SEMI-INFINITE HIGHEST WEIGHT CATEGORIES

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Dedicated to Jens Carsten Jantzen on the occasion of his 70th birthday.

Abstract. We develop axiomatics of highest weight categories and quasi-hereditary algebras in order to incorporate two semi-infinite situations which are in Ringel duality with each other; the underlying posets are either *upper finite* or *lower finite*. We also consider various more general sorts of signed highest weight categories and stratified categories. In the upper finite cases, we give an alternative characterization of these categories in terms of quasi-hereditary and stratified algebras equipped with idempotent-adapted cellular bases. Finally, we explain a general construction which produces many explicit examples of such algebras starting from naturally-occurring locally unital algebras which admit Cartan or triangular decompositions.

Contents

1. Introduction

Highest weight categories were introduced by Cline, Parshall and Scott [CPS1] in order to provide an axiomatic framework encompassing a number of important examples which had previously arisen in representation theory. In the first part of this article, we give a detailed exposition of two semi-infinite generalizations, which we call *lower finite* and *upper finite highest weight categories*. Lower finite highest weight categories were already included in the original work of Cline, Parshall and Scott (although our terminology is different). Well-known examples include the category $\mathcal{R}ep(G)$ of finite-dimensional rational representations of a connected reductive algebraic group. Upper finite highest weight categories have also appeared in the literature in many examples, and an appropriate axiomatic framework was sketched out by Elias and Losev in [ELos, §6.1.2]. However there are plenty of subtleties, so a full treatment seems desirable.

Then, in the next part, we extend Ringel duality to the semi-infinite setting:

$$\left\{\begin{array}{c}\text{lower finite}\\ \text{highest weight categories}\end{array}\right\} \xleftrightarrow{\text{Ringel duality}} \left\{\begin{array}{c}\text{upper finite}\\ \text{highest weight categories}\end{array}\right\}.$$

Other approaches to "semi-infinite Ringel duality" exist in the literature, but these typically require the existence of a $\mathbb{Z}$-grading; e.g., see [Soe] (in a Lie algebra setting) and also [Maz2]. We avoid this by working with finite-dimensional comodules over a coalgebra in the lower finite case, and with locally finite-dimensional modules over a locally finite-dimensional locally unital algebra in the upper finite case.

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Finally, as an application semi-infinite Ringel duality, we give an elementary algebraic characterization of upper finite highest weight categories, showing that any such category is equivalent to the category of locally finite-dimensional modules over an *upper finite based quasi-hereditary algebra*. This is an algebraic formulation of the notion of object-adapted cellular category from [ELau, Definition 2.1], and a generalization of the based quasi-hereditary algebras of [KM, Definition 2.4]. As well as Ringel duality, the proof of this characterization uses a construction from [AST] to construct bases for endomorphism algebras of tilting objects. The observation that the bases arising from [AST] are object-adapted cellular bases was made already by Elias and several others, and appears in recent work of Andersen [And].

Throughout the article, we systematically develop the entire theory in the more general setting of what we call *ε-stratified categories*. The idea of this definition is due to Ágoston, Dlab and Lukács: in [ADL, Definition 1.3] one finds the notion of a stratified algebra of type ε; the category of finite-dimensional left modules over such a finite-dimensional algebra is an example of a ε-stratified category in our sense. The various other generalizations of highest weight category that have been considered in existing iterature fit naturally into our ε-stratified framework.

To explain the contents of the paper in more detail, we start by explaining our precise setup in the finite-dimensional case, since even here it does not seem to have appeared explicitly elsewhere in the literature. Consider a *finite Abelian category*, that is, a category $\mathcal{R}$ equivalent to the category $A\text{-mod}_{\mathrm{fd}}$ of finite-dimensional left A-modules for some finite-dimensional $\Bbbk$-algebra A. Let $\mathbf{B}$ be a finite set indexing a full set of pairwise inequivalent irreducible objects $\{L(b) \mid b \in \mathbf{B}\}$. Let $P(b)$ (resp., $I(b)$) be a projective cover (resp., injective hull) of $L(b)$.

A *stratification* of $\mathcal{R}$ is the data of a function $\rho : \mathbf{B} \to \Lambda$ for some poset $(\Lambda, \leqslant)$. For $\lambda \in \Lambda$, let $\mathcal{R}_{\leqslant\lambda}$ (resp., $\mathcal{R}_{<\lambda}$) be the Serre subcategory of $\mathcal{R}$ generated by the irreducibles $L(b)$ for $b \in \mathbf{B}$ with $\rho(b) \leqslant \lambda$ (resp., $\rho(b) < \lambda$). Define the *stratum* $\mathcal{R}_\lambda$ to be the Serre quotient $\mathcal{R}_{\leqslant\lambda}/\mathcal{R}_{<\lambda}$ with quotient functor $j^\lambda : \mathcal{R}_{\leqslant\lambda} \to \mathcal{R}_\lambda$. For $b \in \mathbf{B}_\lambda := \rho^{-1}(\lambda)$, let $L_\lambda(b) := j^\lambda L(b)$. These give a full set of pairwise inequivalent irreducible objects in $\mathcal{R}_\lambda$. Let $P_\lambda(b)$ (resp., $I_\lambda(b)$) be a projective cover (resp., an injective hull) of $L_\lambda(b)$ in $\mathcal{R}_\lambda$.

The functor j^λ has a left adjoint $j^\lambda_!$ and a right adjoint j^λ_*; see Lemma 2.21. We refer to these as the *standardization* and *costandardization functors*, respectively, following the language of [LW, §2]. Then we introduce the *standard, proper standard, costandard* and *proper costandard* objects of $\mathcal{R}$ for $\lambda \in \Lambda$ and $b \in \mathbf{B}_\lambda$:

$$\Delta(b) := j^\lambda_! P_\lambda(b), \qquad \bar\Delta(b) := j^\lambda_! L_\lambda(b), \qquad \nabla(b) := j^\lambda_* I_\lambda(b), \qquad \bar\nabla(b) := j^\lambda_* L_\lambda(b). \tag{1.1}$$

Equivalently, $\Delta(b)$ (resp., $\nabla(b)$) is the largest quotient of $P(b)$ (resp., the largest subobject of $I(b)$) that belongs to $\mathcal{R}_{\leqslant\lambda}$, and $\bar\Delta(b)$ (resp., $\bar\nabla(b)$) is the largest quotient of $\Delta(b)$ (resp., the largest subobject of $\nabla(b)$) such that all composition factors apart from its irreducible head (resp., its irreducible socle) belong to $\mathcal{R}_{<\lambda}$.

Fix a sign function $\varepsilon : \Lambda \to \{\pm\}$ and define the *ε-standard* and *ε-costandard objects*

$$\Delta_\varepsilon(b) := \begin{cases} \Delta(b) & \text{if } \varepsilon(\rho(b)) = + \\ \bar\Delta(b) & \text{if } \varepsilon(\rho(b)) = - \end{cases}, \qquad \nabla_\varepsilon(b) := \begin{cases} \bar\nabla(b) & \text{if } \varepsilon(\rho(b)) = + \\ \nabla(b) & \text{if } \varepsilon(\rho(b)) = - \end{cases}. \tag{1.2}$$

By a *Δ_ε-flag* (resp., a *∇_ε-flag*) of an object of $\mathcal{R}$, we mean a (necessarily finite) filtration whose sections are of the form $\Delta_\varepsilon(b)$ (resp., $\nabla_\varepsilon(b)$) for $b \in \mathbf{B}$. Then we call $\mathcal{R}$ an *ε-stratified category* if one of the following equivalent properties holds:

($P\Delta_\varepsilon$) For every $b \in \mathbf{B}$, the projective object $P(b)$ has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}$ with $\rho(c) \geqslant \rho(b)$.

($I\nabla_\varepsilon$) For every $b \in \mathbf{B}$, the injective object $I(b)$ has a ∇_ε-flag with sections $\nabla_\varepsilon(c)$ for $c \in \mathbf{B}$ with $\rho(c) \geqslant \rho(b)$.

The fact that these two properties are indeed equivalent was established in [ADL, Theorem 2.2] (under slightly more restrictive hypotheses than here), extending the earlier work of Dlab [Dla1]. We give a self-contained proof in Theorem 3.7 below; see also §6.1 for some elementary examples. An equivalent statement is as follows.

Theorem 1.1 (Dlab,...). *Let $\mathcal{R}$ be a finite Abelian category equipped with a stratification $\rho : \mathbf{B} \to \Lambda$ and $\varepsilon : \Lambda \to \{\pm\}$ be a sign function as above. Then $\mathcal{R}$ is ε-stratified if and only if $\mathcal{R}^{\mathrm{op}}$ is $(-\varepsilon)$-stratified.*

If the stratification function $\rho : \mathbf{B} \to \Lambda$ is a bijection, i.e., each statum $\mathcal{R}_\lambda$ has a unique irreducible object (up to isomorphism), then we can use ρ to identify $\mathbf{B}$ with Λ, and denote the various distinguished objects simply by $L(\lambda), P(\lambda), \Delta_\varepsilon(\lambda), \dots$ for $\lambda \in \Lambda$ instead of by $L(b), P(b), \Delta_\varepsilon(b), \dots$ for $b \in \mathbf{B}$. Then, instead of "ε-stratified category," we call $\mathcal{R}$ an *ε-highest weight category.*

The notion of ε-highest weight category generalizes the original notion of highest weight category from [CPS1]: a *highest weight category* in the sense of *loc. cit.* is an ε-stratified category in which each stratum $\mathcal{R}_\lambda$ is actually simple (cf. Definition 3.5). This stronger assumption means not only that $\mathcal{R}_\lambda$ has a unique irreducible object $L_\lambda(\lambda)$ (up to isomorphism), but also that $L_\lambda(\lambda) = P_\lambda(\lambda) = I_\lambda(\lambda)$, hence, $\Delta(\lambda) = \bar{\Delta}(\lambda)$ and $\nabla(\lambda) = \bar{\nabla}(\lambda)$ for each $\lambda \in \Lambda$. Consequently, the sign function ε plays no role and may be omitted entirely, and the above properties simplify to the following:

($P\Delta$) Each $P(\lambda)$ has a Δ-flag with sections $\Delta(\mu)$ for $\mu \geqslant \lambda$.
($I\nabla$) Each $I(\lambda)$ has a ∇-flag with sections $\nabla(\mu)$ for $\mu \geqslant \lambda$.

In fact, in this context, the equivalence of ($P\Delta$) and ($I\nabla$) was established already in [CPS1]. Moreover, in *loc. cit.*, it is shown that A-$\mathrm{mod}_{\mathrm{fd}}$ is a highest weight category if and only if A is a quasi-hereditary algebra.

The next most important special cases arise when ε is the constant function $+$ or $-$. The idea of a $+$-stratified category originated in the work of Dlab [Dla1] already mentioned, and in another work of Cline, Parshall and Scott [CPS2]. In particular, the "standardly stratified categories" of [CPS2, Definition 2.2.1] are $+$-stratified categories.

We say that a finite Abelian category $\mathcal{R}$ equipped with a stratification $\rho : \mathbf{B} \to \Lambda$ is a *fully stratified category* if it is both a $+$-stratified category and a $-$-stratified category; in that case, it is ε-stratified for all choices of the sign function $\varepsilon : \Lambda \to \{\pm\}$. Such categories arise as categories of modules over the fully stratified algebras introduced in a remark after [ADL, Definition 1.3]. In fact, these sorts of algebras and categories have appeared several times elsewhere in the literature but under different names: they are called "weakly properly stratified" in [Fri1], "exactly properly stratified" in [CZ], and "standardly stratified" in [LW]. The latter seems a particularly confusing choice since it clashes with the established notion from [CPS2] but we completely agree with the sentiment of [LW, Remark 2.2]: fully stratified categories have a well-behaved structure theory. One reason for this is that all of the standardization and costandardization functors in a fully stratified category are exact. We note also that any ε-stratified category with a duality is fully stratified; see Corollary 3.23.

We use the language *signed highest weight category* in place of fully stratified category when the stratification function ρ is a bijection. Such categories are ε-highest weight for all choices of the sign function ε.

There are many classical examples of highest weight categories, including blocks of the BGG category $\mathcal{O}$ for a semisimple Lie algebra, the classical Schur algebra, and Donkin's generalized Schur algebras introduced in [Don2]. Further examples of fully stratified categories and signed highest weight categories which are not highest weight arise in the context of categorification. This includes the pioneering examples of categorified tensor products of finite dimensional irreducible representations for the quantum group attached to $\mathfrak{sl}_k$ from [FKS] (in particular Remark 2.5 therein), and the categorified induced cell modules for Hecke algebras from [MS, 6.5]. Building on these examples and the subsequent work of Webster [Web1], [Web2], Losev and Webster [LW] formulated the important axiomatic definition of a *tensor product categorification.* These are fully stratified categories which have been used to give a categorical interpretation of Lusztig's construction of tensor product of based modules for a quantum group.

The device of incorporating the sign function ε into the definition of ε-stratified or ε-highest weight category is our invention. It seems to be quite convenient as it streamlines many of the subsequent definitions and proofs. It also leads to some interesting new possibilities when it comes to the "tilting theory" which we discuss next.

Assume $\mathcal{R}$ is an ε-stratified category as above. An *ε-tilting object* is an object of $\mathcal{R}$ which has both a Δ_ε-flag and a ∇_ε-flag. Isomorphism classes of indecomposable ε-tilting objects are parametrized in a canonical way by the set $\mathbf{B}$; see Theorem 4.2. The construction of these objects is a non-trivial generalization of Ringel's classical construction via iterated extensions of standard objects: in general we take a mixture of extensions of standard objects at the top for positive strata and extensions of costandard objects at the bottom for negative strata. We denote the indecomposable ε-tilting objects by $\{T_\varepsilon(b) \mid b \in \mathbf{B}\}$.

Now let T be an *ε-tilting generator*, i.e., an ε-tilting object in which every $T_\varepsilon(b)$ appears at least once as a summand. The *Ringel dual* of $\mathcal{R}$ relative to T is the category $\widetilde{\mathcal{R}} := A\text{-mod}_{\mathrm{fd}}$ where $A := \mathrm{End}_{\mathcal{R}}(T)^{\mathrm{op}}$ (so that T is a right A-module). The isomorphism classes of irreducible objects in $\widetilde{\mathcal{R}}$ are in natural bijection with the isomorphism classes of indecomposable summands of T, hence, they are also indexed by the same set $\mathbf{B}$ that indexes the irreducibles in $\mathcal{R}$. Let

$$F := \mathrm{Hom}_{\mathcal{R}}(T, -) : \mathcal{R} \to \widetilde{\mathcal{R}}.$$

This is the *Ringel duality functor*. The following theorem is well known for highest weight categories (where it is due to Ringel [Rin] and Happel [Hap]) and for $+$- and $-$-stratified categories (where it is developed in the framework of standardly stratified algebras in [AHLU]). We prove it for general ε-stratified categories in Theorem 4.11.

Theorem 1.2 (Ringel, Happel, ...). *For $\mathcal{R}$ as above, let $\widetilde{\mathcal{R}}$ be the Ringel dual of $\mathcal{R}$ relative to an ε-tilting generator T. Let Λ^{op} be the opposite poset and $-\varepsilon : \Lambda^{\mathrm{op}} \to \{\pm\}$ be the negation of the original sign function ε.*

(1) The function $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$ defines a stratification of $\widetilde{\mathcal{R}}$ making it into a $(-\varepsilon)$-stratified category. Moreover, each stratum $\widetilde{\mathcal{R}}_\lambda$ of $\widetilde{\mathcal{R}}$ is equivalent to the corresponding stratum $\mathcal{R}_\lambda$ of $\mathcal{R}$.

(2) The functor F defines an equivalence of categories between the category of ∇_ε-filtered objects in $\mathcal{R}$ and the category of $\Delta_{-\varepsilon}$-filtered objects in $\widetilde{\mathcal{R}}$. It sends ε-tilting objects (resp., injective objects) in $\mathcal{R}$ to projective objects (resp., $(-\varepsilon)$-tilting objects) in $\widetilde{\mathcal{R}}$.

(3) If $\mathcal{R}_\lambda$ is of finite global dimension for each λ such that $\varepsilon(\lambda) = -$ then the total derived functor $\mathbb{R}F : D^b(\mathcal{R}) \to D^b(\widetilde{\mathcal{R}})$ is an equivalence between the bounded derived categories.

The original category $\mathcal{R}$ can be recovered from its Ringel dual $\widetilde{\mathcal{R}}$. Indeed, if we let I be an injective cogenerator in $\mathcal{R}$, then $\widetilde{T} := FI$ is a $(-\varepsilon)$-tilting generator in $\widetilde{\mathcal{R}}$ such that $B := \mathrm{End}_{\mathcal{R}}(I)^{\mathrm{op}} \cong \mathrm{End}_{\widetilde{\mathcal{R}}}(\widetilde{T})^{\mathrm{op}}$. Since $\mathcal{R}$ is equivalent to $B\text{-mod}_{\mathrm{fd}}$, it is equivalent to the Ringel dual of $\widetilde{\mathcal{R}}$ relative to $\widetilde{T}$. By Theorem 1.2(3), if $\mathcal{R}_\lambda$ is of finite global dimension for each λ with $\varepsilon(\lambda) = +$ then the Ringel duality functor $\widetilde{F} := \mathrm{Hom}_{\widetilde{\mathcal{R}}}(\widetilde{T}, -)$ in the other direction induces an equivalence $\mathbb{R}\widetilde{F} : D^b(\widetilde{\mathcal{R}}) \to D^b(\mathcal{R})$

We do not consider here derived equivalences in the case of infinite global dimension, but instead refer to [PS], where this and involved t-structures are treated in detail by generalizing the classical theory of co(resolving) subcategories. This requires the use of certain coderived and contraderived categories in place of ordinary derived categories.

Now we shift our attention to the semi-infinite case, which is really the main topic of the article. Following [EGNO], a *locally finite Abelian category* is a category that is equivalent to the category $\mathrm{comod}_{\mathrm{fd}}$-$C$ of finite-dimensional right comodules over some coalgebra C. Let $\mathcal{R}$ be such a category and $\{L(b) \mid b \in \mathbf{B}\}$ be a full set of pairwise inequivalent irreducible objects. Fix also a poset Λ that is *lower finite*, i.e., the intervals $(-\infty, \mu]$ are finite for all $\mu \in \Lambda$, a stratification function $\rho : \mathbf{B} \to \Lambda$ with finite fibers, and a sign function $\varepsilon : \Lambda \to \{\pm\}$. For any lower set (i.e., ideal of the poset) $\Lambda^\downarrow$ in Λ, we can consider the Serre subcategory $\mathcal{R}^\downarrow$ of $\mathcal{R}$ generated by the objects $\{L(b) \mid b \in \mathbf{B}^\downarrow\}$ where $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$. We say that $\mathcal{R}$ is a *lower finite ε-stratified category* if, for every finite lower set $\Lambda^\downarrow$ of Λ, the Serre subcategory $\mathcal{R}^\downarrow$ defined in this way is a finite Abelian category which is ε-stratified by the restriction of ρ; cf. Theorem 3.63. We call it a *lower finite ε-highest weight category* if in addition the stratification function is a bijection.

In a lower finite ε-stratified category, there are ε-standard and ε-costandard objects $\Delta_\varepsilon(b)$ and $\nabla_\varepsilon(b)$; they are the same as the ε-standard and ε-costandard objects of the Serre subcategory $\mathcal{R}^\downarrow$ defined from any finite lower set $\Lambda^\downarrow$ containing $\rho(b)$. As well as (finite) Δ_ε- and ∇_ε-flags, one can consider certain infinite ∇_ε-flags in objects of the ind-competion $\mathrm{Ind}(\mathcal{R})$ (which is the category comod-C of all right C-comodules in the case that $\mathcal{R} = \mathrm{comod}_{\mathrm{fd}}$-$C$). We refer to these as *ascending ∇_ε-flags*; see Definition 3.52 for the precise formulation. Theorem 3.59 establishes a homological criterion for an object to possess an ascending ∇_ε-flag, generalizing the well-known criterion for good filtrations in rational representations of reductive groups [Jan1, Proposition II.4.16]. From this, it follows that the injective hull $I(b)$ of $L(b)$ in $\mathrm{Ind}(\mathcal{R})$ has an ascending ∇_ε-flag. Moreover, the multiplicity of $\nabla_\varepsilon(c)$ as a section of such a flag satisfies

$$(I(b) : \nabla_\varepsilon(c)) = [\Delta_\varepsilon(c) : L(b)],$$

generalizing BGG reciprocity. This leads to an alternative “global” characterization of lower finite ε-stratified categories; see Definition 3.53.

In a lower finite ε-stratified category, there are also ε-tilting objects. Isomorphism classes of the indecomposable ones are labelled by $\mathbf{B}$ just like in the finite case. In fact, for $b \in \mathbf{B}$ the corresponding indecomposable ε-tilting object of $\mathcal{R}$ is the same as the object $T_\varepsilon(b)$ of the Serre subcategory $\mathcal{R}^\downarrow$ defined from any finite lower set $\Lambda^\downarrow$ containing $\rho(b)$. Let $(T_i)_{i\in I}$ be an *ε-tilting generating family* in $\mathcal{R}$. Then we can define the *Ringel dual* $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ relative to $T := \bigoplus_{i\in I} T_i$ (an object in the ind-completion of $\mathcal{R}$): it is the category of A-$\mathrm{mod}_{\mathrm{lfd}}$ locally finite-dimensional left modules over the locally finite-dimensional locally unital algebra

$$A = \Big(\bigoplus_{i,j\in I} \mathrm{Hom}_{\mathcal{R}}(T_i, T_j)\Big)^{\mathrm{op}},$$

where the op denotes that multiplication in A is the opposite of composition in $\mathcal{R}$. Saying that A is *locally unital* means that $A = \bigoplus_{i,j\in I} e_iAe_j$ where $\{e_i \mid i \in I\}$ are the mutually orthogonal idempotents defined by the identity endomorphisms of each T_i, and it is *locally finite-dimensional* if $\dim e_iAe_j < \infty$ for all $i, j \in I$. A *locally finite-dimensional module* is an A-module $V = \bigoplus_{i\in I} e_iV$ with $\dim e_iV < \infty$ for each i. As $e_iAe_j = \mathrm{Hom}_{\mathcal{R}}(T_i, T_j)$ is finite-dimensional, each left ideal Ae_j is a locally finite-dimensional module.

This brings us to the notion of an *upper finite ε-stratified category*, whose definition may be discovered by considering the nature of the categories $\widetilde{\mathcal{R}}$ that can arise as Ringel duals of lower finite ε-stratified categories as just defined. We refer to Definition 3.36 for the intrinsic formulation. In fact, starting from $\mathcal{R}$ that is a lower finite ε-stratified category, the Ringel dual $\widetilde{\mathcal{R}}$ is an upper finite $(-\varepsilon)$-stratified category with stratification defined by reversing the partial order on the poset Λ; see Theorem 4.20 which extends parts (1) and (2) of Theorem 1.2.

In general, in an upper finite ε-stratified category, the underlying poset is required to be upper finite, i.e., all of the intervals $[\lambda, \infty)$ are finite. There are ε-standard and ε-costandard objects, but now these can have infinite length (although composition multiplicities in such objects are finite). On the other hand, the indecomposable projectives and injectives do still have finite Δ_ε-flags and ∇_ε-flags, exactly like in $(P\Delta_\varepsilon)$ and $(I\nabla_\varepsilon)$. Perhaps the most interesting feature is that one can still make sense of ε-tilting objects. These are objects possessing certain infinite flags: both an *ascending* Δ_ε*-flag* and a *descending* ∇_ε*-flag*; see Definition 3.37. This allows to define the *Ringel dual* of an upper finite ε-stratified category; see Definition 4.21 and Theorem 4.22.

For $\widetilde{\mathcal{R}}$ arising as the Ringel dual of a lower finite ε-stratified category $\mathcal{R}$, the indecomposable $(-\varepsilon)$-tilting objects in $\widetilde{\mathcal{R}}$ are the images of the indecomposable injective objects of $\mathcal{R}$ under the Ringel duality functor

$$F := \bigoplus_{i \in I} \mathrm{Hom}_{\mathcal{R}}(T_i, -) : \mathcal{R} \to \widetilde{\mathcal{R}}.$$

Moreover, the Ringel dual of $\widetilde{\mathcal{R}}$ is equivalent to the original category $\mathcal{R}$. The proof of this relies on the following elementary observation: if U is any locally finite-dimensional module over a locally unital algebra A then the endomorphism algebra $\mathrm{End}_A(U)^{\mathrm{op}}$ is the linear dual C^* of a coalgebra C; see Lemma 2.10. Taking U to be a full $(-\varepsilon)$-tilting object in $\widetilde{\mathcal{R}}$, this produces a coalgebra C such that the original category $\mathcal{R}$ is equivalent to $\mathrm{comod}_{\mathrm{fd}}$-$C$. See §6.2 for an explicit example illustrating the semi-infinite Ringel duality construction.

An *upper finite* (resp., *lower finite*) *highest weight category* is an upper finite (resp., lower finite) ε-stratified category all of whose strata are simple. In §5.1, we apply semi-infinite Ringel duality together with arguments from [AST] to give an elementary algebraic characterization of upper finite highest weight categories in terms of the notion of an *upper finite based quasi-hereditary algebra*. In the finite-dimensional setting, these are based quasi-hereditary algebras as defined by Kleshchev and Muth in [KM], who proved that their definition of based quasi-hereditary algebra is equivalent to the original definition of quasi-hereditary algebra from [CPS1]. Our more general algebras are locally finite-dimensional locally unital algebras rather than unital algebras. Viewing them instead as finite-dimensional categories, that is, $\Bbbk$-linear categories with finite-dimensional morphism spaces, the definition translates into the notion of an object-adapted cellular category which was introduced already by Elias and Lauda [ELau, Definition 2.1]. In turn, the Elias-Lauda definition evolved from work of Westbury [Wes], who extended the definition of cellular algebra due to Graham and Lehrer [GL] from finite-dimensional algebras to finite-dimensional categories.

In §5.2, we also introduce *upper finite based* ε*-stratified algebras* and *upper finite based* ε*-quasi-hereditary algebras*, which are the precise algebraic counterparts of upper finite ε-stratified categories and upper finite ε-highest weight categories, respectively.

We say that a fully stratified category is *tilting-rigid* if $T_+(b) \cong T_-(b)$ for all $b \in \mathbf{B}$. Equivalently, the tilting objects $T_\varepsilon(b)$ are isomorphic for all choices of the sign function ε, so that they may all be denoted simply by $T(b)$, as is done for classical highest weight categories. The property of being tilting-rigid is quite strong, for example, it implies that all of the strata are equivalent to categories of finite-dimensional modules over weakly symmetric Frobenius algebras. Many of the naturally-occurring examples of fully stratified categories are tilting-rigid, including the tensor product categorifications from [LW] mentioned earlier. For us, the key point about the tilting-rigid hypothesis is that the Ringel dual of a fully stratified category that is tilting-rigid is again a fully stratified category that is tilting-rigid. This is important in §5.3, when we introduce the final basic notions of *based stratified algebras* and *fibered quasi-hereditary algebras*. These definitions have a similar flavor to the fibered object-adapted cellular categories of [ELau, Definition 2.17]. We show that the category of locally finite-dimensional modules over an upper finite based stratified algebra (resp., an upper finite fibered quasi-hereditary algebra) is an upper finite fully stratified (resp., signed highest weight) category, and conversely any upper finite fully stratified (resp., signed highest weight) category which is also tilting-rigid can be realized in this way.

We also provide another source of naturally-occurring examples of based stratified algebras: any locally unital algebra admitting a *triangular decomposition* in the sense of Definition 5.24 can be endowed with such a structure. The proof of this goes via a related notion of a locally unital algebra with a *Cartan decomposition*; see Definition 5.23. The latter is quite a versatile framework which provides a bridge between based stratified algebras and algebras with triangular decompositions.

The following summarizes some of the connections established between these various types of algebras and their module categories:

$$
\begin{tikzcd}
\text{Upper finite highest weight categories} \arrow[d, shift left] & & \text{Upper finite fully stratified categories} \arrow[d, dashed, shift left, "+\text{tilting-rigid}"] \\
\text{Upper finite based quasi-hereditary algebras} \arrow[u, shift left] \arrow[dr, shift left] & & \text{Upper finite based stratified algebras} \arrow[u, shift left] \arrow[dl, dashed, shift left, "+\text{split Cartan}"] \\
& \text{Locally unital algebras with an upper finite Cartan decomposition} \arrow[ul, dashed, shift left, "+\text{semisimple Cartan}"] \arrow[ur, shift left] \arrow[d, dashed, shift left, "+\text{opposite Borel subalgebras}"] & \\
& \text{Locally unital algebras with an upper finite triangular decomposition} \arrow[u, shift left] &
\end{tikzcd}
$$

In the main body of the text, we also discuss a parallel situation involving *essentially finite* rather than *upper finite* algebras and categories. For example, the finite-dimensional graded algebras with a triangular decomposition studied in [HN, BT] fit naturally into our more general framework of locally unital algebras with an essentially finite triangular decomposition; see Remark 5.25.

As we have already mentioned, the category $\mathcal{R}ep(G)$ for a reductive group G is the archetypal example of a lower finite highest weight category. Its Ringel dual is an upper finite highest weight category. This case has been studied in particular by Donkin (e.g., see [Don2], [Don3]), but Donkin's approach involves truncating to a finite-dimensional algebra from the outset. Other important examples come from blocks of category $\mathcal{O}$ over an affine Lie algebra: in negative level one obtains lower finite highest weight categories, while positive level produces upper finite ones. These and several other prominent examples are outlined in §§6.3–6.7. This includes various Deligne categories which are diagrammatic in nature and come equipped with an evident triangular decomposition, hence, their module categories are upper finite highest weight categories.

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2. Some finiteness properties on Abelian categories

We fix an algebraically closed field $\Bbbk$. All algebras, categories, functors, etc. will be assumed to be linear over $\Bbbk$. We write $\otimes$ for $\otimes_{\Bbbk}$. The naive terms *direct limit* and *inverse limit* will be used for small filtered colimits and limits, respectively. We begin by introducing some language for Abelian categories with various finiteness properties:

$$\begin{tikzcd}
 & \text{Finite Abelian categories} \arrow[d, "\text{Lemma 2.17}"] \arrow[ddl, bend right, "\text{Lemma 2.17}"'] \arrow[ddr, bend left, "\text{Lemma 2.17}"] & \\
 & \text{Essentially finite Abelian categories} \arrow[dl, bend left, "\text{Corollary 2.16}"] \arrow[dr, dashed, "\text{Lemma 2.18}"] & \\
\text{Locally finite Abelian categories} & & \text{Schurian categories}
\end{tikzcd}$$

2.1. **Finite and locally finite Abelian categories.** According to [EGNO, Definition 1.8.5], a *finite Abelian category* is a category that is equivalent to the category $A\text{-mod}_{\text{fd}}$ of finite-dimensional (left) modules over some finite-dimensional algebra A. Equivalently, a finite Abelian category is a category equivalent to the category $\text{comod}_{\text{fd}}\text{-}C$ of finite-dimensional (right) comodules over some finite-dimensional coalgebra C. To explain this in more detail, recall that the dual $A := C^*$ of a finite-dimensional coalgebra C has a natural algebra structure with multiplication $A \otimes A \to A$ that is the dual of the comultiplication $C \to C \otimes C$; for this, one needs to use the canonical isomorphism

$$C^* \otimes C^* \to (C \otimes C)^*, \qquad f \otimes g \mapsto (v \otimes w \mapsto f(v)g(w)) \tag{2.1}$$

to identify $C^* \otimes C^*$ with $(C \otimes C)^*$. Then any right C-comodule can be viewed as a left A-module with action defined from $av := \sum_{i=1}^n a(c_i)v_i$ assuming here that the structure map $\eta : V \to V \otimes C$ sends $v \mapsto \sum_{i=1}^n v_i \otimes c_i$. Conversely, the C-comodule structure on V can be recovered uniquely from the action of A. Thus, the categories $\text{comod}_{\text{fd}}\text{-}C$ and $A\text{-mod}_{\text{fd}}$ are isomorphic.

A *locally finite Abelian category* is a category $\mathcal{R}$ that is equivalent to $\text{comod}_{\text{fd}}\text{-}C$ for a (not necessarily finite-dimensional) coalgebra C. We refer to a choice of C as a *coalgebra realization* of $\mathcal{R}$. The following result of Takeuchi gives an intrinsic characterization of locally finite Abelian categories; see [Tak] and [EGNO, Theorem 1.9.15]. It is a version of [Gab, IV, Theorem 4] adapted to our situation. Note Takeuchi's original paper uses the language "locally finite Abelian" slightly differently (following [Gab]) but his formulation is equivalent to the one here (which follows [EGNO, Definition 1.8.1]). In *loc. cit.* it is shown moreover that C can be chosen so that it is *pointed*, i.e., all of its irreducible comodules are one-dimensional; in that case, C is unique up to isomorphism.

Lemma 2.1. *An essentially small category $\mathcal{R}$ is a locally finite Abelian category if and only if it is Abelian, all of its objects are of finite length, and all of its morphism spaces are finite-dimensional.*

Now we summarize the main properties of the locally finite Abelian category

$$\mathcal{R} = \text{comod}_{\text{fd}}\text{-}C.$$

Fix a full set of pairwise inequivalent irreducible objects $\{L(b) \mid b \in \mathbf{B}\}$ in $\mathcal{R}$. By Schur's Lemma, we have that $\text{End}_{\mathcal{R}}(L(b)) = \Bbbk$ for each $b \in \mathbf{B}$.

The opposite category $\mathcal{R}^{\text{op}}$ is also a locally finite Abelian category. Moreover, a coalgebra realization for it is given by the opposite coalgebra C^{op}. To see this, note that there is a contravariant equivalence

$$* : \text{comod}_{\text{fd}}\text{-}C \to C\text{-comod}_{\text{fd}} \tag{2.2}$$

sending a finite-dimensional right comodule to the dual vector space viewed as a left comodule in the natural way: if $v_1, \dots, v_n$ is a basis for V, with dual basis $f_1, \dots, f_n$ for V^*, and the structure map $V \to V \otimes C$ sends $v_j \mapsto \sum_{i=1}^n v_i \otimes c_{i,j}$ then the dual's structure map $V^* \to C \otimes V^*$ sends $f_i \mapsto \sum_{i=1}^n c_{i,j} \otimes f_j$. Since we have that $C\text{-comod}_{\mathrm{fd}} \cong \mathrm{comod}_{\mathrm{fd}}\text{-}C^{\mathrm{op}}$, we deduce that $\mathcal{R}^{\mathrm{op}}$ is equivalent to $\mathrm{comod}_{\mathrm{fd}}\text{-}C^{\mathrm{op}}$.

In general, $\mathcal{R}$ need not have enough injectives or projectives. To get injectives, we pass to the *ind-completion* $\mathrm{Ind}(\mathcal{R})$; see e.g. [KS, §6.1]. For $V, W \in \mathrm{Ind}(\mathcal{R})$, we write $\mathrm{Ext}^n_{\mathcal{R}}(V, W)$, or sometimes $\mathrm{Ext}^n_C(V, W)$, for $\mathrm{Ext}^n_{\mathrm{Ind}(\mathcal{R})}(V, W)$; it may be computed via an injective resolution of W in the ind-completion. This convention is unambiguous due to [KS, Theorem 15.3.1]. More generally, we can consider the right derived functors $\mathbb{R}^n F$ of any left exact functor $F : \mathrm{Ind}(\mathcal{R}) \to \mathcal{S}$.

Let comod-C be the category of all right C-comodules. Every comodule is the union (hence, the direct limit) of its finite-dimensional subcomodules. Moreover, a comodule V is compact, i.e., the functor $\mathrm{Hom}_C(V, -)$ commutes with direct limits, if and only if it is finite-dimensional. Using this, [KS, Corollary 6.3.5] implies that the canonical functor $\mathrm{Ind}(\mathcal{R}) \to$ comod-C is an equivalence of categories. This means that one can work simply with comod-C in place of $\mathrm{Ind}(\mathcal{R})$, as we do in the next few paragraphs.

The category comod-C is a Grothendieck category: it is Abelian, it possesses all small coproducts, direct colimits of monomorphisms are monomorphisms, and there is a generator. A generating family may be obtained by choosing representatives for the isomorphism classes of finite-dimensional C-comodules. By the general theory of Grothendieck categories, every C-comodule has an injective hull. We use the notation $I(b)$ to denote an injective hull of $L(b)$. The right regular comodule decomposes as

$$C \cong \bigoplus_{b \in \mathbf{B}} I(b)^{\oplus \dim L(b)}. \tag{2.3}$$

By Baer's criterion for Grothendieck categories (e.g., see [KS, Proposition 8.4.7]), arbitrary direct sums of injectives are injective. It follows that an injective hull of $V \in$ comod-C comes from an injective hull of its socle: if $\mathrm{soc}\, V \cong \bigoplus_{s \in S} L(b_s)$ then $\bigoplus_{s \in S} I(b_s)$ is an injective hull of V.

In any Abelian category, we write $[V : L]$ for the *composition multiplicity* of an irreducible object L in an object V. By definition, this is the supremum of the sizes of the sets $\{i = 1, \dots, n | V_i/V_{i-1} \cong L\}$ over all finite filtrations $0 = V_0 < V_1 < \cdots < V_n = V$; possibly, $[V : L] = \infty$. Composition multiplicity is additive on short exact sequences. For any right C-comodule V, we have by Schur's Lemma that

$$[V : L(b)] = \dim \mathrm{Hom}_C(V, I(b)). \tag{2.4}$$

When C is infinite-dimensional, the map (2.1) is not an isomorphism, but one can still use it to make the dual vector space $B := C^*$ into a unital algebra. Since C is the union of its finite-dimensional subcoalgebras, the algebra B is the inverse limit of its finite-dimensional quotients, i.e., the canonical homomorphism $B \to \varprojlim (B/J)$ is an isomorphism where the limit is over all two-sided ideals J of B of finite codimension. These two-sided ideals J form a basis of neighborhoods of 0 making B into a *pseudo-compact topological algebra*; see [Gab, Ch. IV] or [Sim, Definition 2.4]. We refer to the topology on B defined in this way as the *profinite topology*. The coalgebra C can be recovered from B as the *continuous dual*

$$B^\circledast := \left\{ f \in B^* \,\middle|\, f \text{ vanishes on some two-sided ideal } J \text{ of finite codimension} \right\}. \tag{2.5}$$

It has a natural coalgebra structure dual to the algebra structure on B. This is discussed further in [Sim, §3]; see also [EGNO, §1.12] where $B^\circledast$ is called the *finite dual*. We note that any left ideal I of B of finite codimension contains a two-sided ideal J of finite codimension, namely, $J := \mathrm{Ann}_B(B/I)$. So, in the definition (2.5) of continuous dual, "two-sided ideal J of finite codimension" can be replaced by "left ideal I of finite codimension." Similarly for right ideals.

Any right C-comodule V is naturally a left B-module by the same construction as in the finite-dimensional case. We deduce that the category comod-C of all right C-comodules is isomorphic to the full subcategory B-$\mathrm{mod}_{\mathrm{ds}}$ of B-mod consisting of all *discrete* left B-modules, that is, all B-modules which are the unions of their finite-dimensional submodules. In particular, $\mathrm{comod}_{\mathrm{fd}}$-$C$ and B-$\mathrm{mod}_{\mathrm{fd}}$ are identified under this construction. This means that any locally finite Abelian category may be realized as the category of finite-dimensional modules over an algebra which is pseudocompact with respect to the profinite topology; see also [Sim, §3].

The definition of the left C-comodule structure on the linear dual V^* of a right C-comodule V in (2.2) required V to be finite-dimensional in order for it to make sense. If V is an infinite-dimensional right C-comodule, it can be viewed equivalently as a discrete left module over the dual algebra $B := C^*$. Then its dual V^* is a *pseudocompact right B-module*, that is, a B-module isomorphic to the inverse limit of its finite-dimensional quotients. Viewing pseudocompact modules as topological B-modules with respect to the profinite topology (i.e., submodules of finite codimension form a basis of neighborhoods of 0), we obtain the category $\mathrm{mod}_{\mathrm{pc}}$-$B$ of all pseudocompact right B-modules and continuous B-module homomorphisms. The duality functor (2.2) extends to

$$* : B\text{-}\mathrm{mod}_{\mathrm{ds}} \to \mathrm{mod}_{\mathrm{pc}}\text{-}B. \tag{2.6}$$

This is a contravariant equivalence with quasi-inverse given by the functor

$$* : \mathrm{mod}_{\mathrm{pc}}\text{-}B \to B\text{-}\mathrm{mod}_{\mathrm{ds}} \tag{2.7}$$

taking $V \in \mathrm{mod}_{\mathrm{pc}}$-$B$ to its *continuous dual*

$$V^* := \left\{ f \in V^* \;\middle|\; f \text{ vanishes on some submodule of } V \text{ of finite codimension} \right\}.$$

Lemma 2.2. *Suppose that C is a coalgebra and $B := C^*$ is its dual algebra. For any right C-comodule V, composing with the counit $\epsilon : C \to \Bbbk$ defines an isomorphism of left B-modules $\alpha_V : \mathrm{Hom}_C(V, C) \xrightarrow{\sim} V^*$. When $V = C$, this map gives an algebra isomorphism $\mathrm{End}_C(C)^{\mathrm{op}} \cong B$.*

Proof. Let $\eta : V \to V \otimes C$ be the comodule structure map. To show that α_V is an isomorphism, one checks that the map $\beta_V : V^* \to \mathrm{Hom}_C(V, C), f \mapsto (f \bar{\otimes} \mathrm{id}) \circ \eta$ is its two-sided inverse; cf. [Sim, Lemma 4.9]. It remains to show that $\alpha_C : \mathrm{End}_C(C)^{\mathrm{op}} \xrightarrow{\sim} B$ is an algebra homomorphism: for $f, g \in B$ we have that

$$\begin{aligned}\alpha_C(\beta_C(g) \circ \beta_C(f)) &= \epsilon \circ (g \bar{\otimes} \mathrm{id}) \circ \eta \circ (f \bar{\otimes} \mathrm{id}) \circ \eta \\ &= (g \bar{\otimes} \mathrm{id}) \circ (\mathrm{id} \otimes \epsilon) \circ \eta \circ (f \bar{\otimes} \mathrm{id}) \circ \eta = g \circ (f \bar{\otimes} \mathrm{id}) \circ \eta = fg. \qquad \square\end{aligned}$$

2.2. **Locally unital algebras.** We are going to work with certain Abelian categories which are not locally finite, but which nevertheless have some well-behaved finiteness properties. We will define these in the next subsection. First we must review some basic notions about locally unital algebras. These ideas originate in the work of Mitchell [Mit].

A *locally unital algebra* is an associative (but not necessarily unital) algebra A equipped with a distinguished system $\{e_i \mid i \in I\}$ of mutually orthogonal idempotents such that

$$A = \bigoplus_{i,j \in I} e_i A e_j.$$

We say A is *locally finite-dimensional* if each subspace $e_i A e_j$ is finite-dimensional.

A *locally unital homomorphism* (resp. *isomorphism*) between two locally unital algebras is an algebra homomorphism (resp. isomorphism) which takes distinguished idempotents to distinguished idempotents. Also, we say that A is an *idempotent contraction* of B, or B is an *idempotent expansion* of A, if there is an algebra isomorphism $A \xrightarrow{\sim} B$ sending each distinguished idempotent in A to a sum of distinguished idempotents in B. For example, one needs to pass to an idempotent expansion in order to define blocks of a locally finite-dimensional locally unital algebra; see [BD1, (L9)–(L10)].

For a locally unital algebra A, an A-*module* means a left module V as usual such that $V = \bigoplus_{i\in I} e_iV$. A vector $v \in V$ is *homogeneous* if $v \in e_iV$ for some $i \in I$. A module V is

- *locally finite-dimensional* if $\dim e_iV < \infty$ for all $i \in I$;
- *finitely generated* if $V = Av_1 + \cdots + Av_n$ for vectors $v_1, \dots, v_n \in V$ (which may be assumed to be homogeneous) or, equivalently, it is a quotient of the finitely generated projective A-module $Ae_{i_1} \oplus \cdots \oplus Ae_{i_n}$ for $i_1, \dots, i_n \in I$ and $n \in \mathbb{N}$;
- *finitely presented* if there is an exact sequence
$$Ae_{j_1} \oplus \cdots \oplus Ae_{j_m} \to Ae_{i_1} \oplus \cdots \oplus Ae_{i_n} \to V \to 0$$
for $i_1, \dots, i_n, j_1, \dots, j_m \in I$ and $m, n \in \mathbb{N}$.

Let A-mod (resp., A-$\mathrm{mod}_{\mathrm{lfd}}$, resp., A-$\mathrm{mod}_{\mathrm{fg}}$, resp., A-$\mathrm{mod}_{\mathrm{fp}}$) be the category of all A-modules (resp., the locally finite-dimensional ones, resp., the finitely generated ones, resp., the finitely presented ones). Similarly, we define the categories mod-A, $\mathrm{mod}_{\mathrm{lfd}}$-$A$, $\mathrm{mod}_{\mathrm{fg}}$-$A$ and $\mathrm{mod}_{\mathrm{fp}}$-$A$ of right modules.

Remark 2.3. Any locally unital algebra $A = \bigoplus_{i,j\in I} e_iAe_j$ can be viewed as a category with object set I and $\mathrm{Hom}_{\mathcal{A}}(j,i) = e_iAe_j$, with the idempotent $e_i \in A$ corresponding to the identity endomorphism $1_i \in \mathrm{End}_{\mathcal{A}}(i)$. Conversely, any small category $\mathcal{A}$ ($\Bbbk$-linear, of course) gives rise to a corresponding locally unital algebra A, which may be called the *path algebra* of $\mathcal{A}$. In these terms, locally finite-dimensional locally unital algebras correspond to *finite-dimensional categories*, that is, small categories all of whose morphism spaces are finite-dimensional. The notion of idempotent expansion of the algebra A becomes the notion of *thickening* of the category $\mathcal{A}$, which is a sort of "partial Karoubi envelope." Also, a left A-module (resp., a locally finite-dimensional left A-module) is the same as a $\Bbbk$-linear functor from $\mathcal{A}$ to the category $\mathcal{V}ec$ (resp., $\mathcal{V}ec_{\mathrm{fd}}$) of vector spaces (resp., finite-dimensional vector spaces). Similarly, right A-modules are functors from the opposite category $\mathcal{A}^{\mathrm{op}}$.

Lemma 2.4. *An essentially small category $\mathcal{R}$ is equivalent to* A-mod *for some locally unital algebra A if and only if $\mathcal{R}$ is Abelian, it possesses all small coproducts, and it has a projective generating family, i.e., there is a family $(P_i)_{i\in I}$ of compact projective objects such that $V \neq 0 \Rightarrow \mathrm{Hom}_{\mathcal{R}}(P_i, V) \neq 0$ for some $i \in I$.*

Proof. This is similar to [Fre, Exercise 5.F]. One shows that $\mathcal{R}$ is equivalent to A-mod for the locally unital algebra $A = \bigoplus_{i,j\in I} e_iAe_j$ defined by setting $e_iAe_j := \mathrm{Hom}_{\mathcal{R}}(P_i, P_j)$ with multiplication that is the opposite of composition in $\mathcal{R}$. The canonical equivalence $\mathcal{R} \to A$-mod is given by the functor $\bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{R}}(P_i, -)$. □

Lemma 2.5. *Let A be a locally unital algebra. An A-module V is compact if and only if it is finitely presented. Also, for projective modules, the notions of finitely presented and finitely generated coincide.*

Proof. This is well known for modules over a ring, and the usual proof in that setting carries over almost unchanged to the locally unital case. □

Lemma 2.6. *Let A be a locally unital algebra. Any A-module is isomorphic to a direct limit of finitely presented A-modules.*

Proof. As any A-module is the union of its finitely generated submodules, it suffices to show that any finitely generated A-module V is finitely presented. But then V is a quotient of $P = Ae_{i_1} \oplus \cdots \oplus Ae_{i_n}$ by a submodule. This submodule is the union of its finitely generated submodules W, so we have that $V \cong P/\varinjlim W \cong \varinjlim P/W$. This is a direct limit of finitely presented modules. □

The following lemma is fundamental. It is the analog of "adjointness of tensor and hom" in the locally unital setting; see e.g. [BD1, §2.1] for a fuller discussion.

Lemma 2.7. *Let $A = \bigoplus_{i,j\in I} e_iAe_j$ and $B = \bigoplus_{i,j\in J} f_iBf_j$ be locally unital algebras, and let $M = \bigoplus_{i\in I,j\in J} e_iMf_j$ be an (A,B)-bimodule.*

(1) The functor $M\otimes_B - : B\text{-mod} \to A\text{-mod}$ is left adjoint to $\bigoplus_{j\in J}\operatorname{Hom}_A(Mf_j,-)$.
(2) The functor $-\otimes_A M : \text{mod-}A \to \text{mod-}B$ is left adjoint to $\bigoplus_{i\in I}\operatorname{Hom}_B(e_iM,-)$.

For any locally unital algebra A, there is a contravariant equivalence

$$\circledast : A\text{-mod}_{\text{lfd}} \to \text{mod}_{\text{lfd}}\text{-}A \tag{2.8}$$

sending a left module V to $V^\circledast := \bigoplus_{i\in I}(e_iV)^*$, viewed as a right module in the obvious way. The analogous duality functor $\circledast : \text{mod}_{\text{lfd}}\text{-}A \to A\text{-mod}_{\text{lfd}}$ gives a quasi-inverse. The contravariant functor (2.8) also makes sense on arbitrary left (or right) A-modules. It is no longer an equivalence, but we still have that

$$\operatorname{Hom}_A(V, W^\circledast) \cong \operatorname{Hom}_A(W, V^\circledast) \tag{2.9}$$

for any $V \in A$-mod and $W \in$ mod-A. To prove this, apply Lemma 2.7(1) to the $(\Bbbk, A)$-bimodule W to show that $\operatorname{Hom}_A(V, W^\circledast) \cong (W\otimes_A V)^*$, then apply Lemma 2.7(2) to the $(A,\Bbbk)$-bimodule V to show that $(W\otimes_A V)^* \cong \operatorname{Hom}_A(W, V^\circledast)$.

Lemma 2.8. *The dual $V^\circledast$ of a projective (left or right) A-module is an injective (right or left) A-module.*

Proof. Just like in the classic treatment of duality for vector spaces from [Mac, IV.2], (2.9) shows that the covariant functor $\circledast : A\text{-mod} \to (\text{mod-}A)^{\text{op}}$ is left adjoint to the exact covariant functor $\circledast : (\text{mod-}A)^{\text{op}} \to A\text{-mod}$. So it sends projective left A-modules to projectives in $(\text{mod-}A)^{\text{op}}$, which are injective right A-modules. □

Now we assume that A is a locally unital algebra and $U \in A\text{-mod}_{\text{lfd}}$. Let

$$B := \operatorname{End}_A(U)^{\text{op}},$$

which is a unital algebra. Then U is an (A,B)-bimodule and the dual $U^\circledast$ is a (B,A)-bimodule. Let $U_i := e_iU$, so that $U = \bigoplus_{i\in I} U_i$ and $U^\circledast = \bigoplus_{i\in I} U_i^*$.

