$$\Delta(1) = P(1), \qquad \bar\Delta(1) = \begin{array}{c}1\\ \downarrow c\\ 2\end{array}, \qquad \Delta(2) = P(2), \qquad \bar\Delta(2) = L(2),$$

$$\nabla(1) = I(1), \qquad \bar\nabla(1) = L(1), \qquad \nabla(2) = \begin{array}{c}2\\ \downarrow z\\ 2\end{array}, \qquad \bar\nabla(2) = L(2).$$

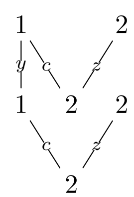
It follows easily that $B\text{-mod}_{\text{fd}}$ is fully stratified.

Next we look at the tilting modules in $B\text{-mod}_{\text{fd}}$. If one takes the sign function $\varepsilon = (\varepsilon_1, \varepsilon_2)$ to be either $(+,+)$ or $(-,+)$ then one finds that the indecomposable ε-tilting modules are $P(1)$ and $P(2)$ with filtrations

$$P(1) = \begin{array}{c}\bar\Delta(1)\\ \bar\Delta(1)\end{array} = \begin{array}{c}\nabla(1)\\ \bar\nabla(2)\oplus\bar\nabla(2)\end{array}, \qquad P(2) = \Delta(2) = \begin{array}{c}\bar\nabla(2)\\ \bar\nabla(2)\end{array}.$$

These cases are not interesting since the Ringel dual category is just $B\text{-mod}_{\text{fd}}$ again. Assume henceforth that $\varepsilon = (-,-)$ or $(+,-)$. Then the indecomposable ε-tilting modules have the following structure:

$T(1) =$



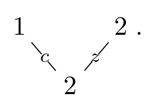
$$= \begin{array}{c}\bar\Delta(2)\oplus\bar\Delta(2)\\ \Delta(1)\end{array} = \begin{array}{c}\nabla(1)\\ \nabla(2)\oplus\nabla(2)\end{array}, \qquad T(2) = P(2).$$

To see this, one just has to check that these modules are indecomposable with the appropriate Δ_ε- and ∇_ε-flags. The minimal projective resolution of $T(1)$ takes the form

$$\cdots \longrightarrow P(2)\oplus P(2) \longrightarrow P(2)\oplus P(2) \longrightarrow P(1)\oplus P(2)\oplus P(2) \longrightarrow T(1) \longrightarrow 0,$$

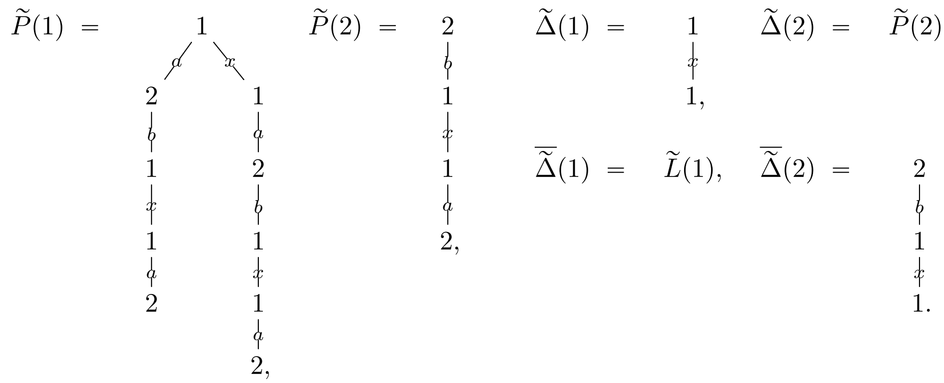
in particular, it is not of finite projective dimension. Observe also that there is a non-split short exact sequence $0 \to X \to T(1) \to X \to 0$ where

$X =$



Let $T := T(1) \oplus T(2)$. We claim that $\operatorname{End}_B(T)^{\text{op}}$ is the algebra A defined above. To prove this, one takes $x : T(1) \to T(1)$ to be an endomorphism whose image and kernel is the submodule X of $T(1)$, $a : T(2) \to T(1)$ to be a homomorphism which includes $T(2)$ as a submodule of $X \subseteq T(1)$, and $b : T(1) \to T(2)$ to be a homomorphism with kernel containing X and image $L(2) \subseteq T(2)$. Hence, $A\text{-mod}_{\text{fd}}$ is the Ringel dual of $B\text{-mod}_{\text{fd}}$ relative to T.

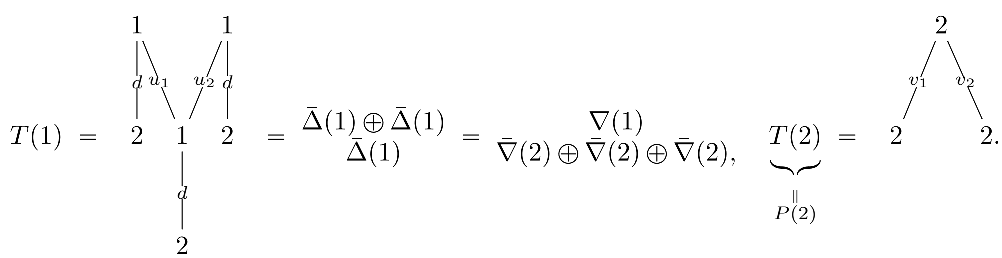
One can also analyze $A\text{-mod}_{\text{fd}}$ directly. Its indecomposable projectives (displayed on the left) and hence the standard and proper standard modules (displayed on the right) have the following structure:



Continuing with $\varepsilon = (-,-)$ or $\varepsilon = (+,-)$, it is easy to check directly from this that $A\text{-mod}_{\text{fd}}$ is $(-\varepsilon)$-stratified, as we knew already due to Theorem 4.11. However, it is not ε-stratified for either choice of ε, so it is *not* fully stratified. We leave it to the reader to compute the indecomposable $(-\varepsilon)$-tilting modules $\widetilde{T}(1)$ and $\widetilde{T}(2)$ in $B\text{-mod}_{\text{fd}}$. Their structure reflects the structure of the injectives $I(1)$ and $I(2)$ in $A\text{-mod}_{\text{fd}}$.