Lemma 2.9. *Suppose that $U = \bigoplus_{i\in I} U_i \in A\text{-mod}_{\text{lfd}}$ and $B := \operatorname{End}_A(U)^{\text{op}}$ are as above. For any $V \in A$-mod, there is a natural isomorphism of right B-modules*

$$\operatorname{Hom}_A(V,U) \xrightarrow{\sim} (U^\circledast \otimes_A V)^*, \qquad \theta \mapsto (f\otimes v \mapsto f(\theta(v))).$$

In particular, taking $V = U$, we get that $(U^\circledast\otimes_A U)^* \cong B$ as (B,B)-bimodules.

Proof. By Lemma 2.7(1) applied to the $(A,\Bbbk)$-bimodule $U^\circledast$, the functor $U^\circledast\otimes_A -$ is left adjoint to $\bigoplus_{i\in I}\operatorname{Hom}_\Bbbk(U_i^*, -)$. Hence,

$$(U^\circledast\otimes_A V)^* = \operatorname{Hom}_\Bbbk(U^\circledast\otimes_A V, \Bbbk) \cong \operatorname{Hom}_A(V, \bigoplus_{i\in I}\operatorname{Hom}_\Bbbk(U_i^*, \Bbbk)) \cong \operatorname{Hom}_A(V,U).$$

This is the natural isomorphism in the statement of the lemma. We leave it to the reader to check that it is a B-module homomorphism. □

Continuing with this setup, let

$$C := U^\circledast \otimes_A U. \tag{2.10}$$

There is a unique way to make C into a coalgebra so that the bimodule isomorphism $B \xrightarrow{\sim} C^*$ from Lemma 2.9 is actually an algebra isomorphism (viewing the dual C^* of a coalgebra as an algebra as in the previous subsection). Explicitly, let $y_1^{(i)},\dots,y_{n_i}^{(i)}$ be a basis for U_i and $x_1^{(i)},\dots,x_{n_i}^{(i)}$ be the dual basis for U_i^*. Let $c_{s,r}^{(i)} := x_r^{(i)}\otimes y_s^{(i)} \in C$. Then the comultiplication $\delta : C \to C\otimes C$ and counit $\epsilon : C \to \Bbbk$ satisfy

$$\delta(c_{r,s}^{(i)}) = \sum_{t=1}^{n_i} c_{r,t}^{(i)}\otimes c_{t,s}^{(i)}, \qquad\qquad \epsilon(c_{r,s}^{(i)}) = \delta_{r,s} \tag{2.11}$$

for each $i \in I$ and $1 \leqslant r,s \leqslant n_i$. For the next lemma, we recall the definition of continuous dual of a pseudocompact topological algebra from (2.5).

Lemma 2.10. *The endomorphism algebra $B = \mathrm{End}_A(U)^{\mathrm{op}}$ of $U \in A\text{-mod}_{\mathrm{lfd}}$ is a pseudocompact topological algebra with respect to the profinite topology, i.e., B is isomorphic to $\varprojlim B/J$ where the inverse limit is over all two-sided ideals J of finite codimension. Moreover, the coalgebra C from (2.10) may be identified with the continuous dual B^*.*

Proof. This follows because $B \cong C^*$ as algebras. □

Now consider the functor $U^{\circledast} \otimes_A - : A\text{-mod} \to B\text{-mod}$. Since U is locally finite-dimensional, it takes finitely generated A-modules to finite-dimensional B-modules. Any A-module V is the direct union of its finitely generated submodules, and $U^{\circledast} \otimes_A -$ commutes with direct limits, so we see that $U^{\circledast} \otimes_A V$ is actually a discrete B-module. This shows that we have a well-defined functor

$$U^{\circledast} \otimes_A - : A\text{-mod} \to B\text{-mod}_{\mathrm{ds}}\,. \tag{2.12}$$

Since $B \cong C^*$, the category $B\text{-mod}_{\mathrm{ds}}$ is isomorphic to comod-C. Consequently, for $V \in A\text{-mod}$, we can view $U^{\circledast} \otimes_A V$ instead as a right C-comodule. Its structure map $\eta : U^{\circledast} \otimes_A V \to U^{\circledast} \otimes_A V \otimes C$ is given explicitly by the formula

$$\eta(x_s^{(i)} \otimes v) = \sum_{r=1}^{n_i} x_r^{(i)} \otimes v \otimes c_{r,s}^{(i)}. \tag{2.13}$$

Recall the definition of the functor $*$ from (2.7).

Lemma 2.11. *Suppose that $U = \bigoplus_{i \in I} U_i \in A\text{-mod}_{\mathrm{lfd}}$ and $B := \mathrm{End}_A(U)^{\mathrm{op}}$ are as above. The functor $U^{\circledast} \otimes_A -$ just constructed is isomorphic to*

$$* \circ \mathrm{Hom}_A(-, U) : A\text{-mod} \to B\text{-mod}_{\mathrm{ds}} \tag{2.14}$$

and left adjoint to the functor

$$\bigoplus_{i \in I} \mathrm{Hom}_B(U_i^*, -) : B\text{-mod}_{\mathrm{ds}} \to A\text{-mod}\,. \tag{2.15}$$

Proof. The fact that (2.12) is left adjoint to (2.15) follows by Lemma 2.7. To see that it is isomorphic to (2.14), take $V \in A\text{-mod}$ and consider the natural isomorphism $\mathrm{Hom}_A(V, U) \cong (U^{\circledast} \otimes_A V)^*$ of right B-modules from Lemma 2.9. As $U^{\circledast} \otimes_A V$ is discrete, its dual is a pseudocompact left B-module, hence, $\mathrm{Hom}_A(V, U)$ is pseudocompact too. Then we apply $*$, using that it is quasi-inverse to $*$, to get that $\mathrm{Hom}_A(V, U)^* \in B\text{-mod}_{\mathrm{ds}}$ is naturally isomorphic to $U^{\circledast} \otimes_A V$. □

2.3. **Schurian categories.** By a *Schurian category*, we mean a category $\mathcal{R}$ that is equivalent to $A\text{-mod}_{\mathrm{lfd}}$ for a locally finite-dimensional locally unital algebra A. This language is new. In fact, this usage of the term "Schurian" is a slight departure from several recent papers of the first author: in [BD1], the phrase "locally Schurian" was used to describe such categories; more precisely, in [BD1], a locally Schurian category referred to a category of the form A-mod (rather than $A\text{-mod}_{\mathrm{lfd}}$) for such locally unital algebras A. We could not use the phrase "Schurian" in *loc. cit.* since that was reserved for a more restrictive notion introduced in [BLW, §2.1]; this more restrictive notion will be discussed in the next subsection, again using different language.

By an *algebra realization* of a Schurian category $\mathcal{R}$, we mean any locally finite-dimensional locally unital algebra A (together with the set I indexing its distinguished idempotents) such that $\mathcal{R}$ is equivalent to $A\text{-mod}_{\mathrm{lfd}}$. We say that A is *pointed* if its set $\{e_i | i \in I\}$ of distinguished idempotents coincides with the set of all primitive idempotents in A. This implies in particular that A is a basic algebra, i.e., all of its irreducible modules are one-dimensional. Any Schurian category $\mathcal{R}$ has a unique (up to isomorphism) pointed algebra realization. We justify this assertion later on; see (2.19).

Let us summarize some of the basic properties of Schurian categories, referring to [BD1, §2] for a more detailed treatment. Assume that A is a locally finite-dimensional locally unital algebra and let

$$\mathcal{R} = A\text{-mod}_{\mathrm{lfd}}\,.$$

Let $\{L(b) \mid b \in \mathbf{B}\}$ be a full set of pairwise inequivalent irreducible objects of $\mathcal{R}$. Schur's Lemma holds: we have that $\mathrm{End}_{\mathcal{R}}(L(b)) = \Bbbk$ for each $b \in \mathbf{B}$.

The opposite category $\mathcal{R}^{\mathrm{op}}$ is also Schurian, and A^{op} gives an algebra realization for it. This follows because $\mathcal{R}^{\mathrm{op}} = (A\text{-mod}_{\mathrm{lfd}})^{\mathrm{op}}$ is equivalent to $\mathrm{mod}_{\mathrm{lfd}}\text{-}A \cong (A^{\mathrm{op}})\text{-mod}_{\mathrm{lfd}}$ using the duality (2.8).

Let $\mathcal{R}_c$ be the (not necessarily Abelian) full subcategory of $\mathcal{R}$ consisting of all compact objects, and $\mathrm{Ind}(\mathcal{R}_c)$ be its ind-completion. The canonical functor $\mathrm{Ind}(\mathcal{R}_c) \to A\text{-mod}$ is an equivalence of categories. To see this, we note that all finitely generated A-modules are locally finite-dimensional as A itself is locally finite-dimensional. Hence, finitely presented A-modules are locally finite-dimensional too, i.e, $A\text{-mod}_{\mathrm{fg}}$ is a subcategory of $A\text{-mod}_{\mathrm{lfd}}$. In view of Lemma 2.5, this is the category $\mathcal{R}_c$. It just remains to apply [KS, Corollary 6.3.5], using Lemma 2.6 when checking the required hypotheses.

The category A-mod is a Grothendieck category. In particular, this means that every A-module has an injective hull in A-mod. Since every A-module is a quotient of a direct sum of projective A-modules of the form Ae_i, the category A-mod also has enough projectives. It is *not* true that an arbitrary A-module has a projective cover, but we will see in Lemma 2.13 below that finitely generated A-modules do.

Like we did in §2.1, we write simply $\mathrm{Ext}^n_{\mathcal{R}}(V,W)$, or sometimes $\mathrm{Ext}^n_A(V,W)$, in place of $\mathrm{Ext}^n_{\mathrm{Ind}(\mathcal{R}_c)}(V,W)$ for any $V,W \in \mathrm{Ind}(\mathcal{R}_c)$. This can be computed either from a projective resolution of V or from an injective resolution of W. We can also consider both right derived functors $\mathbb{R}^n F$ of a left exact functor $F : \mathrm{Ind}(\mathcal{R}_c) \to \mathcal{S}$ and left derived functors $\mathbb{L}_n G$ of a right exact functor $G : \mathrm{Ind}(\mathcal{R}_c) \to \mathcal{S}$. We provide an elementary proof of the following, but note it also follows from [KS, Theorem 15.3.1].

Lemma 2.12. *For $V, W \in \mathcal{R}$ and $n \geqslant 0$, there is a natural isomorphism*

$$\mathrm{Ext}^n_{\mathcal{R}}(V,W) \cong \mathrm{Ext}^n_{\mathcal{R}^{\mathrm{op}}}(W,V).$$

Proof. Using (2.8), we must show that $\mathrm{Ext}^n_A(V,W) \cong \mathrm{Ext}^n_A(W^{\circledast}, V^{\circledast})$ for locally finite-dimensional A-modules V and W. To compute $\mathrm{Ext}^n_A(V,W)$, take a projective resolution

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow V \longrightarrow 0$$

of V in A-mod. By Lemma 2.8, on applying the exact functor $\circledast$, we get an injective resolution

$$0 \longrightarrow V^{\circledast} \longrightarrow P_0^{\circledast} \longrightarrow P_1^{\circledast} \longrightarrow \cdots$$

of $V^{\circledast}$ in mod-A. Since W is locally finite-dimensional, we can use (2.9) to see that $\mathrm{Hom}_A(P_i, W) \cong \mathrm{Hom}_A(W^{\circledast}, P_i^{\circledast})$ for each i. So $\mathrm{Ext}^n_A(V,W) \cong \mathrm{Ext}^n_A(W^{\circledast}, V^{\circledast})$. ☐

Let $I(b)$ be an injective hull of $L(b)$ in A-mod. The dual $(e_iA)^{\circledast}$ of the projective right A-module e_iA is injective in A-mod. Since $\mathrm{End}_A((e_iA)^{\circledast})^{\mathrm{op}} \cong \mathrm{End}_A(e_iA) \cong e_iAe_i$, which is finite-dimensional, the injective module $(e_iA)^{\circledast}$ can be written as a finite direct sum of indecomposable injectives. To determine which ones, we compute its socle: we have that $\mathrm{Hom}_A(L(b), (e_iA)^{\circledast}) \cong \mathrm{Hom}_A(e_iA, L(b)^{\circledast}) \cong (L(b)^{\circledast})e_i = (e_iL(b))^*$, hence,

$$(e_iA)^{\circledast} \cong \bigoplus_{b \in \mathbf{B}} I(b)^{\oplus \dim e_iL(b)}, \tag{2.16}$$

with all but finitely many summands on the right hand side being zero. In particular, this shows for fixed i that $\dim e_iL(b) = 0$ for all but finitely many $b \in \mathbf{B}$. Conversely, for fixed $b \in \mathbf{B}$, we can always choose $i \in I$ so that $e_iL(b) \neq 0$, and deduce that $I(b)$ is a summand of $(e_iA)^{\circledast}$. This means that each indecomposable injective $I(b)$ is a locally finite-dimensional left A-module.

Let $P(b)$ be the dual of the injective hull of the irreducible right A-module $L(b)^{\circledast}$. By dualizing the right module analog of the decomposition (2.16), we get also that

$$Ae_i \cong \bigoplus_{b \in \mathbf{B}} P(b)^{\oplus \dim e_iL(b)}, \tag{2.17}$$

with all but finitely many summands being zero. In particular, $P(b)$ is projective in A-mod, hence, it is a projective cover of $L(b)$ in A-mod.

The composition multiplicities of any A-module satisfy

$$[V : L(b)] = \dim \mathrm{Hom}_A(V, I(b)) = \dim \mathrm{Hom}_A(P(b), V). \tag{2.18}$$

Moreover, V is locally finite-dimensional if and only if $[V : L(b)] < \infty$ for all $b \in \mathbf{B}$. To see this, note that V is locally finite-dimensional if and only if $\dim \mathrm{Hom}_A(Ae_i, V) < \infty$ for each $i \in I$. Using the decompositon (2.17), this is if and only if $\dim \mathrm{Hom}_A(P(b), V) < \infty$ for each $b \in \mathbf{B}$, as claimed.

There is a little more to be said about finitely generated modules. Recall from the previous subsection that a module is finitely generated if $V = Av_1 + \cdots + Av_n$ for homogeneous vectors $v_1, \dots, v_n \in V$. We say that V is *finitely cogenerated* if its dual is finitely generated. It is obvious from these definitions that $\mathrm{Hom}_A(V, W)$ is finite-dimensional either if V is finitely generated and W is locally finite-dimensional, or if V is locally finite-dimensional and W is finitely cogenerated. The following checks that our naive definitions are consistent with the usual notions of finitely generated and cogenerated objects of Grothendieck categories.

Lemma 2.13. *For $V \in A$-mod, the following properties are equivalent:*

(i) V is finitely generated;
(ii) the radical $\mathrm{rad}\, V$, i.e., the sum of its maximal proper submodules, is a superfluous submodule and $\mathrm{hd}\, V := V/\mathrm{rad}\, V$ is of finite length;
(iii) V is a quotient of a finite direct sum of the modules $P(b)$ for $b \in \mathbf{B}$.

Moreover, any finitely generated V has a projective cover.

Proof. We have already observed that $P(b)$ is a projective cover of $L(b)$. The lemma follows by standard arguments given this and the decomposition (2.17). □

Corollary 2.14. *For $V \in A$-mod, the following properties are equivalent:*

(i) V is finitely cogenerated;
(ii) $\mathrm{soc}\, V$ is an essential submodule of finite length;
(iii) V is isomorphic to a submodule of a finite direct sum of modules $I(b)$ for $b \in \mathbf{B}$.

Let us explain why any locally finite-dimensional locally unital algebra is Morita equivalent to a pointed locally finite-dimensional locally unital algebra, as asserted earlier. For $b \in \mathbf{B}$, pick $i(b) \in I$ such that $e_{i(b)} L(b) \neq 0$. In view of (2.17), we find a primitive idempotent $e_b \in e_{i(b)} A e_{i(b)}$ such that $Ae_b \cong P(b)$. Then A is Morita equivalent to

$$B = \bigoplus_{a,b \in \mathbf{B}} e_a A e_b, \tag{2.19}$$

which is pointed. For an explicit equivalence A-mod $\to B$-mod, consider the functor sending an A-module V to $\bigoplus_{b \in \mathbf{B}} e_b V$. In the case that A is pointed, its distinguished idempotents may be indexed by the same set $\mathbf{B}$ as is used to index the isomorphism classes of irreducible objects, so that $P(b) = Ae_b$ and $I(b) = (e_b A)^{\circledast}$. It is also easy to see that if A and B are pointed locally finite-dimensional locally unital algebras which are Morita equivalent, then they are actually isomorphic as locally unital algebras.

2.4. **Essentially finite Abelian categories.** We say that a locally unital algebra $A = \bigoplus_{i,j \in I} e_i A e_j$ is *essentially finite-dimensional* if each right ideal $e_i A$ and each left ideal Ae_j is finite-dimensional. By an *essentially finite Abelian category*, we mean a category that is equivalent to $A\text{-mod}_{\mathrm{fd}}$ for such an A.

Lemma 2.15. *An essentially small category $\mathcal{R}$ is equivalent to $A\text{-mod}_{\mathrm{fd}}$ for a locally unital algebra $A = \bigoplus_{i,j \in I} e_i A e_j$ such that each left ideal Ae_j (resp., each right ideal $e_i A$) is finite-dimensional if and only if $\mathcal{R}$ is a locally finite Abelian category with enough projectives (resp., enough injectives).*

Proof. We just prove the result for left ideals and projectives; the parenthesized statement for right ideals and injectives follows by replacing $\mathcal{R}$ and A with $\mathcal{R}^{\mathrm{op}}$ and A^{op}.

Suppose first that $A = \bigoplus_{i,j\in I} e_i A e_j$ is a locally unital algebra such that each left ideal Ae_j is finite-dimensional. Then A-$\mathrm{mod}_{\mathrm{fd}}$ is a locally finite Abelian category. It has enough projectives because the left ideals Ae_j are finite-dimensional.

Conversely, suppose $\mathcal{R}$ is a locally finite Abelian category with enough projectives. Let $\{L(b)\mid b\in\mathbf{B}\}$ be a full set of pairwise inequivalent irreducible objects, and $P(b)\in\mathcal{R}$ a projective cover of $L(b)$. Define A to be the locally unital algebra $A = \bigoplus_{a,b\in\mathbf{B}} e_a A e_b$ where $e_a A e_b := \mathrm{Hom}_{\mathcal{R}}(P(a), P(b))$ with multiplication that is the opposite of composition in $\mathcal{R}$. This is a pointed locally finite-dimensional locally unital algebra. As in the proof of Lemma 2.4, the functor $\bigoplus_{b\in\mathbf{B}} \mathrm{Hom}_{\mathcal{R}}(P(b), -)$ defines an equivalence $\mathcal{R}\to A$-$\mathrm{mod}_{\mathrm{fd}}$. It remains to note that the ideals Ae_b are finite-dimensional since they are the images under this functor of the projectives $P(b)$, which are of finite length. □

Corollary 2.16. *An essentially small category $\mathcal{R}$ is an essentially finite Abelian category if and only if it is a locally finite Abelian category with enough injectives and projectives.*

Note that $\mathcal{R}$ is essentially finite Abelian if and only if $\mathcal{R}^{\mathrm{op}}$ is essentially finite Abelian. Moreover, if A is an algebra realization for $\mathcal{R}$ then A^{op} is one for $\mathcal{R}^{\mathrm{op}}$ by the obvious duality $* : A\text{-}\mathrm{mod}_{\mathrm{fd}} \to \mathrm{mod}_{\mathrm{fd}}\text{-}A$.

Essentially finite Abelian categories are almost as convenient to work with as finite Abelian categories since one can perform all of the usual constructions of homological algebra without needing to pass to the ind-completion.

Lemma 2.17. *For a category $\mathcal{R}$, the following are equivalent:*

(i) $\mathcal{R}$ is a finite Abelian category;
(ii) $\mathcal{R}$ is a Schurian category with only finitely many isomorphism classes of irreducible objects;
(iii) $\mathcal{R}$ is an essentially finite Abelian category with only finitely many isomorphism classes of irreducible objects;
(iv) $\mathcal{R}$ is a locally finite Abelian category with only finitely many isomorphism classes of irreducible objects and either enough projectives or enough injectives;
(v) $\mathcal{R}$ is both a locally finite Abelian category and a Schurian category.

Proof. Clearly, (i) implies (ii) and (iii). The implication (ii)⇒(i) follows on considering a pointed algebra realization of $\mathcal{R}$. The implication (iii)⇒(iv) follows from Corollary 2.16. The implication (iv)⇒(i) follows from Lemma 2.15. Clearly (ii) and (iv) together imply (v). Finally, to see that (v) implies (ii), it suffices to note that a Schurian category with infinitely many isomorphism classes of irreducible objects cannot be locally finite Abelian: the direct sum of infinitely many non-isomorphic irreducibles is a well-defined object of $\mathcal{R}$ but it is not of finite length. □

Essentially finite Abelian categories with infinitely many isomorphism classes of irreducible objects are *not* Schurian categories. However they are closely related as we explain next.

- If $\mathcal{R}$ is essentially finite Abelian, we define its *Schurian envelope* $\mathrm{Env}(\mathcal{R})$ to be the full subcategory of $\mathrm{Ind}(\mathcal{R})$ consisting of all objects that have finite composition multiplicities.
- If $\mathcal{R}$ is Schurian, let $\mathrm{Fin}(\mathcal{R})$ be the full subcategory of $\mathcal{R}$ consisting of all objects of finite length.

Lemma 2.18. Env *and* Fin *define maps*

$$\left\{\begin{array}{c}\textit{Essentially finite}\\ \textit{Abelian categories}\end{array}\right\} \underset{\mathrm{Fin}}{\overset{\mathrm{Env}}{\rightleftarrows}} \left\{\begin{array}{c}\textit{Schurian categories all of whose indecomposable}\\ \textit{injectives and projectives are of finite length}\end{array}\right\}.$$

These are quasi-inverses in the sense that $\mathrm{Env}(\mathrm{Fin}(\mathcal{R}))$ *is equivalent to $\mathcal{R}$ for $\mathcal{R}$ in the left hand set, and* $\mathrm{Fin}(\mathrm{Env}(\mathcal{R}))$ *is equivalent to $\mathcal{R}$ for $\mathcal{R}$ in the right hand set.*

Proof. This is easy to see in terms of an algebra realization: if $\mathcal{R} = A\text{-mod}_{\mathrm{fd}}$ for an essentially finite-dimensional locally unital algebra A then $\mathrm{Env}(\mathcal{R}) = A\text{-mod}_{\mathrm{lfd}}$, so it is Schurian. Since the indecomposable injectives and projectives in $\mathrm{Env}(\mathcal{R})$ are the same as in $\mathcal{R}$, they have finite length. Conversely, suppose that $\mathcal{R} = A\text{-mod}_{\mathrm{lfd}}$ for a *pointed* locally finite-dimensional locally unital algebra, such that all of the indecomposable injectives and projectives are of finite length. Since A is pointed, this means equivalently that all of the left ideals Ae_i and right ideals e_iA are finite-dimensional. Hence, A is essentially finite-dimensional, and $\mathrm{Fin}(\mathcal{R}) = A\text{-mod}_{\mathrm{fd}}$ is essentially finite Abelian. $\square$

2.5. **Recollement.** We conclude the section with some reminders about "recollement" in our algebraic setting; see [BBD, §1.4] or [CPS1, §2] for further background. We need this here only for Abelian categories $\mathcal{R}$ satisfying finiteness properties as developed above. The recollement formalism provides us with an adjoint triple of functors $(i^*, i, i^!)$ associated to the inclusion $i : \mathcal{R}^{\downarrow} \to \mathcal{R}$ of a Serre subcategory, and an adjoint triple of functors $(j_!, j, j_*)$ associated to the projection $j : \mathcal{R} \to \mathcal{R}^{\uparrow}$ onto a Serre quotient category, with the image of i being the kernel of j. These functors will play an essential role in all subsequent arguments.

First suppose that $\mathcal{R}$ is a finite Abelian, locally finite Abelian, essentially finite Abelian or Schurian category. Assume that we are given a full set $\{L(b) \mid b \in \mathbf{B}\}$ of pairwise inequivalent irreducible objects. Let $\mathbf{B}^{\downarrow}$ be a subset of $\mathbf{B}$ and $\mathcal{R}^{\downarrow}$ be the full subcategory of $\mathcal{R}$ consisting of all the objects V such that $[V : L(b)] \neq 0 \Rightarrow b \in \mathbf{B}^{\downarrow}$. This is a Serre subcategory of $\mathcal{R}$ with irreducible objects $\{L^{\downarrow}(b) \mid b \in \mathbf{B}^{\downarrow}\}$ defined by $L^{\downarrow}(b) := L(b)$.

Lemma 2.19. *In the above setup, the inclusion functor $i : \mathcal{R}^{\downarrow} \to \mathcal{R}$ has a left adjoint i^* and a right adjoint $i^!$:*

$$\begin{tikzcd} \mathcal{R}^{\downarrow} \arrow[r, "i"] & \mathcal{R}. \arrow[l, bend right, "i^!"'] \arrow[l, bend left, "i^*"] \end{tikzcd}$$

The counit of one of these adjunctions and the unit of the other give isomorphisms:

$$i^* \circ i \xrightarrow{\sim} Id_{\mathcal{R}^{\downarrow}} \xrightarrow{\sim} i^! \circ i.$$

In particular, i is fully faithful.

Proof. This is straightforward. Explicitly, i^* (resp., $i^!$) sends an object of $\mathcal{R}$ to the largest quotient (resp., subobject) that belongs to $\mathcal{R}^{\downarrow}$. $\square$

We will use the same notation i, i^* and $i^!$ for the natural extensions of these functors to the appropriate ind-completions. In the case that $\mathcal{R}$ is locally finite Abelian, the Serre subcategory $\mathcal{R}^{\downarrow}$ is locally finite Abelian too, as is clear from Lemma 2.1. We also have the following lemma which explains how to construct an explicit coalgebra realization of $\mathcal{R}^{\downarrow}$ starting from one for $\mathcal{R}$.

Lemma 2.20. *Continuing with the above setup, suppose that $\mathcal{R} = \mathrm{comod}_{\mathrm{fd}}\text{-}C$ for a coalgebra C. Let $C^{\downarrow}$ be the largest right coideal of C belonging to $\mathcal{R}^{\downarrow}$. Then $C^{\downarrow}$ is a subcoalgebra of C. Moreover, $\mathcal{R}^{\downarrow}$ consists of all $V \in \mathrm{comod}_{\mathrm{fd}}\text{-}C$ such that the image of the structure map $\eta : V \to V \otimes C$ is contained in $V \otimes C^{\downarrow}$, i.e., we have that $\mathcal{R}^{\downarrow} = \mathrm{comod}_{\mathrm{fd}}\text{-}C^{\downarrow}$.*

Proof. For a right comodule V with structure map $\eta : V \to V \otimes C$, we can consider $V \otimes C$ as a right comodule with structure map $\mathrm{id} \otimes \delta$. The coassociative and counit axioms imply that η is an injective homomorphism of right comodules. We deduce that all irreducible subquotients of V belong to $\mathcal{R}^{\downarrow}$ if and only if $\eta(V) \subseteq V \otimes C^{\downarrow}$. Applying this with $V = C^{\downarrow}$ shows that $C^{\downarrow}$ is a subcoalgebra. Applying it to $V \in \mathcal{R}$ shows that $V \in \mathcal{R}^{\downarrow}$ if and only if $\eta(V) \subseteq V \otimes C^{\downarrow}$. $\square$

For the remainder of the subsection, we exclude the locally finite Abelian case. So now $\mathcal{R}$ is finite Abelian, essentially finite Abelian or Schurian only, and $\{L(b)\,|\,b \in \mathbf{B}\}$ is a full set of pairwise inequivalent irreducible objects. We claim that the Serre subcategory $\mathcal{R}^\downarrow$ of $\mathcal{R}$ associated to $\mathbf{B}^\downarrow \subseteq \mathbf{B}$ is of the same type as $\mathcal{R}$ again. To explain this, we fix a pointed algebra realization

$$A = \bigoplus_{a,b\in\mathbf{B}} e_a A e_b \tag{2.20}$$

of $\mathcal{R}$, so A is finite-dimensional, essentially finite-dimensional or locally finite-dimensional according to whether $\mathcal{R}$ is finite Abelian, essentially finite Abelian or Schurian. Let $\mathbf{B}^\uparrow := \mathbf{B}\backslash\mathbf{B}^\downarrow$ and

$$A^\downarrow = \bigoplus_{a,b\in\mathbf{B}^\downarrow} \bar{e}_a A^\downarrow \bar{e}_b := A/(e_c \mid c \in \mathbf{B}^\uparrow), \qquad\qquad A^\uparrow := \bigoplus_{a,b\in\mathbf{B}^\uparrow} e_a A e_b, \tag{2.21}$$

where $\bar{x}$ denotes the canonical image of $x \in A$ under the quotient map $A \twoheadrightarrow A^\downarrow$. Then it is clear that $\mathcal{R}^\downarrow$ is equivalent to $A^\downarrow\text{-mod}_{\text{fd}}$ in the finite Abelian or essentially finite Abelian cases, and to $A^\downarrow\text{-mod}_{\text{lfd}}$ in the Schurian case. Moreover, $A^\downarrow$ satsifies the same finiteness properties as A. This proves the claim.

Now we are going to pass to the *Serre quotient* $\mathcal{R}^\uparrow := \mathcal{R}/\mathcal{R}^\downarrow$. This is an Abelian category equipped with an exact *quotient functor* $j : \mathcal{R} \to \mathcal{R}^\uparrow$ satisfying the following universal property: if $h : \mathcal{R} \to \mathcal{C}$ is any exact functor to an Abelian category $\mathcal{C}$ with $hL(b) = 0$ for all $b \in \mathbf{B}^\downarrow$, then there is a unique functor $\bar{h} : \mathcal{R}^\uparrow \to \mathcal{C}$ such that $h = \bar{h} \circ j$. The irreducible objects in $\mathcal{R}^\uparrow$ are $\{L^\uparrow(b)\,|\,b \in \mathbf{B}^\uparrow\}$ where $\mathbf{B}^\uparrow := \mathbf{B}\backslash\mathbf{B}^\downarrow$ and $L^\uparrow(b) := jL(b)$. For a fuller discussion of these statements, see e.g. [Gab].

Fixing a pointed algebra realization of $\mathcal{R}$ as in (2.20), the quotient category $\mathcal{R}^\uparrow$ is realized by the algebra $A^\uparrow$ from (2.21), and the quotient functor j is the obvious "idempotent truncation functor" sending an A-module V to

$$jV := \bigoplus_{a\in\mathbf{B}^\uparrow} e_a V \tag{2.22}$$

with $A^\uparrow$ acting by restricting the action of A. In particular, it follows that $\mathcal{R}^\uparrow$ is of the same type (finite Abelian, essentially finite Abelian or Schurian) as $\mathcal{R}$.

Lemma 2.21. *In the above setup, the quotient functor $j : \mathcal{R} \to \mathcal{R}^\uparrow$ has a left adjoint $j_!$ and and a right adjoint j_*:*

$$\begin{tikzcd} \mathcal{R} \arrow[r, "j"] & \mathcal{R}^\uparrow. \arrow[l, bend right, "j_*"'] \arrow[l, bend left, "j_!"] \end{tikzcd}$$

The counit of one of these adjunctions and the unit of the other give isomorphisms:

$$j \circ j_* \xrightarrow{\sim} Id_{\mathcal{R}^\uparrow} \xrightarrow{\sim} j \circ j_!.$$

In particular, $j_!$ and j_* are fully faithful.

Proof. We again work in terms of the algebra realizations (2.20)–(2.21), so that j is the idempotent truncation functor (2.22). This is isomorphic to the hom functor $\bigoplus_{b\in\mathbf{B}^\uparrow} \mathrm{Hom}_A(Ae_b, -)$, which has the left adjoint

$$j_! := \Big(\bigoplus_{b\in\mathbf{B}^\uparrow} Ae_b\Big) \otimes_{A^\uparrow} - : A^\uparrow\text{-mod} \to A\text{-mod} \tag{2.23}$$

by Lemma 2.7(1). From this, it is clear that the unit of adjunction $\mathrm{Id}_{\mathcal{R}^\uparrow} \to j \circ j_!$ is an isomorphism. On the other hand, j is also isomorphic to the tensor functor $(\bigoplus_{b\in\mathbf{B}^\uparrow} e_b A) \otimes_A -$, so Lemma 2.7(1) also gives that j has the right adjoint

$$j_* := \bigoplus_{a\in\mathbf{B}} \mathrm{Hom}_{A^\uparrow}\Big(\bigoplus_{b\in\mathbf{B}^\uparrow} e_b A e_a, -\Big) : A^\uparrow\text{-mod} \to A\text{-mod}\,. \tag{2.24}$$

Again using this we see that the counit $j \circ j_* \to \mathrm{Id}_{\mathcal{R}^\uparrow}$ is an isomorphism. $\square$

As we did for i, i^* and $i^!$, in the Schurian case, we will use the same notation $j, j_!$ and j_* for the natural extensions of these functors to the ind-completions, i.e., the categories A-mod and $A^\uparrow$-mod. The following lemma describes the effect of these functors on indecomposable projective and injective objects.

Lemma 2.22. *Continuing with the above setup, let $P(b)$ (resp., $I(b)$) and $P^\uparrow(b)$ (resp., $I^\uparrow(b)$) be a projective cover (resp., an injective hull) of $L(b)$ in $\mathcal{R}$ and a projective cover (resp., an injective hull) of $L^\uparrow(b)$ in $\mathcal{R}^\uparrow$. For $b \in \mathbf{B}^\uparrow$, we have that*

$$jP(b) \cong P^\uparrow(b), \qquad jI(b) \cong I^\uparrow(b), \qquad j_!P^\uparrow(b) \cong P(b), \qquad j_*I^\uparrow(b) \cong I(b).$$

Moreover, the adjunction gives isomorphisms

$$\mathrm{Hom}_{\mathcal{R}}(P(b), j_*V) \cong \mathrm{Hom}_{\mathcal{R}^\uparrow}(P^\uparrow(b), V), \quad \mathrm{Hom}_{\mathcal{R}}(j_!V, I(b)) \cong \mathrm{Hom}_{\mathcal{R}^\uparrow}(V, I^\uparrow(b)) \tag{2.25}$$

for $V \in \mathcal{R}^\uparrow$, hence, $[V : L(b)] = [j_*V : L(b)] = [j_!V : L(b)]$ for all $b \in \mathbf{B}^\uparrow$.

Proof. The first part can be checked directly using the explicit realizations (2.22)–(2.24) of these functors. We just go through the argument needed to establish the isomorphism $j_*I^\uparrow(b) \cong I(b)$. We have that $I^\uparrow(b) \cong (e_bA^\uparrow)^\circledast$ and $I(b) \cong (e_bA)^\circledast$, so using (2.9) we get that

$$\begin{aligned} j_*I^\uparrow(b) &\cong \bigoplus_{a\in\mathbf{B}} \mathrm{Hom}_{A^\uparrow}\Big(\bigoplus_{b\in\mathbf{B}^\uparrow} e_bAe_a, (e_bA^\uparrow)^\circledast\Big) \cong \bigoplus_{a\in\mathbf{B}} \mathrm{Hom}_{A^\uparrow}\left(e_bA^\uparrow, \Big(\bigoplus_{b\in\mathbf{B}^\uparrow} e_bAe_a\Big)^\circledast\right) \\ &\cong \bigoplus_{a\in\mathbf{B}} (e_bAe_a)^* = (e_bA)^\circledast \cong I(b). \end{aligned}$$

The second part is immediate given the first part. □

3. Generalizations of highest weight categories

In this section, we define the various generalizations of highest weight categories and derive some of their fundamental properties in the four settings of finite Abelian, essentially finite Abelian, Schurian, and locally finite Abelian categories. The big four definitions in the section are Definitions 3.1, 3.9, 3.36 and 3.53. The reader new to these ideas may find it helpful to assume initially that all of the strata are simple in the sense of Lemma 3.6, when the definitions specialize to the notions of finite, essentially finite, upper finite and lower finite highest weight categories.

3.1. **Stratifications and the associated standard and costandard objects.** Let $(\Lambda, \leqslant)$ be a poset. It is *interval finite* (resp., *upper finite*, resp., *lower finite*) if the interval $[\lambda, \mu] := \{\nu \in \Lambda \mid \lambda \leqslant \nu \leqslant \mu\}$ (resp., $[\lambda, \infty) := \{\nu \in \Lambda \mid \lambda \leqslant \nu\}$, resp., $(-\infty, \mu] := \{\nu \in \Lambda \mid \nu \leqslant \mu\}$) is finite for all $\lambda, \mu \in \Lambda$. A *lower set* (resp., *upper set*) means a subset $\Lambda^\downarrow$ (resp., $\Lambda^\uparrow$) such that $\mu \leqslant \lambda \in \Lambda^\downarrow \Rightarrow \mu \in \Lambda^\downarrow$ (resp., $\mu \geqslant \lambda \in \Lambda^\uparrow \Rightarrow \mu \in \Lambda^\uparrow$).

Definition 3.1. Let $\mathcal{R}$ be an Abelian category of one of the four types discussed in the previous section. A *stratification* $\rho : \mathbf{B} \to \Lambda$ of $\mathcal{R}$ consists of the following data:

(ρ1) An interval finite poset $(\Lambda, \leqslant)$.
(ρ2) A set $\mathbf{B}$ indexing representatives $\{L(b) \mid b \in \mathbf{B}\}$ for the isomorphism classes of irreducible objects in $\mathcal{R}$.
(ρ3) A function $\rho : \mathbf{B} \to \Lambda$ with *finite* fibers $\mathbf{B}_\lambda := \rho^{-1}(\lambda)$.

For each $\lambda \in \Lambda$, let $\mathcal{R}_{\leqslant\lambda}$ and $\mathcal{R}_{<\lambda}$ be the Serre subcategories of $\mathcal{R}$ associated to the subsets $\mathbf{B}_{\leqslant\lambda} := \{b \in \mathbf{B} \mid \rho(b) \leqslant \lambda\}$ and $\mathbf{B}_{<\lambda} := \{b \in \mathbf{B} \mid \rho(b) < \lambda\}$, respectively. We impose the following axiom:

(ρ4) Each of the Abelian subcategories $\mathcal{R}_{\leqslant\lambda}$ has enough projectives and injectives.

In case ρ is a bijection, one can use it to identify $\mathbf{B}$ with Λ, and may simply write $L(\lambda)$ instead of $L(b)$, and similarly for all of the other families of objects introduced below.

Remark 3.2. If $\mathcal{R}$ is finite Abelian, essentially finite Abelian or Schurian then the axiom $(\rho4)$ holds automatically. However, it rules out many situations in which $\mathcal{R}$ is merely locally finite Abelian. For example, the category $\mathcal{R}ep(\mathbb{G}_a)$ of finite-dimensional rational representations of the additive group does not admit a stratification in the above sense.

Given a stratification of $\mathcal{R}$, we write $i_{\leqslant\lambda} : \mathcal{R}_{\leqslant\lambda} \to \mathcal{R}$ and $i_{<\lambda} : \mathcal{R}_{<\lambda} \to \mathcal{R}$ for the inclusion functors. Using Corollary 2.16 and $(\rho4)$ in the locally finite Abelian case, we see that $\mathcal{R}_{\leqslant\lambda}$ and $\mathcal{R}_{<\lambda}$ are finite Abelian if $\mathcal{R}$ is finite Abelian, essentially finite Abelian if $\mathcal{R}$ is locally finite Abelian or essentially finite Abelian, and Schurian if $\mathcal{R}$ is Schurian. Let $\mathcal{R}_\lambda$ be the quotient category $\mathcal{R}_{\leqslant\lambda}/\mathcal{R}_{<\lambda}$ and $j^\lambda : \mathcal{R}_{\leqslant\lambda} \to \mathcal{R}_\lambda$ be the quotient functor. We are in a recollement situation as in Lemmas 2.19 and 2.21:

$$\begin{tikzcd} \mathcal{R}_{<\lambda} \arrow[r, "i_{<\lambda}"] & \mathcal{R}_{\leqslant\lambda} \arrow[l, bend right=50, "i^!_{<\lambda}"'] \arrow[l, bend left=50, "i^*_{<\lambda}"] \arrow[r, "j^\lambda"] & \mathcal{R}_\lambda . \arrow[l, bend right=50, "j^\lambda_*"'] \arrow[l, bend left=50, "j^\lambda_!"] \end{tikzcd} \tag{3.1}$$

As we said already in the introduction, we call $j^\lambda_!$ and j^λ_* the *standardization* and *co-standardization* functors, respectively. The objects

$$\left\{L_\lambda(b) := j^\lambda L(b) \mid b \in \mathbf{B}_\lambda\right\}$$

give a full set of pairwise inequivalent irreducible objects in $\mathcal{R}_\lambda$. Since the set $\mathbf{B}_\lambda$ is finite, Lemma 2.17 implies that $\mathcal{R}_\lambda$ is a finite Abelian category. Let $P_\lambda(b)$ and $I_\lambda(b)$ be a projective cover and an injective hull of $L_\lambda(b)$ in $\mathcal{R}_\lambda$, respectively. Finally, define *standard, costandard, proper standard* and *proper costandard objects* $\Delta(b), \nabla(b), \bar{\Delta}(b)$ and $\bar{\nabla}(b)$ according to (1.1).

Lemma 3.3. *For $b \in \mathbf{B}_\lambda$, $\Delta(b)$ is a projective cover and $\nabla(b)$ is an injective hull of $L(b)$ in $\mathcal{R}_{\leqslant\lambda}$. Also, $\bar{\Delta}(b)$ is the largest quotient of $\Delta(b)$ such that $[\bar{\Delta}(b) : L(b)] = 1$ and all other composition factors are of the form $L(c)$ for $c \in \mathbf{B}_{<\lambda}$. Similarly, $\bar{\nabla}(b)$ is the largest subobject of $\nabla(b)$ such that $[\bar{\nabla}(b) : L(b)] = 1$ and all other composition factors are of the form $L(c)$ for $c \in \mathbf{B}_{<\lambda}$.*

Proof. The first assertion follows by Lemma 2.22. To prove the statement about $\bar{\Delta}(b)$, assume $[\bar{\Delta}(b) : L(c)] \neq 0$. Since $\bar{\Delta}(b) \in \mathcal{R}_{\leqslant\lambda}$, we have $\rho(c) \leqslant \rho(b)$. If $\rho(c) = \rho(b)$ then

$$[\bar{\Delta}(b) : L(c)] = [j^\lambda \bar{\Delta}(b) : L_\lambda(c)] = [L_\lambda(b) : L_\lambda(c)] = \delta_{b,c}.$$

Thus, $\bar{\Delta}(b)$ is such a quotient of $\Delta(b)$. To show that it is the largest such quotient, it suffices to show that the kernel K of $\Delta(b) \twoheadrightarrow \bar{\Delta}(b)$ is finitely generated with head that only involves irreducibles $L(c)$ with $\rho(c) = \rho(b)$. To see this, apply the right exact functor $j^\lambda_!$ to a short exact sequence $0 \to \widehat{K} \to P_\lambda(b) \to L_\lambda(b) \to 0$ to get an epimorphism $j^\lambda_! \widehat{K} \twoheadrightarrow K$. Then observe that $j^\lambda_! \widehat{K}$ is finitely generated as $j^\lambda_!$ is a left adjoint, and its head only involves irreducibles $L(c)$ with $\rho(c) = \rho(b)$. The latter assertion follows because $\operatorname{Hom}_{\mathcal{R}}(j^\lambda_! \widehat{K}, L(c)) \cong \operatorname{Hom}_{R_\lambda}(\widehat{K}, j^\lambda L(c))$ for $c \in \mathbf{B}_{\leqslant\lambda}$. The statement about $\bar{\nabla}(b)$ may be proved similarly. □

Corollary 3.4. *We have that* $\dim \operatorname{Hom}_{\mathcal{R}}(\Delta(b), \bar{\nabla}(c)) = \dim \operatorname{Hom}_{\mathcal{R}}(\bar{\Delta}(b), \nabla(c)) = \delta_{b,c}$ *for all $b, c \in \mathbf{B}$.*

Definition 3.5. Suppose we are given a stratification $\rho : \mathbf{B} \to \Lambda$ of $\mathcal{R}$. For $\lambda \in \Lambda$, we say that the stratum $\mathcal{R}_\lambda$ is *simple* if it is equivalent to the category $\mathcal{V}ec_{\mathrm{fd}}$ of finite-dimensional vector spaces.

Lemma 3.6. *For a stratification $\rho : \mathbf{B} \to \Lambda$ of $\mathcal{R}$, the following are equivalent:*

(i) all of the strata are simple;
(ii) ρ is a bijection and $\Delta(\lambda) = \bar{\Delta}(\lambda)$ for all $\lambda \in \Lambda$;
(iii) ρ is a bijection and $\nabla(\lambda) = \bar{\nabla}(\lambda)$ for all $\lambda \in \Lambda$.

Proof. (i)⇒(ii): Take $\lambda \in \Lambda$. As the stratum $\mathcal{R}_\lambda$ is simple, $\mathbf{B}_\lambda = \{b_\lambda\}$ is a singleton and $P_\lambda(b_\lambda) = L_\lambda(b_\lambda)$. We deduce that ρ is a bijection and $\Delta(b_\lambda) = \bar{\Delta}(b_\lambda)$.

(ii)⇒(i): Take $\lambda \in \Lambda$. Then $\mathcal{R}_\lambda$ has just one irreducible object (up to isomorphism), namely, $j^\lambda \bar{\Delta}(\lambda)$. Since this equals $j^\lambda \Delta(\lambda)$, it is also projective. Hence, $\mathcal{R}_\lambda$ is simple.

(i)⇔(iii): Similar. □

Given a *sign function* $\varepsilon : \Lambda \to \{\pm\}$, we introduce the *$\varepsilon$-standard* and *$\varepsilon$-costandard objects* $\Delta_\varepsilon(b)$ and $\nabla_\varepsilon(b)$ as in (1.2). Corollary 3.4 implies that

$$\dim \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), \nabla_\varepsilon(c)) = \delta_{b,c} \tag{3.2}$$

for $b, c \in \mathbf{B}$. A *Δ_ε-flag* of $V \in \mathcal{R}$ means a finite filtration $0 = V_0 < V_1 < \cdots < V_n = V$ with sections $V_m/V_{m-1} \cong \Delta_\varepsilon(b_m)$ for $b_m \in \mathbf{B}$. Similarly, we define ∇_ε-flags. We denote the exact subcategories of $\mathcal{R}$ consisting of all objects with a Δ_ε-flag or a ∇_ε-flag by $\Delta_\varepsilon(\mathcal{R})$ and $\nabla_\varepsilon(\mathcal{R})$, respectively.

A *Δ-flag* (resp., a *$\bar{\nabla}$-flag*) is a Δ_ε-flag (resp., a ∇_ε-flag) in the special case that $\varepsilon = +$. A *$\bar{\Delta}$-flag* (resp., a *∇-flag*) is a Δ_ε-flag (resp., a ∇_ε-flag) in the special case that $\varepsilon = -$. We denote the exact subcategories of $\mathcal{R}$ consisting of all objects with a Δ-flag, a $\bar{\Delta}$-flag, a ∇-flag or a $\bar{\nabla}$-flag by $\Delta(\mathcal{R})$, $\bar{\Delta}(\mathcal{R})$, $\nabla(\mathcal{R})$ and $\bar{\nabla}(\mathcal{R})$, respectively.

3.2. **Finite and essentially finite ε-stratified categories.** Throughout this subsection, $\mathcal{R}$ is an essentially finite Abelian category equipped with a stratification $\rho : \mathbf{B} \to \Lambda$, and $\varepsilon : \Lambda \to \{\pm\}$ is a sign function. The most important case is when $\mathcal{R}$ is a finite Abelian category, i.e., the index set $\mathbf{B}$ is finite. In this, the "classical case," all of the results in this section are well known, but even here our approach incorporating the sign function ε is not covered fully in the literature.

Let $P(b)$ and $I(b)$ be a projective cover and an injective hull of $L(b)$, respectively. We also need the objects from (1.1)–(1.2). Consider the following two properties:

($\widehat{P\Delta_\varepsilon}$) For each $b \in \mathbf{B}$, there exists a projective object P_b admitting a Δ_ε-flag with $\Delta_\varepsilon(b)$ at the top and other sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}$ with $\rho(c) \geqslant \rho(b)$.

($\widehat{I\nabla_\varepsilon}$) For each $b \in \mathbf{B}$, there exists an injective object I_b admitting a ∇_ε-flag with $\nabla_\varepsilon(b)$ at the bottom and other sections $\nabla_\varepsilon(c)$ for $c \in \mathbf{B}$ with $\rho(c) \geqslant \rho(b)$.

It is easy to see that the property ($P\Delta_\varepsilon$) formulated in the introduction implies ($\widehat{P\Delta_\varepsilon}$), and similarly ($I\nabla_\varepsilon$) implies ($\widehat{I\nabla_\varepsilon}$). The seemingly weaker properties ($\widehat{P\Delta_\varepsilon}$)–($\widehat{I\nabla_\varepsilon}$) are often easier to check in concrete examples. The essence of the following fundamental theorem appeared originally in [ADL], extending earlier work of Dlab [Dla1].

Theorem 3.7. *In the above setup, the four properties ($\widehat{P\Delta_\varepsilon}$), ($\widehat{I\nabla_\varepsilon}$), ($P\Delta_\varepsilon$) and ($I\nabla_\varepsilon$) are equivalent. When these properties hold, the standardization functor $j_!^\lambda$ is exact whenever $\varepsilon(\lambda) = -$, and the costandardization functor j_*^λ is exact whenever $\varepsilon(\lambda) = +$.*

Remark 3.8. When all strata are simple, the properties ($\widehat{P\Delta_\varepsilon}$)–($\widehat{I\nabla_\varepsilon}$) may be written more succinctly as the following:

($\widehat{P\Delta}$) For each $\lambda \in \Lambda$, there exists a projective object P_λ admitting a Δ-flag with $\Delta(\lambda)$ at the top and other sections of the form $\Delta(\mu)$ for $\mu \in \Lambda$ with $\mu \geqslant \lambda$.

($\widehat{I\nabla}$) For each $\lambda \in \Lambda$, there exists an injective object I_λ admitting a ∇-flag with $\nabla(\lambda)$ at the bottom and other sections of the form $\nabla(\mu)$ for $\mu \in \Lambda$ with $\mu \geqslant \lambda$.

Theorem 3.7 shows that these are equivalent to the properties ($P\Delta$)–($I\nabla$) from the introduction, as was used originally by Cline, Parshall and Scott in [CPS1].

We postpone the proof of Theorem 3.7 until a little later in the the subsection. It is important because it justifies the next key definition (εS) and its variations (FS), (εHW), (SHW) and (HW).

Definition 3.9. Let $\mathcal{R}$ be a finite Abelian category (resp., an essentially finite Abelian category) and $\rho : \mathbf{B} \to \Lambda$ be a stratification in the sense of Definition 3.1.

(εS) We say that $\mathcal{R}$ is a *finite* (resp., an *essentially finite*) *ε-stratified category* if one of the equivalent properties $(\widehat{P\Delta}_\varepsilon)$–$(\widehat{I\nabla}_\varepsilon)$ holds for some given choice of sign function $\varepsilon : \Lambda \to \{\pm\}$.

(FS) We say $\mathcal{R}$ is a *finite* (resp., an *essentially finite*) *fully stratified category* if one of these properties holds for all choices of sign function $\varepsilon : \Lambda \to \{\pm\}$.

(εHW) We say $\mathcal{R}$ is a *finite* (resp., an *essentially finite*) *ε-highest weight category* if the stratification function ρ is a bijection, i.e., each stratum has a unique irreducible object (up to isomorphism), and one of the equivalent properties $(\widehat{P\Delta}_\varepsilon)$–$(\widehat{I\nabla}_\varepsilon)$ holds for some given choice of sign function $\varepsilon : \Lambda \to \{\pm\}$.

(SHW) We say $\mathcal{R}$ is a *finite* (resp., an *essentially finite*) *signed highest weight category* if the stratification function ρ is a bijection and one of these properties holds for all choices of sign function.

(HW) We say $\mathcal{R}$ is a *finite* (resp., an *essentially finite*) *highest weight category* if all of the strata are simple (cf. Lemma 3.6) and one of the equivalent properties $(\widehat{P\Delta})$–$(\widehat{I\nabla})$ holds.

Remark 3.10. The language "ε-quasi-hereditary" and "signed quasi-hereditary" in Definition 3.9 is a significant departure from the existing literature, where such categories would be referred to as some form of *properly* stratified category; this terminology goes back to the work of Dlab [Dla2]. A recent exposition which takes a more traditional viewpoint than here can be found in [CZ]. In particular, in [CZ, Definition 2.7.4], one finds five types of finite-dimensional algebra A defined in terms of properties of the category $A\text{-mod}_{\mathrm{fd}}$, namely, standardly stratified algebras, exactly standardly stratified algebras, strongly stratified algebras, properly stratified algebras, and quasi-hereditary algebras. To compare this to our language in Definition 3.9, the category $A\text{-mod}_{\mathrm{fd}}$ in these five cases is $+$-stratified, fully stratified, $+$-highest weight, signed highest weight, and highest weight, respectively. The equivalence of definitions can be seen easily using Lemma 3.22 below. For further reference to the original literature, [CZ, Appendix A.2] is also helpful.

In the setup of (εS), we can view $\{L(b) \mid b \in \mathbf{B}\}$ equivalently as a full set of pairwise inequivalent irreducible objects in $\mathcal{R}^{\mathrm{op}}$. The stratification of $\mathcal{R}$ is also one of $\mathcal{R}^{\mathrm{op}}$. The indecomposable projectives and injectives in $\mathcal{R}^{\mathrm{op}}$ are $I(b)$ and $P(b)$, while the $(-\varepsilon)$-standard and $(-\varepsilon)$-costandard objects in $\mathcal{R}^{\mathrm{op}}$ are $\nabla_\varepsilon(b)$ and $\Delta_\varepsilon(b)$, respectively. So we can reinterpret Theorem 3.7 as the following; this is an extension of Theorem 1.1 from the introduction since we are now including essentially finite Abelian as well as finite Abelian categories.

Theorem 3.11. *$\mathcal{R}$ is ε-stratified (resp., ε-highest weight) if and only if $\mathcal{R}^{\mathrm{op}}$ is $(-\varepsilon)$-stratified (resp., $(-\varepsilon)$-highest weight).*

Now we must prepare for the proof Theorem 3.7. The main step in the argument will be provided by the *homological criterion* for ∇_ε-flags from the next Theorem 3.13. In turn, the proof of this criterion reduces to the following lemma which treats a key special case. The reader wanting to work fully through the proofs should look also at this point at the lemmas in §3.4 below.

Lemma 3.12. *Assume that $\mathcal{R}$ is an essentially finite Abelian category equipped with a stratification ρ and sign function ε, such that property $(\widehat{P\Delta}_\varepsilon)$ holds. Let $\lambda \in \Lambda$ be maximal and $V \in \mathcal{R}$ be an object satisfying the following properties:*

(1) $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$ *for all* $b \in \mathbf{B}$*;*

(2) $\mathrm{soc}\, V \cong L(b_1) \oplus \cdots \oplus L(b_n)$ *for* $b_1, \dots, b_n \in \mathbf{B}_\lambda$*.*

Then V belongs to $\mathcal{R}_{\leqslant\lambda}$ (so that it makes sense to apply the functor j^λ to it), and

$$V \cong \begin{cases} j^\lambda_*(j^\lambda V) & \text{if } \varepsilon(\lambda) = +, \\ \nabla(b_1) \oplus \cdots \oplus \nabla(b_n) & \text{if } \varepsilon(\lambda) = -. \end{cases} \tag{3.3}$$

Moreover, in the case $\varepsilon(\lambda) = +$, the functor j^λ_* is exact. Hence, in both cases, we have that $V \in \nabla_\varepsilon(\mathcal{R})$.

Proof (assuming lemmas from §3.4 below). We first prove (3.3) in case $\varepsilon(\lambda) = -$. Let $W := \nabla(b_1) \oplus \cdots \oplus \nabla(b_n)$. By the maximality of λ and Lemma 3.47, this is an injective hull of soc V. So there is a short exact sequence $0 \to V \to W \to W/V \to 0$. For any $a \in \mathbf{B}$, we apply $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), -)$ and use property (1) to get a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), V) \xrightarrow{f} \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), W) \longrightarrow \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), W/V) \longrightarrow 0. \tag{3.4}$$

If $\rho(a) \neq \lambda$ then $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), W) = 0$ as none of the composition factors of $\Delta_\varepsilon(a)$ are constituents of soc W. If $\rho(a) = \lambda$ then $\Delta_\varepsilon(a) = \bar\Delta(a)$ and any homomorphism $\bar\Delta(a) \to W$ must factor through the unique irreducible quotient $L(a)$ of $\bar\Delta(a)$. So its image is contained in soc $W \subseteq V$, showing that f is an isomorphism. These arguments show that $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), W/V) = 0$ for all $a \in \mathbf{B}$. We deduce that $\operatorname{soc}(W/V) = 0$, hence, $W/V = 0$, which is what we needed.

Now consider (3.3) when $\varepsilon(\lambda) = +$. By Lemma 3.47 again, the injective hull of V is $\nabla(b_1) \oplus \cdots \oplus \nabla(b_n)$, which is an object of $\mathcal{R}_{\leqslant\lambda}$. Hence, $V \in \mathcal{R}_{\leqslant\lambda}$. The unit of adjunction gives us a morphism $g : V \to W := j^\lambda_*(j^\lambda V)$. Since g becomes an isomorphism when we apply j^λ, its kernel belongs to $\mathcal{R}_{<\lambda}$. In view of property (2), we deduce that $\ker g = 0$, so g is a monomorphism. Hence, we can identify V with a subobject of W. To show that g is an epimorphism as well, we apply $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), -)$ to $0 \to V \to W \to W/V \to 0$ to get the short exact sequence (3.4). By adjunction, the middle morphism space is isomorphic to $\operatorname{Hom}_{\mathcal{R}_\lambda}(j^\lambda \Delta_\varepsilon(a), j^\lambda V)$, which is zero if $\rho(a) < \lambda$. If $\rho(a) = \lambda$ then $\Delta_\varepsilon(a) = \Delta(a)$ is the projective cover of $L(a)$ in $\mathcal{R}$ by Lemma 3.47, and $j^\lambda \Delta_\varepsilon(a)$ is the projective cover of $L_\lambda(a)$ in $\mathcal{R}_\lambda$. We deduce that both the first and second morphism spaces in (3.4) are of the same dimension $[V : L(a)] = [j^\lambda V : L_\lambda(a)]$, so f must be an isomorphism. Therefore $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), W/V) = 0$ for all $a \in \mathbf{B}$, hence, $V = W$ and (3.3) is proved.

To complete the proof, we must show that j^λ_* is exact when $\varepsilon(\lambda) = +$. For this, we use induction on composition length to show that j^λ_* is exact on any short exact sequence $0 \to K \to X \to Q \to 0$ in $\mathcal{R}_\lambda$. For the induction step, suppose we are given such an exact sequence with $K, Q \neq 0$. By induction, $j^\lambda_* K$ and $j^\lambda_* Q$ both have filtrations with sections $\bar\nabla(b)$ for $b \in \mathbf{B}_\lambda$. Hence, by Lemma 3.49, we have that $\operatorname{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), j^\lambda_* K) = \operatorname{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), j^\lambda_* Q) = 0$ for all $n \geqslant 1$ and $b \in \mathbf{B}$. As it is a right adjoint, j^λ_* is left exact, so there is an exact sequence

$$0 \longrightarrow j^\lambda_* K \longrightarrow j^\lambda_* X \longrightarrow j^\lambda_* Q. \tag{3.5}$$

Let $Y := j^\lambda_* X / j^\lambda_* K$, so that there is a short exact sequence

$$0 \longrightarrow j^\lambda_* K \longrightarrow j^\lambda_* X \longrightarrow Y \longrightarrow 0. \tag{3.6}$$

To complete the argument, it suffices to show that $Y \cong j^\lambda_* Q$. To establish this, we show that Y satisfies both of the properties (1) and (2); then, by the previous paragraph and exactness of j^λ, we get that $Y \cong j^\lambda_*(j^\lambda Y) \cong j^\lambda_*(X/K) \cong j^\lambda_* Q$, and we are done. To see that Y satisfies (1), we apply $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), -)$ to (3.6) to get an exact sequence

$$\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), j^\lambda_* X) \longrightarrow \operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), Y) \longrightarrow \operatorname{Ext}^2_{\mathcal{R}}(\Delta_\varepsilon(b), j^\lambda_* K).$$

The first Ext^1 is zero by Lemma 3.48. Since we already know that the Ext^2 term is zero, $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), Y) = 0$. To see that Y satisfies (2), note comparing (3.5)–(3.6) that $Y \hookrightarrow j^\lambda_* Q$, and soc $j^\lambda_* Q$ is of the desired form by what we know about its $\bar\nabla_\varepsilon$-flag. $\square$

Theorem 3.13. *Assume that $\mathcal{R}$ is an essentially finite Abelian category equipped with a stratification ρ and sign function ε, such that property $(\widehat{P\Delta_\varepsilon})$ holds. For $V \in \mathcal{R}$, the following properties are equivalent:*

(i) $V \in \nabla_\varepsilon(\mathcal{R})$;
(ii) $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$ *for all* $b \in \mathbf{B}$;
(iii) $\operatorname{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$ *for all* $b \in \mathbf{B}$ *and* $n \geqslant 1$.

If these properties hold, the multiplicity $(V : \nabla_\varepsilon(b))$ of $\nabla_\varepsilon(b)$ as a section of a ∇_ε-flag of V is well-defined independent of the choice of flag, as it equals $\dim \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V)$.

Proof (assuming lemmas from §3.4 below). (iii)⇒(ii): Trivial.

(i)⇒(iii) and the final assertion of the lemma: These follow directly from Lemma 3.49.

(ii)⇒(i): Assume that V satisfies (ii). We prove that it has a ∇_ε-flag by induction on

$$d(V) := \sum_{b \in \mathbf{B}} \dim \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V).$$

The base case when $d(V) = 0$ is trivial as we have then that $V = 0$. For the induction step, let $\lambda \in \Lambda$ be minimal such that $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) \neq 0$ for some $b \in \mathbf{B}$. The Serre subcategory $\mathcal{R}_{\leqslant\lambda}$ is an essentially finite Abelian or Schurian category which also satisfies $(\widehat{P\Delta_\varepsilon})$ thanks to Lemma 3.46(2). Let $W := i^!_{\leqslant\lambda} V$. Because W is a subobject of V, we have by the minimality of λ that $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), W) \neq 0$ only if $b \in \mathbf{B}_\lambda$. Hence, $\operatorname{soc} W \cong L(b_1) \oplus \cdots \oplus L(b_n)$ for $b_1, \dots, b_n \in \mathbf{B}_\lambda$. Thus, W satisfies the hypothesis (2) from Lemma 3.12 (with V and $\mathcal{R}$ there replaced by W and $\mathcal{R}_{\leqslant\lambda}$). To see that it satisfies hypothesis (1), we apply $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), -)$ to a short exact sequence $0 \to W \to V \to Q \to 0$ to get an exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), W) \to \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) \to \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), Q)& \\ \to \operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), W) \to 0 \to \operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), Q) \to \operatorname{Ext}^2_{\mathcal{R}}(\Delta_\varepsilon(b), W)&. \end{aligned}$$

By the definition of W, the socle of Q has no constituent $L(b)$ for $b \in \mathbf{B}_{\leqslant\lambda}$. So for $b \in \mathbf{B}_{\leqslant\lambda}$ the space $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), Q)$ is zero, and we get that $\operatorname{Ext}^1_{\mathcal{R}_{\leqslant\lambda}}(\Delta_\varepsilon(b), W) = 0$. So now we can apply Lemma 3.12 to deduce that $W \in \nabla_\varepsilon(\mathcal{R}_{\leqslant\lambda})$. Hence, $W \in \nabla_\varepsilon(\mathcal{R})$.

In view of Lemma 3.49, we get that $\operatorname{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), W) = 0$ for all $n \geqslant 1$ and $b \in \mathbf{B}$. So, by the above exact sequence again, we have that $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), Q) = 0$ and $d(Q) = d(V) - d(W) < d(V)$. Finally we appeal to the induction hypothesis to deduce that $Q \in \Delta_\varepsilon(\mathcal{R})$. Since we already know that $W \in \Delta_\varepsilon(\mathcal{R})$, this shows that $V \in \Delta_\varepsilon(\mathcal{R})$. □

Corollary 3.14. *In the setup of Theorem 3.13, multiplicities in a ∇_ε-flag of $I(b)$ satisfy* $(I(b) : \nabla_\varepsilon(c)) = [\Delta_\varepsilon(c) : L(b)]$.

Corollary 3.15. *For $\mathcal{R}$ as in Theorem 3.13, let $0 \to U \to V \to W \to 0$ be a short exact sequence. If U and V have ∇_ε-flags then so does W.*

Proof of Theorem 3.7. Suppose that $\mathcal{R}$ satisfies $(\widehat{P\Delta_\varepsilon})$. Since $V = I(b)$ is injective, it satisfies the hypothesis of Theorem 3.13(ii). Hence, by that theorem, $I(b)$ has a ∇_ε-flag and the multiplicity $(I(b) : \nabla_\varepsilon(c))$ of $\nabla_\varepsilon(c)$ as a section of any such flag is given by

$$(I(b) : \nabla_\varepsilon(c)) = \dim \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(c), I(b)) = [\Delta_\varepsilon(c) : L(b)].$$

This is zero unless $\rho(b) \leqslant \rho(c)$. Thus, we have verified that $\mathcal{R}$ satisfies $(I\nabla_\varepsilon)$. Moreover, Lemma 3.12 shows that j^λ_* is exact whenever $\varepsilon(\lambda) = +$, giving half of final assertion made in the statement of the theorem we are trying to prove.

Repeating the arguments in the previous paragraph but with $\mathcal{R}$ replaced by $\mathcal{R}^{\mathrm{op}}$ and ε replaced with $-\varepsilon$ show that $(\widehat{I\nabla_\varepsilon})$ implies $(P\Delta_\varepsilon)$ and that $j^\lambda_!$ is exact whenever $\varepsilon(\lambda) = -$. As we have already observed, it is obvious that $(P\Delta_\varepsilon) \Rightarrow (\widehat{P\Delta_\varepsilon})$ and $(I\nabla_\varepsilon) \Rightarrow (\widehat{I\nabla_\varepsilon})$, so this completes the proof. □

So now Theorem 3.7 is proved and Definition 3.9 is in place. In the remainder of the subsection, we are going to develop some further fundamental properties of these sorts of category. We start off in the most general setup with $\mathcal{R}$ being a finite or essentially finite ε-stratified category. Note some of the proofs that follow also invoke parts of the lemmas from §3.4, as we will point out along the way. In particular, from Lemma 3.45 and the dual statement, deduce that

$$\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), \Delta_\varepsilon(c)) = \mathrm{Ext}^1_{\mathcal{R}}(\nabla_\varepsilon(c), \nabla_\varepsilon(b)) = 0 \tag{3.7}$$

for $b, c \in \mathbf{B}$ with $\rho(b) \nleqslant \rho(c)$. By "dual statement" here, we mean that one takes Lemma 3.45 with $\mathcal{R}$ replaced by $\mathcal{R}^{\mathrm{op}}$ and ε by $-\varepsilon$, which we may do due to Theorem 3.11 and Lemma 2.12, then applies the contravariant isomorphism between $\mathcal{R}$ and $\mathcal{R}^{\mathrm{op}}$ that is the identity on objects and morphisms. In a similar way, the following theorem follows immediately as it is the dual statement to Theorem 3.13.

Theorem 3.16. *Assume that $\mathcal{R}$ is a finite or essentially finite ε-stratified category. For $V \in \mathcal{R}$, the following properties are equivalent:*

(i) $V \in \Delta_\varepsilon(\mathcal{R})$;
(ii) $\mathrm{Ext}^1_{\mathcal{R}}(V, \nabla_\varepsilon(b)) = 0$ *for all* $b \in \mathbf{B}$;
(iii) $\mathrm{Ext}^n_{\mathcal{R}}(V, \nabla_\varepsilon(b)) = 0$ *for all* $b \in \mathbf{B}$ *and* $n \geqslant 1$.

Assuming that these properties hold, the multiplicity $(V : \Delta_\varepsilon(b))$ of $\Delta_\varepsilon(b)$ as a section of a Δ_ε-flag of V is well-defined independent of the choice of flag, as it equals $\dim \mathrm{Hom}_{\mathcal{R}}(V, \nabla_\varepsilon(b))$.

Corollary 3.17. $(P(b) : \Delta_\varepsilon(c)) = [\nabla_\varepsilon(c) : L(b)]$.

Corollary 3.18. *Let $0 \to U \to V \to W \to 0$ be a short exact sequence in a finite or essentially finite ε-stratified category. If V and W have Δ_ε-flags then so does U.*

The following results about truncation to lower and upper sets are extremely useful; some aspects of them were already used in the proof of Theorem 3.13.

Theorem 3.19. *Assume that $\mathcal{R}$ is a finite or essentially finite ε-stratified category. Suppose that $\Lambda^\downarrow$ is a lower set in Λ. Let $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$ and $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the corresponding Serre subcategory of $\mathcal{R}$ with the induced stratification. Then $\mathcal{R}^\downarrow$ is itself a finite or essentially finite ε-stratified category according to whether $\Lambda^\downarrow$ is finite or infinite. Moreover:*

(1) The distinguished objects in $\mathcal{R}^\downarrow$ satisfy $L^\downarrow(b) \cong L(b)$, $P^\downarrow(b) \cong i^* P(b)$, $I^\downarrow(b) \cong i^! I(b)$, $\Delta^\downarrow(b) \cong \Delta(b)$, $\bar\Delta^\downarrow(b) \cong \bar\Delta(b)$, $\nabla^\downarrow(b) \cong \nabla(b)$ and $\bar\nabla^\downarrow(b) \cong \bar\nabla(b)$ for $b \in \mathbf{B}^\downarrow$.
(2) i^* is exact[1] on $\Delta_\varepsilon(\mathcal{R})$ with $i^*\Delta(b) \cong \Delta^\downarrow(b)$ and $i^*\bar\Delta(b) \cong \bar\Delta^\downarrow(b)$ for $b \in \mathbf{B}^\downarrow$; also $i^*\Delta(b) = i^*\bar\Delta(b) = 0$ for $b \notin \mathbf{B}^\downarrow$.
(3) $\mathrm{Ext}^n_{\mathcal{R}}(V, iW) \cong \mathrm{Ext}^n_{\mathcal{R}^\downarrow}(i^*V, W)$ for $V \in \Delta_\varepsilon(\mathcal{R})$, $W \in \mathcal{R}^\downarrow$ and all $n \geqslant 0$.
(4) $i^!$ is exact on $\nabla_\varepsilon(\mathcal{R})$ with $i^!\nabla(b) \cong \nabla^\downarrow(b)$ and $i^!\bar\nabla(b) \cong \bar\nabla^\downarrow(b)$ for $b \in \mathbf{B}^\downarrow$; also $i^!\nabla(b) = i^!\bar\nabla(b) = 0$ for $b \notin \mathbf{B}^\downarrow$.
(5) $\mathrm{Ext}^n_{\mathcal{R}}(iV, W) \cong \mathrm{Ext}^n_{\mathcal{R}^\downarrow}(V, i^!W)$ for $V \in \mathcal{R}^\downarrow, W \in \nabla_\varepsilon(\mathcal{R})$ and all $n \geqslant 0$.
(6) $\mathrm{Ext}^n_{\mathcal{R}}(iV, iW) \cong \mathrm{Ext}^n_{\mathcal{R}^\downarrow}(V, W)$ for $V, W \in \mathcal{R}^\downarrow$ and $n \geqslant 0$.

Proof. Apart from (6), this follows by Lemma 3.46 and its dual. To prove (6), by the same argument as used to prove Lemma 3.46(4), it suffices to show that $(\mathbb{L}_n i^*)V = 0$ for $V \in \mathcal{R}^\downarrow$. Since any such V has finite length it suffices to consider an irreducible object in $\mathcal{R}^\downarrow$, i.e., we must show that $(\mathbb{L}_n i^*)L(b) = 0$ for $b \in \mathbf{B}^\downarrow$. Take a short exact sequence $0 \to K \to \Delta_\varepsilon(b) \to L(b) \to 0$ and apply i^* and Lemma 3.46(3) to get

$$0 \longrightarrow (\mathbb{L}_1 i^*)L(b) \longrightarrow i^*K \longrightarrow i^*\Delta_\varepsilon(b) \longrightarrow i^*L(b) \longrightarrow 0.$$

But $K, \Delta_\varepsilon(b)$ and $L(b)$ all lie in $\mathcal{R}^\downarrow$ so i^* is the identity on them. We deduce that $(\mathbb{L}_1 i^*)L(b) = 0$. Degree shifting easily gives the result for $n > 1$. □

[1]We mean that it sends short exact sequences of objects with Δ_ε-flags to short exact sequences.

Theorem 3.20. *Assume that $\mathcal{R}$ is a finite or essentially finite ε-stratified category. Suppose that $\Lambda^\uparrow$ is an upper set in Λ. Let $\mathbf{B}^\uparrow := \rho^{-1}(\Lambda^\uparrow)$ and $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the corresponding Serre quotient category of $\mathcal{R}$ with the induced stratification. Then $\mathcal{R}^\uparrow$ is itself a finite or essentially finite ε-stratified category according to whether $\Lambda^\uparrow$ is finite or infinite. Moreover:*

1. *For $b \in \mathbf{B}^\uparrow$, the distinguished objects $L^\uparrow(b)$, $P^\uparrow(b)$, $I^\uparrow(b)$, $\Delta^\uparrow(b)$, $\bar{\Delta}^\uparrow(b)$, $\nabla^\uparrow(b)$ and $\bar{\nabla}^\uparrow(b)$ in $\mathcal{R}^\uparrow$ are isomorphic to the images under j of the corresponding objects of $\mathcal{R}$.*
2. *We have that $jL(b) = j\Delta(b) = j\bar{\Delta}(b) = j\nabla(b) = j\bar{\nabla}(b) = 0$ if $b \notin \mathbf{B}^\uparrow$.*
3. *$\operatorname{Ext}^n_{\mathcal{R}}(V, j_*W) \cong \operatorname{Ext}^n_{\mathcal{R}^\uparrow}(jV, W)$ for $V \in \mathcal{R}, W \in \nabla_\varepsilon(\mathcal{R}^\uparrow)$ and all $n \geqslant 0$.*
4. *j_* is exact on $\nabla_\varepsilon(\mathcal{R}^\uparrow)$ with $j_*\nabla^\uparrow(b) \cong \nabla(b)$, $j_*\bar{\nabla}^\uparrow(b) \cong \bar{\nabla}(b)$ and $j_*I^\uparrow(b) \cong I(b)$ for $b \in \mathbf{B}^\uparrow$.*
5. *$\operatorname{Ext}^n_{\mathcal{R}}(j_!V, W) \cong \operatorname{Ext}^n_{\mathcal{R}^\uparrow}(V, jW)$ for $V \in \Delta_\varepsilon(\mathcal{R}^\uparrow)$, $W \in \mathcal{R}$ and all $n \geqslant 0$.*
6. *$j_!$ is exact on $\Delta_\varepsilon(\mathcal{R}^\uparrow)$ with $j_!\Delta^\uparrow(b) \cong \Delta(b)$, $j_!\bar{\Delta}^\uparrow(b) \cong \bar{\Delta}(b)$ and $j_!P^\uparrow(b) = P(b)$ for $b \in \mathbf{B}^\uparrow$.*

Proof. Apart from (4) and (6), this follows from Lemma 3.50 and its dual. For (4) and (6), it suffices to prove (4), since (6) is the equivalent dual statement. The descriptions of $j_*\nabla^\uparrow(b)$, $j_*\bar{\nabla}^\uparrow(b)$ and $j_*I^\uparrow(b)$, follow from Lemma 3.50(1). It remains to prove the exactness. We can actually show slightly more, namely, that $(\mathbb{R}^n j_*)V = 0$ for $V \in \nabla_\varepsilon(\mathcal{R}^\uparrow)$ and $n \geqslant 1$. Take $V \in \nabla_\varepsilon(\mathcal{R}^\uparrow)$. Consider a short exact sequence $0 \to V \to I \to Q \to 0$ in $\mathcal{R}^\uparrow$ with I injective. Apply the left exact functor j_* and consider the resulting long exact sequence:

$$0 \to j_*V \to j_*I \to j_*Q \to (\mathbb{R}^1 j_*)V \to 0.$$

As V has a ∇_ε-flag, we can use (3) to see that $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), j_*V) \cong \operatorname{Hom}_{\mathcal{R}^\uparrow}(j\Delta_\varepsilon(b), V)$ and $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), j_*V) \cong \operatorname{Ext}^1_{\mathcal{R}^\uparrow}(j\Delta_\varepsilon(b), V)$ for every $b \in \mathbf{B}$. Hence, Theorem 3.13, j_*V has a ∇_ε-flag with

$$(j_*V : \nabla_\varepsilon(b)) = \dim \operatorname{Hom}_{\mathcal{R}}(j\Delta_\varepsilon(b), V) = \begin{cases} (V : \nabla^\uparrow_\varepsilon(b)) & \text{if } b \in \mathbf{B}^\uparrow, \\ 0 & \text{otherwise.} \end{cases}$$

Both I and Q have ∇_ε-flags too, so we get similar statements for j_*I and j_*Q. Since $(I : \nabla^\uparrow_\varepsilon(b)) = (V : \nabla^\uparrow_\varepsilon(b)) + (Q : \nabla^\uparrow_\varepsilon(b))$ by the exactness of the original sequence, we deduce that $0 \to j_*V \to j_*I \to j_*Q \to 0$ is exact. Hence, $(\mathbb{R}^1 j_*)V = 0$. This proves the result for $n = 1$. The result for $n > 1$ follows by a degree shifting argument. □

Corollary 3.21. *Let notation be as in Theorem 3.20 and set $\mathbf{B}^\downarrow := \mathbf{B}\backslash\mathbf{B}^\uparrow$.*

1. *For $V \in \nabla_\varepsilon(\mathcal{R})$, there is a short exact sequence $0 \to K \to V \xrightarrow{\gamma} j_*(jV) \to 0$ where γ comes from the unit of adjunction, $j_*(jV)$ has a ∇_ε-flag with sections $\nabla_\varepsilon(b)$ for $b \in \mathbf{B}^\uparrow$, and K has a ∇_ε-flag with sections $\nabla_\varepsilon(c)$ for $c \in \mathbf{B}^\downarrow$.*
2. *For $V \in \Delta_\varepsilon(\mathcal{R})$, there is a short exact sequence $0 \to j_!(jV) \xrightarrow{\delta} V \to Q \to 0$ where δ comes from the counit of adjunction, $j_!(jV)$ has a Δ_ε-flag with sections $\Delta_\varepsilon(b)$ for $b \in \mathbf{B}^\uparrow$ and Q has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}^\downarrow$.*

Proof. We prove only (1), since (2) is just the dual statement. Using (3.7), we can order the ∇_ε-flag of V to get a short exact sequence $0 \to K \to V \to Q \to 0$ such that K has a ∇_ε-flag with sections $\nabla_\varepsilon(b)$ for $b \in \mathbf{B}^\downarrow$ and Q has a ∇_ε-flag with sections $\nabla_\varepsilon(c)$ for $c \in \mathbf{B}^\uparrow$. For each $b \in \mathbf{B}^\uparrow$, the unit of adjunction $\nabla_\varepsilon(b) \to j_*(j\nabla_\varepsilon(b))$ is an isomorphism thanks to Theorem 3.20(4) since it becomes an isomorphism on applying j. Since j_* is exact on $\nabla_\varepsilon(\mathcal{R}^\uparrow)$, we deduce that the the unit of adjunction $Q \to j_*(jQ)$ is an isomorphism too. It remains to note that $jV \cong jQ$, hence, $j_*(jV) \cong j_*(jQ)$. □

Now we proceed to discuss some of the additional features which show up when in one of the more refined settings (FS), (εHW), (SHW) and (HW). By Theorem 3.11, $\mathcal{R}$ is a fully stratified category (resp., a signed highest weight category) if and only if $\mathcal{R}^{\mathrm{op}}$ is a fully stratified category (resp., a signed highest weight category). The following lemma shows that fully stratified categories in our terminology are the same as the "standardly stratified categories" defined by Losev and Webster in [LW, §2].

Lemma 3.22. *The following are equivalent:*

(i) $\mathcal{R}$ *is a fully stratified category;*
(ii) $\mathcal{R}$ *is* ε*-stratified for every choice of sign function* $\varepsilon : \Lambda \to \{\pm\}$*;*
(iii) $\mathcal{R}$ *is* ε*-stratified and* $(-\varepsilon)$*-stratified for some choice of sign function* $\varepsilon : \Lambda \to \{\pm\}$*;*
(iv) $\mathcal{R}$ *is* ε*-stratified for some* $\varepsilon : \Lambda \to \{\pm\}$ *and all of its standardization and costandardization functors are exact;*
(v) $\mathcal{R}$ *is a* $+$*-stratified category and each* $\Delta(b)$ *has a* $\bar{\Delta}$*-flag with sections* $\bar{\Delta}(c)$ *for* c *with* $\rho(c) = \rho(b)$*;*
(vi) $\mathcal{R}$ *is a* $-$*-stratified category and each* $\nabla(b)$ *has a* $\bar{\nabla}$*-flag with sections* $\bar{\nabla}(c)$ *for* c *with* $\rho(c) = \rho(b)$*.*

Proof. (i)⇒(ii)⇒(iii): Obvious.

(iii)⇒(iv): Take ε as in (iii) so that $\mathcal{R}$ is ε-stratified. The standardization functor $j_!^\lambda$ is exact when $\varepsilon(\lambda) = -$ by the last part of Theorem 3.7. Also $\mathcal{R}$ is $(-\varepsilon)$-stratified, so the same result gives that $j_!^\lambda$ is exact when $\varepsilon(\lambda) = +$. Similarly, all of the costandardization functors are exact too.

(iv)⇒(v): Applying the exact standardization functor $j_!^\lambda$ to a composition series of $P_\lambda(b)$, we deduce that $\Delta(b)$ has a $\bar{\Delta}$-flag with sections $\bar{\Delta}(c)$ for c with $\rho(c) = \rho(b)$. Similarly, applying j_*^λ, we get that $\nabla(b)$ has a $\bar{\nabla}$-flag with sections $\bar{\nabla}(c)$ for c with $\rho(c) = \rho(b)$.

To show that $\mathcal{R}$ is $+$-stratified, we check that each $I(b)$ has a $\bar{\nabla}$-flag with sections $\bar{\nabla}(c)$ for c with $\rho(c) \geqslant \rho(b)$. This is immediate if $\varepsilon(b) = +$ since we are assuming that $\mathcal{R}$ is ε-stratified. If $\varepsilon(b) = -$ then $I(b)$ has a ∇-flag with sections $\nabla(c)$ for c with $\rho(c) \geqslant \rho(b)$. Hence, by the previous paragraph, it also has the required sort of $\bar{\nabla}$-flag.

(v)⇒(i): We just need to show that $\mathcal{R}$ is $-$-stratified. We know that each $P(b)$ has a Δ-flag with sections $\Delta(c)$ for c with $\rho(c) \geqslant \rho(b)$. Now use the given $\bar{\Delta}$-flags of each $\Delta(c)$ to see that each $P(b)$ also has the appropriate sort of $\bar{\Delta}$-flag.

(v)⇔(vi): This follows from the above using the observation made earlier that $\mathcal{R}$ is fully stratified if and only if $\mathcal{R}^{\mathrm{op}}$ is fully stratified. □

Corollary 3.23. *Suppose that* $\mathcal{R}$ *is a finite or essentially finite* ε*-stratified category admitting a duality compatible with the stratification, i.e., there is a contravariant equivalence* $\# : \mathcal{R} \to \mathcal{R}$ *such that* $L(b)^\# \cong L(c)$ *implies* $\rho(b) = \rho(c)$ *for* $b, c \in \mathbf{B}$*. Then* $\mathcal{R}$ *is a fully stratified category.*

Proof. Since $\mathcal{R}$ is ε-stratified, $\mathcal{R}^{\mathrm{op}}$ is $(-\varepsilon)$-stratified. Using the duality, we deduce that $\mathcal{R}$ is also $(-\varepsilon)$-stratified. This verifies Lemma 3.22(3). □

Corollary 3.24. *Suppose that* $\mathcal{R}$ *is a finite or essentially finite* ε*-highest weight category admitting a duality* $\#$ *fixing isomorphism classes of irreducible objects. Then* $\mathcal{R}$ *is a signed highest weight category.*

Lemma 3.25. *Suppose that* $\mathcal{R}$ *is a finite or essentially finite fully stratified category. For* $b, c \in \mathbf{B}$ *and* $n \geqslant 0$*, we have that*

$$\mathrm{Ext}^n_{\mathcal{R}}(\bar{\Delta}(b), \bar{\nabla}(c)) \cong \begin{cases} \mathrm{Ext}^n_{\mathcal{R}_\lambda}(L(b), L(c)) & \text{if } \lambda = \mu \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda := \rho(b)$ *and* $\mu := \rho(c)$*.*

Proof. Choose ε so that $\varepsilon(\lambda) = -$, hence, $\bar{\Delta}(b) = \Delta_\varepsilon(b)$. By Lemma 3.22, $\mathcal{R}$ is ε-stratified, so we can apply Theorem 3.19(4) with $\mathcal{R}^\downarrow = \mathcal{R}_{\leqslant\mu}$ to deduce that

$$\mathrm{Ext}^n_{\mathcal{R}}(\bar{\Delta}(b), \bar{\nabla}(c)) \cong \mathrm{Ext}^n_{\mathcal{R}_{\leqslant\mu}}(i^*_{\leqslant\mu}\bar{\Delta}(b), \bar{\nabla}(c)).$$

This is zero unless $\lambda \leqslant \mu$. If $\lambda \leqslant \mu$, it is simply $\mathrm{Ext}^n_{\mathcal{R}_{\leqslant\mu}}(\bar\Delta(b), \bar\nabla(c))$. Now we change ε so that $\varepsilon(\mu) = +$, hence, $\bar\nabla(c) = \nabla_\varepsilon(c)$. Then by Theorem 3.20(3) with $\mathcal{R} = \mathcal{R}_{\leqslant\mu}$ and $\mathcal{R}^\uparrow = \mathcal{R}_\mu$ we get that $\mathrm{Ext}^n_{\mathcal{R}_{\leqslant\mu}}(\bar\Delta(b), \bar\nabla(c)) \cong \mathrm{Ext}^n_{\mathcal{R}_\mu}(j^\mu\bar\Delta(b), L(c))$. This is zero unless $\lambda = \mu$, when $j^\mu\bar\Delta(b) = L(b)$ and we are done. □

The following elementary observation also exploits the fully stratified hypothesis. For example, this implies that if $\mathcal{R}$ is fully stratified and each stratum is itself highest weight then the stratification can be refined to make $\mathcal{R}$ into a highest weight category.

Lemma 3.26. *Suppose that $\mathcal{R}$ is fully stratified with stratification $\rho : \mathbf{B} \to \Lambda$ and that we are given another poset $(\tilde\Lambda, \tilde\leqslant)$ and a finer stratification $\tilde\rho : \mathbf{B} \to \tilde\Lambda$, i.e., we have that $\tilde\rho(a) \mathrel{\tilde\leqslant} \tilde\rho(b) \Rightarrow \rho(a) \leqslant \rho(b)$. Assume the restrictions $\tilde\rho_\lambda : \mathbf{B}_\lambda \to \tilde\Lambda$ define stratifications making all of the original strata $\mathcal{R}_\lambda$ into fully stratified categories with standard and costandard objects denoted $\Delta_\lambda(b)$ and $\nabla_\lambda(b)$ for $b \in \mathbf{B}_\lambda$. Then $\mathcal{R}$ is also fully stratified with respect to the stratification $\tilde\rho : B \to \tilde\Lambda$, and the corresponding standard and costandard objects are $\tilde\Delta(b) = j^\lambda_!\Delta_\lambda(b)$ and $\tilde\nabla(b) = j^\lambda_*\nabla_\lambda(b)$ for $b \in \mathbf{B}$ and $\lambda := \rho(b)$.*

Proof. Take $\lambda \in \Lambda$ and set $\tilde\lambda := \tilde\rho(\lambda)$. The assumption on $\tilde\rho$ implies that the quotient functor $j^{\tilde\lambda} : \mathcal{R} \to \mathcal{R}_{\tilde\lambda}$ factors through $j^\lambda : \mathcal{R} \to \mathcal{R}_\lambda$. Since both $\mathcal{R}$ and $\mathcal{R}_\lambda$ are fully stratified, their standardization and costandardization functors are exact, hence so are $j^{\tilde\lambda}_!$ and $j^{\tilde\lambda}_*$ since they are compositions of exact functors. Similarly, it follows that the standard and costandard objects of $\mathcal{R}_\lambda$ with respect to the stratification $\tilde\rho$ are as stated in the lemma. Using the criterion from Lemma 3.22(iv), it remains to show that each $P(a)$ has a filtration with subquotients $\tilde\Delta(b)$ for $b \in \mathbf{B}$ satisfying $\tilde\rho(b) \mathrel{\tilde\geqslant} \tilde\rho(a)$. As $\mathcal{R}_\lambda$ is fully stratified, $P_\lambda(a)$ has a Δ_λ-flag. Hence, applying the exact functor $j^\lambda_!$, we see that $\Delta(a)$ has a $\tilde\Delta$-flag. Since $P(a)$ has a Δ-flag, the result follows. □

The next results are concerned with global dimension.

Lemma 3.27. *Let $\mathcal{R}$ be a finite ε-stratified category.*

(1) All $V \in \Delta_\varepsilon(\mathcal{R})$ are of finite projective dimension if and only if all negative strata[2] have finite global dimension.

(2) All $V \in \nabla_\varepsilon(\mathcal{R})$ are of finite injective dimension if and only if all positive strata have finite global dimension.

Proof. It suffices to prove (1). Assume that all negative strata have finite global dimension. By [Wei, Exercise 4.1.2], it suffices to show that $\mathrm{pd}\,\Delta_\varepsilon(b) < \infty$ for each $b \in \mathbf{B}$. We proceed by downwards induction on the partial order. For the induction step, consider $\Delta_\varepsilon(b)$ for $b \in \mathbf{B}_\lambda$, assuming that $\mathrm{pd}\,\Delta_\varepsilon(c) < \infty$ for each c with $\rho(c) > \lambda$. We show first that $\mathrm{pd}\,\Delta(b) < \infty$. Using (3.7) with $\varepsilon(\lambda) = -$, we see that there is a short exact sequence $0 \to Q \to P(b) \to \Delta(b) \to 0$ such that Q has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for c with $\rho(c) > \lambda$. By the induction hypothesis, Q has finite projective dimension, hence, so does $\Delta(b)$. This verifies the induction step in the case that $\varepsilon(\lambda) = +$. Instead, suppose that $\varepsilon(\lambda) = -$, i.e., $\Delta_\varepsilon(b) = \bar\Delta(b)$. Let $0 \to P_n \to \cdots \to P_0 \to L_\lambda(b) \to 0$ be a finite projective resolution of $L_\lambda(b)$ in the stratum $\mathcal{R}_\lambda$. Applying $j^\lambda_!$, which is exact thanks to Theorem 3.7, we obtain an exact sequence $0 \to V_n \to \cdots \to V_0 \to \bar\Delta(b) \to 0$ such that each V_m is a direct sum of standard objects $\Delta(c)$ for $c \in \mathbf{B}_\lambda$. The result already established plus [Wei, Exercise 4.1.3] implies that $\mathrm{pd}\,V_m < \infty$ for each m. Arguing like in the proof of [Wei, Theorem 4.3.1], we deduce that $\mathrm{pd}\,\bar\Delta(b) < \infty$ too.

Conversely, suppose that $\mathrm{pd}\,\Delta_\varepsilon(b) < \infty$ for all $b \in \mathbf{B}$. Take $\lambda \in \Lambda$ with $\varepsilon(\lambda) = -$. Suppose first that λ is maximal. Applying the exact functor j^λ to finite projective resolutions of $\bar\Delta(b)$ for each $b \in \mathbf{B}_\lambda$, we obtain finite projective resolutions of $L_\lambda(b)$ in $\mathcal{R}_\lambda$, showing that the stratum $\mathcal{R}_\lambda$ is of finite global dimension. Finally, when λ is not maximal, we let $\Lambda^\downarrow := \Lambda\backslash(\lambda, \infty)$ and $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the corresponding ε-stratified Serre

[2] We mean the strata $\mathcal{R}_\lambda$ for $\lambda \in \Lambda$ such that $\varepsilon(\lambda) = -$.

subcategory of $\mathcal{R}$. Given $b \in \mathbf{B}_\lambda$, take a finite projective resolution $0 \to P_n \to \cdots \to P_0 \to \Delta_\varepsilon(b) \to 0$ of $\Delta_\varepsilon(b)$ in $\mathcal{R}$. Applying i^*, we obtain an exact sequence $0 \to i^*P_n \to \cdots \to i^*P_0 \to \Delta_\varepsilon(b) \to 0$ in $\mathcal{R}^\downarrow$. This sequence is exact due to Theorem 3.19(2); to see this one also needs to use Corollary 3.18 to break the sequence into short exact sequences in $\Delta_\varepsilon(\mathcal{R})$. Since λ is maximal in $\Lambda^\downarrow$, we are reduced to the case already discussed. □

Corollary 3.28. *Suppose that $\mathcal{R}$ is a finite ε-stratified category. If $\mathcal{R}$ is of finite global dimension then all of its strata are of finite global dimension too.*

Corollary 3.29. *Suppose that $\mathcal{R}$ is either a finite $+$-stratified category or a finite $-$-stratified category. If all of the strata are of finite global dimension then $\mathcal{R}$ is of finite global dimension.*

Proof. We just explain this in the case that $\mathcal{R}$ is $-$-stratified; the argument in $+$-stratified case is similar. Lemma 3.27(1) implies that $\bar{\Delta}(b)$ is of finite projective dimension for each $b \in \mathbf{B}$. Moreover, there is a short exact sequence $0 \to K \to \bar{\Delta}(b) \to L(b) \to 0$ where all composition factors of K are of the form $L(c)$ for c with $\rho(c) < \rho(b)$. Ascending induction on the partial order implies that each $L(b)$ has finite projective dimension. □

Remark 3.30. In the fully stratified case, Lemma 3.25 can be used to give a precise bound on the global dimension of $\mathcal{R}$ in Corollary 3.29. Let

$$|\lambda| := \sup\left\{ \frac{\max(\mathrm{gl.dim}\,\mathcal{R}_{\lambda_0}, \dots, \mathrm{gl.dim}\,\mathcal{R}_{\lambda_n})}{2} + n \;\middle|\; \begin{array}{l} n \geqslant 0 \text{ and } \lambda_0, \lambda_1, \dots, \lambda_n \in \Lambda \\ \text{with } \lambda_0 < \lambda_1 < \cdots < \lambda_n = \lambda \end{array} \right\}.$$

By mimicking the proof of [Don4, Proposition A2.3], one shows that $\mathrm{Ext}^n_{\mathcal{R}}(L(b), L(c)) = 0$ for $b, c \in \mathbf{B}$ and any $n > |\rho(b)| + |\rho(c)|$. Hence, $\mathrm{gl.dim}\,\mathcal{R} \leqslant 2\max\{|\lambda| \mid \lambda \in \Lambda\}$.

In particular, Corollary 3.29 recovers the following well-known result, see e.g. [CPS1]. For further detailed remarks about the history of this, and the general notion of highest weight category, we refer to [Don4, §A5] and [DR].

Corollary 3.31. *Finite highest weight categories are of finite global dimension.*

Remark 3.32. With regard to highest weight categories again, Coulembier [Cou] has recently made the following elegant observation: in a finite highest weight category with duality, the partial order on Λ is essentially unique (up to replacing it by a coarser ordering). It would be interesting to extend this result to signed highest weight categories.

3.3. **Upper finite ε-stratified categories.** In this subsection we assume that $\mathcal{R}$ is a Schurian category in the sense of §2.3 and $\rho : \mathbf{B} \to \Lambda$ is a stratification such that the poset Λ is upper finite. Also $\varepsilon : \Lambda \to \{\pm\}$ is a fixed sign function. Let $I(b)$ and $P(b)$ be an injective hull and a projective cover of $L(b)$ in $\mathcal{R}$. Recall (1.1)–(1.2), the properties $(P\Delta_\varepsilon)$–$(I\nabla_\varepsilon)$ and $(P\Delta)$–$(I\nabla)$ from the introduction, and the seemingly stronger properties $(\widehat{P\Delta}_\varepsilon)$–$(\widehat{I\nabla}_\varepsilon)$ and $(\widehat{P\Delta})$–$(\widehat{I\nabla})$ from the previous subsection. Before formulating the main definitions in the upper finite setting, we prove an analog of the homological criterion for ∇_ε-flags from Theorem 3.13. The proof depends on the lemmas proved in §3.4 below, which we used already in the essentially finite Abelian case, together with the following two technical lemmas, which we prove by truncating to finite Abelian quotients.

Lemma 3.33. *Suppose that $\mathcal{R}$ is Schurian with a stratification ρ and sign function ε, and assume that the property $(\widehat{P\Delta}_\varepsilon)$ holds in $\mathcal{R}$. Let $\Lambda^\uparrow$ be a finite upper set in Λ, $\mathbf{B}^\uparrow := \rho^{-1}(\Lambda^\uparrow)$, and $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the corresponding Serre quotient category with the induced stratification. The functor j_* is exact on $\nabla_\varepsilon(\mathcal{R}^\uparrow)$, and it takes objects of $\nabla_\varepsilon(\mathbf{B}^\uparrow)$ to objects of $\nabla_\varepsilon(\mathbf{B})$.*

Proof (assuming lemmas in §3.4 below). We proceed by induction on the length of the ∇_ε-flag. The base case, length one, follows from Lemma 3.50(1). For the induction step, consider a short exact sequence $0 \to K \to X \to Q \to 0$ in $\mathcal{R}^\uparrow$ such that K, X and Q

have ∇_ε-flags with $K, Q \neq 0$. We may assume by induction that j_*K and j_*Q have ∇_ε-flags, and must show that $0 \to j_*K \to j_*X \to j_*Q \to 0$ is exact. Since it is left exact, this follows if we can show that

$$[j_*X : L(b)] = [j_*K : L(b)] + [j_*Q : L(b)]$$

for all $b \in \mathbf{B}$. To see this, let $\Lambda^{\Uparrow}$ be the finite upper set generated by $\Lambda^{\uparrow}$ and b. Let $\mathbf{B}^{\Uparrow} := \rho^{-1}(\Lambda^{\Uparrow})$ and $k : \mathcal{R} \to \mathcal{R}^{\Uparrow}$ be the corresponding Serre quotient. By Lemma 2.22, we have that $[j_*X : L(b)] = [k(j_*X) : kL(b)] = [k(j_*X) : L^{\Uparrow}(b)]$, and similarly for K and Q. Since $\Lambda^{\uparrow}$ is an upper set in $\Lambda^{\Uparrow}$, we can also view $\mathcal{R}^{\uparrow}$ as a quotient of $\mathcal{R}^{\Uparrow}$, and the quotient functor j factors as $j = \bar{\jmath} \circ k$ for another quotient functor $\bar{\jmath} : \mathcal{R}^{\uparrow} \to \mathcal{R}^{\Uparrow}$. We have that $k_* \circ \bar{\jmath}_* \cong j_*$, hence, applying k, we get that $\bar{\jmath}_* \cong k \circ j_*$. It follows that $[k(j_*X) : L^{\Uparrow}(b)] = [\bar{\jmath}_*X : L^{\Uparrow}(b)]$, and similarly for K and Q. It remains to observe that

$$[\bar{\jmath}_*X : L^{\Uparrow}(b)] = [\bar{\jmath}_*K : L^{\Uparrow}(b)] + [\bar{\jmath}_*Q : L^{\Uparrow}(b)].$$

To see this, we note that $\mathcal{R}^{\Uparrow}$ and $\mathcal{R}^{\uparrow}$ are finite ε-highest weight categories due to Lemma 3.50(2) and Theorem 3.7. So we can apply Theorem 3.20(4) to see that the sequence $0 \to \bar{\jmath}_*K \to \bar{\jmath}_*X \to \bar{\jmath}_*Q \to 0$ is exact. $\square$

Lemma 3.34. *Suppose that $\mathcal{R}$ is Schurian with a stratification ρ and sign function ε, and assume that the property ($\widehat{P\Delta_\varepsilon}$) holds in $\mathcal{R}$. Let $V \in \mathcal{R}$ be a finitely cogenerated object such that $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$ for all $b \in \mathbf{B}$. Then we have that $V \in \nabla_\varepsilon(b)$, and the multiplicity $(V : \nabla_\varepsilon(b))$ of $\nabla_\varepsilon(b)$ in any ∇_ε-flag is equal to the dimension of $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V)$.*

Proof (assuming lemmas from §3.4 below). Since V is finitely cogenerated, its injective hull is a finite direct sum of the indecomposable injective objects $I(b)$. This means that we can find a finite upper set $\Lambda^{\uparrow}$ and $\mathbf{B}^{\uparrow} := \rho^{-1}(\Lambda^{\uparrow})$ so that there is a short exact sequence

$$0 \to V \to \bigoplus_{b \in \mathbf{B}^{\uparrow}} I(b)^{\oplus n_b} \to Q \to 0$$

for some $n_b \geqslant 0$. Let $j : \mathcal{R} \to \mathcal{R}^{\uparrow}$ be the corresponding Serre quotient. This is a finite ε-stratified category by Lemma 3.50(2) and Theorem 3.7.