Let $\widetilde{T} := \widetilde{T}(1) \oplus \widetilde{T}(2) \cong T^*$. By the double centralizer property from Corollary 4.12, we have that $B = \operatorname{End}_A(\widetilde{T})^{\text{op}}$, as may also be checked directly. By Theorem 4.15, the functor $\mathbb{R}\operatorname{Hom}_A(\widetilde{T}, -) : D^b(A\text{-mod}_{\text{fd}}) \to D^b(B\text{-mod}_{\text{fd}})$ is an equivalence. Note though that $\mathbb{R}\operatorname{Hom}_B(T, -) : D^b(B\text{-mod}_{\text{fd}}) \to D^b(A\text{-mod}_{\text{fd}})$ is *not* one; this follows using [Kel, Theorem 4.1] since $T(1)$ does not have finite projective dimension.

Finally, we briefly consider $C\text{-mod}_{\text{fd}}$. By similar analysis to $B\text{-mod}_{\text{fd}}$, one checks that it is fully stratified. Then we take $\varepsilon = (-,+)$, and compute the indecomposable ε-tilting modules:



The endomorphism algebra $D := \operatorname{End}_C(T(1) \oplus T(2))^{\text{op}}$ can now be computed. One finds in particular that $\dim Z(D) = 4$, whereas $\dim Z(C) = 3$. This implies that $C\text{-mod}_{\text{fd}}$ and its Ringel dual $D\text{-mod}_{\text{fd}}$ relative to $T(1) \oplus T(2)$ are *not* derived equivalent; see e.g. [KZ, Proposition 6.3.2]. This conclusion is consistent with Theorem 4.15.

5.2. **Category $\mathcal{O}$ for affine Lie algebras.** Perhaps the first naturally occurring examples of finite highest weight categories came from the blocks of the BGG category $\mathcal{O}$ for a semisimple Lie algebra. This context also provides natural examples of finite stratified categories; see [Maz1] for a survey. To get examples of *semi-infinite* highest weight categories, one can consider instead blocks of the category $\mathcal{O}$ for an affine Kac-Moody Lie algebra. We briefly recall the setup referring to [Kac], [Car] for more details.

Let $\mathring{\mathfrak{g}}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ and

$$\mathfrak{g} := \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

be the corresponding affine Kac-Moody algebra. Fix also a Cartan subalgebra $\mathring{\mathfrak{h}}$ contained in a Borel subalgebra $\mathring{\mathfrak{b}}$ of $\mathring{\mathfrak{g}}$. There are corresponding subalgebras $\mathfrak{h}$ and $\mathfrak{b}$ of $\mathfrak{g}$, namely,

$$\mathfrak{h} := \mathring{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d, \qquad \mathfrak{b} := \left(\mathring{\mathfrak{b}} \otimes_{\mathbb{C}} \mathbb{C}[t] + \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} t\mathbb{C}[t]\right) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Let $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ and $\{h_i \mid i \in I\} \subset \mathfrak{h}$ be the simple roots and coroots of $\mathfrak{g}$ and $(\cdot|\cdot)$ be the normalized invariant form on $\mathfrak{h}^*$, all as in [Kac, Ch. 7–8]. The *basic imaginary root* $\delta \in \mathfrak{h}^*$ is the positive root corresponding to the canonical central element $c \in \mathfrak{h}$ under $(\cdot|\cdot)$. The linear automorphisms of $\mathfrak{h}^*$ defined by $s_i : \lambda \mapsto \lambda - \lambda(h_i)\alpha_i$ generate the Weyl group W of $\mathfrak{g}$. Let $\rho \in \mathfrak{h}^*$ be the element satisfying $\rho(h_i) = 1$ for all $i \in I$ and $\rho(d) = 0$. Then define the shifted action of W on $\mathfrak{h}^*$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$ for $w \in W$, $\lambda \in \mathfrak{h}^*$.

We define the *level* of $\lambda \in \mathfrak{h}^*$ to be $(\lambda + \rho)(c) \in \mathbb{C}$. It is *critical* if it equals the level of $\lambda = -\rho$, i.e., it is zero[5]. We usually restrict our attention to integral weights λ, that is, weights $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_i) \in \mathbb{Z}$ for all $i \in I$. The level of an integral weight is either *positive*, *negative* or *critical* (= zero). For any $\lambda \in \mathfrak{h}^*$, we define

$$\tilde{\lambda} := -\lambda - 2\rho. \tag{5.4}$$

Since $w \cdot (-\lambda - 2\rho) = -w \cdot \lambda - 2\rho$, weights λ and μ are in the same orbit under the shifted action of W if and only if so are $\tilde{\lambda}$ and $\tilde{\mu}$. Note also that the level of λ is positive (resp., critical) if and only if the level of $\tilde{\lambda}$ is negative (resp., critical). A crucial fact is that the orbit $W \cdot \lambda$ of an integral weight λ of positive level contains a unique weight $\lambda_{\max}$ such that $\lambda_{\max} + \rho$ is dominant; e.g., see [Kum, Exercise 13.1.E8a and Proposition 1.4.2]. By [Kum, Corollary 1.3.22], this weight is maximal in its orbit with respect to the usual ordering $\leqslant$ on weights, i.e., $\mu \leqslant \lambda$ if $\lambda - \mu \in \bigoplus_{i \in I} \mathbb{N}\alpha_i$. If λ is integral of negative level, we deduce from this discussion that its orbit contains a unique minimal weight $\lambda_{\min}$.

For $\lambda \in \mathfrak{h}^*$, let $\Delta(\lambda)$ be the Verma module with highest weight λ and $L(\lambda)$ be its unique irreducible quotient. Although Verma modules need not be of finite length, the composition multiplicities $[\Delta(\lambda) : L(\mu)]$ are always finite. There is also the *dual Verma module* $\nabla(\lambda)$ which is the restricted dual $\Delta(\lambda)^{\#}$ of $\Delta(\lambda)$, i.e., the sum of the duals of the weight spaces of $\Delta(\lambda)$ with the $\hat{\mathfrak{g}}$-action twisted by the Chevalley antiautomorphism.