Applying j to the above short exact sequence gives us a short exact sequence in $\mathcal{R}^{\uparrow}$. Then we apply the functor $\operatorname{Hom}_{\mathcal{R}^{\uparrow}}(\Delta^{\uparrow}_\varepsilon(b), -)$ to this using also Lemma 3.50(1) to obtain the long exact sequence

$$\begin{aligned}0 \to \operatorname{Hom}_{\mathcal{R}^{\uparrow}}(\Delta^{\uparrow}_\varepsilon(b), jV) \to \operatorname{Hom}_{\mathcal{R}^{\uparrow}}\left(\Delta^{\uparrow}_\varepsilon(b), \textstyle\bigoplus_{b \in \mathbf{B}^{\uparrow}} I^{\uparrow}(b)^{\oplus n_b}\right) \to \operatorname{Hom}_{\mathcal{R}^{\uparrow}}(\Delta^{\uparrow}_\varepsilon(b), jQ)\\ \to \operatorname{Ext}^1_{\mathcal{R}^{\uparrow}}(\Delta^{\uparrow}_\varepsilon(b), jV) \to 0.\end{aligned}$$

From adjunction and Lemma 3.50(1) again, we get a commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_{\mathcal{R}^{\uparrow}}(\Delta^{\uparrow}_\varepsilon(b), jV) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{R}^{\uparrow}}\left(\Delta^{\uparrow}_\varepsilon(b), \bigoplus_{b \in \mathbf{B}^{\uparrow}} I^{\uparrow}(b)^{\oplus n_b}\right) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{R}^{\uparrow}}(\Delta^{\uparrow}_\varepsilon(b), jQ) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) \arrow[r] & \operatorname{Hom}_{\mathcal{R}}\left(\Delta_\varepsilon(b), \bigoplus_{b \in \mathbf{B}^{\uparrow}} I(b)^{\oplus n_b}\right) \arrow[r] & \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), Q) \arrow[r] & 0. \end{tikzcd}$$

The vertical maps are isomorphisms and the bottom row is exact since $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$. Hence the top row is exact. Comparing with the previously displayed long exact sequence, it follows that $\operatorname{Ext}^1_{\mathcal{R}^{\uparrow}}(\Delta^{\uparrow}_\varepsilon(b), jV) = 0$. Now we can apply Theorem 3.13 in the finite ε-stratified category $\mathcal{R}^{\uparrow}$ to deduce that jV has a ∇_ε-flag.

From Lemma 3.33, we deduce that j_*jV has a ∇_ε-flag. Moreover the multiplicity of $\nabla_\varepsilon(b)$ in any ∇_ε-flag in j_*jV is $\dim \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), j_*jV)$ thanks to Lemma 3.49. To complete the proof, we show that the unit of adjunction $f : V \to j_*jV$ is an isomorphism. We know from Lemma 3.50(1) that the unit of adjunction is an isomorphism $I(b) \to$

$j_*jI(b)$ for each $b \in \mathbf{B}^\uparrow$. Since V embeds into a direct sum of such $I(b)$, it follows that f is injective. To show that it is surjective, it suffices to show that

$$[j_*jV : L(b)] = [V : L(b)]$$

for all $b \in \mathbf{B}$. To prove this, we fix a choice of $b \in \mathbf{B}$ then define $\Lambda^{\uparrow\uparrow}$, $\mathbf{B}^{\uparrow\uparrow}$, $k : \mathcal{R} \to \mathcal{R}^{\uparrow\uparrow}$ and $\bar{\jmath} : \mathcal{R}^{\uparrow\uparrow} \to \mathcal{R}^\uparrow$ as in the proof of Lemma 3.33. Since $b \in \mathbf{B}^{\uparrow\uparrow}$, we have that $[V : L(b)] = [kV : L^{\uparrow\uparrow}(b)]$ and $[j_*jV : L(b)] = [k(j_*jV) : L^{\uparrow\uparrow}(b)]$. As in the proof of Lemma 3.33, $k(j_*jV) \cong \bar{\jmath}_*(jV) \cong \bar{\jmath}_*\bar{\jmath}(kV)$. Thus, we are reduced to showing that

$$[\bar{\jmath}_*\bar{\jmath}(kV) : L^{\uparrow\uparrow}(b)] = [kV : L^{\uparrow\uparrow}(b)].$$

This follows because $kV \cong \bar{\jmath}_*\bar{\jmath}(kV)$. To see this, we repeat the arguments in the previous paragraph to show that $kV \in \mathcal{R}^{\uparrow\uparrow}$ has a ∇_ε-flag. Since the unit of adjunction is an isomorphism $\nabla_\varepsilon^{\uparrow\uparrow}(b) \xrightarrow{\sim} \bar{\jmath}_*\bar{\jmath}\nabla_\varepsilon^{\uparrow\uparrow}(b)$ for each $b \in \mathbf{B}^{\uparrow\uparrow}$, we deduce using the exactness from Theorem 3.20(4) that it gives an isomorphism $kV \xrightarrow{\sim} \bar{\jmath}_*\bar{\jmath}(kV)$ too. □

We are ready to proceed to the main definition.

Theorem 3.35. *Theorem 3.7 holds in the present setup too.*

Proof. This is almost the same as the proof of Theorem 3.7 given in the previous subsection. One needs to use Lemma 3.34 in place of Theorem 3.13 to see that $I(b)$ has a ∇_ε-flag with the appropriate multiplicities. For the assertion about exactness of j_*^λ when $\varepsilon(\lambda) = +$, apply Lemma 3.33 working in the category $\mathcal{R}_{\leqslant\lambda}$, which satisfies $(\widehat{P\Delta_\varepsilon})$ due to Lemma 3.46(2). □

Definition 3.36. Let $\mathcal{R}$ be a Schurian category and $\rho : \mathbf{B} \to \Lambda$ be a stratification in the sense of Definition 3.1 such that the poset Λ is upper finite.

- (εS) We say that $\mathcal{R}$ is an *upper finite ε-stratified category* if one of the equivalent properties $(\widehat{P\Delta_\varepsilon})$–$(\widehat{I\nabla_\varepsilon})$ for some given choice of sign function $\varepsilon : \Lambda \to \{\pm\}$.
- (FS) We say that $\mathcal{R}$ is an *upper finite fully stratified category* if one of these properties holds for all choices of sign function $\varepsilon : \Lambda \to \{\pm\}$.
- (εHW) We say that $\mathcal{R}$ is an *upper finite ε-highest weight category* if the stratification function ρ is a bijection, and one of the equivalent properties $(\widehat{P\Delta_\varepsilon})$–$(\widehat{I\nabla_\varepsilon})$ holds for some given choice of sign function $\varepsilon : \Lambda \to \{\pm\}$.
- (SHW) We say that $\mathcal{R}$ is an *upper finite signed highest weight category* if the stratification function is a bijection and one of these properties holds for all choices of sign function.
- (HW) We say that $\mathcal{R}$ is an *upper finite highest weight category* if all of the stata are simple (cf. Lemma 3.6) and one of the equivalent properties $(\widehat{P\Delta})$–$(\widehat{I\nabla})$ holds.

Theorem 3.11 and (3.7) still hold in the same way as before.

Next, we are going to consider two (in fact dual) notions of ascending Δ_ε- and descending ∇_ε-flags, generalizing the finite flags discussed already. One might be tempted to say that an ascending Δ_ε-flag in V is an ascending chain $0 = V_0 < V_1 < V_2 < \cdots$ of subobjects of V with $V = \sum_{n\in\mathbb{N}} V_n$ such that $V_m/V_{m-1} \cong \Delta_\varepsilon(b_m)$, and a descending ∇_ε-flag is a descending chain $V = V_0 > V_1 > V_2 > \cdots$ of subobjects of V such that $\bigcap_{n\in\mathbb{N}} V_n = 0$ and $V_{m-1}/V_m \cong \Delta_\varepsilon(b_m)$, for $b_m \in \mathbf{B}$. These would be serviceable definitions when Λ is countable. In order to avoid this unnecessary restriction, we will work instead with the following more general formulations.

Definition 3.37. Suppose that $\mathcal{R}$ is an upper finite ε-stratified category and $V \in \mathcal{R}$.

- ($\Delta_\varepsilon^{\mathrm{asc}}$) An *ascending Δ_ε-flag* in V is the data of a directed set Ω with smallest element 0 and a direct system $(V_\omega)_{\omega\in\Omega}$ of subobjects of V such that $V_0 = 0$, $\sum_{\omega\in\Omega} V_\omega = V$, and $V_\upsilon/V_\omega \in \Delta_\varepsilon(\mathcal{R})$ for each $\omega < \upsilon$. Let $\Delta_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ be the exact subcategory of $\mathcal{R}$ consisting of all objects V possessing such a flag.

($\nabla_\varepsilon^{\mathrm{dsc}}$) A *descending* ∇_ε*-flag* in V is the data of a directed set Ω with smallest element 0 and an inverse system $(V/V_\omega)_{\omega\in\Omega}$ of quotients of V such that $V_0 = V$, $\bigcap_{\omega\in\Omega} V_\omega = 0$, and $V_\omega/V_\upsilon \in \nabla_\varepsilon(\mathcal{R})$ for each $\omega < \upsilon$. Let $\nabla_\varepsilon^{\mathrm{dsc}}(\mathcal{R})$ be the exact subcategory of $\mathcal{R}$ consisting of all objects V possessing such a flag.

We stress that $\Delta_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ and $\nabla_\varepsilon^{\mathrm{dsc}}(\mathcal{R})$ are subcategories of $\mathcal{R}$: we have *not* passed to the completion $\mathrm{Ind}(\mathcal{R}_c)$.

Lemma 3.38. *Suppose that $\mathcal{R}$ is an upper finite ε-stratified category.*

(1) For $V \in \Delta_\varepsilon^{\mathrm{asc}}(\mathcal{R})$, $W \in \nabla_\varepsilon^{\mathrm{dsc}}(\mathcal{R})$ and $n \geqslant 1$, we have that $\mathrm{Ext}^n_{\mathcal{R}}(V, W) = 0$.

(2) For $V \in \Delta_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ the multiplicity of $\nabla_\varepsilon(b)$ in a Δ_ε-flag may be defined from

$$(V : \Delta_\varepsilon(b)) := \dim \mathrm{Hom}_{\mathcal{R}}(V, \nabla_\varepsilon(b)) = \sup\left\{(V_\omega : \Delta_\varepsilon(b)) \,\middle|\, \omega \in \Omega\right\} < \infty,$$

where $(V_\omega)_{\omega\in\Omega}$ is any choice of ascending Δ_ε-flag.

(3) For $V \in \nabla_\varepsilon^{\mathrm{dsc}}(\mathcal{R})$, the multiplicity of $\Delta_\varepsilon(b)$ in a ∇_ε-flag may be defined from

$$(V : \nabla_\varepsilon(b)) := \dim \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) = \sup\left\{(V/V_\omega : \nabla_\varepsilon(b)) \,\middle|\, \omega \in \Omega\right\} < \infty,$$

where $(V/V_\omega)_{\omega\in\Omega}$ is any choice of descending ∇_ε-flag.

Proof. (1) We first prove this in the special case that $W = \nabla_\varepsilon(b)$. Let $(V_\omega)_{\omega\in\Omega}$ be an ascending Δ_ε-flag in V, so that $V \cong \varinjlim V_\omega$. Since $\mathrm{Ext}^n_{\mathcal{R}}(V_\omega, W) = 0$ by Lemma 3.49, it suffices to show that

$$\mathrm{Ext}^n_{\mathcal{R}}(V, W) \cong \varprojlim \mathrm{Ext}^n_{\mathcal{R}}(V_\omega, W).$$

To see this, like in [Wei, Application 3.5.10], we need to check a Mittag-Leffler condition. We show simply that the natural map $\mathrm{Ext}^{n-1}_{\mathcal{R}}(V_\upsilon, W) \to \mathrm{Ext}^{n-1}_{\mathcal{R}}(V_\omega, W)$ is surjective for each $\omega < \upsilon$ in Ω. Applying $\mathrm{Hom}_{\mathcal{R}}(-, W)$ to the short exact sequence $0 \to V_\omega \to V_\upsilon \to V_\upsilon/V_\omega \to 0$ gives an exact sequence

$$\mathrm{Ext}^{n-1}_{\mathcal{R}}(V_\upsilon, W) \longrightarrow \mathrm{Ext}^{n-1}_{\mathcal{R}}(V_\omega, W) \longrightarrow \mathrm{Ext}^n_{\mathcal{R}}(V_\upsilon/V_\omega, W).$$

It remains to observe that $\mathrm{Ext}^n_{\mathcal{R}}(V_\upsilon/V_\omega, W) = 0$ by Lemma 3.49 again, since we know from the definition of ascending Δ_ε-flag that $V_\upsilon/V_\omega \in \Delta_\varepsilon(\mathcal{R})$.

The dual of the previous paragraph plus Lemma 2.12 gives that $\mathrm{Ext}^n_{\mathcal{R}}(V, W) = 0$ for $n \geqslant 1$, $V = \Delta_\varepsilon(b)$ and $W \in \nabla_\varepsilon^{\mathrm{dsc}}(\mathcal{R})$. Then we can repeat the argument of the previous paragraph yet again, using this assertion in place of Lemma 3.49, to obtain the result we are after for general $V \in \Delta_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ and $W \in \nabla_\varepsilon^{\mathrm{dsc}}(\mathcal{R})$.

(2) This follows from (1) and (3.2) because

$$\mathrm{Hom}_{\mathcal{R}}(V, \nabla_\varepsilon(b)) \cong \mathrm{Hom}_{\mathcal{R}}(\varinjlim V_\omega, \nabla_\varepsilon(b)) \cong \varprojlim \mathrm{Hom}_{\mathcal{R}}(V_\omega, \nabla_\varepsilon(b)),$$

which is finite-dimensional as $\nabla_\varepsilon(b)$, hence, each V_ω, is finitely cogenerated.

(3) Similarly to (2), we have that

$$\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) \cong \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), \varprojlim(V/V_\omega)) \cong \varprojlim \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V/V_\omega),$$

which is finite-dimensional as $\Delta_\varepsilon(b)$ is finitely generated. Then we can apply (1) and (3.2) once again. □

Theorem 3.39. *Assume that $\mathcal{R}$ is an upper finite ε-stratified category. For $V \in \mathcal{R}$, the following are equivalent:*

(i) $V \in \Delta_\varepsilon^{\mathrm{asc}}(\mathcal{R})$;

(ii) $\mathrm{Ext}^1_{\mathcal{R}}(V, \nabla_\varepsilon(b)) = 0$ for all $b \in \mathbf{B}$;

(iii) $\mathrm{Ext}^n_{\mathcal{R}}(V, \nabla_\varepsilon(b)) = 0$ for all $b \in \mathbf{B}$ and $n \geqslant 1$.

Assuming these properties, we have that $V \in \Delta_\varepsilon(\mathcal{R})$ if and only if it is finitely generated.

Proof. (iii)⇒(ii). Trivial.

(i)⇒(iii). This follows from Lemma 3.38.

(ii)⇒(i). Let Ω be the directed set of finite upper sets in Λ. Take $\omega \in \Omega$; it is some finite upper set $\Lambda^\uparrow$. Let $\mathbf{B}^\uparrow := \rho^{-1}(\Lambda^\uparrow)$ and $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the corresponding Serre quotient. Let $V_\omega := j_!(jV)$. By Lemma 3.50(3), $\mathrm{Ext}^1_{\mathcal{R}^\uparrow}(jV, \nabla_\varepsilon(b)) = 0$ for all $b \in \mathbf{B}^\uparrow$. Hence, $V_\omega := j_!(jV) \in \Delta_\varepsilon(\mathcal{R})$ thanks to the dual of Lemma 3.33. Let $f_\omega : V_\omega \to V$ be the morphism induced by the counit of adjunction. We claim for any $b \in \mathbf{B}^\uparrow$ that the map

$$f_\omega(b) : \mathrm{Hom}_{\mathcal{R}}(P(b), V_\omega) \to \mathrm{Hom}_{\mathcal{R}}(P(b), V),\ \theta \mapsto f_\omega \circ \theta$$

is an isomorphism. To see this, we assume that $\mathcal{R} = A\text{-mod}_{\mathrm{lfd}}$ for a pointed locally finite-dimensional locally unital algebra $A = \bigoplus_{a,b\in\mathbf{B}} e_a A e_b$. Then $\mathcal{R}^\uparrow = eAe\text{-mod}_{\mathrm{lfd}}$ where $e = \sum_{a\in\mathbf{B}^\uparrow} e_a$, and $V_\omega = Ae \otimes_{eAe} eV$. In these terms, the map f_ω is the natural multiplication map. For $b \in \mathbf{B}^\uparrow$, this gives an isomorphism $e_b V_\omega \xrightarrow{\sim} e_b V$ with inverse $e_b v \mapsto e_b \otimes e_b v$. This proves the claim.

Now take $\upsilon > \omega$, i.e., another finite upper set $\Lambda^{\uparrow\uparrow} \supset \Lambda^\uparrow$, and let $k : \mathcal{R} \to \mathcal{R}^{\uparrow\uparrow}$ be the associated quotient. The quotient functor $j : \mathcal{R} \to \mathcal{R}^\uparrow$ factors as $j = \bar{\jmath} \circ k$ for another quotient functor $\bar{\jmath} : \mathcal{R}^{\uparrow\uparrow} \to \mathcal{R}^\uparrow$, and we have that

$$V_\omega = (\bar{\jmath} \circ k)_!((\bar{\jmath} \circ k)V) \cong k_!(\bar{\jmath}_!(\bar{\jmath}(kV))), \qquad V_\upsilon = k_!(kV).$$

By Corollary 3.21(2), there is a short exact sequence $0 \to \bar{\jmath}_!(\bar{\jmath}(kV)) \to kV \to Q \to 0$ such that both $\bar{\jmath}_!(\bar{\jmath}(kV))$ and Q belong to $\Delta_\varepsilon(\mathcal{R}^{\uparrow\uparrow})$. Applying $k_!$ and using the exactness from the dual of Lemma 3.33, we get an embedding $f^\upsilon_\omega : V_\omega \hookrightarrow V_\upsilon$ such that $V_\upsilon / V_\omega \cong k_! Q \in \Delta_\varepsilon(\mathcal{R})$. Since the morphisms all came from counits of adjunction, we have that $f_\upsilon \circ f^\upsilon_\omega = f_\omega$.

Now we can show that each f_ω is a monomorphism. It suffices to show that $f_\omega(b) : \mathrm{Hom}_{\mathcal{R}}(P(b), V_\omega) \to \mathrm{Hom}_{\mathcal{R}}(P(b), V)$ is injective for all $b \in \mathbf{B}$. Choose υ in the previous paragraph to be sufficiently large so as to ensure that $b \in \mathbf{B}^{\uparrow\uparrow}$. We explained already that $f_\upsilon(b)$ is an isomorphism. Since $f_\omega = f_\upsilon \circ f^\upsilon_\omega$ and f^υ_ω is a monorphism, it follows that $f_\omega(b)$ is injective too. Thus, identifying V_ω with its image under f_ω, we have defined a direct system $(V_\omega)_{\omega\in\Omega}$ of subobjects of V such that $V_\upsilon / V_\omega \in \Delta_\varepsilon(\mathcal{R})$ for each $\omega < \upsilon$. It remains to observe that $V_\varnothing = 0$ for a trivial reason, and $\sum_{\omega\in\Omega} V_\omega = V$ because we know for each $b \in \mathbf{B}$ that $f_\omega(b)$ is surjective for sufficiently large ω.

Final part: If $V \in \Delta_\varepsilon(\mathcal{R})$, it is obvious that it is finitely generated since each $\Delta_\varepsilon(b)$ is finitely generated. Conversely, suppose that V is finitely generated and has an ascending Δ_ε-flag. To see that it is actually a finite flag, it suffices to show that $\mathrm{Hom}_{\mathcal{R}}(V, \nabla_\varepsilon(b)) = 0$ for all but finitely many $b \in \mathbf{B}$. Say $\mathrm{hd}\, V \cong L(b_1) \oplus \cdots \oplus L(b_n)$. If $V \to \nabla_\varepsilon(b)$ is a non-zero homomorphism, we must have that $\rho(b_i) \leqslant \rho(b)$ for some $i = 1, \dots, n$. Hence, there are only finitely many choices for b as the poset is upper finite. □

Corollary 3.40. *Let $0 \to U \to V \to W \to 0$ be a short exact sequence in $\mathcal{R}$. If V and W belong to $\Delta_\varepsilon(\mathcal{R})$ (resp., to $\Delta^{\mathrm{asc}}_\varepsilon(\mathcal{R})$) so does U.*

Theorem 3.41. *Assume that $\mathcal{R}$ is an upper finite ε-stratified category. For $V \in \mathcal{R}$, the following are equivalent:*

(i) $V \in \nabla^{\mathrm{dsc}}_\varepsilon(\mathcal{R})$;

(ii) $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$ *for all* $b \in \mathbf{B}$;

(iii) $\mathrm{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$ *for all* $b \in \mathbf{B}$ *and* $n \geqslant 1$.

With these properties we have that $V \in \nabla_\varepsilon(\mathcal{R})$ if and only if it is finitely cogenerated.

Proof. This is the equivalent dual statement to Theorem 3.39. □

Corollary 3.42. *Let $0 \to U \to V \to W \to 0$ be a short exact sequence in $\mathcal{R}$. If U and V belong to $\nabla_\varepsilon(\mathcal{R})$ (resp., to $\nabla^{\mathrm{dsc}}_\varepsilon(\mathcal{R})$) so does W.*

The following is the upper finite analog of Theorem 3.19; we have dropped part (6) since the proof of that required objects of $\mathcal{R}^\downarrow$ to have finite length.

Theorem 3.43. *Assume that $\mathcal{R}$ is an upper finite ε-stratified category. Suppose that $\Lambda^\downarrow$ is a lower set in Λ. Let $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$ and $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the corresponding Serre subcategory of $\mathcal{R}$ with the induced stratification. Then $\mathcal{R}^\downarrow$ is an upper finite ε-stratified category. Moreover:*

1. *The distinguished objects in $\mathcal{R}^\downarrow$ satisfy $L^\downarrow(b) \cong L(b)$, $P^\downarrow(b) \cong i^*P(b)$, $I^\downarrow(b) \cong i^!I(b)$, $\Delta^\downarrow(b) \cong \Delta(b)$,$\bar\Delta^\downarrow(b) \cong \bar\Delta(b)$, $\nabla^\downarrow(b) \cong \nabla(b)$ and $\bar\nabla^\downarrow(b) \cong \bar\nabla(b)$ for $b \in \mathbf{B}^\downarrow$.*
2. *i^* is exact on $\Delta_\varepsilon(\mathcal{R})$ with $i^*\Delta(b) \cong \Delta^\downarrow(b)$ and $i^*\bar\Delta(b) \cong \bar\Delta^\downarrow(b)$ for $b \in \mathbf{B}^\downarrow$; also $i^*\Delta(b) = i^*\bar\Delta(b) = 0$ for $b \notin \mathbf{B}^\downarrow$.*
3. *$\operatorname{Ext}^n_{\mathcal{R}}(V, iW) \cong \operatorname{Ext}^n_{\mathcal{R}^\downarrow}(i^*V, W)$ for $V \in \Delta_\varepsilon(\mathcal{R})$, $W \in \mathcal{R}^\downarrow$ and all $n \geqslant 0$.*
4. *$i^!$ is exact on $\nabla_\varepsilon(\mathcal{R})$ with $i^!\nabla(b) \cong \nabla^\downarrow(b)$ and $i^!\bar\nabla(b) \cong \bar\nabla^\downarrow(b)$ for $b \in \mathbf{B}^\downarrow$; also $i^!\nabla(b) = i^!\bar\nabla(b) = 0$ for $b \notin \mathbf{B}^\downarrow$.*
5. *$\operatorname{Ext}^n_{\mathcal{R}}(iV, W) \cong \operatorname{Ext}^n_{\mathcal{R}^\downarrow}(V, i^!W)$ for $V \in \mathcal{R}^\downarrow, W \in \nabla_\varepsilon(\mathcal{R})$ and all $n \geqslant 0$.*

Proof. This follows from Lemma 3.46 and the dual statement. □

Next is the upper finite analog of Theorem 3.20.

Theorem 3.44. *Assume that $\mathcal{R}$ is an upper finite ε-stratified category. Suppose that $\Lambda^\uparrow$ is an upper set in Λ. Let $\mathbf{B}^\uparrow := \rho^{-1}(\Lambda^\uparrow)$ and $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the corresponding Serre quotient category of $\mathcal{R}$ with the induced stratification. Then $\mathcal{R}^\uparrow$ is itself a finite or upper finite ε-stratified category according to whether $\Lambda^\uparrow$ is finite or infinite. Moreover:*

1. *For $b \in \mathbf{B}^\uparrow$, the distinguished objects $L^\uparrow(b)$, $P^\uparrow(b)$, $I^\uparrow(b)$, $\Delta^\uparrow(b)$, $\bar\Delta^\uparrow(b)$, $\nabla^\uparrow(b)$ and $\bar\nabla^\uparrow(b)$ in $\mathcal{R}^\uparrow$ are isomorphic to the images under j of the corresponding objects of $\mathcal{R}$.*
2. *We have that $jL(b) = j\Delta(b) = j\bar\Delta(b) = j\nabla(b) = j\bar\nabla(b) = 0$ if $b \notin \mathbf{B}^\uparrow$.*
3. *$\operatorname{Ext}^n_{\mathcal{R}}(V, j_*W) \cong \operatorname{Ext}^n_{\mathcal{R}^\uparrow}(jV, W)$ for $V \in \mathcal{R}, W \in \nabla^{\mathrm{dsc}}_\varepsilon(\mathcal{R}^\uparrow)$ and all $n \geqslant 0$.*
4. *j_* is exact on $\nabla_\varepsilon(\mathcal{R}^\uparrow)$ with $j_*\nabla^\uparrow(b) \cong \nabla(b)$, $j_*\bar\nabla^\uparrow(b) \cong \bar\nabla(b)$ and $j_*I^\uparrow(b) \cong I(b)$ for $b \in \mathbf{B}^\uparrow$.*
5. *$\operatorname{Ext}^n_{\mathcal{R}}(j_!V, W) \cong \operatorname{Ext}^n_{\mathcal{R}^\uparrow}(V, jW)$ for $V \in \Delta^{\mathrm{asc}}_\varepsilon(\mathcal{R}^\uparrow)$, $W \in \mathcal{R}$ and all $n \geqslant 0$.*
6. *$j_!$ is exact on $\Delta_\varepsilon(\mathcal{R}^\uparrow)$ with $j_!\Delta^\uparrow(b) \cong \Delta(b)$, $j_!\bar\Delta^\uparrow(b) \cong \bar\Delta(b)$ and $j_!P^\uparrow(b) = P(b)$ for $b \in \mathbf{B}^\uparrow$.*

Proof. If $\Lambda^\uparrow$ is finite, this is proved in just the same way as Theorem 3.20. Assume instead that $\Lambda^\uparrow$ is infinite. Then the same arguments prove (1) and (2), but the proofs of the remaining parts need some slight modifications. It suffices to prove (3) and (4), since (5) and (6) are the same results for $\mathcal{R}^{\mathrm{op}}$.

For (3), the argument from the proof of Lemma 3.50(3) reduces to checking that j sends projectives to objects that are acyclic for $\operatorname{Hom}_{\mathcal{R}^\uparrow}(-, W)$. To see this, it suffices to show that $\operatorname{Ext}^n_{\mathcal{R}^\uparrow}(jP(b), W) = 0$ for $n \geqslant 1$ and $b \in \mathbf{B}$, which follows from Lemma 3.38(1).

Finally, for (4), the argument from the proof of Theorem 3.20(4) cannot be used since it depends on $\mathcal{R}^\uparrow$ being essentially finite Abelian. So we provide an alternate argument. Take a short exact sequence $0 \to U \to V \to W \to 0$ in $\nabla_\varepsilon(\mathcal{R}^\uparrow)$. Applying j_*, we get $0 \to j_*U \to j_*V \to j_*W$, and just need to show that the final morphism here is an epimorphism. This follows because, by (3) and Theorem 3.41, j_*U, j_*V and j_*W all have ∇_ε-flags such that $(j_*V : \nabla_\varepsilon(b)) = (j_*U : \nabla_\varepsilon(b)) + (j_*W : \nabla_\varepsilon(b))$ for all $b \in \mathbf{B}$. □

The reader should have no difficulty in transporting Lemma 3.22 and Corollaries 3.23–3.24 to the upper finite setting.

3.4. **Shared lemmas for §§3.2–3.3.** In this subsection, we prove a series of lemmas needed in both §3.2 and in §3.3. Let $\mathcal{R}$ be a category which is *either* essentially finite Abelian (§3.2) *or* Schurian (§3.3). Also let $\rho : \mathbf{B} \to \Lambda$ be a stratification, and $\varepsilon : \Lambda \to \{\pm\}$ be a sign function. In the Schurian case, we assume that the poset Λ is upper finite. In both cases, we assume that property $(\widehat{P\Delta_\varepsilon})$ holds.

Lemma 3.45. *We have that* $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), \Delta_\varepsilon(c)) = 0$ *for* $b, c \in \mathbf{B}$ *such that* $\rho(b) \nleqslant \rho(c)$.

Proof. Using the projective modules P_b given by the assumed property $(\widehat{P\Delta_\varepsilon})$, we can construct the first terms of a projective resolution of $\Delta_\varepsilon(b)$ in the form

$$Q \to \bigoplus_{\substack{a \in \mathbf{B} \\ \rho(a) \geqslant \rho(b)}} P_a^{\oplus n_a} \to P_b \to \Delta_\varepsilon(b) \to 0 \tag{3.8}$$

for some $n_a \geqslant 0$. Now apply $\mathrm{Hom}_{\mathcal{R}}(-, \Delta_\varepsilon(c))$ to get that $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), \Delta_\varepsilon(c))$ is the homology of the complex

$$\mathrm{Hom}_{\mathcal{R}}(P_b, \Delta_\varepsilon(c)) \to \mathrm{Hom}_{\mathcal{R}}\Big(\bigoplus_{\substack{a \in \mathbf{B} \\ \rho(a) \geqslant \rho(b)}} P_a^{\oplus n_a}, \Delta_\varepsilon(c)\Big) \to \mathrm{Hom}_{\mathcal{R}}(Q, \Delta_\varepsilon(c)).$$

The middle term of this already vanishes as $[\Delta_\varepsilon(c) : L(a)] \neq 0 \Rightarrow \rho(a) \leqslant \rho(c)$. □

Lemma 3.46. *Let* $\Lambda^\downarrow$ *be a lower set in* Λ *and* $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$. *Let* $i : \mathcal{R}^\downarrow \to \mathcal{R}$ *be the corresponding Serre subcategory of* $\mathcal{R}$ *equipped with the induced stratification.*

(1) The standard, proper standard and indecomposable projective objects of $\mathcal{R}^\downarrow$ *are the objects* $\Delta(b)$, $\bar{\Delta}(b)$ *and* $i^*P(b)$ *for* $b \in \mathbf{B}^\downarrow$.

(2) The object i^*P_b *is zero unless* $b \in \mathbf{B}^\downarrow$, *in which case it is a projective object admitting a* Δ_ε*-flag with top section* $\Delta_\varepsilon(b)$ *and other sections of the form* $\Delta_\varepsilon(c)$ *for* $c \in \mathbf{B}^\downarrow$ *with* $\rho(c) \geqslant \rho(b)$. *In particular, this shows that* $(\widehat{P\Delta_\varepsilon})$ *holds in* $\mathcal{R}^\downarrow$.

(3) $(\mathbb{L}_n i^*)V = 0$ *for* $V \in \Delta_\varepsilon(\mathcal{R})$ *and* $n \geqslant 1$.

(4) $\mathrm{Ext}^n_{\mathcal{R}}(V, iW) \cong \mathrm{Ext}^n_{\mathcal{R}^\downarrow}(i^*V, W)$ *for* $V \in \Delta_\varepsilon(\mathcal{R})$, $W \in \mathcal{R}^\downarrow$ *and* $n \geqslant 0$.

Proof. (1) For projectives, this follows from the usual adjunction properties. This also shows that i^*P_b is projective, as needed for (2). For standard and proper standard objects, just note that the standardization functors for $\mathcal{R}^\downarrow$ are some of the ones for $\mathcal{R}$.

(2) Consider a Δ_ε-flag of P_b. Using Lemma 3.45, we can rearrange this filtration if necessary so that all of the sections $\Delta_\varepsilon(c)$ with $c \in \mathbf{B}^\downarrow$ appear above the sections $\Delta_\varepsilon(d)$ with $d \in \mathbf{B}\backslash\mathbf{B}^\downarrow$. So there exists a short exact sequence $0 \to K \to P_b \to Q \to 0$ in which Q has a finite filtration with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}^\downarrow$ with $\rho(c) \geqslant \rho(b)$, and K has a finite filtration with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}\backslash\mathbf{B}^\downarrow$. It follows easily that i^*P_b is isomorphic to Q, so it has the appropriate filtration.

(3) It suffices to show that $(\mathbb{L}_n i^*)\Delta_\varepsilon(b) = 0$ for all $b \in \mathbf{B}$ and $n > 0$. Take a short exact sequence $0 \to K \to P_b \to \Delta_\varepsilon(b) \to 0$ and apply i^* to obtain a long exact sequence

$$0 \longrightarrow (\mathbb{L}_1 i^*)\Delta_\varepsilon(b) \longrightarrow i^*K \longrightarrow i^*P_b \longrightarrow i^*\Delta_\varepsilon(b) \longrightarrow 0$$

and isomorphisms $(\mathbb{L}_{n+1} i^*)\Delta_\varepsilon(b) \cong (\mathbb{L}_n i^*)K$ for $n > 0$. We claim that $(\mathbb{L}_1 i^*)\Delta_\varepsilon(b) = 0$. We know that K has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for c with $\rho(c) \geqslant \rho(b)$. We use Lemma 3.45 to order the Δ_ε-flag of K so that it yields a short exact sequence $0 \to L \to K \to Q \to 0$ in which Q has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}^\downarrow$, and L has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}\backslash\mathbf{B}^\downarrow$. It follows that $i^*K = Q$ and there is a short exact sequence $0 \to i^*K \to i^*P_b \to \Delta_\varepsilon(b) \to 0$. Comparing with the long exact sequence, we deduce that $(\mathbb{L}_1 i^*)\Delta_\varepsilon(b) = 0$. Finally some degree shifting using the isomorphisms $(\mathbb{L}_{n+1} i^*)\Delta_\varepsilon(b) \cong (\mathbb{L}_n i^*)K$ gives that $(\mathbb{L}_n i^*)\Delta_\varepsilon(b) = 0$ for $n > 1$ too.

(4) By the adjunction, we have that $\mathrm{Hom}_{\mathcal{R}}(-, iW) \cong \mathrm{Hom}_{\mathcal{R}^\downarrow}(-, W) \circ i^*$, i.e., the result holds when $n = 0$. Also i^* sends projectives to projectives as it is left adjoint to an exact functor. Now the result for $n > 0$ follows by a standard Grothendieck spectral sequence argument; the spectral sequence degenerates due to (3). □

Lemma 3.47. *Suppose that* $\lambda \in \Lambda$ *is maximal and* $b \in \mathbf{B}_\lambda$. *Then* $P(b) \cong \Delta(b)$ *and* $I(b) \cong \nabla(b)$.

Proof. Lemma 3.3 shows that $\Delta(b) \cong i^*_{\leqslant\lambda} P(b)$ and $\nabla(b) \cong i^!_{\leqslant\lambda} I(b)$. To complete the proof for $P(b)$, it remains to observe that $P(b)$ belongs to $\mathcal{R}_{\leqslant\lambda}$, so $i^*_{\leqslant\lambda} P(b) = P(b)$. This follows from $\widehat{P\Delta_\varepsilon}$: the object P_b belongs to $\mathcal{R}_{\leqslant\lambda}$ due to the maximality of λ and $P(b)$ is a summand of it.

The proof for $I(b)$ needs a different approach. From $\nabla(b) \cong i^!_{\leqslant\lambda} I(b)$, we deduce that there is a short exact sequence

$$0 \to \nabla(b) \to I(b) \to Q \to 0 \tag{3.9}$$

with $i^!_{\leqslant\lambda} Q = 0$. Now we must show that $Q = 0$. Take any $a \in \mathbf{B}$. We have that

$$\operatorname{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(a), \nabla(b)) = 0 \tag{3.10}$$

for $n > 0$. This follows using Lemma 3.46(4): it shows that $\operatorname{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(a), \nabla(b)) \cong \operatorname{Ext}^n_{\mathcal{R}_{\leqslant\lambda}}(i^*_{\leqslant\lambda}\Delta_\varepsilon(a), \nabla(b))$ which is zero as $\nabla(b)$ is injective in $\mathcal{R}_{\leqslant\lambda}$. Using this with $n = 1$, we see on applying the functor $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), -)$ to (3.9) that we get an exact sequence

$$\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), I(b)) \to \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), Q) \to 0.$$

If $\rho(a) \neq \lambda$ then the first term of this is zero due to the maximality of λ, showing that $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), Q) = 0$. If $\rho(a) = \lambda$ then we get that $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), Q) = 0$ instead because $i^!_{\leqslant\lambda} Q = 0$. Thus, we have shown that $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), Q) = 0$. This completes the proof in the essentially finite Abelian case because it implies that $\operatorname{soc} Q = 0$.

In the Schurian case, we need to argue a little further because Q need not be finitely cogenerated, so can have zero socle even when it is itself non-zero. Considering the long exact obtained by applying $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), -)$ to (3.9) and using (3.10) with $n = 2$, we get that $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), Q) = 0$. Now we observe that the properties $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), Q) = 0 = \operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), Q)$ for all $a \in \mathbf{B}$ do imply that V is zero. Indeed, we have that $\operatorname{Hom}_{\mathcal{R}}(P, Q) = \operatorname{Ext}^1_{\mathcal{R}}(P, Q) = 0$ for any $P \in \mathcal{R}$ with a Δ_ε-flag. This follows using induction on the length of the flag plus the long exact sequence. Since P_b has a Δ_ε-flag by the hypothesis ($\widehat{P\Delta_\varepsilon}$) and $P(b)$ is a summand of it, we deduce that $\operatorname{Hom}_{\mathcal{R}}(P(b), Q) = 0$ for all $b \in \mathbf{B}$, which certainly implies that $Q = 0$. $\square$

Lemma 3.48. *Assume that $\lambda \in \Lambda$ is maximal and $\varepsilon(\lambda) = +$. For any $V \in \mathcal{R}_\lambda$ and $b \in \mathbf{B}$, we have that* $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), j^\lambda_* V) = 0$.

Proof. If $b \in \mathbf{B}_\lambda$ then $\Delta_\varepsilon(b)$ is projective in $\mathcal{R}_{\leqslant\lambda}$ by Lemma 3.47, so we get the Ext^1-vanishing in this case. For the remainder of the proof, suppose that $b \notin \mathbf{B}_\lambda$. Let I be an injective hull of V in $\mathcal{R}_\lambda$. Applying j^λ_* to a short exact sequence $0 \to V \to I \to Q \to 0$, we get an exact sequence $0 \to j^\lambda_* V \to j^\lambda_* I \to j^\lambda_* Q$. By properties of adjunctions, $j^\lambda_* Q$ is finitely cogenerated and all constituents of its socle are $L(c)$ for $c \in \mathbf{B}_\lambda$. The same is true for $j^\lambda_* I / j^\lambda_* V$ since it embeds into $j^\lambda_* Q$. We deduce that $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), j^\lambda_* I / j^\lambda_* V) = 0$.

Now take an extension $0 \to j^\lambda_* V \to E \to \Delta_\varepsilon(b) \to 0$. Since $j^\lambda_* I$ is injective, we can find morphisms f and g making the following diagram with exact rows commute:

$$\begin{tikzcd}
0 \arrow[r] & j^\lambda_* V \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "f"] & \Delta_\varepsilon(b) \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & j^\lambda_* V \arrow[r] & j^\lambda_* I \arrow[r] & j^\lambda_* I / j^\lambda_* V \arrow[r] & 0.
\end{tikzcd}$$

The previous paragraph implies that $g = 0$. Hence, $\operatorname{im} f \subseteq j^\lambda_* V$. Thus, f splits the top short exact sequence, and we have shown that $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), j^\lambda_* V) = 0$. $\square$

Lemma 3.49. *For $b, c \in \mathbf{B}$ and $n \geqslant 0$, we have that* $\dim \operatorname{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), \nabla_\varepsilon(c)) = \delta_{b,c}\delta_{n,0}$.

Proof. The case $n = 0$ follows from (3.2), so assume that $n > 0$. Suppose that $b \in \mathbf{B}_\lambda$ and $c \in \mathbf{B}_\mu$. By Lemma 3.46(4), we have that

$$\operatorname{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), \nabla_\varepsilon(c)) \cong \operatorname{Ext}^n_{\mathcal{R}_{\leqslant\mu}}(i^*_{\leqslant\mu}\Delta_\varepsilon(b), \nabla_\varepsilon(c)).$$

If $\lambda \not\leqslant \mu$ then $i^*_{\leqslant\mu}\Delta_\varepsilon(b) = 0$ and we get the desired vanishing. Now assume that $\lambda \leqslant \mu$, when we may identify $i^*_{\leqslant\mu}\Delta_\varepsilon(b) = \Delta_\varepsilon(b)$. If $\varepsilon(\mu) = -$ then $\nabla_\varepsilon(c) = \nabla(c)$, and the result follows since $\nabla(c)$ is injective in $\mathcal{R}_{\leqslant\mu}$ by Lemma 3.3. So we may assume also that $\varepsilon(\mu) = +$. If $\lambda = \mu$ then $\Delta(b)$ is projective in $\mathcal{R}_{\leqslant\mu}$ by the same lemma, so again we are done. Finally, we are reduced to $\lambda < \mu$ and $\varepsilon(\mu) = +$, and need to show that $\mathrm{Ext}^n_{\mathcal{R}_{\leqslant\mu}}(\Delta_\varepsilon(b), \bar\nabla(c)) = 0$ for $n > 0$. If $n = 1$, we get the desired conclusion from Lemma 3.48 applied in the subcategory $\mathcal{R}_{\leqslant\mu}$ (allowed due to Lemma 3.46(2)). Then for $n \geqslant 2$ we use a standard degree shifting argument: let $P := i^*_{\leqslant\mu}P_b$. By Lemma 3.46(2), P is projective in $\mathcal{R}_{\leqslant\mu}$, and there is a short exact sequence $0 \to K \to P \to \Delta_\varepsilon(b) \to 0$ such that K has a Δ_ε-flag with sections $\Delta_\varepsilon(a)$ for a with $\lambda \leqslant \rho(a) \leqslant \mu$. Applying $\mathrm{Hom}_{\mathcal{R}_{\leqslant\mu}}(-, \bar\nabla(c))$ we obtain $\mathrm{Ext}^n_{\mathcal{R}_{\leqslant\mu}}(\Delta_\varepsilon(b), \bar\nabla(c)) \cong \mathrm{Ext}^{n-1}_{\mathcal{R}_{\leqslant\mu}}(K, \bar\nabla(c))$, which is zero by induction. □

Lemma 3.50. *Let $\Lambda^\uparrow$ be an upper set in Λ and $\mathbf{B}^\uparrow := \rho^{-1}(\Lambda^\uparrow)$. Let $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the corresponding Serre quotient category of $\mathcal{R}$ equipped with the induced stratification.*

1. *For $b \in \mathbf{B}^\uparrow$, the objects $P^\uparrow(b)$, $I^\uparrow(b)$, $\Delta^\uparrow(b)$, $\bar\Delta^\uparrow(b)$, $\nabla^\uparrow(b)$ and $\bar\nabla^\uparrow(b)$ in $\mathcal{R}^\uparrow$ are the images under j of the corresponding objects of $\mathcal{R}$. Moreover, we have that $j_!\Delta^\uparrow(b) \cong \Delta(b)$, $j_!\bar\Delta^\uparrow(b) \cong \bar\Delta(b)$, $j_!P^\uparrow(b) \cong P(b)$ and $j_*\nabla^\uparrow(b) \cong \nabla(b)$, $j_*\bar\nabla^\uparrow(b) \cong \bar\nabla(b)$, $j_*I^\uparrow(b) \cong I(b)$.*
2. *For any $b \in \mathbf{B}$, the object jP_b has a Δ_ε-flag with top section $\Delta^\uparrow_\varepsilon(b)$ and other sections of the form $\Delta^\uparrow_\varepsilon(c)$ for $c \in \mathbf{B}^\uparrow$ with $\rho(c) \geqslant \rho(b)$. In particular, this show that $(\widehat{P\Delta_\varepsilon})$ holds in $\mathcal{R}^\uparrow$.*
3. *$\mathrm{Ext}^n_{\mathcal{R}}(V, j_*W) \cong \mathrm{Ext}^n_{\mathcal{R}^\uparrow}(jV, W)$ for $V \in \mathcal{R}$, $W \in \nabla_\varepsilon(\mathcal{R}^\uparrow)$ and $n \geqslant 0$.*

Proof. (1) By Lemma 2.22, $P^\uparrow(b) = jP(b)$ for each $b \in \mathbf{B}^\uparrow$. Now take $b \in \mathbf{B}_\lambda$ for $\lambda \in \Lambda^\uparrow$. Let $j^\lambda : \mathcal{R}_{\leqslant\lambda} \to \mathcal{R}_\lambda$ be the quotient functor as usual, and denote the analogous functor for $\mathcal{R}^\uparrow$ by $k^\lambda : \mathcal{R}^\uparrow_{\leqslant\lambda} \to \mathcal{R}^\uparrow_\lambda$. The universal property of quotient category gives us an exact functor $\bar{j} : \mathcal{R}_\lambda \to \mathcal{R}^\uparrow_\lambda$ making the diagram

$$\begin{array}{ccc} \mathcal{R}_{\leqslant\lambda} & \xrightarrow{j} & \mathcal{R}^\uparrow_{\leqslant\lambda} \\ {\scriptstyle j^\lambda}\downarrow & & \downarrow{\scriptstyle k^\lambda} \\ \mathcal{R}_\lambda & \xrightarrow[\bar{j}]{} & \mathcal{R}^\uparrow_\lambda \end{array}$$

commute. In fact, $\bar{j}$ is an equivalence of categories because it sends the indecomposable projective $j^\lambda P(b)$ in $\mathcal{R}_\lambda$ to the indecomposable projective $k^\lambda P^\uparrow(b)$ in $\mathcal{R}^\uparrow_\lambda$ for each $b \in \mathbf{B}_\lambda$. We deduce that there is an isomorphism of functors $j_! \circ k^\lambda_! \circ \bar{j} \cong j^\lambda_!$. Applying this to $P_\lambda(b)$ and to $L_\lambda(b)$ gives that $j_!\Delta^\uparrow(b) \cong \Delta(b)$ and $j_!\bar\Delta^\uparrow(b) \cong \bar\Delta(b)$. Also by adjunction properties we have that $j_!P^\uparrow(b) \cong P(b)$. Similarly, applying it to $I_\lambda(b)$ and to $L_\lambda(b)$ gives that $j_*\nabla^\uparrow(b) \cong \nabla(b)$ and $j_*\bar\nabla^\uparrow(b) \cong \bar\nabla(b)$. Also by adjunction properties we have that $j_*I^\uparrow(b) \cong I(b)$. It just remains to apply j to the isomorphisms constructed thus far and use $j \circ j_* \cong \mathrm{Id}_{\mathcal{R}^\uparrow} \cong j \circ j_!$.

(2) This follows from (1) and the exactness of j, using also that $j\Delta_\varepsilon(b) = 0$ if $b \notin \mathbf{B}^\uparrow$.

(3) The adjunction gives an isomorphism $\mathrm{Hom}_{\mathcal{R}}(-, j_*W) \cong \mathrm{Hom}_{\mathcal{R}^\uparrow}(-, W) \circ j$. This proves the result when $n = 0$. For $n > 0$, the functor j is exact, so all that remains is to check that j sends projectives to objects that are acyclic for $\mathrm{Hom}_{\mathcal{R}^\uparrow}(-, W)$. By (2), the functor j sends projectives in $\mathcal{R}$ to objects with a Δ_ε-flag. It remains to note that $\mathrm{Ext}^1_{\mathcal{R}^\uparrow}(X, W) = 0$ for $X \in \Delta_\varepsilon(\mathcal{R}^\uparrow), W \in \nabla_\varepsilon(\mathcal{R}^\uparrow)$. This follows from the analog of Lemma 3.49 for $\mathcal{R}^\uparrow$, which is valid due to (2). □

3.5. **Lower finite ε-stratified categories.** In this subsection, Λ is a lower finite poset, $\mathcal{R}$ is a locally finite Abelian category equipped with a stratification $\rho : \mathbf{B} \to \Lambda$, and we fix a sign function $\varepsilon : \Lambda \to \{\pm\}$. For $b \in \mathbf{B}$, define $\Delta(b), \bar\Delta(b), \nabla(b)$ and $\bar\nabla(b)$ as in (1.1), and recall the notation (1.2). Also let $I(b)$ be an injective hull of $L(b)$ in $\mathrm{Ind}(\mathcal{R})$.

Lemma 3.51. *For any $b \in \mathbf{B}$, the objects $\Delta(b), \bar{\Delta}(b), \nabla(b)$ and $\bar{\nabla}(b)$ are of finite length, as is $i^!_{\leqslant\lambda} I(b)$ for any $\lambda \in \Lambda$.*

Proof. For $\lambda \in \Lambda$, the set $\mathbf{B}_{\leqslant\lambda}$ is finite. Given this, Lemma 2.17 together with axiom (ρ4) from Definition 3.1 imply that $\mathcal{R}_{\leqslant\lambda}$ is a finite Abelian category. Since $\Delta(b), \bar{\Delta}(b), \nabla(b)$ and $\bar{\nabla}(b)$ are objects in $\mathcal{R}_{\leqslant\lambda}$ for $\lambda := \rho(b)$, they have finite length. Also, by properties of adjunctions, $i^!_{\leqslant\lambda} I(b)$ is zero unless $b \in \mathbf{B}_{\leqslant\lambda}$, and it is an injective hull of $L(b)$ in $\mathcal{R}_{\leqslant\lambda}$ if $b \in \mathbf{B}_{\leqslant\lambda}$. So it is also of finite length. □

We need to consider another sort of infinite good filtration in objects of $\mathrm{Ind}(\mathcal{R})$, which we call *ascending ∇_ε-flag*. Usually (e.g., if Λ is countable), it is sufficient to restrict attention to ascending ∇_ε-flags that are given simply by a chain of subobjects $0 = V_0 < V_1 < V_2 < \cdots$ such that $V = \sum_{n\in\mathbb{N}} V_n$ and $V_m/V_{m-1} \cong \nabla_\varepsilon(b_m)$ for some $b_m \in \mathbf{B}$. Here is the general definition which avoids this restriction.

Definition 3.52. An *ascending ∇_ε-flag* in an object $V \in \mathrm{Ind}(\mathcal{R})$ is the data of a direct system $(V_\omega)_{\omega\in\Omega}$ of subobjects of V such that $V = \sum_{\omega\in\Omega} V_\omega$ and each V_ω has a ∇_ε-flag. An *ascending ∇-flag* (resp., $\bar{\nabla}$-flag) means an ascending ∇_ε-flag in the special case that $\varepsilon = -$ (resp., $\varepsilon = +$).

Now we can formulate the main definition in the lower finite setting. The way we are about to formulate this is different from the way it was explained in the introduction; the equivalence of the two formulations is established in Theorem 3.63 below. In the definition, we will refer to the following two properties.

($\widehat{I\nabla^{\mathrm{asc}}_\varepsilon}$) For every $b \in \mathbf{B}$, there exists a finitely cogenerated injective object $I_b \in \mathrm{Ind}(\mathcal{R})$ admitting an ascending ∇_ε-flag $(V_\omega)_{\omega\in\Omega}$ in which every V_ω has a ∇_ε-flag with $\nabla_\varepsilon(b)$ at the bottom and other sections $\nabla_\varepsilon(c)$ for $c \in \mathbf{B}$ with $\rho(c) \geqslant \rho(b)$.

($\widehat{I\nabla^{\mathrm{asc}}}$) For every $\lambda \in \Lambda$, there exists a finitely cogenerated injective object $I_\lambda \in \mathrm{Ind}(\mathcal{R})$ admitting an ascending ∇-flag $(V_\omega)_{\omega\in\Omega}$ in which every V_ω has a ∇-flag with $\nabla(\lambda)$ at the bottom and other sections $\nabla(\mu)$ for $\mu \in \Lambda$ with $\mu \geqslant \lambda$.

Definition 3.53. Let $\mathcal{R}$ be a finite Abelian category equipped with a stratification $\rho : \mathbf{B} \to \Lambda$ as in Definition 3.1 such that the poset Λ is lower finite.

(εS) We say that $\mathcal{R}$ is a *lower finite ε-stratified category* if property ($\widehat{I\nabla^{\mathrm{asc}}_\varepsilon}$) holds for some given choice of sign function $\varepsilon : \Lambda \to \{\pm\}$.

(FS) We say that $\mathcal{R}$ is a *lower finite fully stratified category* if property ($\widehat{I\nabla^{\mathrm{asc}}_\varepsilon}$) holds for all choices of sign function $\varepsilon : \Lambda \to \{\pm\}$.

(εHW) We say that $\mathcal{R}$ is a *lower finite ε-highest weight category* if the stratification function ρ is a bijection and the property ($\widehat{I\nabla^{\mathrm{asc}}_\varepsilon}$) holds for some given choice of sign function $\varepsilon : \Lambda \to \{\pm\}$.

(SHW) We say that $\mathcal{R}$ is a *lower finite signed highest weight category* if the stratification function ρ is a bijection and the property ($\widehat{I\nabla^{\mathrm{asc}}_\varepsilon}$) holds for all choices of sign function.

(HW) We say that $\mathcal{R}$ is a *lower finite highest weight category* if all strata are simple (cf. Lemma 3.6) and the property ($\widehat{I\nabla^{\mathrm{asc}}}$) holds.

Our next goal is adapt Theorem 3.19 to such categories.

Lemma 3.54. *Suppose $\mathcal{R}$ is a lower finite ε-stratified category and take $b, c \in \mathbf{B}$ with $\rho(b) \nleqslant \rho(c)$. Then we have that Then $\mathrm{Ext}^1_{\mathcal{R}}(\nabla_\varepsilon(c), \nabla_\varepsilon(b)) = 0$.*

Proof. Since I_b has $L(b)$ in its socle, there is an injective resolution $0 \to \nabla_\varepsilon(b) \to I_b \to J \to \cdots$ in $\mathrm{Ind}(\mathcal{R})$. Let $(V_\omega)_{\omega\in\Omega}$ be an ascending ∇_ε-flag in I_b in which every V_ω has a ∇_ε-flag with $\nabla_\varepsilon(b)$ at the bottom and other sections $\nabla_\varepsilon(a)$ for a with $\rho(a) \geqslant \rho(b)$. Then $\nabla_\varepsilon(b) \hookrightarrow I_b$ too. Moreover, $I_b/\nabla_\varepsilon(b) = \sum_{\omega\in\Omega}(V_\omega/\nabla_\varepsilon(b))$, so its socle only involves constituents $L(a)$ with $\rho(a) \geqslant \rho(b)$. So J is a direct sum of I_a with $\rho(a) \geqslant \rho(b)$. The

Ext^1-vanishing now follows on applying $\mathrm{Hom}_{\mathcal{R}}(\nabla_\varepsilon(c), -)$ to the resolution and taking homology. □

Here is the first half of the analog of Theorem 3.19.

Theorem 3.55. *Suppose $\mathcal{R}$ is a lower finite ε-stratified category. Let $\Lambda^\downarrow$ be a finite lower set, $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$, and $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the corresponding Serre subcategory of $\mathcal{R}$ with the induced stratification. Then $\mathcal{R}^\downarrow$ is a finite ε-stratified category with distinguished objects $L^\downarrow(b) \cong L(b)$, $I^\downarrow(b) \cong i^! I(b)$, $\Delta^\downarrow(b) \cong \Delta(b)$, $\bar{\Delta}^\downarrow(b) \cong \bar{\Delta}(b)$, $\nabla^\downarrow(b) \cong \nabla(b)$ and $\bar{\nabla}^\uparrow(b) \cong \bar{\nabla}(b)$ for $b \in \mathbf{B}^\downarrow$.*

Proof. The identification of the distinguished objects of $\mathcal{R}^\downarrow$ is straightforward. In particular, the objects $\nabla_\varepsilon(b)$ in $\mathcal{R}^\downarrow$ are just the same as the ones in $\mathcal{R}$ indexed by $b \in \mathbf{B}^\downarrow$, while the indecomposable injectives in $\mathrm{Ind}(\mathcal{R}^\downarrow)$ are the objects $i^! I(b)$ for $b \in \mathbf{B}^\downarrow$. To complete the proof, we need to prove the following for each $b \in \mathbf{B}^\downarrow$:

(1) $i^! I_b$ has finite length, i.e., it actually lies in $\mathcal{R}^\downarrow$;
(2) $i^! I_b$ satisfies the property $(\widehat{I\nabla_\varepsilon})$.

The first of these implies that $\mathcal{R}^\downarrow$ is a locally finite Abelian category with finitely many isomorphism classes of irreducible objects and with enough injectives. Hence, $\mathcal{R}^\downarrow$ is a finite Abelian category by the discussion after Corollary 2.16. Then (2) checks that it is ε-stratified as in Definition 3.9.

Since I_b is finitely cogenerated, it is a finite direct sum of indecomposable injectives, so Lemma 3.51 implies that $i^!_{\leqslant\lambda} I_b$ has finite length. This proves (1).

For (2), take $b \in \mathbf{B}^\downarrow$. Let $(V_\omega)_{\omega\in\Omega}$ be an ascending ∇_ε-flag in I_b as in Definition 3.53, and fix also a ∇_ε-flag in each V_ω with bottom section $\nabla_\varepsilon(b)$ and other sections $\nabla_\varepsilon(c)$ for c with $\rho(c) \geqslant \rho(b)$. For $\lambda \in \Lambda^\downarrow$, let $m(\lambda, \omega)$ be the sum of the multiplicities of the objects $\nabla_\varepsilon(c)$ $(c \in \mathbf{B}_\lambda)$ as sections of the ∇_ε-flag of V_ω. Let $m(\lambda) := \sup\{m(\lambda,\omega) \mid \omega \in \Omega\}$. We claim that $m(\lambda) < \infty$. To see this, suppose for a contradiction that it is not the case. Choose λ minimal so that $m(\lambda) = \infty$. Then for any $n \in \mathbb{N}$, we can find $\omega \in \Omega$ such that the sum of the multiplicities of the objects $\nabla_\varepsilon(c)$ $(c \in \mathbf{B}_\lambda)$ as sections of the ∇_ε-flag of V_ω is greater than n. Using Lemma 3.51 and the minimality of λ, we can rearrange this flag if necessary so that the only other sections appearing *below* these ones are of the form $\nabla_\varepsilon(d)$ for $d \in \mathbf{B}_{<\lambda}$. Then we deduce that $\sum_{c\in\mathbf{B}_\lambda} [i^!_{\leqslant\lambda} V_\omega : L(c)] > n$. Since n can be chosen to be arbitrarily large, this contradicts (1).

Then, using Lemma 3.54 again, we rearrange the ∇_ε-flag in each V_ω if necessary to deduce that there are short exact sequences $0 \to V'_\omega \to V_\omega \to V''_\omega \to 0$ such that V'_ω has a ∇_ε-flag with sections $\nabla_\varepsilon(c)$ for c with $\rho(b) \leqslant \rho(c) \in \Lambda^\downarrow$ and V''_ω has a ∇_ε-flag with sections $\nabla_\varepsilon(d)$ for d with $\rho(d) \notin \Lambda^\downarrow$. Moreover, the finiteness property established in the previous paragraph means that the length of the ∇_ε-flag of V'_ω is bounded by $\sum_{\lambda\in\Lambda^\downarrow} m(\lambda)$ independent of $\omega \in \Omega$. Consequently, we can find some sufficiently large ω in the directed set Ω so that $V'_\upsilon = V'_\omega$ for all $\upsilon > \omega$. Then $i^! I_b = V'_\omega$ for this ω. This proves (2). □

Corollary 3.56. *In a lower finite ε-stratified category $\mathcal{R}$, we have for each $b, c \in \mathbf{B}$ that*

$$\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), \nabla_\varepsilon(c)) = 0.$$

Proof. Given b, c, let $\Lambda^\downarrow$ be the finite lower set of Λ that they generate. Let $\mathcal{R}^\downarrow$ be the corresponding finite ε-stratified subcategory of $\mathcal{R}$. Since $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), \nabla_\varepsilon(c)) = \mathrm{Ext}^1_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), \nabla_\varepsilon(c))$, it is zero thanks to Theorem 3.13. □

Suppose that $V \in \mathrm{Ind}(\mathcal{R})$ has an ascending ∇_ε-flag $(V_\omega)_{\omega\in\Omega}$. Corollary 3.56 implies as usual that the multiplicity $(V_\omega : \nabla_\varepsilon(b))$ of $\nabla_\varepsilon(b)$ as a section of an ∇_ε-flag of V_ω is well-defined for every ω. Since $\Delta_\varepsilon(b)$ is finitely generated, we have that

$$\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), \varinjlim V_\omega) \cong \varinjlim \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V_\omega).$$

We deduce that

$$(V : \nabla_\varepsilon(b)) := \sup\left\{(V_\omega : \nabla_\varepsilon(b)) \,\middle|\, \omega \in \Omega\right\} = \dim \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) \in \mathbb{N} \cup \{\infty\}, \tag{3.11}$$

which is independent of the particular choice of ∇_ε-flag. Having made sense of these multiplicities, we let $\nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ be the exact subcategory consisting of all objects V that possess an ascending ∇_ε-flag such that $(V : \nabla_\varepsilon(b)) < \infty$ for all $b \in \mathbf{B}$. For example, the object I_b from Definition 3.53 belongs to $\nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ since soc I_b is of finite length.

Corollary 3.57. *Assume that $\mathcal{R}$ is a lower finite ε-stratified category. For $V \in \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ and $b \in \mathbf{B}$, we have that $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0$.*

Proof. Let $(V_\omega)_{\omega\in\Omega}$ be an ascending ∇_ε-flag in V. Take an extension $V \hookrightarrow E \twoheadrightarrow \Delta_\varepsilon(b)$. We can find a subobject E_1 of E of finite length such that $V + E_1 = V + E$; this follows easily by induction on the length of $\Delta_\varepsilon(b)$ as explained at the start of the proof of [CPS1, Lemma 3.8(a)]. Since $V \cap E_1$ is of finite length, there exists $\omega \in \Omega$ with $V \cap E_1 \subseteq V_\omega$. Then we have that $V \cap E_1 = V_\omega \cap E_1$ and

$$(V_\omega + E_1)/V_\omega \cong E_1/V_\omega \cap E_1 = E_1/V \cap E_1 \cong (V + E_1)/V = (V + E)/V \cong \Delta_\varepsilon(b).$$

Thus, there is a short exact sequence $0 \to V_\omega \to V_\omega + E_1 \to \Delta_\varepsilon(b) \to 0$. By Corollary 3.56, this splits, so we can find a subobject $E_2 \cong \Delta_\varepsilon(b)$ of $V_\omega + E_1$ such that $V_\omega + E_1 = V_\omega \oplus E_2$. Then $V + E = V + E_1 = V + V_\omega + E_1 = V + V_\omega + E_2 = V + E_2 = V \oplus E_2$, and our original short exact sequence splits. □

Corollary 3.58. *In the notation of Theorem 3.55, if $V \in \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ then $i^! V \in \nabla_\varepsilon(\mathcal{R}^\downarrow)$.*

Proof. Take a short exact sequence $0 \to i^! V \to V \to Q \to 0$. Note that

$$\operatorname{Hom}_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), i^! V) \cong \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V)$$

is finite-dimensional for each $b \in \mathbf{B}^\downarrow$. Since $\mathcal{R}^\downarrow$ is finite Abelian, it follows that $i^! V \in \mathcal{R}^\downarrow$ (rather than $\operatorname{Ind}(\mathcal{R}^\downarrow)$). Moreover, $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), Q) = 0$ for $b \in \mathbf{B}^\downarrow$. So, on applying $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), -)$ and considering the long exact sequence using Corollary 3.57, we get that $\operatorname{Ext}^1_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), i^! V) = \operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), i^! V) = 0$ for all $b \in \mathbf{B}^\downarrow$. Thus, by Theorem 3.13, we have that $i^! V \in \nabla_\varepsilon(\mathcal{R}^\downarrow)$. □

The following homological criterion for ascending ∇_ε-flags is similar to the homological criterion for good filtrations from [Jan1, Proposition II.4.16]. It generalizes Theorem 3.13.

Theorem 3.59. *Assume that $\mathcal{R}$ is a lower finite ε-stratified category. For $V \in \operatorname{Ind}(\mathcal{R})$ such that $\dim \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) < \infty$ for all $b \in \mathbf{B}$, the following are equivalent:*

(i) V has an ascending ∇_ε-flag;
(ii) $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0)$ for all $b \in \mathbf{B}$;
(iii) $\operatorname{Ext}^n_{\mathcal{R}}(\Delta_\varepsilon(b), V) = 0)$ for all $b \in \mathbf{B}$ and $n \geqslant 1$.

Proof. (ii)⇒(i): Let Ω be the directed set consisting of all finite lower sets in Λ. Take $\omega \in \Omega$. It is a finite lower set $\Lambda^\downarrow \subseteq \Lambda$, so we can associate a corresponding finite ε-stratified subcategory $\mathcal{R}^\downarrow$ as in Theorem 3.55. Letting $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the inclusion, we set $V_\omega := i^! V$. By Corollary 3.58, we have that $V_\omega \in \nabla_\varepsilon(\mathcal{R})$. So we have the required data $(V_\omega)_{\omega\in\Omega}$ of an ascending ∇_ε-flag in V. Finally, we let $V' := \sum_{\omega\in\Omega} V_\omega$ and complete the proof by showing that $V = V'$. To see this, apply $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), -)$ to the short exact sequence $0 \to V' \to V \to V/V' \to 0$ using Corollary 3.57, to deduce that there is a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V') \longrightarrow \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V) \longrightarrow \operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), V/V') \longrightarrow 0$$

for every $b \in \mathbf{B}$. But any homomorphism $\Delta_\varepsilon(b) \to V$ has image contained in V_ω for sufficiently large ω, hence, also in V'. Thus, the first morphism in this short exact sequence is an isomorphism, and $\operatorname{Hom}_{\mathcal{R}}(\Delta(b), V/V') = 0$ for all $b \in \mathbf{B}$. This implies that $V/V' = 0$ as required.

(i)⇒(ii): This is Corollary 3.57.

(iii)⇒(ii): Trivial.

(i)⇒(iii): This follows from Theorem 3.62(2). The forward reference causes no issues since we will only appeal to the equivalence of (i) and (ii) prior to that point. □

Corollary 3.60. *In a lower finite ε-stratified category, each indecomposable injective object $I(b)$ belongs to $\nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ and $(I(b) : \nabla_\varepsilon(c)) = [\Delta_\varepsilon(c) : L(b)]$ for each $b, c \in \mathbf{B}$.*

Proof. The first part is immediate from the homological criterion of Theorem 3.59. For the second part, we get from (3.11) that $(I(b) : \nabla_\varepsilon(c)) = \dim \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(c) : L(b)$. □

Corollary 3.61. *Let $0 \to U \to V \to W \to 0$ be a short exact sequence in a lower finite ε-stratified category. If $U, V \in \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ then $W \in \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ too. Moreover*

$$(V : \nabla_\varepsilon(b)) = (U : \nabla_\varepsilon(b)) + (W : \nabla_\varepsilon(b)).$$

The second half of our analog of Theorem 3.19 is as follows.

Theorem 3.62. *Suppose $\mathcal{R}$ is a lower finite ε-stratified category. Let $\Lambda^\downarrow$ be a finite lower set, $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda)$, and $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the inclusion of the corresponding finite ε-stratified subcategory of $\mathcal{R}$ as in Theorem 3.55.*

(1) $(\mathbb{R}^n i^!)V = 0$ for $n \geqslant 1$ and either $V \in \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ or $V \in \mathcal{R}^\downarrow$.
(2) $\mathrm{Ext}^n_{\mathcal{R}}(iV, W) \cong \mathrm{Ext}^n_{\mathcal{R}^\downarrow}(V, i^! W)$ for $V \in \mathcal{R}^\downarrow, W \in \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ and all $n \geqslant 0$.
(3) $\mathrm{Ext}^n_{\mathcal{R}}(iV, iW) \cong \mathrm{Ext}^n_{\mathcal{R}^\downarrow}(V, W)$ for $V, W \in \mathcal{R}^\downarrow$ and all $n \geqslant 0$.

Proof. (1) Assume first that $V \in \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$. Let I be an injective hull of $\mathrm{soc}\, V$. Note that I is of the form $\bigoplus_{a\in\mathbf{B}} I(a)^{\oplus n_a}$ for $0 \leqslant n_a \leqslant (V : \nabla_\varepsilon(a)) < \infty$. It has an ascending ∇_ε-flag by Corollary 3.60. Moreover, $\dim \mathrm{Hom}_{\mathcal{R}}(\Delta(b), I) = \sum_{a\in\mathbf{B}} n_a [\Delta_\varepsilon(b) : L(a)] < \infty$, hence, $I \in \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$.

Now consider the short exact sequence $0 \to V \to I \to Q \to 0$. By Corollary 3.61, we have that $Q \in \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ too. Applying $i^!$ and considering the long exact sequence, we see that to prove that $(\mathbb{R}^1 i^!)V = 0$ it suffices to show that the last morphism in the exact sequence $0 \to i^! V \to i^! I \to i^! Q$ is an epimorphism. Once that has been proved we can use degree shifting to establish the desired vanishing for all higher n; it is important for the induction step that we have already established that $Q \in \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ just like V.

To prove the surjectivity, look at $0 \to i^! I / i^! V \to i^! Q \to C \to 0$. Both $i^! I$ and $i^! V$ have ∇_ε-flags by Corollary 3.58. Hence, so does $i^! I / i^! V$, and on applying $\mathrm{Hom}_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), -)$ for $b \in \mathbf{B}^\downarrow$, we get a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), i^! I / i^! V) \longrightarrow \mathrm{Hom}_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), i^! Q) \longrightarrow \mathrm{Hom}_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), C) \longrightarrow 0.$$

The first space here has dimension

$$(i^! I : \nabla_\varepsilon(b)) - (i^! V : \nabla_\varepsilon(b)) = (I : \nabla_\varepsilon(b)) - (V : \nabla_\varepsilon(b)) = (Q : \nabla_\varepsilon(b)) = (i^! Q : \nabla_\varepsilon(b)),$$

which is the dimension of the second space. This shows that the first morphism is an isomorphism. Hence, $\mathrm{Hom}_{\mathcal{R}^\downarrow}(\Delta_\varepsilon(b), C) = 0$. This implies that $C = 0$ as required.

Finally let $V \in \mathcal{R}^\downarrow$. Then V is of finite length, so it suffices just to consider the case that $V = L(b)$ for $b \in \mathbf{B}^\downarrow$. Then we consider the short exact sequence $0 \to L(b) \to \nabla_\varepsilon(b) \to Q \to 0$. Applying $i^!$ and using the vanishing established so far gives $0 \to i^! L(b) \to i^! \nabla_\varepsilon(b) \to i^! Q \to (\mathbb{R}^1 i^!) L(b) \to 0$ and isomorphisms $(\mathbb{R}^n i^!) Q \cong (\mathbb{R}^{n+1} i^!) L(b)$ for $n \geqslant 1$. But $i^!$ is the identity on $L(b), \nabla_\varepsilon(b)$ and Q, so this immediately yields $(\mathbb{R}^1 i^!) L(b) = 0$, and then $(\mathbb{R}^n i^!) L(b) = 0$ for higher n by degree shifting.

(2), (3) These follow by the usual Grothendieck spectral sequence argument starting from the adjunction isomorphism $\mathrm{Hom}_{\mathcal{R}^\downarrow}(iV, -) \cong \mathrm{Hom}_{\mathcal{R}}(V, -) \circ i^!$. One just needs (1) and the observation that $i^!$ sends injectives to injectives. □

The following is an alternate characterization of "lower finite ε-stratified category." In the introduction, we used this as the definition.

Theorem 3.63. *Let $\mathcal{R}$ be a locally finite Abelian category. Suppose we are given a set $\mathbf{B}$ indexing representatives $\{L(b) \mid b \in \mathbf{B}\}$ for the isomorphism classes of irreducible objects in $\mathcal{R}$, a lower finite poset Λ, a function $\rho : \mathbf{B} \to \Lambda$ with finite fibers, and a sign function $\varepsilon : \Lambda \to \{\pm\}$. This is the data of a lower finite ε-stratified (resp., ε-highest weight) category in the sense of Definition 3.53 if and only if the Serre subcategory $\mathcal{R}^{\downarrow}$ of $\mathcal{R}$ associated to $\mathbf{B}^{\downarrow} := \rho^{-1}(\Lambda^{\downarrow})$ is a finite ε-stratified (resp., ε-highest weight) category with respect to the induced stratification for each finite lower set $\Lambda^{\downarrow} \subseteq \Lambda$.*

Proof. ($\Rightarrow$): This is the content of Theorem 3.55.

($\Leftarrow$): We have the data of a stratification as in Definition 3.1, with the axiom (ρ4) holding since we are given in particular that $\mathcal{R}_{\leqslant\lambda}$ is a finite Abelian category for each $\lambda \in \Lambda$. Then we can repeat the proof of the implication (ii)$\Rightarrow$(i) of Theorem 3.59 in the given category $\mathcal{R}$; the arguments given above only actually used the conclusions of Theorem 3.55 (which we are assuming) rather than Definition 3.53. Since $I(b)$ satisfies the homological criterion of Theorem 3.59(ii), we deduce that $I(b) \in \nabla_{\varepsilon}^{\mathrm{asc}}(\mathcal{R})$. Moreover, $(I(b) : \nabla_{\varepsilon}(c)) = [\Delta_{\varepsilon}(c) : L(b)]$ which is zero unless $\rho(c) \geqslant \rho(b)$. Hence, $\mathcal{R}$ is a lower finite ε-stratified category. □

Corollary 3.64. *$\mathcal{R}$ is a lower finite ε-stratified category if and only if $\mathcal{R}^{\mathrm{op}}$ is a lower finite $(-\varepsilon)$-stratified category.*

Proof. This follows from Theorem 3.63 plus Theorem 3.11. □

Corollary 3.65. *Let $\mathcal{R}$ be a locally finite Abelian category equipped with a stratification $\rho : \mathbf{B} \to \Lambda$ such that the poset Λ is lower finite. It is a lower finite fully stratified category if and only if each $I(b)$ has an ascending ∇-flag involving sections $\nabla(c)$ for c with $\rho(c) \geqslant \rho(b)$ and each $\nabla(b)$ has a $\bar{\nabla}$-flag with sections $\bar{\nabla}(c)$ for $\rho(c) = \rho(b)$.*

Proof. Use Theorem 3.63 and Lemma 3.22. □

Remark 3.66. The category $\mathcal{R}$ is a lower finite highest weight category in the sense of Definition 3.53 if and only if $\mathrm{Ind}(\mathcal{R})$ is a highest weight category in the original sense of [CPS1] with a weight poset that is lower finite.

4. Tilting modules and semi-infinite Ringel duality

We now develop the theory of tilting objects and Ringel duality. Even in the finite case, we are not aware of a complete exposition of these results in the existing literature in the general ε-stratified setting.

4.1. **Tilting objects in the finite and lower finite cases.** In this subsection, $\mathcal{R}$ will be a finite or lower finite ε-stratified category; see Definitions 3.9 and 3.53. By an *ε-tilting object*, we mean an object of the following full subcategory of $\mathcal{R}$:

$$\mathcal{T}ilt_{\varepsilon}(\mathcal{R}) := \Delta_{\varepsilon}(\mathcal{R}) \cap \nabla_{\varepsilon}(\mathcal{R}). \tag{4.1}$$

The following shows that the additive subcategory $\mathcal{T}ilt_{\varepsilon}(\mathcal{R})$ of $\mathcal{R}$ is a Karoubian subcategory.

Lemma 4.1. *Direct summands of ε-tilting objects are ε-tilting objects.*

Proof. This follows easily from the homological criteria from Theorems 3.13 and 3.16. In the lower finite case, one needs to pass first to a finite ε-stratified subcategory containing the object in question using Theorem 3.55. □

The next goal is to construct and classify ε-tilting objects. Our exposition of this is based roughly on [Don4, Appendix], which in turn goes back to the work of Ringel [Rin]. There are some additional complications in the ε-stratified setting.

Theorem 4.2. *Assume that $\mathcal{R}$ is a finite or lower finite ε-stratified category. For $b \in \mathbf{B}$ with $\rho(b) = \lambda$, there is an indecomposable object $T_\varepsilon(b) \in \mathcal{T}ilt_\varepsilon(\mathcal{R})$ satisfying the following properties:*

(i) $T_\varepsilon(b)$ has a Δ_ε-flag with bottom section isomorphic to $\Delta_\varepsilon(b)$;
(ii) $T_\varepsilon(b)$ has a ∇_ε-flag with top section isomorphic to $\nabla_\varepsilon(b)$;
(iii) $T_\varepsilon(b) \in \mathcal{R}_{\leqslant\lambda}$ and $j^\lambda T_\varepsilon(b) \cong \begin{cases} P_\lambda(b) & \text{if } \varepsilon(\lambda) = + \\ I_\lambda(b) & \text{if } \varepsilon(\lambda) = - \end{cases}$.

These properties determine $T_\varepsilon(b)$ uniquely up to isomorphism: if U is any indecomposable object of $\mathcal{T}ilt_\varepsilon(\mathcal{R})$ satisfying any one of the properties (i)–(iii) then $U \cong T_\varepsilon(b)$; hence, it satisfies the other two properties as well.

Proof. By replacing $\mathcal{R}$ by the Serre subcategory associated to a sufficiently large but finite lower set $\Lambda^\downarrow$ in Λ, chosen so as to contain λ and (for the uniqueness statement) all $\rho(b)$ for b such that $[T : L(b)] \neq 0$, one reduces to the case that $\mathcal{R}$ is a finite ε-stratified category. This reduction depends only on Theorem 3.55. Thus, we may assume henceforth that Λ is finite.

Existence: The main step is to construct an indecomposable object $T_\varepsilon(b) \in \mathcal{T}ilt_\varepsilon(\mathcal{R})$ such that (iii) holds. The argument for this proceeds by induction on $|\Lambda|$. If $\lambda \in \Lambda$ is minimal, we set $T_\varepsilon(b) := \Delta(b)$ if $\varepsilon(\lambda) = +$ or $\nabla(b)$ if $\varepsilon(\lambda) = -$. Since $\bar\Delta(b) = L(b) = \bar\nabla(b)$ by the minimality of λ, this has both a Δ_ε- and a ∇_ε-flag. It is indecomposable, and we get (iii) from Lemma 2.22.

For the induction step, suppose that λ is not minimal and pick $\mu < \lambda$ that is minimal. Let $\Lambda^\uparrow := \Lambda \backslash \{\mu\}, \mathbf{B}^\uparrow := \rho^{-1}(\Lambda^\uparrow)$, and $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the corresponding Serre quotient. By induction, there is an indecomposable object $T_\varepsilon^\uparrow(b) \in \mathcal{T}ilt_\varepsilon(\mathcal{R}^\uparrow)$ satisfying the analogue of (iii). Now there are two cases according to whether $\varepsilon(\mu) = +$ or $-$.

Case $\varepsilon(\mu) = +$: For any $V \in \mathcal{R}$, let $d_+(V) := \sum_{c \in \mathbf{B}_\mu} \dim \operatorname{Ext}^1_{\mathcal{R}}(\Delta(c), V)$. We recursively construct $n \geqslant 0$ and $T_0, T_1, \dots, T_n$ so that $d_+(T_0) > d_+(T_1) > \cdots > d_+(T_n) = 0$ and the following properties hold for all m:

(1) $T_m \in \Delta_\varepsilon(\mathcal{R})$;
(2) $j^\lambda T_m \cong P_\lambda(b)$ if $\varepsilon(\lambda) = +$ or $I_\lambda(b)$ if $\varepsilon(\lambda) = -$;
(3) $\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), T_m) = 0$ for all $a \in \mathbf{B} \backslash \mathbf{B}_\mu$.

To start with, set $T_0 := j_! T_\varepsilon^\uparrow(b)$. This satisfies all of the above properties: (1) follows from Theorem 3.20(6); (2) follows because j^λ factors through j and we know that $T_\varepsilon^\uparrow(b)$ satisfies the analogous property; (3) follows by Theorem 3.20(5). For the recursive step, assume that we are given T_m satisfying (1), (2) and (3) and $d_+(T_m) > 0$. We can find $c \in \mathbf{B}_\mu$ and a non-split extension

$$0 \longrightarrow T_m \longrightarrow T_{m+1} \longrightarrow \Delta(c) \longrightarrow 0. \tag{4.2}$$

This constructs T_{m+1}. We claim that $d_+(T_{m+1}) < d_+(T_m)$ and that T_{m+1} satisfies (1), (2) and (3) too. Part (1) is clear from the definition. For (2), we just apply the exact functor j^λ to the exact sequence (4.2), noting that $j^\lambda \Delta(c) = 0$. For (3), take $a \in \mathbf{B} \backslash \mathbf{B}_\mu$ and apply the functor $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(a), -)$ to the short exact sequence (4.2) to get

$$\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), T_m) \longrightarrow \operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), T_{m+1}) \longrightarrow \operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), \Delta(c)).$$

The first and last term are zero by hypothesis and (3.7), implying $\operatorname{Ext}^1_{\mathcal{R}}(T_{m+1}, \nabla_\varepsilon(a)) = 0$. It remains to show $d_+(T_{m+1}) < d_+(T_m)$. For $a \in \mathbf{B}_\mu$, we have $\operatorname{Ext}^1_{\mathcal{R}}(\Delta(a), \Delta(c)) = 0$ by (3.7), so again we have an exact sequence

$$\operatorname{Hom}_{\mathcal{R}}(\Delta(a), \Delta(c)) \xrightarrow{f} \operatorname{Ext}^1_{\mathcal{R}}(\Delta(a), T_m) \longrightarrow \operatorname{Ext}^1_{\mathcal{R}}(\Delta(a), T_{m+1}) \longrightarrow 0.$$

This shows that $\dim \operatorname{Ext}^1_{\mathcal{R}}(\Delta(a), T_{m+1}) \leqslant \dim \operatorname{Ext}^1_{\mathcal{R}}(\Delta(a), T_m)$, and we just need to observe that the inequality is actually a strict one in the case $a = c$. To see this, note that the first morphism f is non-zero in the case $a = c$ as $f(\mathrm{id}_{\Delta(c)}) \neq 0$ due to the fact that the original short exact sequence was not split. This completes the claim. We have now defined an object $T_n \in \Delta_\varepsilon(\mathcal{R})$ such that $j^\lambda T_n \cong P_\lambda(b)$ if $\varepsilon(\lambda) = +$ or $I_\lambda(b)$ if $\varepsilon(\lambda) = -$, and moreover $\mathrm{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), T_n) = 0$ for all $a \in \mathbf{B}$. By Theorem 3.13, we deduce that $T_n \in \nabla_\varepsilon(\mathcal{R}_{\leqslant\lambda})$ too, hence, it is an ε-tilting object. Decompose T_n into indecomposables $T_n = T_{n,1} \oplus \cdots \oplus T_{n,r}$. Then each $T_{n,i}$ is also an ε-tilting object. Since $j^\lambda T_n$ is indecomposable, we must have that $j^\lambda T_n = j^\lambda T_{n,i}$ for some unique i. Then we set $T_\varepsilon(b) := T_{n,i}$ for this i. This gives us the desired indecomposable ε-tilting object.

Case $\varepsilon(\mu) = -$: Let $d_-(V) := \sum_{c \in \mathbf{B}_\mu} \dim \mathrm{Ext}^1_{\mathcal{R}}(V, \nabla(c))$. This time, one recursively constructs $T_0 :=:= j_* T^\uparrow_\varepsilon(b), T_1, \dots, T_n$ so that $d_-(T_0) > \cdots > d_-(T_n) = 0$ and

(1′) $T_m \in \nabla_\varepsilon(\mathcal{R})$;
(2′) $j^\lambda T_m \cong P_\lambda(b)$ if $\varepsilon(\lambda) = +$ or $I_\lambda(b)$ if $\varepsilon(\lambda) = -$;
(3′) $\mathrm{Ext}^1_{\mathcal{R}}(T_m, \nabla_\varepsilon(a)) = 0$ for all $a \in \mathbf{B} \backslash \mathbf{B}_\mu$.

Since this is this is just the dual construction to the case $\varepsilon(\mu) = +$ already treated, i.e., it is the same construction in the opposite category, we omit the details. Then, we the end, we decompose T_n into indecomposables $T_n = T_{n,1} \oplus \cdots \oplus T_{n,r}$. By Theorem 3.16 each $T_{n,i}$ is an ε-tilting object. Since $j^\lambda T_n$ is indecomposable, we must have that $j^\lambda T_n = j^\lambda T_{n,i}$ for some unique i, and finally set $T_\varepsilon(b) := T_{n,i}$ for this i.

This completes the construction of $T_\varepsilon(b)$ in general. We have shown it satisfies (iii). Let us show that it also satisfies (i) and (ii). For (i), we know by (iii) that $T_\varepsilon(b)$ belongs to $\mathcal{R}_{\leqslant\lambda}$, and it has a Δ_ε-flag. By (3.7), we may order this flag so that the sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}_\lambda$ appear at the bottom. Thus, there is a short exact sequence $0 \to K \to T_\varepsilon(b) \to Q \to 0$ such that K has a Δ_ε-flag with sections $\Delta_\varepsilon(c)$ for $c \in \mathbf{B}_\lambda$ and $j^\lambda Q = 0$. Then $j^\lambda K \cong j^\lambda T_\varepsilon(b)$. If $\varepsilon(\lambda) = +$, this is $P_\lambda(b)$. Since j^λ is exact and $j^\lambda \Delta(c) = P_\lambda(c)$ for each $c \in \mathbf{B}_\lambda$, we must have that $K \cong \Delta(b)$, and (1) follows. Instead, if $\varepsilon(\lambda) = -$, the bottom section of the $\bar{\nabla}$-flag of K must be $\bar{\nabla}(b)$ since $j^\lambda K \cong I_\lambda(b)$ has irreducible socle $L_\lambda(b)$, giving (i) in this case too. The proof of (ii) is similar.

Uniqueness: Let $T := T_\varepsilon(b)$ and U be some other indecomposable object of $\mathcal{T}ilt_\varepsilon(\mathcal{R})$ satisfying one of the properties (i)–(iii). We must prove that $T \cong U$. By the argument from the previous paragaph, we may assume actually that U satisfies either (i) or (ii). We just explain how to see this in the case that U satisfies (i); the dual argument treats the case that U satisfies (ii). So there are short exact sequences $0 \to \Delta_\varepsilon(b) \xrightarrow{f} U \to Q_1 \to 0$ and $0 \to \Delta_\varepsilon(b) \xrightarrow{g} T \to Q_2 \to 0$ such that Q_1, Q_2 have Δ_ε-flags. Applying $\mathrm{Hom}_{\mathcal{R}}(-, T)$ to the first and using $\mathrm{Ext}^1_{\mathcal{R}}(Q_1, T) = 0$, we get that $\mathrm{Hom}_{\mathcal{R}}(U, T) \twoheadrightarrow \mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), T)$. Hence, g extends to a homomorphism $\bar{g} : U \to T$. Similarly, f extends to $\bar{f} : T \to U$. We have constructed morphisms making the triangles in the following diagram commute:

$$\begin{tikzcd} & U \arrow[d, shift right, "\bar{g}"'] \\ \Delta_\varepsilon(b) \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & T \arrow[u, shift right, "\bar{f}"'] \end{tikzcd}$$

Since $\bar{f} \circ \bar{g} \circ f = f$, we deduce that $\bar{f} \circ \bar{g}$ is not nilpotent. Since U is indecomposable, Fitting's Lemma implies $\bar{f} \circ \bar{g}$ is an isomorphism. Similarly, so is $\bar{g} \circ \bar{f}$. Hence, $U \cong T$. $\square$

Remark 4.3. Let $b \in \mathbf{B}_\lambda$. When $\varepsilon(\lambda) = +$, Theorem 4.2 implies that $(T_\varepsilon(b) : \Delta_\varepsilon(b)) = 1$ and $(T_\varepsilon(b) : \Delta_\varepsilon(c)) = 0$ for all other $c \in \mathbf{B}_\lambda$. Similarly, when $\varepsilon(\lambda) = -$, we have that $(T_\varepsilon(b) : \nabla_\varepsilon(b)) = 1$ and $(T_\varepsilon(b) : \nabla_\varepsilon(c)) = 0$ for all other $c \in \mathbf{B}_\lambda$.

The following corollaries show that ε-tilting objects behave well with respect to passage to lower and upper sets, extending Theorems 3.19, 3.55 and 3.20. Really, this follows easily from those theorems plus the characterization of tilting objects in Theorem 4.2; the situation is just like [Don4, Lemma A4.5].

Corollary 4.4. *Let $\mathcal{R}$ be a finite or lower finite ε-stratified category and $\mathcal{R}^{\downarrow}$ the finite ε-stratified subcategory associated to a finite lower set $\Lambda^{\downarrow}$ of Λ. For $b \in \mathbf{B}^{\downarrow} := \rho^{-1}(\Lambda^{\downarrow})$, the corresponding indecomposable ε-tilting object of $\mathcal{R}^{\downarrow}$ is $T_\varepsilon(b)$ (the same as in $\mathcal{R}$).*

Corollary 4.5. *Assume $\mathcal{R}$ is a finite ε-stratified category and let $\Lambda^{\uparrow}$ be an upper set in Λ with quotient $j : \mathcal{R} \to \mathcal{R}^{\uparrow}$. Let $b \in \mathbf{B}^{\uparrow} := \rho^{-1}(\Lambda^{\uparrow})$. The corresponding indecomposable ε-tilting object $T^{\uparrow}_\varepsilon(b)$ of $\mathcal{R}^{\uparrow}$ satisfies $T^{\uparrow}_\varepsilon(b) \cong jT_\varepsilon(b)$. Also $jT_\varepsilon(b) = 0$ if $b \notin \mathbf{B}^{\uparrow}$.*

The next result is concerned with tilting resolutions.

Definition 4.6. Assume that $\mathcal{R}$ is a finite or lower finite ε-stratified category. An *ε-tilting resolution* $d : T_\bullet \to V$ of $V \in \mathcal{R}$ is the data of an exact sequence

$$\cdots \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} V \longrightarrow 0$$

such that

(TR1) $T_m \in \mathcal{T}ilt_\varepsilon(\mathcal{R})$ for each $m = 0, 1, \dots$;
(TR2) $\operatorname{im} d_m \in \nabla_\varepsilon(\mathcal{R})$ for $m \gg 0$.

Similarly, an *ε-tilting coresolution* $d : V \to T^\bullet$ of $V \in \mathcal{R}$ is the data of an exact sequence

$$0 \longrightarrow V \xrightarrow{d^0} T^0 \xrightarrow{d^1} T^1 \xrightarrow{d^2} \cdots$$

such that

(TC1) $T^m \in \mathcal{T}ilt_\varepsilon(\mathcal{R})$ for $m = 0, 1, \dots$;
(TC2) $\operatorname{coim} d^m \in \Delta_\varepsilon(\mathcal{R})$ for $m \gg 0$.

We say it is a *finite* resolution (resp., coresolution) if there is some n such that $T_m = 0$ (resp., $T^m = 0$) for $m > n$. Note in the finite case that the second axiom is redundant.

Lemma 4.7. *If $d : T_\bullet \to V$ is an ε-tilting resolution of $V \in \mathcal{R}$ then* $\operatorname{im} d_m \in \nabla_\varepsilon(\mathcal{R})$ *for* all $m \geqslant 0$. *In particular, $V \in \nabla_\varepsilon(\mathcal{R})$.*

Proof. It suffices to show that for any exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in a finite or lower finite ε-stratified category $B \in \nabla_\varepsilon(\mathcal{R})$ and $\operatorname{im} f \in \nabla_\varepsilon(\mathcal{R})$ implies $\operatorname{im} g \in \nabla_\varepsilon(\mathcal{R})$. Since $\operatorname{im} f = \ker g$, there is a short exact sequence $0 \to \operatorname{im} f \to B \to \operatorname{im} g \to 0$. Now apply Corollary 3.15 (or Corollary 3.61). □

Corollary 4.8. *If $d : V \to T^\bullet$ is an ε-tilting coresolution of $V \in \mathcal{R}$ then* $\operatorname{im} d^m \in \Delta_\varepsilon(\mathcal{R})$ *for* all $m \geqslant 0$. *In particular, $V \in \Delta_\varepsilon(\mathcal{R})$.*

Proof. An ε-tilting coresolution of V in $\mathcal{R}$ is the same thing as a $(-\varepsilon)$-tilting resolution of V in $\mathcal{R}^{\mathrm{op}}$. Hence, the corollary is the equivalent dual statement to Lemma 4.7. □

Theorem 4.9. *Let $\mathcal{R}$ be a finite or lower finite ε-stratified category and take $V \in \mathcal{R}$.*

(1) V has an ε-tilting resolution if and only if $V \in \nabla_\varepsilon(\mathcal{R})$.
(2) V has an ε-tilting coresolution if and only if $V \in \Delta_\varepsilon(\mathcal{R})$.

Proof. We just prove (1), since (2) is the equivalent dual statement. If V has an ε-tilting resolution, then we must have that $V \in \nabla_\varepsilon(\mathcal{R})$ thanks to Lemma 4.7. For the converse, we claim for $V \in \nabla_\varepsilon(\mathcal{R})$ that there is a short exact sequence $0 \to S_V \to T_V \to V \to 0$ with $S_V \in \nabla_\varepsilon(\mathcal{R})$ and $T_V \in \mathcal{T}ilt_\varepsilon(\mathcal{R})$. Given the claim, one can construct an ε-tilting resolution of V by "Splicing" (e.g., see [Wei, Figure 2.1]), to complete the proof.

To prove the claim, we argue by induction on the length $\sum_{b \in \mathbf{B}} (V : \nabla_\varepsilon(b))$ of a ∇_ε-flag of V. If this number is one, then $V \cong \nabla_\varepsilon(b)$ for some $b \in \mathbf{B}$, and there is a short exact sequence $0 \to S_V \to T_V \to V \to 0$ with $S_V \in \nabla_\varepsilon(b)$ and $T_V := T_\varepsilon(b)$ due to Theorem 4.2(ii). If it is greater than one, then there is a short exact sequence $0 \to U \to V \to W \to 0$ such that U and W have strictly shorter ∇_ε-flags. By induction, there are short exact sequences $0 \to S_U \to T_U \to U \to 0$ and $0 \to S_W \to T_W \to W \to 0$ with $S_U, S_W \in \nabla_\varepsilon(\mathcal{R})$ and $T_U, T_W \in \mathcal{T}ilt_\varepsilon(\mathcal{R})$. It remains to show that these short exact sequences can be assembled to produce the desired short exact sequence for V. The argument is like in the proof of the Horseshoe Lemma in [Wei, Lemma 2.2.8].

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_U \arrow[r] \arrow[d] & T_U \arrow[r, "i"] \arrow[d] & U \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & S_V \arrow[r] \arrow[d] & T_V \arrow[r, "j"] \arrow[d] & V \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & S_W \arrow[r] \arrow[d] & T_W \arrow[r, "k"] \arrow[d] \arrow[ur, dashed, "\hat{k}"] & W \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{4.3}$$

Since $\mathrm{Ext}^1_{\mathcal{R}}(T_W, U) = 0$, we can lift $k : T_W \to W$ to $\hat{k} : T_W \to V$ so that $k = g \circ \hat{k}$. Let $T_V := T_U \oplus T_W$ and $j : T_V \to V$ be $\mathrm{diag}(fi, \hat{k})$. This gives us a split short exact sequence in the middle column in (4.3), such that the right hand squares commute. Then we let $S_V := \ker j$, and see that there are induced maps making the left hand column and middle row into short exact sequences such that the left hand squares commute too. $\square$

4.2. **Finite Ringel duality.** In this subsection, we review the theory of Ringel duality for finite ε-stratified categories. Our exposition is based in part on [Don4, Appendix], which gives a self-contained treatment in the highest weight setting, and [AHLU], where the $+$-highest weight case is considered assuming $\Lambda = \{1 < \cdots < n\}$. Throughout, we assume that $\mathcal{R}$ is a finite ε-stratified category with the usual stratification $\rho : \mathbf{B} \to \Lambda$, and Λ^{op} denotes Λ viewed as a poset with the opposite partial order.

Definition 4.10. Let $\mathcal{R}$ be a finite ε-stratified category with stratification $\rho : \mathbf{B} \to \Lambda$. By an *$\varepsilon$-tilting generator* in $\mathcal{R}$, we mean $T \in \mathcal{T}ilt_\varepsilon(\mathcal{R})$ such that T has a summand isomorphic to $T_\varepsilon(b)$ for each $b \in \mathbf{B}$. Given such an object, we define the *Ringel dual* of $\mathcal{R}$ relative to T to be the finite Abelian category $\widetilde{\mathcal{R}} := A\text{-mod}_{\mathrm{fd}}$ where $A := \mathrm{End}_{\mathcal{R}}(T)^{\mathrm{op}}$. We also define the two (covariant) *Ringel duality functors*

$$F := \mathrm{Hom}_{\mathcal{R}}(T, -) : \mathcal{R} \to \widetilde{\mathcal{R}}, \tag{4.4}$$

$$G := * \circ \mathrm{Hom}_{\mathcal{R}}(-, T) : \mathcal{R} \to \widetilde{\mathcal{R}}. \tag{4.5}$$

Note for the second of these that $\mathrm{Hom}_{\mathcal{R}}(V, T)$ is naturally a finite-dimensional right A-module for $V \in \mathcal{R}$, hence, its dual is a left A-module.

Theorem 4.11. *In the setup of Definition 4.10, the Ringel dual $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ relative to T is a finite $(-\varepsilon)$-stratified category with stratification defined from $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$ and distinguished objects satisfying*

$$\widetilde{P}(b) \cong FT_\varepsilon(b), \qquad \widetilde{I}(b) \cong GT_\varepsilon(b), \qquad \widetilde{L}(b) \cong \mathrm{hd}\ \widetilde{P}(b) \cong \mathrm{soc}\ \widetilde{I}(b),$$
$$\widetilde{\Delta}_{-\varepsilon}(b) \cong F\nabla_\varepsilon(b), \qquad \widetilde{\nabla}_{-\varepsilon}(b) \cong G\Delta_\varepsilon(b), \qquad \widetilde{T}_{-\varepsilon}(b) \cong FI(b) \cong GP(b).$$

The restrictions $F : \nabla_\varepsilon(\mathcal{R}) \to \Delta_{-\varepsilon}(\widetilde{\mathcal{R}})$ and $G : \Delta_\varepsilon(\mathcal{R}) \to \nabla_{-\varepsilon}(\widetilde{\mathcal{R}})$ are equivalences; in fact, they induce isomorphisms

$$\mathrm{Ext}^n_{\mathcal{R}}(V_1, V_2) \cong \mathrm{Ext}^n_{\widetilde{\mathcal{R}}}(FV_1, FV_2), \qquad \mathrm{Ext}^n_{\mathcal{R}}(W_1, W_2) \cong \mathrm{Ext}^n_{\widetilde{\mathcal{R}}}(GW_1, GW_2), \tag{4.6}$$

for all $V_i \in \nabla_\varepsilon(\mathcal{R})$, $W_i \in \Delta_\varepsilon(\mathcal{R})$ and $n \geqslant 0$.