All of the modules just introduced are objects in the category $\mathcal{O}$ consisting of all $\mathfrak{g}$-modules M which are semisimple over $\mathfrak{h}$ with finite-dimensional weight spaces and such that the set of weights of M is contained in the lower set generated by a finite subset of $\mathfrak{h}^*$; see [Kum, Section 2.1]. In the literature, one also often encounters a larger category $\widehat{\mathcal{O}}$ consisting of the $\mathfrak{g}$-modules M which are semisimple over $\mathfrak{h}$ and locally finite-dimensional over $\mathfrak{b}$. The latter category is ind-complete, and moreover every module in $\widehat{\mathcal{O}}$ is the union of its finitely generated submodules, all of which belong to the subcategory $\mathcal{O}$. It follows that the inclusion functor $\mathcal{O} \to \widehat{\mathcal{O}}$ induces an equivalence of categories between the ind-completion $\mathrm{Ind}(\mathcal{O})$ and the category $\widehat{\mathcal{O}}$.

Let $\sim$ be the equivalence relation on $\mathfrak{h}^*$ generated by $\lambda \sim \mu$ if there exists a positive root γ and $n \in \mathbb{Z}$ such that $2(\lambda + \rho|\gamma) = n(\gamma|\gamma)$ and $\lambda - \mu = n\gamma$. For a $\sim$-equivalence class Λ, let $\mathcal{O}_\Lambda$ (resp., $\widehat{\mathcal{O}}_\Lambda$) be the full subcategory of $\mathcal{O}$ (resp., $\widehat{\mathcal{O}}$) consisting of all $M \in \mathcal{O}$ (resp., $M \in \widehat{\mathcal{O}}$) such that $[M : L(\lambda)] \neq 0 \Rightarrow \lambda \in \Lambda$. Note that $\widehat{\mathcal{O}}_\Lambda$ is the ind-completion of $\mathcal{O}_\Lambda$. In view of the linkage principle from [KK, Theorem 2], these subcategories may be called the *blocks* of $\mathcal{O}$ and of $\widehat{\mathcal{O}}$, respectively. In particular, by [DGK, Theorem 4.2], any $M \in \mathcal{O}$ decomposes uniquely as a direct sum $M = \bigoplus_{\Lambda \in \mathfrak{h}^*/\sim} M_\Lambda$ with $M_\Lambda \in \mathcal{O}_\Lambda$. Note though that $\mathcal{O}$ is not the coproduct of its blocks in the strict sense since it is

[5]Many authors define the level to be $\lambda(c)$, in which case the critical level is $-\check{h}$, where $\check{h}$ is the dual Coxeter number.

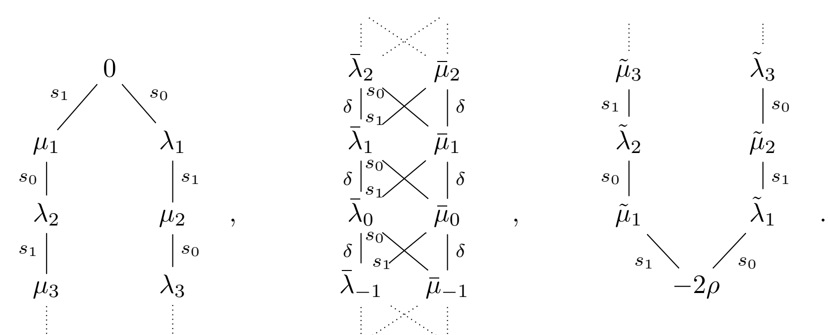
possible to find $M \in \mathcal{O}$ such that M_Λ is non-zero for infinitely many different Λ. The situation is more satisfactory for the ind-completions: $\widehat{\mathcal{O}}$ is the product of its blocks since by [Soe, Theorem 6.1] the functor

$$\prod_{\Lambda \in \mathfrak{h}^*/\sim} \widehat{\mathcal{O}}_\Lambda \to \widehat{\mathcal{O}}, \qquad (M_\Lambda)_{\Lambda \in \mathfrak{h}^*/\sim} \mapsto \bigoplus_{\Lambda \in \mathfrak{h}^*/\sim} M_\Lambda \tag{5.5}$$

is an equivalence of categories. Note also that $[\Delta(\lambda) : L(\mu)] \neq 0$ implies that the level of λ equals that of μ, since the scalars by which c acts on $L(\lambda)$ and $L(\mu)$ must agree. Consequently, we can talk simply about the level of a block.

A general combinatorial description of the $\sim$-equivalence classses Λ can be found for instance in [Fie3, Lemma 3.9]. To keep things simpler, we restrict ourselves from now on to integral blocks. In non-critical levels, one gets exactly the W-orbits $W \cdot \lambda$ of the integral weights of non-critical level. In critical level, one needs to incorporate also the translates by $\mathbb{Z}\delta$. From this description, it follows that the poset $(\Lambda, \leqslant)$ underlying an integral block $\mathcal{O}_\Lambda$ is upper finite with unique maximal element $\lambda_{\max}$ if $\mathcal{O}_\Lambda$ is of positive level, and lower finite with unique minimal element $\lambda_{\min}$ if $\mathcal{O}_\Lambda$ is of negative level. In case of the critical level, the poset is neither upper finite nor lower finite, but it is always interval finite.

Example 5.1. Here we give some explicit examples of posets which can occur for $\mathfrak{g} = \hat{\mathfrak{sl}}_2$, the Kac-Moody algebra for the Cartan matrix $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. The labelling set for the principal block is $W \cdot 0 = \{\lambda_k, \mu_k \mid k \geqslant 0\}$ where $\lambda_k := -\frac{1}{2}k(k+1)\alpha_0 - \frac{1}{2}k(k-1)\alpha_1$ and $\mu_k := -\frac{1}{2}k(k-1)\alpha_0 - \frac{1}{2}k(k+1)\alpha_1$. This is a block of positive level with maximal element $\lambda_0 = \mu_0 = 0$. Applying the map (5.4), we deduce that $W \cdot (-2\rho) = \{\tilde{\lambda}_k, \tilde{\mu}_k \mid k \geqslant 0\}$. This is the labelling set for a block of negative level with minimal element $\tilde{\lambda}_0 = \tilde{\mu}_0 = -2\rho$. Finally, we have that $W \cdot (\alpha_0 - \rho) \sqcup W \cdot (\alpha_1 - \rho) = \{\bar{\lambda}_k, \bar{\mu}_k \mid k \in \mathbb{Z}\}$ where $\bar{\lambda}_k := (k+1)\alpha_0 + k\alpha_1 - \rho$ and $\bar{\mu}_k := k\alpha_0 + (k+1)\alpha_1 - \rho$. This is the labelling set for a block of critical level, and it is neither upper nor lower finite. The following pictures illustrate these three situations:



Recall the definitions of upper finite and lower finite highest weight categories from Definitions 3.42 and 3.64, respectively.

Theorem 5.2. *Let $\mathcal{O}_\Lambda$ be an integral block of $\mathcal{O}$ of non-critical level. Then it is an upper finite or lower finite highest weight category according to whether the level is positive or negative, respectively. In both cases, the standard and costandard objects are the Verma modules $\Delta(\lambda)$ and the dual Verma modules $\nabla(\lambda)$, respectively.*

Proof. First, we prove the result for an integral block $\mathcal{O}_\Lambda$ of positive level. As explained above, the poset Λ is upper finite in this case. Let $\lambda_{\max}$ be its unique maximal weight.

Claim 1: In the positive level case, $\mathcal{O}_\Lambda$ is the full subcategory of $\widehat{\mathcal{O}}_\Lambda$ consisting of all modules M such that $[M : L(\lambda)] < \infty$ for all $\lambda \in \Lambda$. To prove this, given $M \in \mathcal{O}_\Lambda$, it is obvious that all of its composition multiplicities are finite since M has finite-dimensional weight spaces. Conversely, suppose that all of the composition multiplicities of $M \in \widehat{\mathcal{O}}_\Lambda$ are finite. All weights of M lie in the lower set generated by $\lambda_{\max}$. Moreover, for $\lambda \leqslant \lambda_{\max}$, the dimension of the λ-weight space of M is

$$\dim M_\lambda = \sum_{\mu \in \Lambda} [M : L(\mu)] \dim L(\mu)_\lambda.$$

Since the poset is upper finite, there are only finitely many $\mu \in \Lambda$ such that the λ-weight space $L(\mu)_\lambda$ is non-zero, and these weight spaces are finite-dimensional, so we deduce that $\dim M_\lambda < \infty$. This proves the claim.

Now we observe that the Verma module $M(\lambda_{\max})$ with maximal possible highest weight is projective in $\widehat{\mathcal{O}}_\Lambda$. From this and a standard argument involving translation functors through walls (see e.g. [Nei]) and the combinatorics from [Fie1, §4] (see also the introduction of [Fie2]), it follows that each of the irreducible modules $L(\lambda)$ $(\lambda \in \Lambda)$ has a projective cover $P(\lambda)$ in $\widehat{\mathcal{O}}_\Lambda$. Moreover, these projective covers have (finite) Δ-flags as in the axiom $(P\Delta)$. In particular, this shows that each $P(\lambda)$ actually belongs to $\mathcal{O}_\Lambda$. All that is left to complete the proof of the theorem in the positive level case is to show that $\mathcal{O}_\Lambda$ is a locally Schurian category. Let $A := \bigoplus_{\lambda,\mu \in \Lambda} \operatorname{Hom}_{\mathfrak{g}}(P(\lambda), P(\mu))$ with multiplication that is the opposite of composition. Since the multiplicities $[P(\mu) : L(\lambda)]$ are finite, A is a locally finite-dimensional locally unital algebra. Using Lemma 2.3, we deduce that $\widehat{\mathcal{O}}_\Lambda$ is equivalent to the category A-mod of all left A-modules. As explained in the discussion after (2.25), $A\text{-mod}_{\mathrm{lfd}}$ is the full subcategory of A-mod consisting of all modules with finite composition multiplicities. Combining this with Claim 1, we deduce that the equivalence between $\widehat{\mathcal{O}}_\Lambda$ and A-mod restricts to an equivalence between $\mathcal{O}_\Lambda$ and $A\text{-mod}_{\mathrm{lfd}}$. Hence, $\mathcal{O}_\Lambda$ is a locally Schurian category.

We turn our attention to an integral block $\mathcal{O}_\Lambda$ of negative level. In this case, we know already that the poset Λ is lower finite with a unique minimal element $\lambda_{\min}$.

Claim 2. In the negative level case, the category $\mathcal{O}_\Lambda$ is the full subcategory of $\widehat{\mathcal{O}}_\Lambda$ consisting of all modules of finite length. For this, it is obvious that any module in $\widehat{\mathcal{O}}_\Lambda$ of finite length belongs to $\mathcal{O}_\Lambda$. Conversely, any object in $\mathcal{O}_\Lambda$ is of finite length thanks to the formula [Kum, 2.1.11 (1)], taking λ therein to be $\lambda_{\min}$.