Before the proof, we give some applications. The first is a *double centralizer property.* It implies that our situation fits into the setup from [Wak, (A1), (A2)], and T is an example of a *Wakamatsu module.*

Corollary 4.12. *Suppose that the ε-stratified category $\mathcal{R}$ in Theorem 4.11 is the category B-$\mathrm{mod}_{\mathrm{fd}}$ for a finite-dimensional algebra B, so that T is a (B, A)-bimodule. Let $\widetilde{T} := T^*$ be the dual (A, B)-bimodule. Then the following holds.*

(1) *$\widetilde{T}$ is a $(-\varepsilon)$-tilting generator in $\widetilde{\mathcal{R}}$ such that $B \cong \mathrm{End}_A(\widetilde{T})^{\mathrm{op}}$. Hence, the Ringel dual of $\widetilde{\mathcal{R}}$ relative to $\widetilde{T}$ is isomorphic to the original category $\mathcal{R}$.*
(2) *Denote the Ringel duality functors $\widetilde{F}$ and $\widetilde{G}$ for $\widetilde{\mathcal{R}}$ relative to $\widetilde{T}$ instead by $G_* := \mathrm{Hom}_{\widetilde{\mathcal{R}}}(\widetilde{T}, -) : \widetilde{\mathcal{R}} \to \mathcal{R}$ and $F^* := * \circ \mathrm{Hom}_{\widetilde{\mathcal{R}}}(-, \widetilde{T}) : \widetilde{\mathcal{R}} \to \mathcal{R}$, respectively. We have that $F^* \cong T \otimes_A -$ and $G \cong \widetilde{T} \otimes_B -$. Hence, (F^*, F) and (G, G_*) are adjoint pairs.*

Proof. (1) Note that GB is a $(-\varepsilon)$-tilting generator since $GP(b) \cong \widetilde{T}_{-\varepsilon}(b)$ for $b \in \mathbf{B}$. Actually, $GB = \mathrm{Hom}_B(B, T)^* \cong T^* = \widetilde{T}$. Thus, $\widetilde{T}$ is a $(-\varepsilon)$-tilting generator in $\widetilde{\mathcal{R}}$. Its opposite endomorphism algebra is B since G defines an algebra isomorphism

$$B \cong \mathrm{End}_B(B)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_A(GB)^{\mathrm{op}} \cong \mathrm{End}_A(\widetilde{T})^{\mathrm{op}}.$$

(2) As F^* is right exact and commutes with direct sums, a standard argument using the Five Lemma shows that it is isomorphic to $F^*(A) \otimes_A - \cong T \otimes_A -$. Thus, F^* is left adjoint to F. Similarly, $G \cong \widetilde{T} \otimes_B -$ is left adjoint to G_*. ☐

The next corollary describes the strata $\widetilde{\mathcal{R}}_\lambda$ of the Ringel dual category. Let $\tilde{\jmath}^\lambda_! : \widetilde{\mathcal{R}}_\lambda \to \widetilde{\mathcal{R}}_{\geqslant\lambda}$ and $\tilde{\jmath}^\lambda_* : \widetilde{\mathcal{R}}_\lambda \to \widetilde{\mathcal{R}}_{\geqslant\lambda}$ be its standardization and costandardization functors.

Corollary 4.13. *For $\lambda \in \Lambda$, the strata $\mathcal{R}_\lambda$ and $\widetilde{\mathcal{R}}_\lambda$ are equivalent. More precisely:*

(1) *If $\varepsilon(\lambda) = +$ the functor $F_\lambda := \tilde{\jmath}^\lambda \circ F \circ j^\lambda_* : \mathcal{R}_\lambda \to \widetilde{\mathcal{R}}_\lambda$ is a well-defined equivalence.*
(2) *If $\varepsilon(\lambda) = -$ the functor $G_\lambda := \tilde{\jmath}^\lambda \circ G \circ j^\lambda_! : \mathcal{R}_\lambda \to \widetilde{\mathcal{R}}_\lambda$ is a well-defined equivalence.*

Proof. We just prove (1), since (2) is similar. So assume that $\varepsilon(\lambda) = +$. Note that the definition of F_λ makes sense: j^λ_* is exact by Theorem 3.7, so it sends objects of $\mathcal{R}_\lambda$ to objects of $\mathcal{R}$ which have filtrations with sections $\nabla_\varepsilon(b)$ for $b \in \mathbf{B}_\lambda$. Then F sends such objects into $\Delta_{-\varepsilon}(\widetilde{\mathcal{R}}_{\geqslant\lambda})$, on which $\tilde{\jmath}^\lambda$ is defined. This shows moreover that F_λ is exact. Adopting the setup of Corollary 4.12, we can also define

$$F^*_\lambda := j^\lambda \circ F^* \circ \tilde{\jmath}^\lambda_! : \widetilde{\mathcal{R}}_\lambda \to \mathcal{R}_\lambda,$$

and get that F^*_λ is well-defined by similar arguments. We complete the proof by showing that F_λ and F^*_λ are quasi-inverse equivalences. Note that F^*_λ is left adjoint to F_λ. The counit of adjunction gives us a natural transformation $F^*_\lambda \circ F_\lambda \to \mathrm{Id}_{\mathcal{R}_\lambda}$. We claim this is an isomorphism. Since both functors are exact, it suffices to prove this on irreducible objects: we have $F^*_\lambda(F_\lambda L_\lambda(b)) \cong F^*_\lambda \widetilde{L}_\lambda(b) \cong L_\lambda(b)$. Similar argument shows that the unit of adjunction is an isomorphism in the other direction. ☐

Corollary 4.14. *Let $\mathcal{R}$ be a finite ε-stratified category.*

(1) *All $V \in \nabla_\varepsilon(\mathcal{R})$ have finite ε-tilting resolutions if and only if all positive strata are of finite global dimension.*
(2) *All $V \in \Delta_\varepsilon(\mathcal{R})$ have finite ε-tilting coresolutions if and only if all negative strata are of finite global dimension.*

Proof. We just explain the proof of (1). By Theorem 4.11, all $V \in \nabla_\varepsilon(\mathcal{R})$ have finite ε-tilting resolutions if and only if all $\widetilde{V} \in \Delta_{-\varepsilon}(\widetilde{\mathcal{R}})$ have finite projective resolutions. By Lemma 3.27(1), this is equivalent to all negative strata of the $(-\varepsilon)$-stratified category $\widetilde{\mathcal{R}}$ are of finite global dimension. Equivalently, by Corollary 4.13, all positive strata of the ε-stratified category $\mathcal{R}$ are of finite global dimension. ☐

For the final application, let $\mathbb{R}F$ and $\mathbb{L}G$ be the total derived functors of F respectively G. These give triangulated functors between the bounded derived categories $D^b(\mathcal{R})$ and

$D^b(\widetilde{\mathcal{R}})$. The following theorem is a consequence of Happel's general tilting theory for finite dimensional algebras from [Hap].

Theorem 4.15. *Let $\widetilde{\mathcal{R}}$ be the Ringel dual of a finite ε-stratified category $\mathcal{R}$. Assume that all negative strata (resp., all positive strata) of $\mathcal{R}$ are of finite global dimension. Then $\mathbb{R}F : D^b(\mathcal{R}) \to D^b(\widetilde{\mathcal{R}})$ (resp., $\mathbb{L}G : D^b(\mathcal{R}) \to D^b(\widetilde{\mathcal{R}})$) is an equivalence of triangulated categories. Moreover, if $\mathcal{R}$ is of finite global dimension, then so is $\widetilde{\mathcal{R}}$.*

Proof. Assuming $\mathcal{R}$ has finite global dimension, this all follows by [Hap, Lemma 2.9, Theorem 2.10]; the hypotheses there hold thanks to Corollary 4.14. To get the derived equivalence without assuming $\mathcal{R}$ has finite global dimension, we cite instead Keller's exposition of Happel's result in [Kel, Theorem 4.1], since it assumes slightly less; the hypotheses (a) and (c) there hold due to Corollary 4.14(2) and Lemma 3.27(1). $\square$

Corollary 4.16. *If $\mathcal{R}$ is $+$-highest weight (resp., $-$-highest weight) and $\widetilde{\mathcal{R}}$ is the Ringel dual relative to a $+$-tilting generator (resp., a $-$tilting generator), then $\mathbb{R}F : D^b(\mathcal{R}) \to D^b(\widetilde{\mathcal{R}})$ (resp., $\mathbb{L}G : D^b(\mathcal{R}) \to D^b(\widetilde{\mathcal{R}})$) is an equivalence.*

Proof of Theorem 4.11. This follows the same steps as in [Don4, pp.158–160]. Assume without loss of generality that $\mathcal{R} = B\text{-mod}_{\text{fd}}$ for a finite-dimensional algebra B. For each $b \in \mathbf{B}$, let $e_b \in A = \operatorname{End}_A(T)^{\text{op}}$ be an idempotent such that $Te_b \cong T_\varepsilon(b)$. Then $\widetilde{P}(b) := Ae_b$ is an indecomposable projective A-module and the modules

$$\left\{\widetilde{L}(b) := \text{hd } \widetilde{P}(b) \,\middle|\, b \in \mathbf{B}\right\}$$

give a full set of pairwise inequivalent irreducible A-modules. Since $\widetilde{\mathcal{R}}$ is a finite Abelian category, it is immediate that $\rho : \mathbf{B} \to \Lambda^{\text{op}}$ defines a stratification of it. Let $\widetilde{\Delta}_{-\varepsilon}(b)$ and $\widetilde{\nabla}_{-\varepsilon}(b)$ be the $(-\varepsilon)$-standard and $(-\varepsilon)$-costandard objects of $\widetilde{\mathcal{R}}$ defined from this stratification. Set $V(b) := F\nabla_\varepsilon(b)$.

Step 1: *For $b \in \mathbf{B}$ we have that $\widetilde{P}(b) \cong FT_\varepsilon(b)$.* This follows immediately from the equality $\operatorname{Hom}_B(T,T)e_b = \operatorname{Hom}_B(T, Te_b)$.

Step 2: *The functor F is exact on $\nabla_\varepsilon(\mathcal{R})$.* This is the usual Ext^1-vanishing between Δ_ε- and ∇_ε-filtered objects.

Step 3: *For $a, b \in \mathbf{B}$, we have that $[V(b) : \widetilde{L}(a)] = (T_\varepsilon(a) : \Delta_\varepsilon(b))$.* The left hand side is $\dim e_a V(b) = \dim e_a \operatorname{Hom}_B(T, \nabla_\varepsilon(b)) \cong \dim \operatorname{Hom}_B(T_\varepsilon(a), \nabla_\varepsilon(b))$, which equals the right hand side.

Step 4: *$V(b)$ is a non-zero quotient of $\widetilde{P}(b)$, thus,* hd $V(b) = \widetilde{L}(b)$. By Theorem 4.2(i), $\nabla_\varepsilon(b)$ is a quotient of $T_\varepsilon(b)$, hence $V(b)$ is quotient of $\widetilde{P}(b)$ by Step 2. It is non-zero by Step 3.

Step 5: *We have that $V(b) \cong \widetilde{\Delta}_{-\varepsilon}(b)$.* Let $\lambda := \rho(b)$. We treat the cases $\varepsilon(\lambda) = +$ and $\varepsilon(\lambda) = -$ separately. If $\varepsilon(\lambda) = +$ we must show that $V(b)$ is the largest quotient of $\widetilde{P}(b)$ with the property that $[V(b) : \widetilde{L}(a)] \neq 0 \Rightarrow \rho(a) \geqslant \rho(b)$. We have already observed in Step 4 that $V(b)$ is a quotient of $\widetilde{P}(b)$. Also $(T_\varepsilon(a) : \Delta_\varepsilon(b)) \neq 0 \Rightarrow \rho(b) \leqslant \rho(a)$ by Theorem 4.2(iii). Using Step 3, this imples that $V(b)$ has the property $[V(b) : \widetilde{L}(a)] \neq 0 \Rightarrow \rho(a) \geqslant \rho(b)$. It remains to show that any strictly larger quotient of $\widetilde{P}(b)$ fails this condition. To see this, since $\varepsilon(\lambda) = +$, a ∇_ε-flag in $T_\varepsilon(b)$ has $\nabla_\varepsilon(b)$ at the top and other sections $\nabla_\varepsilon(c)$ for c with $\rho(c) < \rho(b)$. In view of Step 4, any strictly larger quotient of $\widetilde{P}(b)$ than $V(b)$ therefore has an additional composition factor $\widetilde{L}(c)$ arising from the head of $V(c)$ for some c with $\rho(c) < \rho(b)$.

Instead, if $\varepsilon(\lambda) = -$, we use the characterization of $\widetilde{\Delta}_{-\varepsilon}(b)$ from Lemma 3.3: we must show that $V(b)$ is the largest quotient of $\widetilde{P}(b)$ with the property that $[V(b) : \widetilde{L}(b)] = 1$ and $[V(b) : \widetilde{L}(a)] \neq 0 \Rightarrow \rho(a) > \rho(b)$ for $a \neq b$. Since $\varepsilon(\lambda) = -$, we have that

$(T_\varepsilon(b) : \nabla_\varepsilon(b)) = 1$ and $(T_\varepsilon(b) : \nabla_\varepsilon(a)) \neq 0 \Rightarrow \rho(a) < \rho(b)$ for $a \neq b$. Hence, using Step 3 again, the quotient $V(b)$ of $\widetilde{P}(b)$ has the required properties. A ∇_ε-flag in $T_\varepsilon(b)$ has $\nabla_\varepsilon(b)$ at the top and other sections $\nabla_\varepsilon(c)$ for c with $\rho(c) \leqslant \rho(b)$. So any strictly larger quotient of $\widetilde{P}(b)$ than $V(b)$ has a composition factor $\widetilde{L}(c)$ arising from the head of $V(c)$ for c with $\rho(c) \leqslant \rho(b)$. In case $c = b$, this violates the requirement that the quotient has $\widetilde{L}(b)$ appearing with multiplicity one; otherwise, it violates the requirement that all other composition factors of the quotient are of the form $\widetilde{L}(a)$ with $\rho(a) > \rho(b)$.

Step 6: *$\widetilde{\mathcal{R}}$ is a finite $(-\varepsilon)$-stratified category.* In view of Step 5, it suffices to show that $\widetilde{P}(b)$ has a filtration with sectons $V(c)$ for c with $\rho(c) \leqslant \rho(b)$. Since $T_\varepsilon(b)$ has a ∇_ε-flag with sections $\nabla_\varepsilon(c)$ for c with $\rho(c) \leqslant \rho(b)$, this follows using Steps 1 and 2.

Step 7: *For any $U \in \mathcal{T}ilt_\varepsilon(\mathcal{R})$ and $V \in \mathcal{R}$, the map $f : \operatorname{Hom}_B(U, V) \to \operatorname{Hom}_A(FU, FV)$ induced by F is an isomorphism.* It suffices to prove this when $U = T$, so that the right hand space is $\operatorname{Hom}_A(A, FV)$ and $FV = \operatorname{Hom}_B(T, V)$. This special case follows because f is the inverse of the isomorphism $\operatorname{Hom}_A(A, FV) \to FV, \theta \mapsto \theta(1)$.

Step 8: *For any $V, W \in \nabla_\varepsilon(\mathcal{R})$ and $n \geqslant 0$, the functor F induces a linear isomorphism $\operatorname{Ext}^n_{\mathcal{R}}(V, W) \xrightarrow{\sim} \operatorname{Ext}^n_{\widetilde{\mathcal{R}}}(FV, FW)$.* Take an ε-tilting resolution $d : T_\bullet \to V$ in the sense of Definition 4.6, which exists thanks to Theorem 4.9. The functor F takes this resolution to a complex

$$\cdots \to FT_1 \to FT_0 \to FV \to 0.$$

In fact, this complex is exact. To see this, take $m \geqslant 0$ and consider the short exact sequence $0 \to \ker d_m \to T_m \to \operatorname{im} d_m \to 0$. All of $\ker d_m$, T_m and $\operatorname{im} d_m$ have ∇_ε-flags due to Lemma 4.7. Hence, thanks to Step 2, we get a short exact sequence

$$0 \longrightarrow F(\ker d_m) \xrightarrow{i} FT_m \xrightarrow{p} F(\operatorname{im} d_m) \longrightarrow 0$$

on applying F. Since F is left exact, the canonical map $F(\operatorname{im} d_m) \to FT_{m-1}$ is a monomorphism. Its image is all $\theta : T \to T_{m-1}$ with image contained in $\operatorname{im} d_m$. As p is an epimorphism, any such θ can be written as $d_m \circ \phi$ for $\phi : T \to T_m$, i.e., $\theta \in \operatorname{im}(Fd_m)$. Thus, $F(\operatorname{im} d_m) \cong \operatorname{im}(Fd_m)$, and $0 \to \ker(Fd_m) \longrightarrow FT_m \to \operatorname{im}(Fd_m) \to 0$ is exact, as required. In view of Step 1, we have constructed a projective resolution of FV in $\widetilde{\mathcal{R}}$:

$$\cdots \longrightarrow FT_1 \longrightarrow FT_0 \longrightarrow FV \longrightarrow 0.$$

Next, we use the projective resolution just constructed to compute $\operatorname{Ext}^n_{\widetilde{\mathcal{R}}}(FV, FI)$ for any injective $I \in \mathcal{R}$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{\mathcal{R}}(V, I) \arrow[r] \arrow[d, "f"] & \operatorname{Hom}_{\mathcal{R}}(T_0, I) \arrow[r] \arrow[d, "f_0"] & \operatorname{Hom}_{\mathcal{R}}(T_1, I) \arrow[r] \arrow[d, "f_1"] & \cdots \\
0 \arrow[r] & \operatorname{Hom}_{\widetilde{\mathcal{R}}}(FV, FI) \arrow[r] & \operatorname{Hom}_{\widetilde{\mathcal{R}}}(FT_0, FI) \arrow[r] & \operatorname{Hom}_{\widetilde{\mathcal{R}}}(FT_1, FI) \arrow[r] & \cdots
\end{tikzcd}$$

with vertical maps induced by F. The maps $f_0, f_1, \dots$ are isomorphisms due to Step 7. Also the top row is exact as I is injective. We deduce that the bottom row is exact at the positions $\operatorname{Hom}_{\widetilde{\mathcal{R}}}(FT_m, FI)$ for all $m \geqslant 1$. It is exact at positions $\operatorname{Hom}_{\widetilde{\mathcal{R}}}(FV, FI)$ and $\operatorname{Hom}_{\widetilde{\mathcal{R}}}(FT_0, FI)$ as $\operatorname{Hom}_{\widetilde{\mathcal{R}}}(-, FI)$ is left exact. Thus, the bottom row is exact everywhere. So the map f is an isomorphism too and $\operatorname{Ext}^n_{\widetilde{\mathcal{R}}}(FV, FI) = 0$ for $n > 0$.

Finally, take a short exact sequence $0 \to W \to I \to Q \to 0$ in $\mathcal{R}$ with I injective. We have that $Q \in \nabla_\varepsilon(\mathcal{R})$ by Corollary 3.15. Hence, using Step 2 and the previous paragraph, there is a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{R}}(V, W) \arrow[r, hook] \arrow[d, "f_1"] & \operatorname{Hom}_{\mathcal{R}}(V, I) \arrow[r] \arrow[d, "f_2"] & \operatorname{Hom}_{\mathcal{R}}(V, Q) \arrow[r, two heads] \arrow[d, "f_3"] & \operatorname{Ext}^1_{\mathcal{R}}(V, W) \arrow[d, "f_4"] \\
\operatorname{Hom}_{\widetilde{\mathcal{R}}}(FV, FW) \arrow[r, hook] & \operatorname{Hom}_{\widetilde{\mathcal{R}}}(FV, FI) \arrow[r] & \operatorname{Hom}_{\widetilde{\mathcal{R}}}(FV, FQ) \arrow[r, two heads] & \operatorname{Ext}^1_{\widetilde{\mathcal{R}}}(FV, FW)
\end{tikzcd}$$

with exact rows. As f_2 is an isomorphism, we get that f_1 is injective. Since this is proved for all W, this means that f_3 is injective too. Then a diagram chase gives that f_1 is surjective, hence, f_3 is surjective and f_4 is an isomorphism. Degree shifting now gives the isomorphisms $\operatorname{Ext}^n_{\mathcal{R}}(V,W) \xrightarrow{\sim} \operatorname{Ext}^n_{\widetilde{\mathcal{R}}}(FV,FW)$ for $n \geqslant 2$ as well.

Step 9: *We have that $\widetilde{T}_{-\varepsilon}(b) \cong FI(b)$.* By Steps 5 and 8, we get that

$$\operatorname{Ext}^1_{\widetilde{\mathcal{R}}}(\widetilde{\Delta}_{-\varepsilon}(a), FI(b)) \cong \operatorname{Ext}^1_{\mathcal{R}}(\nabla_\varepsilon(a), I(b)) = 0$$

for all $a \in \mathbf{B}$. Hence, by the homological criterion for $\nabla_{-\varepsilon}$-flags in the $(-\varepsilon)$-stratified category $\widetilde{\mathcal{R}}$, the A-module $FI(b)$ has a $\nabla_{-\varepsilon}$-flag. It also has a $\Delta_{-\varepsilon}$-flag with bottom section isomorphism to $\widetilde{\Delta}_{-\varepsilon}(b)$ due to Steps 2 and 5. So $FI(b) \in \mathcal{T}ilt_{-\varepsilon}(\widetilde{\mathcal{R}})$. It is indecomposable as $\operatorname{End}_{\widetilde{\mathcal{R}}}(FI(b)) \cong \operatorname{End}_{\mathcal{R}}(I(b))$ by Step 8, which is local. Therefore $FI(b) \cong \widetilde{T}_{-\varepsilon}(b)$ due to Theorem 4.2(i).

Step 10: *The restriction $F : \nabla_\varepsilon(\mathcal{R}) \to \Delta_{-\varepsilon}(\widetilde{\mathcal{R}})$ is an equivalence of categories.* It is full and faithful by Step 8. It remains to show that it is dense, i.e., for any $V \in \Delta_{-\varepsilon}(\widetilde{\mathcal{R}})$ there exists $V' \in \nabla_\varepsilon(\mathcal{R})$ with $FV' \cong V$. The proof of this goes by induction on the length of a $\Delta_{-\varepsilon}$-flag of V. If this length is one, we are done by Step 5. For the induction step, consider V fitting into a short exact sequence $0 \to U \to V \to W \to 0$ for shorter $U, W \in \Delta_{-\varepsilon}(\mathcal{R}')$. By induction there are $U', W' \in \nabla_\varepsilon(\mathcal{R})$ such that $FU' \cong U$ and $FW' \cong W$. Then we use the isomorphism $\operatorname{Ext}^1_{\widetilde{\mathcal{R}}}(FW', FU') \cong \operatorname{Ext}^1_{\mathcal{R}}(W', U')$ from Step 8 to see that there is an extension V' of U' and W' in $\mathcal{R}$ such that $FV' \cong V$.

Step 11: *The dual right B-module T^* to T is a $(-\varepsilon)$-tilting generator in $\mathcal{R}^{\mathrm{op}} = \operatorname{mod}_{\mathrm{fd}}\text{-}B$ such that $\operatorname{End}_B(T^*)^{\mathrm{op}} = A^{\mathrm{op}}$. Moreover, letting $F^{\mathrm{op}} := \operatorname{Hom}_B(T^*, -) : \operatorname{mod}_{\mathrm{fd}}\text{-}B \to \operatorname{mod}_{\mathrm{fd}}\text{-}A$ be the corresponding Ringel duality functor, we have that $G \cong * \circ F^{\mathrm{op}} \circ *$.* The first statement is clear from Theorem 3.11, observing that $\operatorname{End}_B(T^*)^{\mathrm{op}} \cong \operatorname{End}_B(T)$ since $* : B\text{-}\operatorname{mod}_{\mathrm{fd}} \to \operatorname{mod}_{\mathrm{fd}}\text{-}B$ is a contravariant equivalence. It remains to observe that $* \circ F^{\mathrm{op}} \circ * \cong * \circ \operatorname{Hom}_B(T^*, (-)^*) \cong * \circ \operatorname{Hom}_B(-, T) = G$.

Step 12: *The restriction $G : \Delta_\varepsilon(\mathcal{R}) \to \nabla_{-\varepsilon}(\widetilde{\mathcal{R}})$ is an equivalence of categories inducing isomorphisms as in (4.6), such that $GT_\varepsilon(b) \cong \widetilde{I}(b)$, $G\Delta_\varepsilon(b) \cong \widetilde{\nabla}_{-\varepsilon}(b)$ and $GP(b) \cong \widetilde{T}_{-\varepsilon}(b)$.* This follows from Step 11 together with the analogs of Steps 1, 5, 8, 9 and 10 with $\mathcal{R}^{\mathrm{op}} = \operatorname{mod}_{\mathrm{fd}}\text{-}B, \widetilde{\mathcal{R}}^{\mathrm{op}} = \operatorname{mod}_{\mathrm{fd}}\text{-}A$ and F^{op} replacing $\mathcal{R} = B\text{-}\operatorname{mod}_{\mathrm{fd}}, \widetilde{\mathcal{R}} = A\text{-}\operatorname{mod}_{\mathrm{fd}}$ and F, respectively. □

4.3. **Tilting objects in the upper finite case.** Throughout the subsection, $\mathcal{R}$ will be an upper finite ε-stratified category. We are going to extend the definition of tilting objects to this situation. Using the notions of ascending Δ_ε-flags and descending ∇_ε-flags from Definition 3.37, we set

$$\mathcal{T}ilt_\varepsilon(\mathcal{R}) := \Delta_\varepsilon^{\mathrm{asc}}(\mathcal{R}) \cap \nabla_\varepsilon^{\mathrm{dsc}}(\mathcal{R}). \tag{4.7}$$

We emphasize that objects of $\mathcal{T}ilt_\varepsilon(\mathcal{R})$ are in particular objects of $\mathcal{R}$, so all of their composition multiplicities are finite. Like in Lemma 4.1, $\mathcal{T}ilt_\varepsilon(\mathcal{R})$ is a Karoubian subcategory of $\mathcal{R}$. In general, in the upper finite setting, an ε-tilting object in $\mathcal{R}$ has both an infinite ascending Δ_ε-flag and an infinite descending ∇_ε-flag. For a baby example of this phenomenon, see (6.11) below.

Theorem 4.17. *Assume that $\mathcal{R}$ is an upper finite ε-stratified category. For $b \in \mathbf{B}$ with $\rho(b) = \lambda$, there is an indecomposable object $T_\varepsilon(b) \in \mathcal{T}ilt_\varepsilon(\mathcal{R})$ satisfying the following properties:*

(i) $T_\varepsilon(b)$ has an ascending Δ_ε-flag with bottom section[3] isomorphic to $\Delta_\varepsilon(b)$;
(ii) $T_\varepsilon(b)$ has a descending ∇_ε-flag with top section[4] isomorphic to $\nabla_\varepsilon(b)$;

[3]We mean that there is an ascending Δ_ε-flag $(V_\omega)_{\omega\in\Omega}$ in which Ω has a smallest non-zero element 1 such that $V_1 \cong \Delta_\varepsilon(b)$.
[4]Similarly, we mean that $V/V_1 \cong \nabla_\varepsilon(b)$.

(iii) $T_\varepsilon(b) \in \mathcal{R}_{\leqslant\lambda}$ *and* $j^\lambda T_\varepsilon(b) \cong \begin{cases} P_\lambda(b) & \text{if } \varepsilon(\lambda) = + \\ I_\lambda(b) & \text{if } \varepsilon(\lambda) = - \end{cases}$.

These properties determine $T_\varepsilon(b)$ uniquely up to isomorphism: if T is any indecomposable object of $\mathcal{T}ilt_\varepsilon(\mathcal{R})$ satisfying any one of the properties (i)–(iii) then $T \cong T_\varepsilon(b)$; hence, it satisfies the other two properties as well.

Proof. Existence: Replacing $\mathcal{R}$ by $\mathcal{R}_{\leqslant\lambda}$ if necessary and using Theorem 3.43, we reduce to the special case that λ is the largest element of the poset Λ. Assuming this, the first step in the construction of $T_\varepsilon(b)$ is to define a direct system $(V_\omega)_{\omega\in\Omega}$ of objects of $\mathcal{R}$. This is indexed by the directed set Ω of all finite upper sets in Λ. Let $V_\varnothing := 0$. Then take $\varnothing \neq \omega \in \Omega$ and denote it instead by $\Lambda^\uparrow$. Letting $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the corresponding finite ε-stratified quotient of $\mathcal{R}$, we set $V_\omega := j_! T_\varepsilon^\uparrow(b)$. By Theorem 3.44(6), this has a Δ_ε-flag. Given also $\omega < \upsilon \in \Omega$, i.e., another upper set $\Lambda^{\uparrow\uparrow}$ containing $\Lambda^\uparrow$, let $k : \mathcal{R} \to \mathcal{R}^{\uparrow\uparrow}$ be the corresponding quotient. Then j factors as $j = \bar{\jmath} \circ k$ for an induced quotient functor $\bar{\jmath} : \mathcal{R}^{\uparrow\uparrow} \to \mathcal{R}^\uparrow$. Since $\bar{\jmath} T_\varepsilon^{\uparrow\uparrow}(b) \cong T_\varepsilon^\uparrow(b)$ by Corollary 4.5, we deduce from Corollary 3.21(2) that there is a short exact sequence

$$0 \longrightarrow \bar{\jmath}_! T_\varepsilon^\uparrow(b) \longrightarrow T_\varepsilon^{\uparrow\uparrow}(b) \longrightarrow Q \longrightarrow 0$$

such that Q has a Δ_ε-flag with sections $\Delta_\varepsilon^{\uparrow\uparrow}(c)$ for c with $\rho(c) \in \Lambda^{\uparrow\uparrow}\backslash\Lambda^\uparrow$. Applying $k_!$ and using the exactness from Theorem 3.44(6) again, we deduce that there is an embedding $f_\omega^\upsilon : V_\omega \hookrightarrow V_\upsilon$ with coker $f_\omega^\upsilon \in \Delta_\varepsilon(\mathcal{R})$. Thus, we have a direct system $(V_\omega)_{\omega\in\Omega}$. Now let $T_\varepsilon(b) := \varinjlim V_\omega \in \operatorname{Ind}(\mathcal{R}_c)$. Using the induced embeddings $f_\omega : V_\omega \hookrightarrow T_\varepsilon(b)$, we identify each V_ω with a subobject of $T_\varepsilon(b)$. We have shown for $\omega < \upsilon$ that $V_\upsilon/V_\omega \in \Delta_\varepsilon(\mathcal{R})$ and, moreover, $jV_\upsilon = jV_\omega$ where $j : \mathcal{R} \to \mathcal{R}^\uparrow$ is the quotient associated to ω.

In this paragraph, we show that $T_\varepsilon(b)$ actually lies in $\mathcal{R}$ rather than $\operatorname{Ind}(\mathcal{R}_c)$, i.e., all of the composition multiplicities $[T_\varepsilon(b) : L(c)]$ are finite. To see this, take $c \in \mathbf{B}$. Let $\omega = \Lambda^\uparrow \in \Omega$ be some fixed finite upper set such that $\rho(c) \in \Lambda^\uparrow$, and $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the quotient functor as usual. Then for any $\upsilon \geqslant \omega$ we have that

$$[V_\upsilon : L(c)] = [jV_\upsilon : L^\uparrow(c)] = [jV_\omega : L^\uparrow(c)] = [V_\omega : L(c)].$$

Hence, $[T_\varepsilon(b) : L(c)] = [V_\omega : L(c)] < \infty$.

So now we have defined $T_\varepsilon(b) \in \mathcal{R}$ together with an ascending Δ_ε-flag $(V_\omega)_{\omega\in\Omega}$. The smallest non-empty element of Ω is $\omega := \{\lambda\}$, and $V_\omega = j_!^\lambda P_\lambda(b) = \Delta_\varepsilon(b)$ if $\varepsilon(\lambda) = +$, or $j_!^\lambda I_\lambda(b)$ if $\varepsilon(\lambda) = -$. Since $j^\lambda T_\varepsilon(b) = j^\lambda V_\omega$, we deduce that (iii) holds. Also by construction $T_\varepsilon(b)$ has an ascending Δ_ε-flag. To see that it has a descending ∇_ε-flag, take any $a \in \mathbf{B}$. Let $\omega = \Lambda^\uparrow \in \Omega$ be such that $\rho(a) \in \Lambda^\uparrow$. Then $\Delta_\varepsilon(a) = j_! \Delta_\varepsilon^\uparrow(a)$ and $jT_\varepsilon(b) = jV_\omega = T_\varepsilon^\uparrow(b)$, so by Theorem 3.44(5) we get that

$$\operatorname{Ext}^1_{\mathcal{R}}(\Delta_\varepsilon(a), T_\varepsilon(b)) \cong \operatorname{Ext}^1_{\mathcal{R}^\uparrow}(\Delta_\varepsilon^\uparrow(a), T_\varepsilon^\uparrow(b)) = 0.$$

By Theorem 3.41, this shows that $T_\varepsilon(b) \in \nabla_\varepsilon^{\mathrm{dsc}}(\mathcal{R})$.

Note finally that $T_\varepsilon(b)$ is indecomposable. This follows because $jT_\varepsilon(b)$ is indecomposable for every $j : \mathcal{R} \to \mathcal{R}^\uparrow$ (adopting the usual notation). Indeed, by the construction we have that $jT_\varepsilon(b) \cong T_\varepsilon^\uparrow(b)$ This completes the construction of the indecomposable object $T_\varepsilon(b) \in \mathcal{T}ilt_\varepsilon(\mathcal{R})$. We have shown that it satisfies (iii), and it follows easily that it also satisfies (i) and (ii).

Uniqueness: Since (iii) implies (i) and (ii), it suffices to show that any indecomposable $U \in \mathcal{T}ilt_\varepsilon(\mathcal{R})$ satisfying either (i) or (ii) is isomorphic to the object $T := T_\varepsilon(b)$ just constructed. We explain this just in the case of (i), since the argument for (ii) is similar. We take a short exact sequence $0 \to \Delta_\varepsilon(b) \to T \to Q \to 0$ with $Q \in \Delta_\varepsilon^{\mathrm{asc}}(\mathcal{R})$. Using the Ext-vanishing from Lemma 3.38, we deduce like in the proof of Theorem 4.2 that the inclusion $f : \Delta_\varepsilon(b) \hookrightarrow T$ extends to $\bar{f} : U \to T$. In fact, $\bar{f}$ is an isomorphism. To see this, take a finite upper set $\Lambda^\uparrow$ containing λ and consider the quotient $j : \mathcal{R} \to \mathcal{R}^\uparrow$ as usual. Both jU and jT are isomorphic to $T_\varepsilon^\uparrow(b)$ by the uniqueness in Theorem 4.2. The proof there implies that any homomorphism $jT \to jU$ which restricts to an isomorphism on the subobject $\Delta^\uparrow_\varepsilon(b)$ is an isomorphism. We deduce that $j\bar f$ is an isomorphism. Since holds for all choices of $\Lambda^\uparrow$, it follows that $\bar f$ itself is an isomorphism. □

Corollary 4.18. *Any object of $\mathcal{T}ilt_\varepsilon(\mathcal{R})$ is isomorphic to $\bigoplus_{b\in\mathbf{B}} T_\varepsilon(b)^{\oplus n_b}$ for unique multiplicities $n_b \in \mathbb{N}$. Conversely, any such direct sum belongs to $\mathcal{T}ilt_\varepsilon(\mathcal{R})$.*

Proof. Let us first show that any direct sum $U := \bigoplus_{b\in\mathbf{B}} T_\varepsilon(b)^{\oplus n_b}$ belongs to $\mathcal{T}ilt_\varepsilon(\mathcal{R})$. The only issue is to see that U actually belongs to $\mathcal{R}$ rather than $\operatorname{Ind}(\mathcal{R}_c)$, i.e., it has finite composition multiplicities. But for a given $c \in \mathbf{B}$, the multiplicity $[T_\varepsilon(b) : L(c)]$ is zero unless $\rho(c) \leqslant \rho(b)$. There are only finitely many such $b \in \mathbf{B}$, so $[U : L(c)] = \sum_{b\in\mathbf{B}} n_b[T_\varepsilon(b) : L(c)] < \infty$.

Now take any $U \in \mathcal{T}ilt_\varepsilon(\mathcal{R})$. Let Ω be the directed set of all finite upper sets in Λ. Take $\omega \in \Omega$, say it is the finite upper set $\Lambda^\uparrow$. Let $j : \mathcal{R} \to \mathcal{R}^\uparrow$ be the quotient functor as usual. Then we have that $jU \in \mathcal{T}ilt_\varepsilon(\mathcal{R}^\uparrow)$, so it decomposes as a finite direct sum as $jU \cong \bigoplus_{b\in\mathbf{B}^\uparrow} T^\uparrow_\varepsilon(b)^{\oplus n_b(\omega)}$ for $n_b(\omega) \in \mathbb{N}$. There is a corresponding direct summand $T_\omega \cong \bigoplus_{b\in\mathbf{B}^\uparrow} T_\varepsilon(b)^{\oplus n_b(\omega)}$ of U. Then $T = \varinjlim T_\omega$. Moreover, for $b \in \mathbf{B}^\uparrow$, the multiplicities $n_b(\omega)$ are stable in the sense that $n_b(\upsilon) = n_b(\omega)$ for all $\upsilon > \omega$. We deduce that $U \cong \bigoplus_{b\in\mathbf{B}} T_\varepsilon(b)^{\oplus n_b}$ where $n_b := n_b(\omega)$ for any sufficiently large ω. □

There are also obvious analogs of Corollaries 4.4 and 4.5 in the upper finite setting.

4.4. **Semi-infinite Ringel duality.** Throughout the subsection, Λ will be a lower finite poset and $\varepsilon : \Lambda \to \{\pm\}$ is a sign function. The opposite poset Λ^{op} is upper finite. The goal is to extend Ringel duality to include stratifications indexed by Λ or Λ^{op}. The situation is not as symmetric as in the finite case and demands different constructions when going from lower finite to upper finite or from upper finite to lower finite. If we start with a lower finite ε-stratified category, the Ringel dual is an upper finite $(-\varepsilon)$-stratified category:

Definition 4.19. Let $\mathcal{R}$ be a lower finite ε-stratified category with stratification $\rho : \mathbf{B} \to \Lambda$. An *$\varepsilon$-tilting generating family* is a family $(T_i)_{i\in I}$ of ε-tilting objects in $\mathcal{R}$ such that every $T_\varepsilon(b)$ is isomorphic to a summand of T_i for some $i \in I$. Define the *Ringel dual* of $\mathcal{R}$ relative to $T := \bigoplus_{i\in I} T_i \in \operatorname{Ind}(\mathcal{R})$ to be the Schurian category $\widetilde{\mathcal{R}} := A\text{-mod}_{\mathrm{lfd}}$, where $A = \bigoplus_{i,j\in I} e_iAe_j$ is the locally finite-dimensional locally unital algebra with $e_iAe_j := \operatorname{Hom}_{\mathcal{R}}(T_i, T_j)$ and multiplication that is the opposite of composition in $\mathcal{R}$. Identifying $\operatorname{Ind}(\widetilde{\mathcal{R}}_c)$ with A-mod as explained in (2.3), we have the *Ringel duality functor*

$$F := \bigoplus_{i\in I} \operatorname{Hom}_{\mathcal{R}}(T_i, -) : \operatorname{Ind}(\mathcal{R}) \to \operatorname{Ind}(\widetilde{\mathcal{R}}_c). \tag{4.8}$$

Theorem 4.20. *In the setup of Definition 4.19, $\widetilde{\mathcal{R}}$ is an upper finite $(-\varepsilon)$-stratified category with stratification defined from $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$. Its distinguished objects satisfy*

$$\widetilde{P}(b) \cong FT_\varepsilon(b), \qquad \widetilde{L}(b) \cong \operatorname{hd} \widetilde{P}(b),$$
$$\widetilde{\Delta}_{-\varepsilon}(b) \cong F\nabla_\varepsilon(b), \qquad \widetilde{T}_{-\varepsilon}(b) \cong FI(b).$$

The restriction $F : \nabla^{\mathrm{asc}}_\varepsilon(\mathcal{R}) \to \Delta^{\mathrm{asc}}_{-\varepsilon}(\widetilde{\mathcal{R}})$ is an equivalence of categories.

The proof will be explained later. In the other direction, if we start from an upper finite $(-\varepsilon)$-stratified category, the Ringel dual is a lower finite ε-stratified category:

Definition 4.21. Let $\widetilde{\mathcal{R}}$ be an upper finite $(-\varepsilon)$-stratified category with stratification $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$. A *$(-\varepsilon)$-tilting generator* is an object $\widetilde{T} \in \mathcal{T}ilt_{-\varepsilon}(\widetilde{\mathcal{R}})$ such that $\widetilde{T}_{-\varepsilon}(b)$ is a summand of $\widetilde{T}$ for every $b \in \mathbf{B}$. By Lemma 2.10, the algebra $B := \operatorname{End}_{\widetilde{\mathcal{R}}}(\widetilde{T})^{\mathrm{op}}$ is a pseudocompact topological algebra with respect to the profinite topology; let C be the coalgebra that is its continuous dual. Then the *Ringel dual* of $\mathcal{R}$ relative to $\widetilde{T}$ is the category $\mathcal{R} := B\text{-mod}_{\mathrm{fd}} \cong \mathrm{comod}_{\mathrm{fd}}\text{-}C$. Recalling the continuous duality functor $\circledast$ from (2.7) and the definition of the functor (2.14), we define the *Ringel duality functor*

$$F^* := \circledast \circ \mathrm{Hom}_{\widetilde{\mathcal{R}}}(-, \widetilde{T}) : \mathrm{Ind}(\widetilde{\mathcal{R}}_c) \to \mathrm{Ind}(\mathcal{R}). \tag{4.9}$$

Theorem 4.22. *In the setup of Definition 4.21, $\mathcal{R}$ is a lower finite ε-stratified category with stratification defined from $\rho : \mathbf{B} \to \Lambda$. Its distinguished objects satisfy*

$$I(b) \cong F^*\widetilde{T}_{-\varepsilon}(b), \qquad L(b) \cong \mathrm{soc}\, I(b),$$
$$\nabla_\varepsilon(b) \cong F^*\widetilde{\Delta}_{-\varepsilon}(b), \qquad T_\varepsilon(b) \cong F^*\widetilde{P}(b).$$

The restriction $F^* : \Delta^{\mathrm{asc}}_{-\varepsilon}(\widetilde{\mathcal{R}}) \to \nabla^{\mathrm{asc}}_{\varepsilon}(\mathcal{R})$ is an equivalence of categories.

Remark 4.23. In Definitions 4.19 and 4.21, we have only defined an analog of the functor F from (4.4) when going from lower finite to upper finite, and an analog of the functor G from (4.5) when going from upper finite to lower finite. In order to define precise analogs of F and G in the other directions, one would need to work systematically everywhere with pro-completions rather than ind-completions, which we have assiduously avoided.

The following two corollaries give the analogs of the double centralizer property from Corollary 4.12 in the semi-infinite setting.

Corollary 4.24. *Let notation be as in Definition 4.19. Assume in addition that $\mathcal{R} =$ $\mathrm{comod}_{\mathrm{fd}}\text{-}C$ for a coalgebra C. Let $B := C^*$ be the dual algebra, so that T is a (B, A)-bimodule. Let $\widetilde{T} := T^{\circledast}$ be the dual (A, B)-bimodule.*

(1) $\widetilde{T}$ is a $(-\varepsilon)$-tilting generator in $\widetilde{\mathcal{R}}$ such that $B \cong \mathrm{End}_{\widetilde{\mathcal{R}}}(\widetilde{T})^{\mathrm{op}}$. Thus, the Ringel dual of $\widetilde{\mathcal{R}}$ relative to $\widetilde{T}$ is isomorphic to the original category $\mathcal{R}$.
(2) The functor F^* from (4.9) is isomorphic to the functor $T \otimes_A -$ defined as in (2.12). Moreover, (F^*, F) is an adjoint pair thanks to Lemma 2.11.

Proof. By Lemma 2.2, we have that $\mathrm{Hom}_C(T_i, C) \cong T_i^*$ as right B-modules, hence, $FC \cong \widetilde{T}$ as an (A, B)-bimodule. Since every $I(b)$ appears as a summand of the regular comodule, and $FI(b) \cong \widetilde{T}_{-\varepsilon}(b)$ by Theorem 4.20, we deduce that $\widetilde{T}$ is a $(-\varepsilon)$-tilting generator in $\widetilde{\mathcal{R}}$. To see that $B = \mathrm{End}_A(\widetilde{T})^{\mathrm{op}}$, we use the fact that F is an equivalence on ∇-filtered objects to deduce that

$$\mathrm{End}_A(\widetilde{T})^{\mathrm{op}} \cong \mathrm{End}_A(FC)^{\mathrm{op}} \cong \mathrm{End}_C(C)^{\mathrm{op}} \cong B,$$

using Lemma 2.2 again for the final algebra isomorphism. This establishes (1). For (2), we get that $F^* = \circledast \circ \mathrm{Hom}_A(-, \widetilde{T}) \cong T \otimes_A -$. Lemma 2.11 now implies that F^* is left adjoint to F. □

Corollary 4.25. *Let notation be as in Definition 4.21, and assume in addition that $\widetilde{R} = A\text{-mod}_{\mathrm{lfd}}$ for a locally finite-dimensional locally unital algebra $A = \bigoplus_{i,j \in I} e_i A e_j$. Let $T := \widetilde{T}^{\circledast}$, which is a (B, A)-bimodule. Set $T_i := Te_i \in B\text{-mod}_{\mathrm{fd}}$. Then $(T_i)_{i \in I}$ is an ε-tilting generating family in $\mathcal{R}$ such that $A \cong \left(\bigoplus_{i,j \in I} \mathrm{Hom}_{\mathcal{R}}(T_i, T_j)\right)^{\mathrm{op}}$. Thus, the Ringel dual of $\mathcal{R}$ relative to T is isomorphic to the category $\widetilde{\mathcal{R}}$.*

Proof. Note that $T_i = F^*(Ae_i)$. So Theorem 4.22 implies that $(T_i)_{i \in I}$ is an ε-tilting generating family in $\mathcal{R}$. Moreover,

$$\mathrm{Hom}_{\mathcal{R}}(T_i, T_j) = \mathrm{Hom}_{\mathcal{R}}(F^*(Ae_i), F^*(Ae_j)) \cong \mathrm{Hom}_A(Ae_i, Ae_j).$$

The corollary now follows. □

Corollary 4.13 carries over to the semi-infinite case. We leave this to the reader. We have not investigated derived equivalences or any analog of Theorem 4.15 in this setting.

Proof of Theorem 4.20. We may assume that $\mathcal{R} = \text{comod}_{\text{fd}}\text{-}C$ for a coalgebra C. Let $B := C^*$ be the dual algebra, so that $\mathcal{R}$ is identified also with $B\text{-mod}_{\text{fd}}$. We can replace the ε-tilting generating family $(T_i)_{i\in I}$ with any other such family. This just has the effect of transforming A into a Morita equivalent locally unital algebra. Consequently, without loss of generality, we may assume that $I = \mathbf{B}$ and $(T_i)_{i\in I} = (T_\varepsilon(b))_{b\in\mathbf{B}}$. Then

$$A = \bigoplus_{a,b\in\mathbf{B}} \text{Hom}_B(T_\varepsilon(a), T_\varepsilon(b))$$

is a pointed locally finite-dimensional locally unital algebra, with primitive idempotents $\{e_b \mid b \in \mathbf{B}\}$. Let $\widetilde{P}(b) := Ae_b$ and $\widetilde{L}(b) := \text{hd}\ \widetilde{P}(b)$. Then $\widetilde{\mathcal{R}} = A\text{-mod}_{\text{lfd}}$ is a Schurian category, the A-modules $\{\widetilde{L}(b) \mid b \in \mathbf{B}\}$ give a full set of pairwise inequivalent irreducible objects in $\widetilde{\mathcal{R}}$, and $\widetilde{P}(b)$ is a projective cover of $\widetilde{L}(b)$ in $\text{Ind}(\widetilde{\mathcal{R}}_c) = A\text{-mod}$. It is immediate that $\rho : \mathbf{B} \to \Lambda^{\text{op}}$ gives a stratification of $\widetilde{\mathcal{R}}$. Let $\widetilde{\Delta}_{-\varepsilon}(b)$ and $\widetilde{\nabla}_{-\varepsilon}(b)$ be its $(-\varepsilon)$-standard and $(-\varepsilon)$-costandard objects. Also let $V(b) := F\nabla_\varepsilon(b)$. Now one checks that Steps 1–6 from the proof of Theorem 4.11 carry over to the present situation with very minor modifications. We will not rewrite these steps here, but cite them freely below. In particular, Step 6 establishes that $\widetilde{\mathcal{R}}$ is an upper finite $(-\varepsilon)$-stratified category. Also, $F\nabla_\varepsilon(b) \cong \widetilde{\Delta}_{-\varepsilon}(b)$ by Step 5. It just remains to show that

- F restricts to an equivalence of categories between $\nabla_\varepsilon^{\text{asc}}(\mathcal{R})$ and $\Delta_{-\varepsilon}^{\text{asc}}(\widetilde{\mathcal{R}})$;
- $FI(b) \cong \widetilde{T}_{-\varepsilon}(b)$.

This requires some different arguments compared to the ones from Steps 7–10 in the proof of Theorem 4.11.

Let Ω be the directed poset consisting of all finite lower sets in Λ. Take $\omega \in \Omega$, say it is the lower set $\Lambda^\downarrow$. Let $\nabla_\varepsilon(\mathcal{R}, \omega)$ be the full subcategory of $\nabla_\varepsilon(\mathcal{R})$ consisting of the ∇_ε-filtered objects with sections $\nabla_\varepsilon(b)$ for $b \in \mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$. Similarly, we define the subcategory $\Delta_{-\varepsilon}(\widetilde{\mathcal{R}}, \omega)$ of $\Delta_{-\varepsilon}(\widetilde{\mathcal{R}})$. By Steps 2 and 5, F restricts to a well-defined functor

$$F : \nabla_\varepsilon(\mathcal{R}, \omega) \to \Delta_{-\varepsilon}(\widetilde{\mathcal{R}}, \omega). \tag{4.10}$$

We claim that this is an equivalence of categories. To prove it, let $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the finite ε-stratified subcategory of $\mathcal{R}$ associated to $\Lambda^\downarrow$. Let $e := \sum_{b\in\mathbf{B}^\downarrow} e_b \in A$. Then $T^\downarrow := \bigoplus_{b\in\mathbf{B}^\downarrow} T_\varepsilon(b)$ is an ε-tilting generator in $\mathcal{R}^\downarrow$. As $\text{End}_{\mathcal{R}^\downarrow}(T^\downarrow)^{\text{op}} = eAe$, the Ringel dual of $\mathcal{R}^\downarrow$ relative to $T^\downarrow$ is the quotient category $\widetilde{\mathcal{R}}^\downarrow := eAe\text{-mod}_{\text{fd}}$ of $\widetilde{R}$. Let $F^\downarrow := \text{Hom}_C(T^\downarrow, -)$ be the corresponding Ringel duality functor. We also know that $\widetilde{\mathcal{R}}^\downarrow$ is the finite $(-\varepsilon)$-stratified quotient of $\widetilde{\mathcal{R}}$ associated to $\Lambda^\downarrow$ (which is a finite upper set in Λ^{op}). Let $\jmath : \widetilde{\mathcal{R}} \to \widetilde{\mathcal{R}}^\downarrow$ be the quotient functor, i.e., the functor defined by multiplication by the idempotent e. For a right C-comodule V, we have that

$$F^\downarrow(i^! V) \cong \bigoplus_{b\in\mathbf{B}^\downarrow} \text{Hom}_C(T_\varepsilon(b), i^! V) \cong e \bigoplus_{b\in\mathbf{B}} \text{Hom}_C(T_\varepsilon(b), V) \cong \jmath(FV).$$

This shows that

$$F^\downarrow \circ i^! \cong \jmath \circ F. \tag{4.11}$$

By Theorem 4.11, $F^\downarrow$ gives an equivalence $\nabla_\varepsilon(\mathcal{R}^\downarrow) \to \Delta_{-\varepsilon}(\widetilde{\mathcal{R}}^\downarrow)$. Also $i^! : \nabla_\varepsilon(\mathcal{R}, \omega) \to \nabla_\varepsilon(\mathcal{R}^\downarrow)$ and $\jmath : \Delta_{-\varepsilon}(\widetilde{\mathcal{R}}, \omega) \to \Delta_{-\varepsilon}(\widetilde{\mathcal{R}}^\downarrow)$ are equivalences. This is clear for $i^!$. To see it for $\jmath$, one shows using Theorem 3.44 that the left adjoint $\jmath_!$ gives a quasi-inverse equivalence. Putting these things together, we deduce that (4.10) is an equivalence as claimed.