From Claim 2 and Lemma 2.1, it follows that $\mathcal{R} := \mathcal{O}_\Lambda$ is a locally finite Abelian category. By [Fie1, Theorem 2.7] the Serre subcategory $\mathcal{R}^\downarrow$ of $\mathcal{R}$ associated to $\Lambda^\downarrow$ is a finite highest weight category for each finite lower set $\Lambda^\downarrow$ of Λ. We deduce that $\mathcal{R}$ is a lower finite highest weight category using Lemma 3.62. □

Let $\mathcal{O}_\Lambda$ be an integral block of non-critical level. Since $\widehat{\mathcal{O}}_\Lambda$ is a Grothendieck category, it has enough injectives. However it does not have enough projectives in general. This is apparent from the general theory developed in §3.3 and §3.5; see also [Soe, Remark 6.5]. In fact, the situation in positive versus negative levels is quite different:

- In the positive level case, we have noted already that the irreducible modules have projectives covers, namely, the modules $\{P(\lambda) \mid \lambda \in \Lambda\}$ constructed in the proof of Theorem 5.2. Their restricted duals $I(\lambda) := P(\lambda)^\#$ are the indecomposable injective modules in $\widehat{\mathcal{O}}_\Lambda$; they possess (finite) ∇-flags as in axiom $(I\nabla)$ so belong to $\mathcal{O}_\Lambda$.
- In the negative level case, each irreducible module $L(\lambda)\,(\lambda \in \Lambda)$ has an injective hull $I(\lambda)$ in $\widehat{\mathcal{O}}_\Lambda$ which possesses a (possibly infinite) ascending ∇-flag in the sense of

Definition 3.51. But, $\widehat{\mathcal{O}}_\Lambda$ usually does not have any projectives at all (although one could construct such modules in the pro-completion of $\mathcal{O}_\Lambda$ as done e.g. in [Fie2]).

The following results about tilting modules are consequences of the general theory developed in §4.1 and §4.3. They already appeared in an equivalent form in [Soe].

- In the negative level case, tilting modules are objects in $\mathcal{O}_\Lambda$ admitting both a (finite) Δ-flag and a (finite) ∇-flag. The isomorphism classes of indecomposable tilting modules in $\mathcal{O}_\Lambda$ are parametrized by their highest weights. They may also be constructed by applying translation functors to the Verma module $\Delta(\lambda_{\min})$.
- In the positive level case, tilting modules are objects in $\mathcal{O}_\Lambda$ which admit a (possibly infinite) ascending Δ-flag and a (possibly infinite) descending ∇-flag in the sense of Definition 3.35. Again, the isomorphism classes of indecomposable tilting modules are parametrized by their highest weights.

In both cases, our characterization of the indecomposable tilting module $T(\lambda)$ of highest weight λ is slightly different from the one given in [Soe, Definition 6.3]. From our definition, one sees immediately that $T(\lambda)^\# \cong T(\lambda)$.

Remark 5.3. Elsewhere in the literature dealing with positive level, it is common to pass to a different category of modules, e.g., to the Whittaker category in [BY] or to truncated versions of $\mathcal{O}$ in [SVV, Section 3], before contemplatin tilting modules.

Our next result is concerned with the Ringel duality between integral blocks of positive and negative level. This depends crucially on a special case of the *Arkhipov-Soergel equivalence* from [Ark], [Soe]. Let S be Arkhipov's semi-regular bimodule, which is the bimodule S_γ of [Soe] with $\gamma := 2\rho$ as in [Soe, Lemma 7.1]. For $\lambda \in \mathfrak{h}^*$, let $T(\lambda)$ be the indecomposable tilting module from [Soe, Definition 6.3]. Also let $P(\lambda)$ be a projective cover of $L(\lambda)$ in $\widehat{\mathcal{O}}$ whenever such an object exists, cf. [Soe, Remark 6.5(2)].

Theorem 5.4 (Arkhipov, Soergel)**.** *Tensoring with the semi-regular bimodule defines an equivalence $S \otimes_{U(\mathfrak{g})} - : \Delta(\mathcal{O}) \to \nabla(\mathcal{O})$ between the exact subcategories of $\mathcal{O}$ consisting of objects with (finite) Δ- and ∇-flags, respectively, such that*

(1) $S \otimes_{U(\mathfrak{g})} \Delta(\lambda) \cong \nabla(\tilde{\lambda})$;

(2) $S \otimes_{U(\mathfrak{g})} P(\lambda) \cong T(\tilde{\lambda})$ (assuming $P(\lambda)$ exists).

Corollary 5.5. *Assume that $\mathcal{O}_\Lambda$ is an integral block of negative level. Let $\widetilde{\mathcal{R}}$ be the Ringel dual of $\mathcal{R} := \mathcal{O}_\Lambda$ relative to some choice of $T = \bigoplus_{i\in I} T_i$ as in Definition 4.19, and let F be the Ringel duality functor from (4.9). Also let $\widetilde{\Lambda} := \{\tilde{\lambda} \mid \lambda \in \Lambda\}$. Then there is an equivalence of categories $E : \widetilde{\mathcal{R}} \to \mathcal{O}_{\widetilde{\Lambda}}$ such that $E \circ F : \nabla(\mathcal{O}_\Lambda) \to \Delta(\mathcal{O}_{\widetilde{\Lambda}})$ is a quasi-inverse to the Arkhipov-Soergel equivalence $S \otimes_{U(\mathfrak{g})} - : \Delta(\mathcal{O}_{\widetilde{\Lambda}}) \to \nabla(\mathcal{O}_\Lambda)$.*

Proof. Note to start with that $\mathcal{O}_{\widetilde{\Lambda}}$ is an integral block of positive level. Moreover, the map $\Lambda^{\mathrm{op}} \to \widetilde{\Lambda}, \lambda \mapsto \tilde{\lambda}$ is an order isomorphism.

Choose a quasi-inverse D to $S \otimes_{U(\mathfrak{g})} - : \Delta(\mathcal{O}_{\widetilde{\Lambda}}) \to \nabla(\mathcal{O}_\Lambda)$, and set $P_i := DT_i$. By Theorem 5.4(2), $(P_i)_{i\in I}$ is a projective generating family for $\widehat{\mathcal{O}}_{\widetilde{\Lambda}}$. Moreover, recalling that $\widetilde{\mathcal{R}}$ is the category A-mod$_{\mathrm{lfd}}$ where $A := \bigoplus_{i,j\in I} \mathrm{Hom}_{\mathcal{O}_\Lambda}(T_i, T_j)$, the functor D induces an isomorphism via which we can identify A with $\bigoplus_{i,j\in I} \mathrm{Hom}_{\mathcal{O}_{\widetilde{\Lambda}}}(P_i, P_j)$.

As explained in the proof of Theorem 5.2, the functor

$$H := \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{O}_{\widetilde{\Lambda}}}(P_i, -) : \mathcal{O}_{\widetilde{\Lambda}} \to A\text{-mod}_{\mathrm{lfd}}$$

is an equivalence of categories. Moreover, we have that

$$H \circ D = \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{O}_{\widetilde{\Lambda}}}(P_i, D(-)) \cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{O}_\Lambda}(S \otimes_{U(\mathfrak{g})} P_i, -) \cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{O}_\Lambda}(T_i, -) = F.$$

Letting E be a quasi-inverse equivalence to H, it follows that $E \circ F \cong D$. □

Remark 5.6. In the setup of Corollary 5.5, the Arkhipov-Soergel equivalence extends to an equivalence $S \otimes_{U(\mathfrak{g})} - : \Delta^{\mathrm{asc}}(\mathcal{O}_{\tilde{\Lambda}}) \to \nabla^{\mathrm{asc}}(\mathcal{O}_\Lambda)$, which is a quasi-inverse to $E \circ F : \nabla^{\mathrm{asc}}(\mathcal{O}_\Lambda) \to \Delta^{\mathrm{asc}}(\mathcal{O}_{\tilde{\Lambda}})$. These functors interchange the indecomposable injectives in $\widehat{\mathcal{O}}_\Lambda$ with the indecomposable tiltings in $\mathcal{O}_{\tilde{\Lambda}}$.

Finally we discuss the situation for an integral critical block $\mathcal{O}_\Lambda$. As we have already explained, in this case the poset Λ is neither upper nor lower finite. In fact, these blocks do not fit into the framework of this article at all, since the Verma modules have infinite length and there are no projectives. One sees this already for the Verma module $\Delta(-\rho)$ for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, which has composition factors $L(-\rho - m\delta)$ for $m \geqslant 0$, each appearing with multiplicity equal to the number of partitions of m; see e.g. [AF1, Theorem 4.9(1)]. However, there is an auto-equivalence $\Sigma := L(\delta) \otimes - : \widehat{\mathcal{O}}_\Lambda \to \widehat{\mathcal{O}}_\Lambda$, which makes it possible to pass to the *restricted category* $\widehat{\mathcal{O}}_\Lambda^{\mathrm{res}}$, which we define next.

Let A_n be the vector space of natural transformations $\Sigma^n \to \mathrm{Id}$. This gives rise to a graded algebra $A := \bigoplus_{n\in\mathbb{Z}} A_n$. Then the restricted category $\widehat{\mathcal{O}}_\Lambda^{\mathrm{res}}$ is the full subcategory of $\widehat{\mathcal{O}}_\Lambda$ consisting of all modules which are annihilated by the induced action of A_n for $n \neq 0$; cf. [AF1, §4.3]. The irreducible modules in the restricted category are the same as in $\widehat{\mathcal{O}}_\Lambda$ itself. There are also the *restricted Verma modules*

$$\Delta(\lambda)^{\mathrm{res}} := \Delta(\lambda) \Big/ \sum_{\eta \in A_{\neq 0}} \mathrm{im}\left(\eta_{\Delta(\lambda)} : \Sigma^n \Delta(\lambda) \to \Delta(\lambda)\right) \tag{5.6}$$

from [AF1, §4.4]. In other words, $\Delta(\lambda)^{\mathrm{res}}$ is the largest quotient of $\Delta(\lambda)$ that belongs to the restricted category. Similarly, the *restricted dual Verma module* $\nabla(\lambda)^{\mathrm{res}}$ is the largest submodule of $\nabla(\lambda)$ that belongs to the restricted category.

The restricted category $\widehat{\mathcal{O}}_\Lambda^{\mathrm{res}}$ is no longer indecomposable: by [AF2, Theorem 5.1] it decomposes further as

$$\widehat{\mathcal{O}}_\Lambda^{\mathrm{res}} = \prod_{\overline{\Lambda} \in \Lambda/W} \widehat{\mathcal{O}}_{\overline{\Lambda}}^{\mathrm{res}} \tag{5.7}$$

where Λ/W denotes the orbits of W under the dot action. For instance, the poset Λ for the critical level displayed in Example 5.1 splits into two orbits $W \cdot (\alpha_0 - \rho)$ and $W \cdot (\alpha_1 - \rho)$ (i.e., one removes the edges labelled by δ). In the most singular case, $\widehat{\mathcal{O}}_{-\rho}^{\mathrm{res}}$ is a product of simple blocks; in particular, $\Delta^{\mathrm{res}}(-\rho) = L(-\rho) = \nabla^{\mathrm{res}}(-\rho)$.

Conjecture 5.7. Let $\widehat{\mathcal{O}}_{\overline{\Lambda}}^{\mathrm{res}}$ be a regular (in the sense of [AF2]) integral critical block. Let $\mathcal{O}_{\overline{\Lambda}}^{\mathrm{res}} := \mathrm{Fin}\left(\widehat{\mathcal{O}}_{\overline{\Lambda}}^{\mathrm{res}}\right)$ be the full subcategory consisting of all modules of finite length. Then $\mathcal{O}_{\overline{\Lambda}}^{\mathrm{res}}$ is an essentially finite highest weight category with standard and costandard objects $\Delta(\lambda)^{\mathrm{res}}$ and $\nabla(\lambda)^{\mathrm{res}}$ for $\lambda \in \overline{\Lambda}$. Moreover, the indecomposable projective modules in $\mathcal{O}_{\overline{\Lambda}}^{\mathrm{res}}$ are also its indecomposable tilting modules, and therefore $\mathcal{O}_{\overline{\Lambda}}^{\mathrm{res}}$ is tilting-bounded and Ringel self-dual.

This conjecture is true for the basic example of a critical block from Example 5.1 thanks to [Fie3, Theorem 6.6]; the same category arises as the principal block of category $\mathcal{O}$ for $\mathfrak{gl}_{1|1}(\mathbb{C})$ discussed in §5.5 below. The conjecture is also consistent with the so-called *Feigin-Frenkel conjecture* [AF1, Conjecture 4.7], which says that composition multiplicities of restricted Verma modules are related to the periodic Kazhdan-Lusztig polynomials from [Lus] (and Jantzen's generic decomposition patterns from [Jan2]). These polynomials depend on the relative position of the given pair of weights and, when not too close to walls, they vanish for weights that are far apart. This is consistent with the conjectured existence of indecomposable projectives of finite length in regular blocks of the restricted category.