Now we can show that F restricts to an equivalence $F : \nabla_\varepsilon^{\text{asc}}(\mathcal{R}) \to \Delta_{-\varepsilon}^{\text{asc}}(\widetilde{\mathcal{R}})$. Take $V \in \nabla_\varepsilon^{\text{asc}}(\mathcal{R})$. Then V has a distinguished ascending ∇_ε-flag $(V_\omega)_{\omega\in\Omega}$ indexed by the set Ω of finite lower sets in Λ. This is defined by setting $V_\omega := i^! V$ in the notation of the previous paragraph; see the proof of Theorem 3.59. As each comodule $T_\varepsilon(b)$ is finite-dimensional, hence, compact, the functor F commutes with direct limits. Hence, $FV \cong \varinjlim(FV_\omega)$. In fact, $(FV_\omega)_{\omega\in\Omega}$ is the data of an ascending $\Delta_{-\varepsilon}$-flag in $FV \in \widetilde{\mathcal{R}}$. To see this, we have that $FV_\omega \in \Delta_{-\varepsilon}(\widetilde{\mathcal{R}})$ by the previous paragraph. For $\omega < \upsilon$ the quotient V_υ/V_ω has a ∇_ε-flag thanks to Corollary 3.61, so $FV_\upsilon/FV_\omega \cong F(V_\upsilon/V_\omega)$ has a $\Delta_{-\varepsilon}$-flag. We still need to show that FV is locally finite-dimensional. For this, we prove that $\dim \mathrm{Hom}_A(FV, \widetilde{I}(b)) < \infty$ for each $b \in \mathbf{B}$. Since $\widetilde{I}(b)$ has a finite $\nabla_{-\varepsilon}$-flag, this reduces to checking that $\dim \mathrm{Hom}_A(FV, \widetilde{\nabla}_{-\varepsilon}(b)) < \infty$ for each b, which holds because the multiplicities $(V_\omega : \nabla_\varepsilon(b))$ are bounded by the definition of the category $\nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$.

At this point, we have proved that F restricts to a well-defined functor

$$F : \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R}) \to \Delta_{-\varepsilon}^{\mathrm{asc}}(\widetilde{\mathcal{R}}).$$

We prove that this is an equivalence by showing that the left adjoint $F^* := T \otimes_A -$ to F gives a quasi-inverse. The left mate of (4.11) gives an isomorphism

$$i \circ (F^\downarrow)^* \cong F^* \circ \tilde{\jmath}_!. \tag{4.12}$$

Combining this with Corollary 4.12, we deduce that F^* restricts to a quasi-inverse of the equivalence (4.10) for each $\omega \in \Omega$. Also, F^* commutes with direct limits, and again any $\widetilde{V} \in \Delta_{-\varepsilon}^{\mathrm{asc}}(\widetilde{\mathcal{R}})$ has a distinguished ascending $\Delta_{-\varepsilon}$-flag $(\widetilde{V}_\omega)_{\omega\in\Omega}$ as we saw in the proof of Theorem 3.39. These facts are enough to show that F^* restricts to a well-defined functor $F^* : \Delta_{-\varepsilon}^{\mathrm{asc}}(\widetilde{\mathcal{R}}) \to \nabla_\varepsilon^{\mathrm{asc}}(\mathcal{R})$ which is quasi-inverse to F.

Finally, we check that $FI(b) \cong \widetilde{T}_{-\varepsilon}(b)$. Let $V := I(b)$ and $(V_\omega)_{\omega\in\Omega}$ be its distinguished ascending ∇_ε-flag indexed by the set Ω of finite lower sets in Λ. Using the same notation as above, for ω that is a lower set $\Lambda^\downarrow$ satisfying $\rho(b) \in \Lambda^\downarrow$, we know that V_ω is an injective hull of $L(b)$ in $\mathcal{R}^\downarrow$. Hence, by Theorem 4.11, $F^\downarrow V_\omega \cong \widetilde{T}^\downarrow_{-\varepsilon}(b) \in \widetilde{\mathcal{R}}^\downarrow$. From this, we see that the ascending $\Delta_{-\varepsilon}$-flag (FV_ω) in $FI(b)$ coincides with the distinsguished ascending $\Delta_{-\varepsilon}$-flag in $\widetilde{T}_{-\varepsilon}(b)$ from the construction from the proof of Theorem 4.17. □

Proof of Theorem 4.22. We may assume that $\widetilde{\mathcal{R}} = A\text{-mod}_{\mathrm{lfd}}$ for a pointed locally finite-dimensional locally unital algebra $A = \bigoplus_{a,b\in\mathbf{B}} e_a A e_b$, so that $\widetilde{T}$ is a locally finite-dimensional left A-module. Let $T := \widetilde{T}^{\circledast}$ and $C := T \otimes_A T^{\circledast}$, which we view as a coalgebra according to (2.11). By Lemma 2.10, this coalgebra is the continuous dual of $B = \mathrm{End}_A(\widetilde{T})^{\mathrm{op}}$. We may identify $\mathcal{R}$ with $\mathrm{comod}_{\mathrm{fd}}\text{-}C$, which is a locally finite Abelian category. Applying Lemma 2.11, we can also identify the Ringel duality functor F^* with the functor $T \otimes_A - : A\text{-mod} \to \mathrm{comod}\text{-}C$, the comodule structure map of $T \otimes_A V$ being defined as in (2.13). Let

$$I(b) := F^*\widetilde{T}_{-\varepsilon}(b), \qquad \nabla_\varepsilon(b) := F^*\widetilde{\Delta}_{-\varepsilon}(b), \qquad L(b) := \mathrm{soc}\ I(b). \tag{4.13}$$

Each $I(b)$ is an indecomposable injective right C-comodule, and $\{L(b)\,|\,b \in \mathbf{B}\}$ is a full set of pairwise inequivalent irreducible comodules. Since $\widetilde{\Delta}_{-\varepsilon}(b) \hookrightarrow \widetilde{T}_{-\varepsilon}(b)$, and F^* is exact on $\Delta_{-\varepsilon}^{\mathrm{asc}}(\widetilde{\mathcal{R}})$ by the original definition of F^* and the Ext^1-vanishing from Lemma 3.38, we see that $\nabla_\varepsilon(b) \hookrightarrow I(b)$. Thus, we also have that $L(b) = \mathrm{soc}\ \nabla_\varepsilon(b)$.

Now let $\Lambda^\downarrow$ be a finite lower set in Λ. Set $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$, and let $\tilde{\jmath} : \widetilde{\mathcal{R}} \to \widetilde{\mathcal{R}}^\downarrow$ be the corresponding Serre quotient of $\widetilde{\mathcal{R}}$. Since $\Lambda^\downarrow$ is a finite upper set in Λ^{op}, this is a finite $(-\varepsilon)$-stratified category thanks to Theorem 3.44. In fact, $\widetilde{\mathcal{R}}^\downarrow = eAe\text{-mod}_{\mathrm{lfd}}$ where $e := \sum_{b\in\mathbf{B}^\downarrow} e_b$, and $\tilde{\jmath}$ is the functor defined by multiplying by e. By the upper finite analog of Corollary 4.5, $e\widetilde{T}$ is a $(-\varepsilon)$-tilting generator in $\widetilde{\mathcal{R}}^\downarrow$. Let $B^\downarrow := \mathrm{End}_{eAe}(e\widetilde{T})^{\mathrm{op}}$ be its (finite-dimensional) endomorphism algebra. Then $\mathcal{R}^\downarrow := B^\downarrow\text{-mod}_{\mathrm{fd}}$ is the Ringel dual of $\widetilde{\mathcal{R}}^\downarrow$ relative to $e\widetilde{T}$. By the finite Ringel duality from Theorem 4.11, $\mathcal{R}^\downarrow$ is a finite ε-stratified category. Let $(F^\downarrow)^* : \widetilde{\mathcal{R}}^\downarrow \to \mathcal{R}^\downarrow$ be its Ringel duality functor.

The functor $\tilde{\jmath}$ defines an algebra homomorphism

$$\pi : B \to B^\downarrow. \tag{4.14}$$

We claim that π is surjective. To prove this, consider the short exact sequence

$$0 \longrightarrow Ae \otimes_{eAe} e\widetilde{T} \longrightarrow \widetilde{T} \longrightarrow Q \longrightarrow 0 \tag{4.15}$$

which comes from the upper finite counterpart of Lemma 3.21(2); thus, $Q \in \Delta^{\mathrm{asc}}_{-\varepsilon}(\mathcal{R})$ and all of its sections are of the form $\widetilde{\Delta}_{-\varepsilon}(b)$ for $b \notin \mathbf{B}^{\downarrow}$. Applying the functor $\mathrm{Hom}_A(-,\widetilde{T})$ and using that $\mathrm{Ext}^1_A(Q,\widetilde{T}) = 0$, we deduce that the natural restriction map $\mathrm{Hom}_A(\widetilde{T},\widetilde{T}) \to \mathrm{Hom}_A(Ae \otimes_{eAe} e\widetilde{T},\widetilde{T})$ is surjective. Since $\mathrm{Hom}_A(Ae \otimes_{eAe} e\widetilde{T},\widetilde{T}) \cong \mathrm{Hom}_{eAe}(e\widetilde{T},e\widetilde{T})$, this proves the claim.

From the claim, we see that the functor $i : \mathcal{R}^{\downarrow} \to \mathcal{R}$ defined by restriction through the epimorphism π includes $\mathcal{R}^{\downarrow}$ as an Abelian subcategory of $\mathcal{R}$. We claim moreover that that

$$i \circ (F^{\downarrow})^* \cong F^* \circ \tilde{\jmath}_!. \tag{4.16}$$

This can be proved in the same way as (4.12) above, but the following alternative argument is more convenient in the present setting: consider the finite-dimensional dual coalgebra

$$C^{\downarrow} := (B^{\downarrow})^*.$$

The dual map π^* to (4.14) defines a coalgebra homomorphism $C^{\downarrow} \to C$. Moreover, if we identify $C^{\downarrow}$ with $Te \otimes_{eAe} e\widetilde{T}$ like in Lemma 2.9 then π^* corresponds to the obvious coalgebra homomorphism $Te \otimes_{eAe} e\widetilde{T} \to T \otimes_A \widetilde{T}$ induced by the inclusion $Te \otimes e\widetilde{T} \hookrightarrow T \otimes \widetilde{T}$. Since π is surjective, the dual map π^* is injective, so it identifies $C^{\downarrow}$ with a subcoalgebra of C. Now the functor $(F^{\downarrow})^*$ is $Te \otimes_{eAe} - : eAe\text{-mod} \to \text{comod-}C^{\downarrow}$, and we get (4.16) since $T \otimes_A Ae \otimes_{eAe} V \cong Te \otimes_{eAe} V$ for any eAe-module V.

From (4.16) and Theorem 3.44(6), we see that ε-costandard objects of $\mathcal{R}^{\downarrow}$ are the comodules $\{\nabla_{\varepsilon}(b) | b \in \mathbf{B}^{\downarrow}\}$ defined by (4.13). Representatives for the isomorphism classes of irreducible objects in $\mathcal{R}^{\downarrow}$ are given by the socles $\{L(b) \mid b \in \mathbf{B}^{\downarrow}\}$ of these costandard objects. In fact, $\mathcal{R}^{\downarrow}$ is the Serre subcategory of $\mathcal{R}$ generated by $\{L(b) \mid b \in \mathbf{B}^{\downarrow}\}$. To prove this, by Lemma 2.20, it suffices to show that $C^{\downarrow}$ is the largest right coideal of C such that all of its irreducible subquotients are of the form $\{L(b) \mid b \in \mathbf{B}^{\downarrow}\}$. Apply F^* to (4.15), using the exactness noted before, to get a short exact sequence

$$0 \longrightarrow Te \otimes_{eAe} e\widetilde{T} \longrightarrow T \otimes_A \widetilde{T} \longrightarrow F^*Q \longrightarrow 0.$$

Since $C^{\downarrow} = Te \otimes_{eAe} e\widetilde{T}$ and $C = T \otimes_A \widetilde{T}$, this shows that $C/C^{\downarrow} \cong F^*Q$. To finish the argument we show that all irreducible constituents of $\mathrm{soc}\,(F^*Q)$ are of the form $L(b)$ for $b \notin \mathbf{B}^{\downarrow}$. Fix an ascending $\Delta_{-\varepsilon}$-flag $(V_\omega)_{\omega\in\Omega}$ in Q. As F^* commutes with direct limits and is exact on $\Delta_{-\varepsilon}$-flags, we deduce that F^*Q is the union of subobjects of the form F^*V_ω. Now the sections in a Δ_{ε}-flag in V_ω are $\widetilde{\Delta}_{-\varepsilon}(b)$ for $b \notin \mathbf{B}^{\downarrow}$, hence, F^*V_ω has a ∇_{ε}-flag with sections $\nabla_{\varepsilon}(b)$ for $b \notin \mathbf{B}^{\downarrow}$. It follows that $\mathrm{soc}\,(F^*V_\omega)$ is of the desired form for each ω, hence, the socle of F^*Q is too.

We can now complete the proof of the theorem. Theorem 3.63 implies that $\mathcal{R}$ is a lower finite ε-stratified category. Theorem 4.11 once again shows for any choice of $\Lambda^{\downarrow}$ that the ε-tilting object of $\mathcal{R}^{\downarrow}$ indexed by $b \in \mathbf{B}^{\downarrow}$ is

$$T^{\downarrow}_{\varepsilon}(b) := F^{*\downarrow}(\tilde{\jmath}\widetilde{P}(b)) \cong F^*(\tilde{\jmath}_!(\tilde{\jmath}\widetilde{P}(b))) \cong F^*\widetilde{P}(b).$$

This is also the ε-tilting object $T_{\varepsilon}(b)$ of $\mathcal{R}$ due to Corollary 4.4. Also, for $a, b \in \mathbf{B}^{\downarrow}$, we have that

$$\mathrm{Hom}_C(T_{\varepsilon}(a), T_{\varepsilon}(b)) \cong \mathrm{Hom}_{C^{\downarrow}}(T^{\downarrow}_{\varepsilon}(a), T^{\downarrow}_{\varepsilon}(b)) \cong \mathrm{Hom}_{eAe}(e\widetilde{P}(a), e\widetilde{P}(b)) \cong e_a A e_b.$$

These things are true for all choices of $\Lambda^{\downarrow}$, so we see that the Ringel dual of $\mathcal{R}$ relative to $\bigoplus_{b\in\mathbf{B}} T_{\varepsilon}(b)$ is the original category $A\text{-mod}_{\mathrm{lfd}}$. This puts us in the situation of Corollary 4.24, and finally we invoke that corollary (whose proof did not depend on Theorem 4.22) to establish that $F^* : \Delta^{\mathrm{asc}}_{-\varepsilon}(\widetilde{\mathcal{R}}) \to \nabla^{\mathrm{asc}}_{\varepsilon}(\mathcal{R})$ is an equivalence. $\square$

4.5. **The essentially finite case.** In this subsection, we let $\mathcal{R}$ be an essentially finite ε-stratified category with stratification $\rho : \mathbf{B} \to \Lambda$. As usual, Λ^{op} denotes the opposite poset. Since Λ is interval finite, unions of lower sets of the form $(-\infty, \lambda]$ are upper finite. If $\mathcal{R}^{\downarrow}$ is the Serre subcategory of $\mathcal{R}$ associated to such an upper finite lower set then its Schurian envelope $\mathrm{Env}(\mathcal{R}^{\downarrow})$ in the sense of Lemma 2.18 is an upper finite ε-stratified category. This follows from Theorem 3.19.

For $b \in \mathbf{B}$, we define the corresponding ε-tilting object $T_\varepsilon(b) \in \mathrm{Env}(\mathcal{R})$ as follows: pick any upper finite lower set $\Lambda^{\downarrow}$ such that $\rho(b) \in \Lambda^{\downarrow}$, let $\mathcal{R}^{\downarrow}$ be the corresponding Serre subcategory of $\mathcal{R}$, then let $T_\varepsilon(b)$ be the ε-tilting object in $\mathrm{Env}(\mathcal{R}^{\downarrow})$ from Theorem 4.17. This is well-defined independent of the choice of $\Lambda^{\downarrow}$ by the uniqueness part of Theorem 4.17. We will only consider Ringel duality in the essentially finite case under the hypothesis that $\mathcal{R}$ is *ε-tilting-bounded*, meaning that the matrix

$$\big(\dim \mathrm{Hom}_{\mathcal{R}}(T_\varepsilon(a), T_\varepsilon(b))\big)_{a,b\in\mathbf{B}} \tag{4.17}$$

has only finitely many non-zero entries in every row and in every column. This condition implies in particular that each $T_\varepsilon(b)$ is of finite length, i.e., it belongs to $\mathcal{R}$ rather than $\mathrm{Env}(\mathcal{R})$.

Remark 4.26. Most of the interesting examples of essentially finite highest weight categories which arise "in nature" seem to satisfy the tilting-bounded hypothesis, although there is no reason for this to be the case from the recursive construction of Theorem 4.17. We refer the reader to Remark 6.2 for an explicit essentially finite example which is not tilting-bounded.

Assuming $\mathcal{R}$ is ε-tilting-bounded, we define

$$\mathcal{T}ilt_\varepsilon(\mathcal{R}) := \Delta_\varepsilon(\mathcal{R}) \cap \nabla_\varepsilon(\mathcal{R}) \tag{4.18}$$

just like in (4.1). Theorem 4.2 carries over easily, to show that $\{T_\varepsilon(b) \mid b \in \mathbf{B}\}$ gives a full set of the indecomposable objects in the Karoubian category $\mathcal{T}ilt_\varepsilon(\mathcal{R})$. The construction of Theorem 4.9 also carries over unchanged. So all objects of $\nabla_\varepsilon(\mathcal{R})$ have ε-tilting resolutions and all objects of $\Delta_\varepsilon(\mathcal{R})$ have ε-cotilting resolutions.

Definition 4.27. Assume $\mathcal{R}$ is an essentially finite ε-stratified category with stratification $\rho : \mathbf{B} \to \Lambda$. Assume in addition that $\mathcal{R}$ is ε-tilting bounded. An *ε-tilting generating family* in $\mathcal{R}$ means a family $(T_i)_{i\in I}$ of objects $T_i \in \mathcal{R}$ such that each T_i is a direct sum of the objects $T_\varepsilon(b)$ and every $T_\varepsilon(b)$ appears as a summand of at least one and at most finitely many different T_i. Given such a family, we define the *Ringel dual* of $\mathcal{R}$ relative to $T := \bigoplus_{i\in I} T_i \in \mathrm{Env}(\mathcal{R})$ to be the category $\widetilde{\mathcal{R}} := A\text{-mod}_{\mathrm{fd}}$ where $A := \left(\bigoplus_{i,j\in I} \mathrm{Hom}_{\mathcal{R}}(T_i, T_j)\right)^{\mathrm{op}}$. Also define the two Ringel duality functors

$$F := \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{R}}(T_i, -) : \mathcal{R} \to \widetilde{\mathcal{R}}, \tag{4.19}$$

$$G := * \circ \mathrm{Hom}_{\mathcal{R}}(-, T) : \mathcal{R} \to \widetilde{\mathcal{R}}. \tag{4.20}$$

Theorem 4.28. *In the setup of Definition 4.27, $\widetilde{\mathcal{R}}$ is an essentially finite $(-\varepsilon)$-stratified category with stratification defined from $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$. Moreover, $\widetilde{\mathcal{R}}$ is $(-\varepsilon)$-tilting-bounded. Its distinguished objects satisfy*

$$\widetilde{P}(b) \cong FT_\varepsilon(b), \qquad \widetilde{I}(b) \cong GT_\varepsilon(b), \qquad \widetilde{L}(b) \cong \mathrm{hd}\, \widetilde{P}(b) \cong \mathrm{soc}\, \widetilde{I}(b),$$
$$\widetilde{\Delta}_{-\varepsilon}(b) \cong F\nabla_\varepsilon(b), \qquad \widetilde{\nabla}_{-\varepsilon}(b) \cong G\Delta_\varepsilon(b), \qquad \widetilde{T}_{-\varepsilon}(b) \cong FI(b) \cong GP(b).$$

The restrictions $F : \nabla_\varepsilon(\mathcal{R}) \to \Delta_{-\varepsilon}(\widetilde{\mathcal{R}})$ and $G : \Delta_\varepsilon(\mathcal{R}) \to \nabla_{-\varepsilon}(\widetilde{\mathcal{R}})$ are equivalences.

Proof. We may assume that $\mathcal{R} = B\text{-mod}_{\mathrm{fd}}$ for an essentially finite-dimensional locally unital algebra $B = \bigoplus_{k,l\in J} f_k B f_l$. The assumption that $\mathcal{R}$ is ε-tilting-bounded implies that $\sum_{i\in I} \dim \mathrm{Hom}_{\mathcal{R}}(T_i, T_j) < \infty$ and $\sum_{j\in I} \dim \mathrm{Hom}_{\mathcal{R}}(T_i, T_j) < \infty$ for each $i, j \in I$.

Thus, the locally unital algebra A is also essentially finite-dimensional, i.e., $\widetilde{\mathcal{R}}$ is essentially finite Abelian. For $b \in \mathbf{B}$, pick $i(b) \in I$ and a primitive idempotent $e_b \in e_{i(b)} A e_{i(b)}$ such that $T_{i(b)} e_b \cong T_\varepsilon(b)$. Then $\widetilde{P}(b) := Ae_b$ is an indecomposable projective A-module, and

$$\left\{ \widetilde{L}(b) := \operatorname{hd} \widetilde{P}(b) \;\middle|\; b \in \mathbf{B} \right\}$$

is a full set of pairwise inequivalent irreducibles. It is immediate that $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$ defines a stratification of $\widetilde{\mathcal{R}}$. One checks that Steps (1)–(12) from the proof of Theorem 4.11 all go through essentially unchanged in the present setting. This completes the proof but for one point: we must observe finally that $\widetilde{\mathcal{R}}$ is $(-\varepsilon)$-tilting-bounded. This follows because the analog of the matrix (4.17) for $\widetilde{\mathcal{R}}$ is the Cartan matrix

$$\left(\dim \operatorname{Hom}_B(P(a), P(b)) \right)_{a,b \in \mathbf{B}}$$

of $\mathcal{R}$. Its rows and columns have only finitely many non-zero entries as B is essentially finite-dimensional. □

Corollary 4.29. *Suppose that the ε-stratified category $\mathcal{R}$ in Theorem 4.28 is B-$\mathrm{mod}_{\mathrm{fd}}$ for an essentially finite-dimensional locally unital algebra $B = \bigoplus_{k,l \in J} f_k B f_l$, so that $T = \bigoplus_{k \in J} f_k T$ is a (B, A)-bimodule. Let $\widetilde{T} := T^{\circledast}$, which is an (A, B)-bimodule. Then the following holds.*

(1) The modules $(\widetilde{T}_k := \widetilde{T} f_k)_{k \in J}$ constitute a $(-\varepsilon)$-tilting generating family in $\widetilde{\mathcal{R}}$ such that $B \cong \left(\bigoplus_{k,l \in J} \operatorname{Hom}_A(\widetilde{T}_k, \widetilde{T}_l) \right)^{\mathrm{op}}$. Hence, the Ringel dual of $\widetilde{\mathcal{R}}$ relative to $\widetilde{T}$ is isomorphic to the original category $\mathcal{R}$.

(2) Denote the Ringel duality functors $\widetilde{F}$ and $\widetilde{G}$ for $\widetilde{\mathcal{R}}$ with respect to $\widetilde{T}$ instead by $G_* := \bigoplus_{k \in J} \operatorname{Hom}_A(\widetilde{T}_k, -) : \widetilde{\mathcal{R}} \to \mathcal{R}$ and $F^* := * \circ \operatorname{Hom}_A(-, \widetilde{T}) : \widetilde{\mathcal{R}} \to \mathcal{R}$, respectively. We have that $F^* \cong T \otimes_A -$ and $G \cong \widetilde{T} \otimes_B -$. Hence, (F^*, F) and (G, G_*) are adjoint pairs.

Proof. For (1), note that $(G(Bf_k))_{k \in J}$ is a $(-\varepsilon)$-tilting generating family since $GP(b) \cong \widetilde{T}_{-\varepsilon}(b)$ for $b \in \mathbf{B}$. Actually, $G(Bf_k) = \operatorname{Hom}_B(Bf_k, T)^* \cong (f_k T)^* = \widetilde{T} f_k = \widetilde{T}_k$. Thus, $(\widetilde{T}_k)_{k \in J}$ is a $(-\varepsilon)$-tilting generating family in $\widetilde{\mathcal{R}}$. To obtain the isomorphism between B and the locally finite endomorphism algebra of $\bigoplus_{k \in J} \widetilde{T}_k$, apply the functor G to the canonical isomorphism $B \cong \left(\bigoplus_{k,l \in J} \operatorname{Hom}_B(Bf_k, Bf_l) \right)^{\mathrm{op}}$. The proof of (2) is the same as in the proof of Corollary 4.12. □

We leave it to the reader to adapt Corollary 4.13 to the present setting.

5. Generalizations of quasi-hereditary algebras

In this section, we give some applications of semi-infinite Ringel duality. First, we use it to show that any upper finite highest weight category can be realized as A-$\mathrm{mod}_{\mathrm{lfd}}$ for an upper finite based quasi-hereditary algebra A. The latter notion, which is Definition 5.1, already exists in the literature in some equivalent forms. When A is finite-dimensional, it gives an alternative algebraic characterization of the usual notion of quasi-hereditary algebra. Then, in §5.2, we introduce further notions of based ε-stratified algebras and based ε-quasi-hereditary algebras, which correspond to ε-stratified categories and ε-highest weight categories, respectively. In §5.3, we introduce based stratified algebras and fibered quasi-hereditary algebras, which are related to the notions of fully stratified and signed highest weight categories, respectively. Finally, in §5.4, we relate based stratified algebras to locally unital algebras with Cartan decompositions (a new idea) and triangular decompositions (which have already appeared in some form in the literature).

5.1. **Based quasi-hereditary algebras.** The following definition is a translation of [ELau, Definition 2.1] from the framework of $\Bbbk$-linear categories to that of locally unital algebras. Also, for finite-dimensional algebras, it is equivalent to [KM, Definition 2.4]. These assertions will be explained in more detail in Remark 5.5 below.

Definition 5.1. An *upper finite* (resp., *essentially finite*) *based quasi-hereditary algebra* is a locally finite-dimensional (resp., essentially finite-dimensional) locally unital algebra $A = \bigoplus_{i,j\in I} e_i A e_j$ with the following additional data:

(QH1) A subset $\Lambda \subseteq I$ indexing the *special idempotents* $\{e_\lambda \mid \lambda \in \Lambda\}$;
(QH2) A partial order $\leqslant$ on the set Λ which is upper finite (resp., interval finite);
(QH3) Sets $Y(i,\lambda) \subset e_i A e_\lambda$, $X(\lambda, j) \subset e_\lambda A e_j$ for $\lambda \in \Lambda, i, j \in I$.

Let $Y(\lambda) := \bigcup_{i\in I} Y(i,\lambda)$ and $X(\lambda) := \bigcup_{j\in I} X(\lambda, j)$. We impose the following axioms:

(QH4) The products yx for all $(y,x) \in \bigcup_{\lambda\in\Lambda} Y(\lambda) \times X(\lambda)$ give a basis for A.
(QH5) For $\lambda, \mu \in \Lambda$, the sets $Y(\lambda,\mu)$ and $X(\mu,\lambda)$ are empty unless $\lambda \leqslant \mu$.
(QH6) We have that $Y(\lambda,\lambda) = X(\lambda,\lambda) = \{e_\lambda\}$ for each $\lambda \in \Lambda$.

We say simply that A is a *locally unital based quasi-hereditary algebra* if it is either an upper finite or an essentially finite based quasi-hereditary algebra.

The archetypal example of a (unital) based quasi-hereditary algebra as in Definition 5.1 is of course the classical Schur algebra $S(n,r)$ with its basis of *codeterminants* as constructed by Green in [Gre]. In general, we refer to the basis for A from (QH4) as the *idempotent-adapted cellular basis*. In the presence of an additional anti-involution interchanging the sets $X(\lambda)$ and $Y(\lambda)$, it is a cellular basis in the general sense of [GL], [Wes], but Definition 5.1 is considerably more restrictive than the general notion of cellular algebra/category, hence, the additional “idempotent-adapted” in our terminology (which mirrors the “object-adapted” from [ELau, Definition 2.1]). In fact, as we will explain more fully below, for finite-dimensional algebras, Definition 5.1 is equivalent to the usual notion of quasi-hereditary algebra. Also note that the idempotent-adapted cellular basis for A is far from being unique, indeed, one can replace any $Y(i,\lambda)$ or $X(\lambda,i)$ by another basis that spans the same subspace up to “higher terms.”

Remark 5.2. It is clear from (QH4) that $A = \bigoplus_{\lambda\in\Lambda} A e_\lambda A$. Hence, A is Morita equivalent to the idempotent truncation $\bigoplus_{\lambda,\mu\in\Lambda} e_\lambda A e_\mu$. This means that if one is prepared to pass to a Morita equivalent algebra then one can assume without loss of generality that the sets Λ and I in Definition 5.1 are actually equal, i.e., all distinguished idempotents are special. However, in naturally-occurring examples, one often encounters situations in which the set I is stricly larger than Λ.

Lemma 5.3. *Let A be a locally unital based quasi-hereditary algebra. For $\lambda \in \Lambda$, any element f of the two-sided ideal $A e_\lambda A$ can be written as a linear combination of elements of the form yx for $y \in Y(\mu)$, $x \in X(\mu)$ and $\mu \geqslant \lambda$.*

Proof. We first consider the upper finite case. We proceed by downward induction on the partial order on Λ. By considering the cellular basis, we may assume that $f = y_1 x_1 y_2 x_2$ for $y_1 \in Y(\mu_1), x_1 \in X(\mu_1, \lambda)$, $y_2 \in Y(\lambda, \mu_2), x_2 \in X(\mu_2)$ and $\mu_1, \mu_2 \geqslant \lambda$. If $\mu_r > \lambda$ for some $r \in \{1,2\}$, then we have that $f \in A e_{\mu_r} A$ for this r, and get done by induction. If $\mu_1 = \mu_2 = \lambda$ then $x_1 = e_\lambda = y_2$ and $f = y_1 x_2$, as required.

The essentially finite case is similar. Assuming that $f \in e_i A e_j$ for $i, j \in I$, the assumption that A is essentially finite-dimensional implies that there are only finitely many $\mu \in \Lambda$ such that $e_i A e_\mu \neq 0$ or $e_\mu A e_j \neq 0$. Letting Λ' be the finite set of all such μ, we can then proceed by downward induction on the partial order on Λ' as in the previous paragraph. $\square$

Corollary 5.4. *Let $\Lambda^\uparrow$ be an upper set in Λ. The two-sided ideal $J_{\Lambda^\uparrow}$ of A generated by $\{e_\lambda \mid \lambda \in \Lambda^\uparrow\}$ has basis $\left\{yx \;\middle|\; (y,x) \in \bigcup_{\lambda\in\Lambda^\uparrow} Y(\lambda) \times X(\lambda)\right\}$.*

Proof. Let J be the subspace of A with basis given by the products yx for $y \in Y(\lambda), x \in X(\lambda)$ and $\lambda \in \Lambda^\uparrow$. For any such element $yx \in J$, we have that $yx = ye_\lambda x$, hence, $yx \in J_{\Lambda^\uparrow}$. This shows that $J \subseteq J_{\Lambda^\uparrow}$. Conversely, any element of $J_{\Lambda^\uparrow}$ is a linear combination of elements of $Ae_\lambda A$ for $\lambda \in \Lambda^\uparrow$. In turn, Lemma 5.3 shows that any element of $Ae_\lambda A$ for $\lambda \in \Lambda^\uparrow$ is a linear combination of elements yx for $y \in Y(\mu), x \in X(\mu)$ and $\mu \geqslant \lambda$. Since $\Lambda^\uparrow$ is an upper set, all of these elements yx belong to J, hence, $J_{\Lambda^\uparrow} \subseteq J$. □

Remark 5.5. In the upper finite case, Definition 5.1 is equivalent to the notion of object-adapted cellular category[5] from [ELau, Definition 2.1]. This can be seen from Corollary 5.4 and [ELau, Lemmas 2.6–2.8]; we have imposed the additional assumption that the underlying categories are finite-dimensional. When A is a finite-dimensional algebra, i.e., A is unital rather than locally unital, Definition 5.1 is equivalent to the notion of based quasi-hereditary algebra from [KM, Definition 2.4]. To see this, one takes our set Λ indexing the special idempotents to be the set I from [KM]; this set indexes mutually orthogonal idempotents $e_i \in A$ according to [KM, Lemma 2.8]. Then we take our set I to be the set $\Lambda \sqcup \{0\}$, i.e., we add one more element indexing one more idempotent $e_0 := 1_A - \sum_{\lambda\in\Lambda} e_\lambda$. Kleshchev and Muth established the equivalence of their notion of based quasi-hereditary algebra with the original notion of quasi-hereditary algebra from [CPS1] (providing the partial order on Λ is actually a total order); we will reprove this equivalence in a different way below. See also [DuR] which established a similar result using a related notion of standardly based algebra.

Let A be a locally unital based quasi-hereditary algebra. For $\lambda \in \Lambda$, let $A_{\leqslant\lambda}$ be the quotient of A by the two-sided ideal generated by the idempotents e_μ for $\mu \nleqslant \lambda$. For $y \in A$, we often write simply $\bar{y}$ for the image of y in $A_{\leqslant\lambda}$. Corollary 5.4 implies that

$$A_{\leqslant\lambda} = \bigoplus_{i,j\in I} \bar{e}_i A_{\leqslant\lambda} \bar{e}_j \tag{5.1}$$

is an upper finite based quasi-hereditary algebra in its own right, with special idempotents indexed by elements of the lower set $(-\infty, \lambda]$ and idempotent-adapted cellular basis given by the products $\bar{y}\bar{x}$ for $y \in Y(i,\mu), x \in X(\mu, j)$, $i, j \in I$ and $\mu \in (-\infty, \lambda]$. Define the *standard* and *costandard modules* associated to $\lambda \in \Lambda$ from

$$\Delta(\lambda) := A_{\leqslant\lambda}\bar{e}_\lambda \qquad\qquad \nabla(\lambda) = (\bar{e}_\lambda A_{\leqslant\lambda})^{\circledast}. \tag{5.2}$$

These are left A-modules which are projective and injective as $A_{\leqslant\lambda}$-modules, respectively. The modules $\Delta(\lambda)$ may also be called *cell modules* and the modules $\nabla(\lambda)$ *dual cell modules*. The vectors $\{y\bar{e}_\lambda \mid y \in Y(\lambda)\}$ give the *standard basis* for $\Delta(\lambda)$. Similarly, the vectors $\{\bar{e}_\lambda x \mid x \in X(\lambda)\}$ give a basis for the right A-module $\bar{e}_\lambda A$; the dual basis to this is the *costandard basis* $\{(\bar{e}_\lambda x)^\vee \mid x \in X(\lambda)\}$ for $\nabla(\lambda)$. In the essentially finite case $\Delta(\lambda)$ and $\nabla(\lambda)$ are finite-dimensional, but in the upper finite case they are merely locally finite-dimensional.

Theorem 5.6. *Let A be an upper finite (resp., an essentially finite) based quasi-hereditary algebra as above. For $\lambda \in \Lambda$, the standard module $\Delta(\lambda)$ has a unique irreducible quotient denoted $L(\lambda)$. The modules $\{L(\lambda) \mid \lambda \in \Lambda\}$ give a complete set of pairwise inequivalent irreducible A-modules. Moreover, the category $\widetilde{\mathcal{R}} := A\text{-mod}_{\mathrm{lfd}}$ (resp., $\widetilde{\mathcal{R}} := A\text{-mod}_{\mathrm{fd}}$) is an upper finite (resp., an essentially finite) highest weight category. Its standard and costandard objects $\Delta(\lambda)$ and $\nabla(\lambda)$ are as defined by (5.2), with the partial order on Λ being the given one.*

Proof. For $\lambda \in \Lambda$, let P_λ be the left ideal Ae_λ. We claim that P_λ has a Δ-flag with $\Delta(\lambda)$ at the top and other sections of the form $\Delta(\mu)$ for $\mu > \lambda$. To prove this, fix some λ and set $P := P_\lambda$ for short. This module has basis yx for $\mu \geqslant \lambda$, $y \in Y(\mu)$ and $x \in X(\mu, \lambda)$. Let $\{\mu_1, \dots, \mu_n\}$ be the finite set $\{\mu \in [\lambda, \infty) \mid X(\mu, \lambda) \neq \varnothing\}$ ordered so

[5]Strictly speaking, the notion in [ELau] corresponds to what we would call an upper finite based quasi-hereditary algebra *with duality*, since it assumes the presence of an additional symmetry.

that $\mu_r \leqslant \mu_s \Rightarrow r \leqslant s$; in particular, $\mu_1 = \lambda$. Let P_r be the subspace of P spanned by all yx for $s = r+1, \dots, n, y \in Y(\mu_s)$ and $x \in X(\mu_s, \lambda)$. In fact, each P_r is a submodule of P, and this defines a filtration $P = P_0 > P_1 > \cdots > P_n = 0$. Moreover, there is an A-module isomorphism

$$\theta : \bigoplus_{x \in X(\mu_r, \lambda)} \Delta(\mu_r) \xrightarrow{\sim} P_{r-1}/P_r \tag{5.3}$$

sending the basis vector $y\bar{e}_{\mu_r}$ $(y \in Y(\mu_r))$ in the xth copy of $\Delta(\mu_r)$ to $yx + P_r \in P_{r-1}/P_r$. To prove this, θ is clearly a linear isomorphism, so we just need to check that it is an A-module homomorphism. Take $y \in Y(j, \mu_r)$ and $u \in e_i A e_j$. Expand uy in terms of the cellular basis as $\sum_p c_p y_p + \sum_q c'_q y'_q x'_q$ for scalars c_p, c'_q, $y_p \in Y(i, \mu_r)$, $y'_q \in Y(i, \nu_q), x'_q \in X(\nu_q, \mu_r)$ and $\nu_q > \mu_r$. Then we have that $uy\bar{e}_{\mu_r} = \sum_p c_p y_p \bar{e}_{\mu_r}$ and $uyx + P_r = \sum_p c_p y_p x + P_r$, since the "higher terms" $y'_q x'_q$ act as zero on both $\bar{e}_{\mu_r}$ and $x + P_r$. This is all that is needed to prove that θ intertwines the actions of u on $\Delta(\mu_r)$ and P_{r-1}/P_r. Since $P_0/P_1 \cong \Delta(\lambda)$, the claim is now proved.

Now we can classify the irreducible A-modules. The first step for this is to show that $\Delta(\lambda)$ has a unique irreducible quotient. To see this, note that the "weight space" $e_\lambda \Delta(\lambda)$ is one-dimensional with basis $\bar{e}_\lambda$, due to the fact that $Y(\lambda, \lambda) = \{e_\lambda\}$. This is a cyclic vector, so any proper submodule of $\Delta(\lambda)$ must intersect $e_\lambda \Delta(\lambda)$ trivially. It follows that the sum of all proper submodules is proper, so $\Delta(\lambda)$ has a unique irreducible quotient $L(\lambda)$. Since $e_\lambda L(\lambda)$ is one-dimensional and all other μ with $e_\mu L(\lambda) \neq 0$ satisfy $\mu < \lambda$, the modules $\{L(\lambda) \mid \lambda \in \Lambda\}$ are pairwise inequivalent. To see that they give a full set of irreducible A-modules, let L be any irreducible A-module. In view of Remark 5.2, there exists $\lambda \in \Lambda$ such that $e_\lambda L \neq 0$. Then L is a quotient of $P_\lambda = Ae_\lambda$. By the claim established in the previous paragraph, it follows that L is a quotient of $\Delta(\mu)$ for some $\mu \geqslant \lambda$, i.e., $L \cong L(\mu)$.

Thus, we have shown that the modules $\{L(\lambda) \mid \lambda \in \Lambda\}$ give a full set of pairwise inequivalent irreducible A-modules. Now consider the stratification of $\widetilde{\mathcal{R}}$ arising from the partial order on the index set Λ. In the recollement situation of (3.1), the Serre subcategory $\widetilde{\mathcal{R}}_{\leqslant\lambda}$ may be identified with $A_{\leqslant\lambda}\text{-mod}_{\text{lfd}}$ (resp., $A_{\leqslant\lambda}\text{-mod}_{\text{fd}}$), and the Serre quotient $\widetilde{\mathcal{R}}_\lambda = \widetilde{\mathcal{R}}_{\leqslant\lambda}/\widetilde{\mathcal{R}}_{<\lambda}$ is $\bar{A}_\lambda\text{-mod}_{\text{fd}}$ where $\bar{A}_\lambda := \bar{e}_\lambda A_{\leqslant\lambda} \bar{e}_\lambda$. The algebra $\bar{A}_\lambda$ has basis $\bar{e}_\lambda$, i.e., it is a copy of the ground field $\Bbbk$. This shows that all strata are simple in the sense of Lemma 3.6. Moreover, the standard and costandard objects in the general sense of (1.1) are obtained by applying the standardization functor $j_!^\lambda := A_{\leqslant\lambda}\bar{e}_\lambda \otimes_{\bar{A}_\lambda} -$ and the costandardization functor $j_*^\lambda := \bigoplus_{i \in I} \operatorname{Hom}_{\bar{A}_\lambda}(\bar{e}_\lambda A_{\leqslant\lambda} \bar{e}_i, -)$ to the irreducible $\bar{A}_\lambda$-module $\bar{A}_\lambda$. Clearly, the resulting modules are isomorphic to $\Delta(\lambda)$ and $\nabla(\lambda)$ as defined by (5.2). To complete the proof, it remains to observe that the axiom $(\widehat{P\Delta})$ follows from the claim established in the opening paragraph of the proof. □

The following theorem gives a converse to Theorem 5.6. The proof is an application of Ringel duality together with the general construction of cellular bases for endomorphism algebras of tilting modules explained in [AST]. We will give a self-contained proof of the latter result in the next subsection, when we generalize it to ε-stratified categories.

Theorem 5.7. *Let $\widetilde{\mathcal{R}}$ be an upper finite (resp., essentially finite) highest weight category with weight poset Λ, and let $A = \bigoplus_{i,j \in I} e_i A e_j$ be an algebra realization of it. In the essentially finite case, assume in addition that $\widetilde{\mathcal{R}}$ is tilting-bounded. There is an idempotent expansion $A = \bigoplus_{i,j \in \hat{I}} e_i A e_j$ with $\Lambda \subseteq \hat{I}$, and subsets $Y(i, \lambda) \subset e_i A e_\lambda, X(\lambda, j) \subset e_\lambda A e_j$ for all $\lambda \in \Lambda$ and $i, j \in \hat{I}$, making A into an upper finite (resp., an essentially finite) based quasi-hereditary algebra with respect to the given ordering on Λ.*

Proof. Since we are allowed to pass to an idempotent expansion of A, i.e., to refine the given set distinguished idempotents, we may as well assume that the idempotents e_i $(i \in I)$ given initially are all primitive. Then for each $\lambda \in \Lambda$, there exists some $i(\lambda) \in I$

such that the left ideal $Ae_{i(\lambda)}$ corresponds under the equivalence between $A\text{-mod}_{\mathrm{lfd}}$ (resp., $A\text{-mod}_{\mathrm{fd}}$) and $\widetilde{\mathcal{R}}$ to the projective cover $\widetilde{P}(\lambda)$ of the irreducible object $\widetilde{L}(\lambda) \in \widetilde{\mathcal{R}}$. In this way, we obtain an embedding $i : \Lambda \hookrightarrow I$. Henceforth, we will identify the set Λ with a subset of I via the map thus chosen.

In the language of Definition 4.21 (resp., Definition 4.27), let $\mathcal{R}$ be the Ringel dual of $\widetilde{\mathcal{R}}$ with respect to some choice of $\widetilde{T}$. The weight poset for $\mathcal{R}$ is Λ^{op}, i.e., the poset Λ with the opposite ordering, which is lower finite. Then let $T_i := (e_i\widetilde{T})^*$. By Corollary 4.24 (resp., Corollary 4.29 applied with the roles of $\mathcal{R}$ and B interchanged with $\widetilde{R}$ and A), $(T_i)_{i\in I}$ is a $(-\varepsilon)$-tilting generating family for $\mathcal{R}$ such that the original algebra $A = \bigoplus_{i,j\in I} e_iAe_j$ is isomorphic as a locally unital algebra to $\left(\bigoplus_{i,j\in I} \mathrm{Hom}_{\mathcal{R}}(T_i, T_j)\right)^{\mathrm{op}}$.

Now we need to produce the finite sets $Y(i,\lambda) \subset \mathrm{Hom}_{\mathcal{R}}(T_i, T_\lambda)$ and $X(\lambda, j) \subset \mathrm{Hom}_{\mathcal{R}}(T_\lambda, T_j)$. We want $Y(\lambda,\lambda)$ and $X(\lambda,\lambda)$ to consist just of the identity endomorphism $e_\lambda \in \mathrm{End}_{\mathcal{R}}(T_\lambda)$, $Y(\lambda,\mu)$ and $X(\mu,\lambda)$ to be empty if $\lambda \nleqslant \mu$, and the morphisms $\{x \circ y \mid y \in Y(i,\lambda), x \in X(\lambda, j), \lambda \in \Lambda\}$ should give a basis for $\mathrm{Hom}_{\mathcal{R}}(T_i, T_j)$. But all of this follows by an application of [AST, Theorem 3.1]: to obtain bases of this form we just have to choose $Y(i,\lambda)$ to be any lift of a basis of $\mathrm{Hom}_{\mathcal{R}}(T_i, \nabla(\lambda))$ and $X(\lambda, j)$ to be any lift of a basis of $\mathrm{Hom}_{\mathcal{R}}(\Delta(\lambda), T_j)$; see (5.7) below for the helpful picture. We can clearly choose these lifts so that $Y(\lambda,\lambda) = \{e_\lambda\} = X(\lambda,\lambda)$. Moreover, $Y(\lambda,\mu)$ and $X(\mu,\lambda)$ are empty when $\lambda \nleqslant \mu$ because $\mathrm{Hom}_{\mathcal{R}}(T(\lambda), \nabla(\mu))$ and $\mathrm{Hom}_{\mathcal{R}}(\Delta(\lambda), T(\mu))$ are zero for such λ, μ (remembering that for $\mathcal{R}$ we are working with the opposite ordering on Λ). We refer to *loc. cit.* (or Lemma 5.15 below) for further explanations. $\square$

Remark 5.8. In particular, Theorems 5.6–5.7 show that any finite highest weight category can be realized as $A\text{-mod}_{\mathrm{fd}}$ for a unital based quasi-hereditary algebra A, and conversely all such module categories are finite highest weight categories. This recovers a special case of [KM, Proposition 3.5]. It is only a special case because Kleshchev and Muth work over more general ground rings than us[6].

5.2. **Based ε-stratified and ε-quasi-hereditary algebras.** In this subsection, we upgrade the results of §5.1 to ε-stratified and ε-highest weight categories. The main new definition is as follows.

Definition 5.9. An *upper finite* (resp., *essentially finite*) *based ε-stratified algebra* is a locally finite-dimensional (resp., essentially finite-dimensional) locally unital algebra $A = \bigoplus_{i,j\in I} e_iAe_j$ with the following additional data:

- (εS1) A subset $\mathbf{B} \subseteq I$ indexing the *special idempotents* $\{e_b \mid b \in \mathbf{B}\}$.
- (εS2) An upper finite (resp., interval finite) poset $(\Lambda, \leqslant)$ and sign function $\varepsilon : \Lambda \to \{\pm\}$.
- (εS3) A stratification function $\rho : \mathbf{B} \to \Lambda$ with finite fibers $\mathbf{B}_\lambda := \rho^{-1}(\lambda)$.
- (εS4) Sets $Y(i,b) \subset e_iAe_b$ and $X(b,j) \subset e_bAe_j$ for all $b \in \mathbf{B}$ and $i,j \in I$.

Let $Y(b) := \bigcup_{i\in I} Y(i,b)$ and $X(b) := \bigcup_{j\in I} X(b,j)$. There are then four axioms, the first three of which are as follows:

- (εS5) The products yx for all $(y,x) \in \bigcup_{b\in\mathbf{B}} Y(b) \times X(b)$ give a basis for A.
- (εS6) For $a,b \in \mathbf{B}$, the sets $Y(a,b)$ and $X(b,a)$ are empty unless $\rho(a) \leqslant \rho(b)$.
- (εS7) The following hold for all $\lambda \in \Lambda$ and $a,b \in \mathbf{B}_\lambda$:
  - if $\varepsilon(\lambda) = -$ then $Y(a,b) = \{e_a\}$ when $a = b$ and $Y(a,b) = \varnothing$ otherwise;
  - if $\varepsilon(\lambda) = +$ then $X(a,b) = \{e_a\}$ when $a = b$ and $X(a,b) = \varnothing$ otherwise.

To formulate the fourth axiom, let $e_\lambda := \sum_{b\in\mathbf{B}_\lambda} e_b$ for short. Let $A_{\leqslant\lambda}$ be the quotient of A by the two-sided ideal generated by $\{e_\mu \mid \mu \nleqslant \lambda\}$ and $\bar{A}_\lambda := \bar{e}_\lambda A_{\leqslant\lambda}\bar{e}_\lambda$ (where $\bar{y} \in A_{\leqslant\lambda}$ denotes the image of $y \in A$ as usual). Then:

[6]They also consider graded analogs (both $\mathbb{Z}$- and $\mathbb{Z}/2-$). These can be fitted into our general approach by the general construction explained in Remark 5.25 below.

(εS8) For each $\lambda \in \Lambda$, the finite-dimensional algebra $\bar{A}_\lambda$ is a basic algebra, and its set of primitive idempotents is $\{\bar{e}_b \mid b \in \mathbf{B}_\lambda\}$.

We also have the following, which is nothing more than Definition 5.9 rewritten in the special case that the stratification function ρ is a bijection.