It seems to us that the Feigin-Frenkel conjecture might have a geometric explanation in terms of a sequence of equivalences of categories similar to [FG, (7)]. Ultimately this should connect $\mathcal{O}^{\text{res}}_{\overline{\Lambda}}$ with representations of the quantum group analog of Jantzen's thickened Frobenius kernel G_1T. The latter are already known by [AJS, §17] to be essentially finite highest weight categories controlled[6] by the periodic Kazhdan-Lusztig polynomials. Also, in these categories, tilting modules are projective, hence, the Ringel self-duality would be an obvious consequence.

5.3. **Rational representations.** As we noted already after Definition 3.64, the definition of lower finite highest weight category originated in the work of Cline, Parshall and Scott [CPS1]. As well as the BGG category $\mathcal{O}$ already mentioned, their work was motivated by the representation theory of a connected reductive algebraic group G in positive characteristic, as developed for example in [Jan1]: the (monoidal!) category $\text{Rep}(G)$ of finite-dimensional rational representations of G is a lower finite highest weight category. We give a little more detail in the next paragraph.

Fixing a maximal torus T contained in a Borel subgroup B of G, the weight poset Λ is the set $X^+(T)$ of dominant characters of T with respect to B; Jantzen's convention is that the roots of B are negative. The costandard objects $\nabla(\lambda)$ are the induced modules $H^0(G/B, \mathcal{L}_\lambda)$. For the partial order $\leqslant$, one can use the usual ordering on $X^+(T)$, or the more refined Bruhat order of [Jan1, §II.6.4]. This makes $\text{Rep}(G)$ into a lower finite highest weight category by [Jan1, Proposition II.4.18] and [Jan1, Proposition II.6.13]. In fact, in the case of $\text{Rep}(G)$, all of the general results about ascending ∇-flags found in §3.5 were known already before the time of [CPS1], e.g., they are discussed in Donkin's book [Do1] (and called there *good filtrations*). Tilting modules for G were studied in [Do3]. However, our general formulation of semi-infinite Ringel duality from §4.4 is not mentioned there explicitly there: Donkin instead took the approach pioneered in [Do2] of truncating to a finite lower set before taking Ringel duals.

A key observation in this setting is that tensor products of tilting modules are tilting; this is the Donkin-Mathieu-Wang theorem [Do1], [Mat], [Wan]. It suggests the following construction. Let $(T_i)_{i\in I}$ be a family of tilting modules in $\text{Rep}(G)$ such that every indecomposable tilting module is isomorphic to a summand of a tensor product $T_{\boldsymbol{i}} := T_{i_1} \otimes \cdots \otimes T_{i_r}$ for some finite (possibly empty) sequence $\boldsymbol{i} = (i_1, \ldots, i_r)$ of elements of I. Let $\langle I\rangle$ denote the set of all such sequences. Then define $\mathcal{A}$ to be the category with objects $\langle I\rangle$ and $\text{Hom}_{\mathcal{A}}(\boldsymbol{i}, \boldsymbol{j}) := \text{Hom}_G(T_{\boldsymbol{i}}, T_{\boldsymbol{j}})$, with composition induced by composition in $\text{Rep}(G)$. The category $\mathcal{A}$ has an evident strict monoidal structure; the tensor product of objects is by concatenation of sequences. Recalling Remark 2.4, the locally Schurian category $\mathcal{F}un(\mathcal{A}^{\text{op}}, \text{Vec}_{\text{fd}})$ is the Ringel dual of $\text{Rep}(G)$ relative to $(T_{\boldsymbol{i}})_{\boldsymbol{i}\in\langle I\rangle}$. It is naturally an upper finite highest weight category.

The simplest interesting example comes from $G = SL_2$. For this, we may take $I := \{\bullet\}$ and $T_\bullet$ to be the natural two-dimensional representation. The strict monoidal category $\mathcal{A}$ in this case has an explicit diagrammatical description: it is the *Temperley-Lieb category* at parameter 2; see e.g. [GW]. The case that $G = GL_n$ is also much studied. If one assumes that $p := \text{char}\ \Bbbk$ is either 0 or satisfies $p > n$, then one can take $I := \{\uparrow, \downarrow\}$, $T_\uparrow$ to be the natural G-module V, and $T_\downarrow$ to be its dual V^*. The resulting strict monoidal category $\mathcal{A}$ is the *oriented Brauer category* $\mathcal{OB}(n)$ modulo one relation corresponding to the fact that $\bigwedge^{n+1} V = 0$; see e.g. [Bru, Theorem 1.10]. When $0 < p \leqslant n$ one needs to

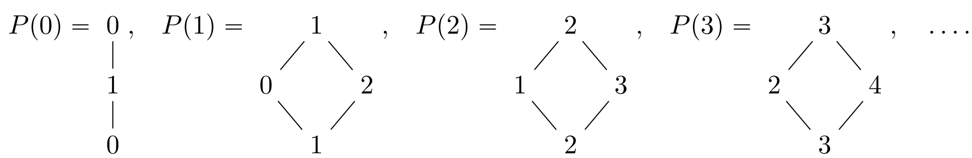
[6]One needs to assume that the root of unity ℓ is odd and bigger than or equal to the Coxeter number.

include all of the exterior powers $\bigwedge^m V$ and $\bigwedge^m V^*$ for $1 \leqslant m \leqslant n$ in order to get an appropriate generating family of tilting modules.

All of this works also when G is replaced by the corresponding quantum group $U_q(\mathfrak{g})$, possibly at a root of unity (taking the Lusztig form). When at a root of unity over the ground field is $\mathbb{C}$, it turns out that the indecomposable projectives and injectives in the category of rational representations of $U_q(\mathfrak{g})$ are all finite-dimensional, i.e., the category is Schurian. Tiltings are also finite-dimensional, indeed, the category is tilting-bounded in the sense of §4.5. The first example of this nature comes from $U_q(\mathfrak{sl}_2)$ at a root of unity over $\mathbb{C}$. In this case, see e.g. [AT, Theorem 3.12, Definition 3.3], the principal block is Morita equivalent to the locally unital algebra A defined as the path algebra of the quiver

$$0 \underset{b_0}{\overset{a_0}{\rightleftarrows}} 1 \underset{b_1}{\overset{a_1}{\rightleftarrows}} 2 \underset{b_2}{\overset{a_2}{\rightleftarrows}} 3 \cdots \qquad \text{with relations } a_{i+1}a_i = b_i b_{i+1} = a_i b_i - b_{i+1}a_{i+1} = 0.$$

The appropriate partial order on the weight poset $\mathbb{N}$ is the natural order $0 < 1 < \cdots$. The indecomposable projectives have the following structure:



The tilting objects are $T(0) := L(0)$ and $T(n) := P(n-1)$ for $n \geqslant 1$. From this, it is easy to see that the Ringel dual is described by the same quiver with one additional relation, namely, that $b_0 a_0 = 0$ (and of course the order is reversed).

5.4. **Tensor product categorifications.** Until quite recently, most of the naturally occurring examples were highest weight categories (like the ones described in the previous two subsections). But the work of Webster [Web1], [Web2] and Losev and Webster [LW] has brought to prominence a very general source of examples that are fully stratified but seldom highest weight.

Fundamental amongst these new examples are the categorifications of tensor products of irreducible highest weight modules of symmetrizable Kac-Moody Lie algebras. Rather than attempting to repeat the definition of these here, we refer the reader to [LW]. All of these examples are finite fully stratified categories possessing a duality as in Corollary 3.22. Another interesting feature of them is that $T_+(b) \cong T_-(b)$ for all $b \in \mathbf{B}$: both families of tilting objects are constructed by the same inductive procedure using the categorification functors. Consequently, the Ringel dual is both standardly and costandardly statified, i.e., it is also fully stratified. In fact, the Ringel dual category is always another tensor product categorification[7] (reverse the order of the tensor product).

In [Web1], Webster also introduced some more general tensor product categorifications, including ones which categorify the tensor product of an integrable lowest weight module tensored with an integrable highest weight module; see also [BD1, Construction 4.13]. The latter are particularly important since they may be realized as generalized cyclotomic quotients of the Kac-Moody 2-category. They are upper finite fully stratified categories, and their Ringel duals are lower finite fully stratified categories.

The category of locally finite-dimensional representations of the Deligne category $\underline{\mathrm{Rep}}(GL_\delta)$ arises as a special case of these lowest weight tensored highest weight tensor

[7]This was noted in Remark 3.10 of the `arxiv` version of [LW] but the authors removed this remark in the published version (along with Remark 2.7 which was misleading).

product categorifications; see [Bru, Theorem 1.11]. We just note that $\underline{\mathrm{Rep}}(GL_\delta)$ is the additive Karoubi envelope of the oriented Brauer category $\mathcal{OB}(\delta)$ for $\delta \in \mathbb{C}$, which was mentioned already in the previous subsection. Its Ringel dual is equivalent to the Abelian envelope $\underline{\mathrm{Rep}}^{ab}(GL_\delta)$ of Deligne's category constructed by Entova, Hinich and Serganova [EHS], which is a (monoidal!) lower finite highest weight category. In [E-A], it is shown that $\underline{\mathrm{Rep}}^{ab}(GL_\delta)$ categorifies a highest weight tensored lowest weight representation, which is the dual result to that of [Bru]. This example will be discussed further in a sequel to this article, where we give an explicit description of the blocks of $\underline{\mathrm{Rep}}^{ab}(GL_\delta)$ via Khovanov's arc (co)algebra, thereby proving a conjecture formulated in the introduction of [BS2].

5.5. **Representations of Lie superalgebras.** Finally, we mention briefly an interesting source of essentially finite highest weight categories: the analogs of the BGG category $\mathcal{O}$ for classical Lie superalgebras. A detailed account in the case of the Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{C})$ can be found in [BLW]. Its category $\mathcal{O}$ gives an essentially finite highest weight category which is neither lower finite nor upper finite. Moreover, it is tilting-bounded as in §4.5, so that the Ringel dual category is also an essentially finite highest weight category. There is one very easy special case: the principal block of category $\mathcal{O}$ for $\mathfrak{gl}_{1|1}(\mathbb{C})$ is equivalent to the category of finite-dimensional modules over the essentially finite-dimensional locally unital algebra which is the path algebra of the following quiver

$$\cdots -1 \underset{b_{-1}}{\overset{a_{-1}}{\rightleftarrows}} 0 \underset{b_0}{\overset{a_0}{\rightleftarrows}} 1 \underset{b_1}{\overset{a_1}{\rightleftarrows}} 2 \cdots \text{ with relations } a_{i+1}a_i = b_ib_{i+1} = a_ib_i - b_{i+1}a_{i+1} = 0,$$

see e.g. [BS1, p. 380]. This is very similar to the $U_q(\mathfrak{sl}_2)$-example from §5.3, but now the poset $\mathbb{Z}$ is neither lower nor upper finite, and it is Ringel self-dual.

One gets similar examples from $\mathfrak{osp}_{m|2n}(\mathbb{C})$, as discussed for example in [BW] and [ES]. The simplest non-trivial case of $\mathfrak{osp}_{3|2}(\mathbb{C})$ produces the path algebra of a D_∞ quiver (replacing than the A_∞ quiver above); see [ES, §II].

The "strange" families $\mathfrak{p}_n(\mathbb{C})$ and $\mathfrak{q}_n(\mathbb{C})$ also exhibit similar structures. The former has not yet been investigated systematically (although basic aspects of the finite-dimensional finite-dimensional representations and category $\mathcal{O}$ were recently studied in [B+9] and [CC], respectively). For the latter, we refer to [BD2] and the references therein. In fact, the integral blocks for $\mathfrak{q}_n(\mathbb{C})$ are fully stratified (rather than highest weight) categories; this observation is due to Frisk [Fri2].

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