Definition 5.10. An *upper finite* (resp., *essentially finite*) *based ε-quasi-hereditary algebra* is a locally finite-dimensional (resp., essentially finite-dimensional) locally unital algebra $A = \bigoplus_{i,j\in I} e_i A e_j$ with the following additional data:

(εQH1) A subset $\Lambda \subseteq I$ indexing the *special idempotents* $\{e_\lambda \mid \lambda \in \Lambda\}$.
(εQH2) An partial order $\leqslant$ on Λ which is upper finite (resp., interval finite) and a sign function $\varepsilon : \Lambda \to \{\pm\}$.
(εQH3) Sets $Y(i,\lambda) \subset e_i A e_\lambda$, $X(\lambda, j) \subset e_\lambda A e_j$ for $\lambda \in \Lambda$, $i, j \in I$.

Let $Y(\lambda) := \bigcup_{i\in I} Y(i,\lambda)$ and $X(\lambda) := \bigcup_{j\in I} X(\lambda, j)$. The axioms are as follows:

(εQH4) The products yx for all $(y,x) \in \bigcup_{\lambda\in\Lambda} Y(\lambda) \times X(\lambda)$ give a basis for A.
(εQH5) For $\lambda, \mu \in \Lambda$, the sets $Y(\lambda,\mu)$ and $X(\mu,\lambda)$ are empty unless $\lambda \leqslant \mu$.
(εQH6) If $\varepsilon(\lambda) = -$ then $Y(a,b) = \{e_a\}$, and if $\varepsilon(\lambda) = +$ then $X(a,b) = \{e_a\}$.
(εQH7) The finite-dimensional algebra $\bar{A}_\lambda$ defined as in Definition 5.9 is basic and local.

From now on, we just formulate the results for based ε-stratified algebras, since based ε-quasi-hereditary algebras are a special case. In fact, the development below parallels the treatment in the previous subsection, but there are some additional subtleties. Remark 5.2 remains true: one can always pass to a Morita equivalent algebra in which all of the distinguished idempotents are special. The analog of Lemma 5.3 is as follows.

Lemma 5.11. *Let A be a locally unital based ε-stratified algebra. For $\lambda \in \Lambda$, any element f of the two-sided ideal $Ae_\lambda A$ can be written as a linear combination of elements of the form yx for $y \in Y(a), x \in X(a)$ and $a \in \mathbf{B}$ with $\rho(a) \geqslant \lambda$.*

Proof. This is similar to the proof of Lemma 5.3. We just explain in the upper finite case. We may assume that $f = y_1x_1y_2x_2$ for $y_1 \in Y(a_1), x_1 \in X(a_1, b), y_2 \in Y(b, a_2), x_2 \in X(a_2)$, $b \in \mathbf{B}_\lambda$ and $a_1, a_2 \in \mathbf{B}$ with $\rho(a_1), \rho(a_2) \geqslant \lambda$. If $\rho(a_1) > \lambda$ or $\rho(a_2) > \lambda$, we are done by induction. If $\rho(a_1) = \rho(a_2) = \lambda$, there are two cases according to whether $\varepsilon(\lambda) = +$ or $\varepsilon(\lambda) = -$. The arguments for these are similar, so we just go through the former case when $\varepsilon(\lambda) = +$. Then we have that $a_1 = b$ and $x_1 = e_b$. Hence $f = y_1y_2x_2$. Then we use the basis again to expand y_1y_2 as a linear combination of terms y_3x_3 for $y_3 \in Y(a_3), x_3 \in X(a_3, a_2)$ and $a_3 \in \mathbf{B}$ with $\rho(a_3) \geqslant \lambda$. If $\rho(a_3) > \lambda$, we can then rewrite $y_3x_3x_2$ in the desired form by induction. If $\rho(a_3) = \lambda$ then we get that $a_3 = a_2$ and $x_3 = e_{a_2}$, so $y_3x_3x_2 = y_3x_2$ as required. $\square$

Corollary 5.12. *Let $\Lambda^\uparrow$ be an upper set in Λ. The two-sided ideal $J_{\Lambda^\uparrow}$ of A generated by $\{e_\lambda \mid \lambda \in \Lambda^\uparrow\}$ has basis $\left\{yx \,\middle|\, (y,x) \in \bigcup_{b\in\rho^{-1}(\Lambda^\uparrow)} Y(b) \times X(b)\right\}$.*

Let A be a locally unital based ε-stratified algebra. Take $\lambda \in \Lambda$ and consider the basic algebra $\bar{A}_\lambda = \bar{e}_\lambda A_{\leqslant\lambda} \bar{e}_\lambda$ from Definition 5.9. It has basis $\left\{\bar{y} \,\middle|\, y \in \bigcup_{a,b\in\mathbf{B}_\lambda} Y(a,b)\right\}$ if $\varepsilon(\lambda) = +$ or $\left\{\bar{x} \,\middle|\, x \in \bigcup_{a,b\in\mathbf{B}_\lambda} X(a,b)\right\}$ if $\varepsilon(\lambda) = -$. For $b \in \mathbf{B}_\lambda$, let

$$P_\lambda(b) := \bar{A}_\lambda \bar{e}_b, \qquad I_\lambda(b) := (\bar{e}_b \bar{A}_\lambda)^\circledast, \qquad L_\lambda(b) := \operatorname{hd} P_\lambda(b) \cong \operatorname{soc} I_\lambda(b). \tag{5.4}$$

Then we define standard, proper standard, costandard and proper costandard modules:

$$\Delta(b) := A_{\leqslant\lambda}\bar{e}_b \cong j_!^\lambda P_\lambda(b), \qquad \bar{\Delta}(b) := j_!^\lambda L_\lambda(b), \tag{5.5}$$

$$\nabla(b) := (\bar{e}_b A_{\leqslant\lambda})^\circledast \cong j_*^\lambda I_\lambda(b), \qquad \bar{\nabla}(b) := j_*^\lambda L_\lambda(b), \tag{5.6}$$

where now the standardization functor $j_!^\lambda$ is defined from $j_!^\lambda := A_{\leqslant\lambda}\bar{e}_\lambda \otimes_{\bar{A}_\lambda} -$, and the costandardization functor j_*^λ is $\bigoplus_{i\in I} \operatorname{Hom}_{\bar{A}_\lambda}(\bar{e}_\lambda A_{\leqslant\lambda}\bar{e}_i, -)$. Adopt the shorthands $\Delta_\varepsilon(b)$ and $\nabla_\varepsilon(b)$ from (1.2) too. The module $\Delta_\varepsilon(b)$ has a standard basis indexed by the set $Y(b)$. In the case that $\varepsilon(\lambda) = +$, when $\Delta_\varepsilon(b) = \Delta(b)$, this basis is $\{y\bar{e}_b \mid y \in Y(b)\}$. In the case that $\varepsilon(\lambda) = -$, when $\Delta_\varepsilon(b) = \bar{\Delta}(b)$, let $\tilde{e}_b$ be the canonical image of $\bar{e}_b$ under the natural quotient map $\Delta(b) \twoheadrightarrow \bar{\Delta}(b)$. Then the basis is $\{y\tilde{e}_b \mid y \in Y(b)\}$. (One can also construct a costandard basis for $\nabla_\varepsilon(b)$ indexed by $X(b)$ by taking a certain dual basis, but we will not need this here.)

Lemma 5.13. *If $\varepsilon(\lambda) = -$ then the standardization functor $j_!^\lambda$ is exact.*

Proof. It suffices to show that $A_{\leqslant\lambda}\bar{e}_\lambda$ is projective as a right $\bar{A}_\lambda$-module. This follows because there is an isomorphism of right $\bar{A}_\lambda$-modules

$$\bigoplus_{a\in\mathbf{B}_\lambda, y\in Y(a)} \bar{e}_a\bar{A}_\lambda \xrightarrow{\sim} A_{\leqslant\lambda}\bar{e}_\lambda$$

sending the vector $\bar{e}_a$ in the yth copy of $\bar{e}_a\bar{A}_\lambda$ to $\bar{y} \in A_{\leqslant\lambda}\bar{e}_\lambda$. To see this, the module $A_{\leqslant\lambda}\bar{e}_\lambda$ has basis given by products $\bar{y}\bar{x}$ for $y \in Y(a), x \in X(a,b)$ and $a, b \in \mathbf{B}_\lambda$. As $\varepsilon(\lambda) = -$, the projective right $\bar{A}_\lambda$-module $\bar{e}_a\bar{A}_\lambda$ has basis $\bar{x}$ for $x \in X(a,b)$ and $b \in \mathbf{B}_\lambda$. $\square$

Theorem 5.14. *Let A be an upper finite (resp., essentially finite) based ε-stratified algebra as above. For $b \in \mathbf{B}$, the standard module $\Delta(b)$ has a unique irreducible quotient denoted $L(b)$. The modules $\{L(b) \mid b \in \mathbf{B}\}$ give a complete set of pairwise inequivalent irreducible A-modules. Moreover, $\widetilde{\mathcal{R}} := A\text{-mod}_{\mathrm{lfd}}$ (resp., $\widetilde{\mathcal{R}} := A\text{-mod}_{\mathrm{fd}}$) is an upper finite (resp., essentially finite) ε-stratified category with stratification $\rho : \mathbf{B} \to \Lambda$. Its standard, proper standard, costandard and proper costandard objects are as defined by (5.5)–(5.6).*

Proof. For $b \in \mathbf{B}$, let P_b be the left ideal Ae_b. We claim that P_b has a Δ_ε-flag with $\Delta_\varepsilon(b)$ at the top and other sections of the form $\Delta_\varepsilon(a)$ for $a \in \mathbf{B}$ with $\rho(a) \geqslant \rho(b)$. To prove this, suppose that $b \in \mathbf{B}_\lambda$ and set $P := P_b$ for short. Let $\{\mu_1, \dots, \mu_n\}$ be the set $\{\mu \in [\lambda, \infty) \mid Y(\mu, \lambda) \neq \varnothing\}$ ordered so that $\mu_r \leqslant \mu_s \Rightarrow r \leqslant s$; in particular, $\mu_1 = \lambda$. Let P_r be the subspace of P with basis given by all yx for $y \in Y(a), x \in X(a,b)$ and $a \in \mathbf{B}_{\mu_{r+1}} \cup \dots \cup \mathbf{B}_{\mu_n}$. This defines a filtration $P = P_0 > P_1 \geqslant \dots \geqslant P_n = 0$. Now we show that each P_{r-1}/P_r has a Δ_ε-flag with sections of the form $\Delta_\varepsilon(a)$ for $a \in \mathbf{B}_{\mu_r}$. There are two cases:

Case one: $\varepsilon(\mu_r) = +$. In this case, there is an A-module isomorphism

$$\theta : \bigoplus_{a\in\mathbf{B}_{\mu_r}} \bigoplus_{x\in X(a,b)} \Delta(a) \xrightarrow{\sim} P_{r-1}/P_r$$

sending the basis vector $y\bar{e}_a$ ($y \in Y(a)$) in the xth copy of $\Delta(a)$ to $yx + P_r \in P_{r-1}/P_r$. This follows from properties of the idempotent-adapted cellular basis and is similar to the proof of (5.3).

Case two: $\varepsilon(\mu_r) = -$. Note that P_{r-1}/P_r is naturally an $A_{\leqslant\mu_r}$-module. Let $Q := \bar{e}_{\mu_r}(P_{r-1}/P_r)$. This is an A_{μ_r}-module with basis $\{\bar{x} + P_r \mid a \in \mathbf{B}_{\mu_r}, x \in X(a,b)\}$. We claim that the natural multiplication map

$$A_{\leqslant\mu_r}\bar{e}_{\mu_r} \otimes_{A_{\mu_r}} Q \to P_{r-1}/P_r, \qquad y\bar{e}_{\mu_r} \otimes (x + P_r) \mapsto yx + P_r$$

is an isomorphism. This follows because the module on the left is spanned by the vectors $\bar{y} \otimes (x + P_r)$ for $a \in \mathbf{B}_{\mu_r}, y \in Y(a), x \in X(a,b)$, and the images of these vectors under the multiplication map give a basis for the module on the right. Hence, $P_{r-1}/P_r \cong j_!^{\mu_r}Q$. We deduce that it has a Δ_ε-flag with sections of the form $\bar{\Delta}(a)$ ($a \in \mathbf{B}_{\mu_r}$) on applying the standardization functor to a composition series for Q, using the exactness from Lemma 5.13.

We can now complete the proof of the claim. The only thing left is to check that the top section of the Δ_ε-flag we have constructed so far is isomorphic to $\Delta_\varepsilon(b)$. For this, note that $P_0/P_1 \cong \Delta(b)$, which is $\Delta_\varepsilon(b)$ if $\varepsilon(\lambda) = +$. If $\varepsilon(\lambda) = -$ then $\Delta(b) \cong j_!^\lambda P_\lambda(b)$, which has a Δ_ε-flag with top section $\Delta_\varepsilon(b) \cong j_!^\lambda L_\lambda(b)$ by the exactness of $j_!^\lambda$.

Using the claim just established, we can now classify the irreducible A-modules. For $b \in \mathbf{B}_\lambda$, the standard module $\Delta(b)$ is $j_!^\lambda P_\lambda(b)$, which has irreducible head $L(b)$. This follows by the usual properties of adjunctions and the quotient functor $j^\lambda : A_{\leqslant\lambda}\text{-mod}_{\mathrm{lfd}} \to \bar{A}_\lambda\text{-mod}_{\mathrm{fd}}, V \mapsto \bar{e}_\lambda V$. This argument shows that $L(b)$ is the unique (up to isomorphism) irreducible $A_{\leqslant\lambda}$-module such that $j^\lambda L(b) \cong L_\lambda(b)$. From this description, it is clear that the modules $\{L(b) \mid b \in \mathbf{B}\}$ are pairwise inequivalent. To see that they give a full set of irreducible A-modules, let L be any irreducible A-module. By the analog of Remark 5.2, there exists $b \in \mathbf{B}$ such that $e_b L \neq 0$. Then L is a quotient of $P_b = Ae_b$. Finally, using the claim, we deduce that L is a quotient of $\Delta(a)$ for some $a \in \mathbf{B}$ with $\rho(a) \geqslant \rho(b)$.

Now we can complete the proof of the theorem. Consider the stratification $\rho : \mathbf{B} \to \Lambda$ of $\widetilde{\mathcal{R}}$. We are in the recollement situation of (3.1), with $\widetilde{\mathcal{R}}_{\leqslant\lambda} = A_{\leqslant\lambda}\text{-mod}_{\mathrm{lfd}}$ (resp., $\widetilde{\mathcal{R}}_{\leqslant\lambda} = A_{\leqslant\lambda}\text{-mod}_{\mathrm{fd}}$) and $\mathcal{R}_\lambda = \bar{A}_\lambda\text{-mod}_{\mathrm{fd}}$. Since (5.5)–(5.6) agrees with (1.1), the standard, proper standard, costandard and proper costandard modules are the correct objects. Moreover, the claim established at the start of the proof verifies the property $(\widehat{P\Delta}_\varepsilon)$ as required by Definition 3.36 (resp., Definition 3.9). □

The goal in the remainder of the subsection is to prove a converse to Theorem 5.14. The proof relies on the following, which extends the construction of [AST, Theorem 3.1] to ε-stratified categories.

Lemma 5.15. *Let $\mathcal{R}$ be a lower finite (resp., an essentially finite) ε-stratified category with stratification $\rho : \mathbf{B} \to \Lambda$. In the essentially finite case, assume in addition that it is ε-tilting-bounded. Fix an embedding $\iota_b : \Delta_\varepsilon(b) \hookrightarrow T_\varepsilon(b)$ and a projection $\pi_b : T_\varepsilon(b) \twoheadrightarrow \nabla_\varepsilon(b)$ for each $b \in B$. Take $M \in \Delta_\varepsilon(\mathcal{R})$ and $N \in \nabla_\varepsilon(\mathcal{R})$. Choose*

$$Y_b \subset \mathrm{Hom}_{\mathcal{R}}(M, T_\varepsilon(b)), \qquad X_b \subset \mathrm{Hom}_{\mathcal{R}}(T_\varepsilon(b), N)$$

so that $\left\{\bar{y} := \pi_b \circ y \,\middle|\, y \in Y_b\right\}$ is a basis for $\mathrm{Hom}_{\mathcal{R}}(M, \nabla_\varepsilon(b))$ and $\left\{\bar{x} := x \circ \iota_b \,\middle|\, x \in X_b\right\}$ is a basis for $\mathrm{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), N)$, as illustrated by the diagram:

$$\begin{tikzcd}
& \Delta_\varepsilon(b) \arrow[d, hook, "\iota_b"'] \arrow[dr, "\bar{x}"] & \\
M \arrow[r, "y"] \arrow[dr, "\bar{y}"'] & T_\varepsilon(b) \arrow[r, "x"'] \arrow[d, two heads, "\pi_b"] & N . \\
& \nabla_\varepsilon(b) &
\end{tikzcd} \tag{5.7}$$

Then the morphisms $x \circ y$ for all $(y, x) \in \bigcup_{b \in \mathbf{B}} Y_b \times X_b$ give a basis for $\mathrm{Hom}_{\mathcal{R}}(M, N)$.

Proof. We proceed by induction on the size of the finite set

$$\Lambda(M, N) := \{\lambda \in \Lambda \mid \text{there exists } b \in \mathbf{B}_\lambda \text{ with } [M : L(b)] \neq 0 \text{ or } [N : L(b)] \neq 0\}.$$

The base case is when $|\Lambda(M, N)| = 0$, which is trivial since then $M = N = 0$. For the induction step, let $\lambda \in \Lambda(M, N)$ be maximal. Replacing $\mathcal{R}$ by the Serre subcategory of $\mathcal{R}$ associated to the lower set of Λ generated by $\Lambda(M, N)$, we may assume that λ is actually maximal in Λ. Then we let Let $\Lambda^\downarrow := \Lambda \backslash \{\lambda\}$, $\mathbf{B}^\downarrow := \rho^{-1}(\Lambda^\downarrow)$, and $i : \mathcal{R}^\downarrow \to \mathcal{R}$ be the natural inclusion of the corresponding Serre subcategory of $\mathcal{R}$. Let $j : \mathcal{R} \to \mathcal{R}_\lambda$ be the quotient functor.

In this paragraph, we treat the special case $N \in \mathcal{R}^\downarrow$. Let $M^\downarrow := i^* M$. Note by the choice of λ that $|\Lambda(M^\downarrow, N)| < |\Lambda(M, N)|$. By (3.7) and Theorem 3.19(2), we have that $M^\downarrow \in \Delta_\varepsilon(\mathcal{R}^\downarrow)$, and there is a short exact sequence $0 \to K \to M \to M^\downarrow \to 0$ where K has a Δ_ε-flag with sections of the form $\Delta_\varepsilon(b)$ for $b \in \mathbf{B}_\lambda$. It follows that the natural inclusion $\mathrm{Hom}_{\mathcal{R}}(M^\downarrow, N) \hookrightarrow \mathrm{Hom}_{\mathcal{R}}(M, N)$ is an isomorphism. For $b \in \mathbf{B}^\downarrow$, all of the morphisms $\{y : M \to T_\varepsilon(b) \mid y \in Y_b\}$ factor through $M^\downarrow$ too. Hence, we can apply the induction hypothesis to deduce that the morphisms $x \circ y$ for all $(y, x) \in \bigcup_{b \in \mathbf{B}^\downarrow} Y_b \times X_b$

give a basis for $\operatorname{Hom}_{\mathcal{R}}(M^{\downarrow}, N) = \operatorname{Hom}_{\mathcal{R}}(M, N)$. Since $X_b = \varnothing$ for $b \in \mathbf{B}_\lambda$, this is just what is needed.

Now suppose that $N \notin \mathcal{R}^{\downarrow}$ and let $N^{\downarrow} := i^! N \in \mathcal{R}^{\downarrow}$. We have that $|\Lambda(M, N^{\downarrow})| < |\Lambda(M, N)|$. By (3.7) and Theorem 3.19(3), we have that $N^{\downarrow} \in \nabla_\varepsilon(\mathcal{R}^{\downarrow})$, and there is a short exact sequence $0 \to N^{\downarrow} \to N \xrightarrow{\pi} Q \to 0$ where Q has a ∇_ε-flag with sections of the form $\nabla_\varepsilon(b)$ for $b \in \mathbf{B}_\lambda$. Applying $\operatorname{Hom}_{\mathcal{R}}(M, -)$ to this and using Theorem 3.16(2) gives a short exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{R}}(M, N^{\downarrow}) \to \operatorname{Hom}_{\mathcal{R}}(M, N) \to \operatorname{Hom}_{\mathcal{R}}(M, Q) \to 0.$$

For $b \in \mathbf{B}^{\downarrow}$, the morphisms $\{x : T_\varepsilon(b) \to N \mid x \in X_b\}$ have image contained in $N^{\downarrow}$ and are lifts of a basis for $\operatorname{Hom}_{\mathcal{R}^{\downarrow}}(\Delta_\varepsilon(b), N^{\downarrow})$. By induction, we get that $\operatorname{Hom}_{\mathcal{R}}(M, N^{\downarrow})$ has basis given by the compositions $x \circ y$ for all $(y, x) \in \bigcup_{b \in \mathbf{B}^{\downarrow}} Y_b \times X_b$. In view of this and the above short exact sequence, we are therefore reduced to showing that the morphisms $\pi \circ x \circ y$ for $(y, x) \in \bigcup_{b \in \mathbf{B}_\lambda} Y_b \times X_b$ give a basis for $\operatorname{Hom}_{\mathcal{R}}(M, Q)$. The exact quotient functor j defines isomorphisms $\operatorname{Hom}_{\mathcal{R}}(M, Q) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{R}_\lambda}(jM, jQ)$, $\operatorname{Hom}_{\mathcal{R}}(M, \nabla_\varepsilon(b)) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{R}_\lambda}(jM, j\nabla_\varepsilon(b))$ and $\operatorname{Hom}_{\mathcal{R}}(\Delta_\varepsilon(b), N) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{R}_\lambda}(j\Delta_\varepsilon(b), jN)$ for $b \in \mathbf{B}_\lambda$. Moreover, $j\pi : jN \to jQ$ is an isomorphism. Thus, we are reduced to showing that the morphisms $jx \circ jy$ give a basis for $\operatorname{Hom}_{\mathcal{R}_\lambda}(jM, jN)$ for all $(y, x) \in \bigcup_{b \in \mathbf{B}_\lambda} Y_b \times X_b$. Note the morphisms $\{jy \mid y \in Y_b\}$ and $\{jx \mid x \in X_b\}$ appearing here are lifts of bases for $\operatorname{Hom}_{\mathcal{R}_\lambda}(jM, j\nabla_\varepsilon(b))$ and $\operatorname{Hom}_{\mathcal{R}_\lambda}(j\Delta_\varepsilon(b), jN)$, respectively.

To complete the proof, we need to separate into the two cases $\varepsilon(\lambda) = +$ and $\varepsilon(\lambda) = -$. The arguments are similar (in fact, dual), so we just explain the former. In this case, for $b \in \mathbf{B}_\lambda$, we have that $j\nabla_\varepsilon(b) = L_\lambda(b)$ and $j\Delta_\varepsilon(b) = P_\lambda(b) = jT_\varepsilon(b)$ by Theorem 4.2(3). The module $M' := jM$ is projective in $\mathcal{R}_\lambda$. Also let $N' := jN$. We are trying to show that the morphisms $x' \circ y'$ for all $(y', x') \in \bigcup_{b \in \mathbf{B}_\lambda} Y'_b \times X'_b$ give a basis for $\operatorname{Hom}_{\mathcal{R}_\lambda}(M', N')$, where $Y'_b \subset \operatorname{Hom}_{\mathcal{R}_\lambda}(M', P_\lambda(b))$ is a set lifting a basis of $\operatorname{Hom}_{\mathcal{R}_\lambda}(M', L_\lambda(b))$ and X'_b is a basis of $\operatorname{Hom}_{\mathcal{R}_\lambda}(P_\lambda(b), N')$. Since M' is projective, the proof of this reduces to the case that $M' = P_\lambda(b)$, when the assertion is clear. □

Theorem 5.16. *Let $\widetilde{\mathcal{R}}$ be an upper finite (resp., an essentially finite) ε-stratified category with the usual stratification $\rho : \mathbf{B} \to \Lambda$. In the essentially finite case, assume in addition that it is ε-tilting bounded. Let $A = \bigoplus_{i,j \in I} e_i A e_j$ be an algebra realization of $\widetilde{\mathcal{R}}$. There is an idempotent expansion $A = \bigoplus_{i,j \in \widehat{I}} e_i A e_j$ with $\mathbf{B} \subseteq \widehat{I}$, and finite sets $Y(i, b) \subset e_i A e_b, X(b, j) \subset e_b A e_j$ for all $b \in \mathbf{B}$ and $i, j \in \widehat{I}$, making A into an upper finite (resp., essentially finite) based ε-stratified algebra with ρ as its stratification function.*

Proof. As in the first paragraph of the proof of Theorem 5.7, we may assume that $\mathbf{B} \subseteq I$ and that Ae_b ($b \in \mathbf{B}$) corresponds under the equivalence between $A\text{-mod}_{\mathrm{lfd}}$ (resp., $A\text{-mod}_{\mathrm{fd}}$) and $\widetilde{\mathcal{R}}$ to the indecomposable projective object $\widetilde{P}(b) \in \widetilde{\mathcal{R}}$. Let $\mathcal{R}$ be the Ringel dual of $\widetilde{\mathcal{R}}$ with respect to some choice of $\widetilde{T}$. The stratification for $\mathcal{R}$ is $\rho : \mathbf{B} \to \Lambda^{\mathrm{op}}$, where Λ^{op} is the poset Λ with the opposite ordering, and its sign function is $-\varepsilon$. Let $T_i := F^*(Ae_i) = (e_i \widetilde{T})^* \in \mathcal{R}$ for each $i \in I$. By Corollary 4.25 (resp., Corollary 4.29), the original algebra $A = \bigoplus_{i,j \in I} e_i A e_j$ is isomorphic as a locally unital algebra to $\left(\bigoplus_{i,j \in I} \operatorname{Hom}_{\mathcal{R}}(T_i, T_j)\right)^{\mathrm{op}}$.

Noting that $T_b \cong T_{-\varepsilon}(b)$ for each $b \in \mathbf{B}$, we apply Lemma 5.15 to the objects $M = T_i, N = T_j$ of $\mathcal{R}$ to produce finite sets $Y(i, b) \subset \operatorname{Hom}_{\mathcal{R}}(T_i, T_b)$ and $X(b, j) \subset \operatorname{Hom}_{\mathcal{R}}(T_b, T_j)$ such that the compositions $x \circ y$ for all $(y, x) \in \bigcup_{b \in \mathbf{B}} Y(i, b) \times X(b, j)$ give a basis for $\operatorname{Hom}_{\mathcal{R}}(T_i, T_j)$. We can certainly choose these lifts so that e_b, the identity endomorphism of T_b, belongs to $Y(b, b)$ and $X(b, b)$ for each $b \in \mathbf{B}$. The axioms (εS5)–(εS7) from Definition 5.9 are satisfied. For example, to check (εS7) in the case that $\varepsilon(\lambda) = -$, we have for $a, b \in \mathbf{B}_\lambda$ that $\dim \operatorname{Hom}_{\mathcal{R}}(T_{-\varepsilon}(a), \nabla_{-\varepsilon}(b)) = \delta_{a,b}$ thanks to Remark 4.3.

It just remains to check the final axiom (εS8). Take $\lambda \in \Lambda$. The set $\Lambda_{\leqslant\lambda}$ is a lower set in Λ, hence, an upper set in Λ^{op}. Let $\mathbf{B}_{\leqslant\lambda} := \rho^{-1}(\Lambda_{\leqslant\lambda})$ and $j^{\leqslant\lambda} : \mathcal{R} \to \mathcal{R}_{\leqslant\lambda}$ be the corresponding Serre quotient, i.e., it is the quotient of $\mathcal{R}$ by the Serre subcategory generated by all $L(b)$ with $\rho(b) \nleqslant \lambda$. Applying the quotient functor $j^{\leqslant\lambda}$ to the isomorphism $A \xrightarrow{\sim} \bigoplus_{i,j\in I} \mathrm{Hom}_{\mathcal{R}}(T_i, T_j)$ produces a surjective homomorphism from A to $\bigoplus_{i,j\in I} \mathrm{Hom}_{\mathcal{R}_{\leqslant\lambda}}(j^{\leqslant\lambda}T_i, j^{\leqslant\lambda}T_j)$. We claim that this factors through the quotient algebra $A_{\leqslant\lambda}$ to induce an isomorphism

$$A_{\leqslant\lambda} \xrightarrow{\sim} \bigoplus \mathrm{Hom}_{\mathcal{R}_{\leqslant\lambda}}(j^{\leqslant\lambda}T_i, j^{\leqslant\lambda}T_j).$$

To see this, note that all e_μ for $\mu \nleqslant \lambda$ are sent to zero, so we certainly have an induced homomorphism from $A_{\leqslant\lambda}$. Then we just need to compare bases: the proof of Lemma 5.11, which depends just on the first three axioms already established, shows that $A_{\leqslant\lambda}$ has basis yx for all $(y, x) \in \bigcup_{\mu\leqslant\lambda} \bigcup_{a,b\in\mathbf{B}_\mu} Y_a \times X_a$. The corresponding morphisms $j^{\leqslant\lambda}x \circ j^{\leqslant\lambda}y$ are linearly independent thanks to the analog of Lemma 5.15 in the category $\mathcal{R}_{\leqslant\lambda}$. This proves the claim. Finally, from the claim and letting $\mathcal{R}_\lambda$ be the Serre subcategory of $\mathcal{R}_{\leqslant\lambda}$ generated by the irreducible objects $j^{\leqslant\lambda}L(b)$ for $b \in \mathbf{B}_\lambda$, we truncate by the idempotent $\bar{e}_\lambda$ and its image to see that

$$\bar{A}_\lambda \cong \mathrm{End}_{\mathcal{R}_\lambda}\Big(\bigoplus_{b\in\mathbf{B}_\lambda} j^{\leqslant\lambda}T_\varepsilon(b)\Big)^{\mathrm{op}}. \tag{5.8}$$

If $\varepsilon(\lambda) = +$ then $j^{\leqslant\lambda}T_\varepsilon(b)$ is the injective hull $I_\lambda(b)$ of $L(b)$ in $\mathcal{R}_\lambda$, thanks to Theorem 4.2(3) (remembering that for $\mathcal{R}$ the sign function is $-\varepsilon$). So the algebra on the right hand side of (5.8) is basic with the given set of primitive idempotents. If $\varepsilon(\lambda) = -$, use instead that $j^{\leqslant\lambda}T_\varepsilon(b)$ is the projective cover $P_\lambda(b)$ of $L(b)$ in $\mathcal{R}_\lambda$. □

5.3. **Based stratified algebras and fibered quasi-hereditary algebras.** In this subsection, we introduce more symmetric notions of based algebras which are similar to Definitions 5.9 and 5.10 but remove the dependency of the bases on the sign function ε. These definitions were inspired by [ELau, Definition 2.17] (which also introduced the terminology "fibered" in an analogous setting with locally unital algebras replaced by categories). We then prove analogs of the two theorems in the previous subsection for this more symmetric notion; the proof of the second one requires the additional assumption of tilting-rigidity, which is interesting in its own right.

Definition 5.17. An *upper finite* (resp., *essentially finite*) *based stratified algebra* is a locally finite-dimensional (resp., essentially finite-dimensional) locally unital algebra $A = \bigoplus_{i,j\in I} e_iAe_j$ with the following additional data:

(BS1) A subset $\mathbf{B} \subseteq I$ indexing the *special idempotents* $\{e_b \mid b \in \mathbf{B}\}$.

(BS2) An upper finite (resp., interval finite) poset $(\Lambda, \leqslant)$.

(BS3) A stratification function $\rho : \mathbf{B} \to \Lambda$ with finite fibers $\mathbf{B}_\lambda := \rho^{-1}(\lambda)$.

(BS4) Sets $Y(i, a) \subset e_iAe_a$, $H(a, b) \subset e_aAe_b$, $X(b, j) \subset e_bAe_j$ for $\lambda \in \Lambda$, $a, b \in \mathbf{B}_\lambda$, $i, j \in I$.

Let $Y(a) := \bigcup_{i\in I} Y(i, a)$ and $X(b) := \bigcup_{j\in I} X(b, j)$. The axioms are as follows:

(BS5) The products yhx for all $(y, h, x) \in \bigcup_{\lambda\in\Lambda} \bigcup_{a,b\in\mathbf{B}_\lambda} Y(a) \times H(a, b) \times X(b)$ give a basis for A.

(BS6) For $a, b \in \mathbf{B}$, the sets $Y(a, b)$ and $X(b, a)$ are empty unless $\rho(a) \leqslant \rho(b)$.

(BS7) For $a, b \in \mathbf{B}_\lambda$, we have that $Y(a, b) = X(a, b) = \{e_a\}$ when $a = b$ and $Y(a, b) = X(a, b) = \varnothing$ otherwise.

(BS8) The same final axiom (εS8) as in Definition 5.9.

Here is the same definition rewritten in the special case that the stratification function ρ is a bijection.

Definition 5.18. An *upper finite* (resp., *essentially finite*) *fibered quasi-hereditary algebra* is a locally finite-dimensional (resp., essentially finite-dimensional) locally unital algebra $A = \bigoplus_{i,j \in I} e_i A e_j$ with the following additional data:

(FQH1) A subset $\Lambda \subseteq I$ indexing the *special idempotents* $\{e_\lambda \mid \lambda \in \Lambda\}$.
(FQH2) An upper finite (resp., interval finite) partial order $\leqslant$ on the set Λ.
(FQH3) Sets $Y(i,\lambda) \subset e_i A e_\lambda$, $H(\lambda) \subset e_\lambda A e_\lambda$, $X(\lambda, j) \subset e_\lambda A e_j$ for $\lambda \in \Lambda$, $i, j \in I$.

Let $Y(\lambda) := \bigcup_{i \in I} Y(i,\lambda)$ and $X(\lambda) := \bigcup_{j \in I} X(\lambda, j)$. The axioms are as follows.

(FQH4) The products yhx for all $(y,h,x) \in \bigcup_{\lambda \in \Lambda} Y(\lambda) \times H(\lambda) \times X(\lambda)$ give a basis for A.
(FQH5) For $\lambda, \mu \in \Lambda$, the sets $Y(\lambda,\mu)$ and $X(\mu,\lambda)$ are empty unless $\lambda \leqslant \mu$.
(FQH6) We have that $Y(\lambda,\lambda) = X(\lambda,\lambda) = \{e_\lambda\}$.
(FQH7) The finite-dimensional algebra $\bar{A}_\lambda$ defined as in Definition 5.9 is basic and local.

In the remainder of the subsection, we just explain the results for based stratified algebras, since fibered quasi-hereditary algebras are a special case. For the next lemma, we adopt the shorthands

$$YH(i,b) := \{yh \mid a \in \mathbf{B}_{\rho(b)}, y \in Y(i,a), h \in H(a,b)\}, \tag{5.9}$$

$$HX(b,j) := \{hx \mid c \in \mathbf{B}_{\rho(b)}, h \in H(b,c), x \in X(c,j)\}. \tag{5.10}$$

Also set $YH(b) := \bigcup_{i \in I} YH(i,b)$ and $HX(b) := \bigcup_{j \in I} HX(b,j)$.

Lemma 5.19. *Suppose that A is a locally unital based stratified algebra as in Definition 5.17. Also let $\varepsilon : \Lambda \to \{\pm\}$ be any choice of sign function. Then A is a locally unital based ε-stratified algebra with the required sets $Y(i,b)$ and $X(b,j)$ from Definition 5.9 being the sets $YH(i,b)$ and $X(b,j)$ in the present setup if $\varepsilon(\lambda) = +$, or the sets $Y(i,b)$ and $HX(b,j)$ in the present setup if $\varepsilon(\lambda) = -$.*

Proof. This follows on comparing Definitions 5.9 and 5.17. □

This means that the results from the previous subsection apply to based stratified algebras too. In particular, we define the standard, proper standard, costandard and proper costandard modules in the same way as (5.5)–(5.6). The module $\Delta(b)$ has basis $\{y\bar{e}_b \mid y \in YH(b)\}$ and the module $\bar{\Delta}(b)$ has standard basis $\{y\tilde{e}_b \mid y \in Y(b)\}$. Similarly, one can introduce costandard bases for $\nabla(b)$ and $\bar{\nabla}(b)$ indexed by the sets $HX(b)$ and $X(b)$, respectively. Note also that the algebra $\bar{A}_\lambda$ has basis $\left\{\bar{h} \mid h \in \bigcup_{a,b \in \mathbf{B}_\lambda,} H(a,b)\right\}$.

Theorem 5.20. *Let A be an upper finite (resp., essentially finite) based stratified algebra as in Definition 5.17. For $b \in \mathbf{B}$, the standard module $\Delta(b)$ has a unique irreducible quotient denoted $L(b)$. The modules $\{L(b) \mid b \in \mathbf{B}\}$ give a complete set of pairwise inequivalent irreducible A-modules. Moreover, $\mathcal{R} := A\text{-mod}_{\mathrm{lfd}}$ (resp., $\mathcal{R} := A\text{-mod}_{\mathrm{fd}}$) is an upper finite (resp., essentially finite) fully stratified category with stratification $\rho : \mathbf{B} \to \Lambda$. Its standard, proper standard, costandard and proper costandard objects are as defined by (5.5–5.6).*

Proof. Given Lemma 5.19, this follows from Theorem 5.14 applied twice, once with $\varepsilon = +$ and once with $\varepsilon = -$. □

Recall from the introduction that a fully stratified category $\mathcal{R}$ is called *tilting-rigid* if $T(b) := T_+(b) \cong T_-(b)$ for all $b \in \mathbf{B}$. The definition makes sense in any of our usual settings (finite, essentially finite, upper finite or lower finite). The module $T(b)$ then has a Δ_ε-flag and a ∇_ε-flag for all choices of the sign function $\varepsilon : \Lambda \to \{\pm\}$, hence, $T(b) \cong T_\varepsilon(b)$ for all ε, as was asserted already in the introduction. Moreover, by Theorem 4.2(3) or Theorem 4.17(3) (depending on the particular setting we are in), we deduce that

$$P_\lambda(b) \cong I_\lambda(b) \tag{5.11}$$

for all $\lambda \in \Lambda$ and $b \in \mathbf{B}_\lambda$. Consequently, if $\bar{A}_\lambda$ is a finite-dimensional algebra giving a realization of the stratum $\mathcal{R}_\lambda$, then $\bar{A}_\lambda$ is a *weakly symmetric Frobenius algebra*; see [GHK, Theorem 4.4.5]. Also, the Ringel dual $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ (in the appropriate sense depending on the setting) can be defined either taking $\varepsilon = +$ or $\varepsilon = -$, with either choice producing the same Ringel dual category due to the tilting-rigid assumption. Using properties of the Ringel duality functor, which is the same functor in both cases $\varepsilon = +$ and $\varepsilon = -$, it follows that $\widetilde{\mathcal{R}}$ is again a fully stratified category which is tilting-rigid.

In the remainder of the subsection, we are going to prove a converse to Theorem 5.20. As in the previous two subsections, our argument depends on Ringel duality. First, we need to upgrade Lemma 5.15; note for this we need to assume tilting-rigidity.

Lemma 5.21. *Let $\mathcal{R}$ be a lower finite (resp., essentially finite) fully stratified category with stratification $\rho : \mathbf{B} \to \Lambda$. Assume also that $\mathcal{R}$ is tilting-rigid (resp., tilting-rigid and tilting-bounded). Fix embeddings $\iota_a : \Delta(a) \hookrightarrow T(a)$, $\bar{\iota}_b : \bar{\Delta}(b) \hookrightarrow T(b)$ and projections $\bar{\pi}_a : T(a) \twoheadrightarrow \bar{\nabla}(a), \pi_b : T(b) \twoheadrightarrow \nabla(b)$ for $a, b \in \mathbf{B}$. Take $M \in \Delta(\mathcal{R})$ and $N \in \nabla(\mathcal{R})$. Choose*

$$Y_a \subset \mathrm{Hom}_{\mathcal{R}}(M, T(a)), \;\; H(a,b) \subset \mathrm{Hom}_{\mathcal{R}}(T(a), T(b)), \;\; X_b \subset \mathrm{Hom}_{\mathcal{R}}(T(b), N)$$

so that $\left\{\bar{y} := \bar{\pi}_a \circ y \,\middle|\, y \in Y_a\right\}$ is a basis for $\mathrm{Hom}_{\mathcal{R}}(M, \bar{\nabla}(a))$, $\left\{\bar{h} := \pi_b \circ h \circ \iota_a \,\middle|\, h \in H(a,b)\right\}$ is a basis for $\mathrm{Hom}_{\mathcal{R}}(\Delta(a), \nabla(b))$, and $\left\{\bar{x} := x \circ \bar{\iota}_b \,\middle|\, x \in X_b\right\}$ is a basis for $\mathrm{Hom}_{\mathcal{R}}(\Delta(b), N)$, as illustrated by the diagram:

$$\begin{tikzcd}
 & \Delta(a) \arrow[r, "\bar{h}"] \arrow[d, hook, "\iota_a"'] & \nabla(b) & \\
M \arrow[r, "y"] \arrow[dr, "\bar{y}"'] & T(a) \arrow[r, "h"'] \arrow[d, two heads, "\bar{\pi}_a"] & T(b) \arrow[r, "x"] \arrow[u, two heads, "\pi_b"'] & N \\
 & \bar{\nabla}(a) & \bar{\Delta}(b) \arrow[u, hook, "\bar{\iota}_b"] \arrow[ur, "\bar{x}"'] &
\end{tikzcd} \cdot \tag{5.12}$$

Then the morphisms $x \circ h \circ y$ for all $(y, h, x) \in \bigcup_{\lambda \in \Lambda} \bigcup_{a,b \in \mathbf{B}_\lambda} Y_a \times H(a,b) \times X_b$ give a basis for $\mathrm{Hom}_{\mathcal{R}}(M, N)$.

Proof. This follows by the same strategy as was used in the proof of Lemma 5.15. The only substantial difference is in the final paragraph of the proof. By that point, we have reduced to showing for projective and injective objects $M', N' \in \mathcal{R}_\lambda$, respectively, that the morphisms $x' \circ h' \circ y'$ for all $(y', h', x') \in \bigcup_{a,b \in \mathbf{B}_\lambda} Y'_a \times H'(a,b) \times X'_b$ give a basis for $\mathrm{Hom}_{\mathcal{R}_\lambda}(M', N')$, where $Y'_a \subset \mathrm{Hom}_{\mathcal{R}_\lambda}(M', P_\lambda(a))$ is a set lifting a basis of $\mathrm{Hom}_{\mathcal{R}_\lambda}(M', L_\lambda(a))$, $H'(a,b)$ is a basis for $\mathrm{Hom}_{\mathcal{R}_\lambda}(P_\lambda(a), I_\lambda(b))$, and $X'_b \subset \mathrm{Hom}_{\mathcal{R}_\lambda}(I_\lambda(b), N')$ is a set lifting a basis of $\mathrm{Hom}_{\mathcal{R}_\lambda}(L_\lambda(b), N')$. Using that M' is projective and N' is injective, the proof of this reduces to the case that $M' = P_\lambda(a)$ and $N' = I_\lambda(b)$, when the assertion is clear. □

Theorem 5.22. *Let $\widetilde{\mathcal{R}}$ be an upper finite (resp., essentially finite) fully stratified category with stratification $\rho : \mathbf{B} \to \Lambda$. Assume that $\widetilde{\mathcal{R}}$ is tilting-rigid (resp., tilting-rigid and tilting-bounded). Let $A = \bigoplus_{i,j \in I} e_i A e_j$ be an algebra realization of $\widetilde{\mathcal{R}}$. There is an idempotent expansion $A = \bigoplus_{i,j \in \hat{I}} e_i A e_j$ with $\mathbf{B} \subseteq \hat{I}$, and finite sets $Y(i,a) \subset e_i A e_a, H(a,b) \subset e_a A e_b, X(b,j) \subset e_b A e_j$ for all $a, b \in \mathbf{B}$ and $i, j \in \hat{I}$, making A into an upper finite (resp., essentially finite) based stratified algebra.*

Proof. This is similar to the proof of Theorem 5.16, using Lemma 5.21 in place of Lemma 5.15, plus the observation that the Ringel dual $\mathcal{R}$ of $\widetilde{\mathcal{R}}$ is also tilting-rigid. □

5.4. **Cartan and triangular decompositions.** Theorems 5.7, 5.16 and 5.22 give general theoretical tools for constructing based stratified algebras or quasi-hereditary algebras of various types. However, in practice, it can be difficult to construct an appropriate idempotent-adapted cellular basis explicitly. One reason that this is difficult is due to the insistence from the final axioms of Definitions 5.9, 5.10, 5.17 and 5.18 that the algebra $\bar{A}_\lambda$ is basic. In this subsection, we consider two more related concepts which are more flexible in practice. First we have locally unital algebras with Cartan decompositions:

Definition 5.23. Let $A = \bigoplus_{i,j\in I} e_i A e_j$ be a locally unital algebra. A *Cartan decomposition* of A is the data of a subset $\Gamma \subseteq I$ and a triple $(A^\flat, A^\circ, A^\sharp)$ consisting of subspaces $A^\flat = \bigoplus_{i\in I,\gamma\in\Gamma} e_i A^\flat e_\gamma$, $A^\sharp = \bigoplus_{\gamma\in\Gamma, j\in I} e_\gamma A^\sharp e_j$ (not subalgebras!) and a locally unital subalgebra $A^\circ = \bigoplus_{\gamma\in\Gamma} e_\gamma A^\circ e_\gamma$ such that the following axioms hold:

(CD1) $A^\circ A^\flat \subseteq A^\flat = A^\flat A^\circ$, $A^\circ A^\sharp = A^\sharp \supseteq A^\sharp A^\circ$.
(CD2) The natural multiplication map $A^\flat \otimes_{A^\circ} A^\sharp \to A$ is a linear isomorphism.
(CD3) $A^\flat$ is projective as a right A°-module and $A^\sharp$ is projective as a left A°-module.
(CD4) We have that $\bigoplus_{\gamma\in\Gamma} e_\gamma A^\flat e_\gamma = \bigoplus_{\gamma\in\Gamma} e_\gamma A^\sharp e_\gamma = A^\circ$.
(CD5) There is a partial order $\leqslant$ on Γ such that $e_\beta A^\flat e_\gamma = e_\gamma A^\sharp e_\beta = 0$ unless $\beta \leqslant \gamma$.

We call it an *upper finite* Cartan decomposition if A is locally finite-dimensional and the poset $(\Gamma, \leqslant)$ is upper finite, and an *essentially finite* Cartan decomposition if A is essentially finite-dimensional and the poset is interval finite.

Note in the setup of Definition 5.23 that the algebra $A = \bigoplus_{i,j\in I} e_i A e_j$ is Morita equivalent to $\bigoplus_{\beta,\gamma\in\Gamma} e_\beta A e_\gamma$. If we replace A by this Morita equivalent algebra, we obtain a locally unital algebra with a Cartan decomposition in the sense of Definition 5.23 in which the sets I and Γ are actually *equal*. This equality is assumed from the outset in the next definition of locally unital algebra with a triangular decomposition:

Definition 5.24. Let $A = \bigoplus_{\beta,\gamma\in\Gamma} e_\beta A e_\gamma$ be a locally unital algebra. A *triangular decomposition* of A is the data of a triple (A^-, A°, A^+) consisting of locally unital subalgebras $A^- = \bigoplus_{\beta,\gamma\in\Gamma} e_\beta A^- e_\gamma$, $A^+ = \bigoplus_{\beta,\gamma\in\Gamma} e_\beta A^+ e_\gamma$ and $A^\circ = \bigoplus_{\gamma\in\Gamma} e_\gamma A^\circ e_\gamma$ such that the axioms (TD1)–(TD4) below hold. To write the axioms down, let $\mathbb{K} := \bigoplus_{\gamma\in\Gamma} \Bbbk e_\gamma$.

(TD1) The subspaces $A^\flat := A^- A^\circ$, $A^\sharp := A^\circ A^+$ are locally unital subalgebras of A.
(TD2) The natural multiplication map $A^- \otimes_{\mathbb{K}} A^\circ \otimes_{\mathbb{K}} A^+ \to A$ is a linear isomorphism.
(TD3) We have that $\bigoplus_{\gamma\in\Gamma} e_\gamma A^- e_\gamma = \bigoplus_{\gamma\in\Gamma} e_\gamma A^+ e_\gamma = \mathbb{K}$.
(TD4) There is a partial order $\leqslant$ on Γ such that $e_\beta A^- e_\gamma = e_\gamma A^+ e_\beta = 0$ unless $\beta \leqslant \gamma$.

We call it an *upper finite* triangular decomposition if A is locally finite-dimensional and the poset $(\Gamma, \leqslant)$ is upper finite, and an *essentially finite* triangular decomposition if A is essentially finite-dimensional and the poset is interval finite.

Remark 5.25. A special case of Definition 5.24 is related to work of Holmes and Nakano [HN] on $\mathbb{Z}$-graded algebras with a triangular decomposition. To explain the connection, given a unital algebra $\tilde{A} = \bigoplus_{\gamma\in\Gamma} \tilde{A}_\gamma$ graded by an Abelian group Γ, there is an associated locally unital algebra $A = \bigoplus_{\beta,\gamma\in\Gamma} e_\beta A e_\gamma$ with $e_\beta A e_\gamma := \tilde{A}_{\beta-\gamma}$ and multiplication induced by multiplication in $\tilde{A}$ in the natural way. Moreover, any left $\tilde{A}$-module V can be viewed as a left A-module with $e_\gamma V := V_\gamma$; this defines an isomorphism from the usual category $\tilde{A}$-grmod of Γ-graded $\tilde{A}$-modules and grading-preserving morphisms to the category A-mod of locally unital A-modules. If we start with $\tilde{A}$ that is a finite-dimensional $\mathbb{Z}$-graded algebra with a triangular decomposition $(\tilde{A}^-, \tilde{A}^\circ, \tilde{A}^+)$ as in [HN] (see also [BT, Definition 3.1]) then the locally unital algebra A and the triple (A^-, A°, A^+) of subalgebras obtained from this construction give a locally unital algebra with an essentially finite triangular decomposition in the sense of Definition 5.24, with $\Gamma := \mathbb{Z}$ ordered in the natural way.

Having a triangular decomposition is stronger than having a Cartan decomposition:

Lemma 5.26. *Suppose that A has a triangular decomposition as in Definition 5.24. Viewing the subalgebras $A^\flat := A^-A^\circ$ and $A^\sharp := A^\circ A^+$ merely as subspaces, we obtain the data of a Cartan decomposition as in Definition 5.23 with $I = \Gamma$.*

Proof. This is obvious on comparing the definitions. $\square$

There are many naturally-occurring examples of locally unital algebras with triangular decompositions. We have already mentioned in Remark 5.25 the essentially finite examples of a periodic nature from [HN, BT]. Another source of examples admitting upper finite triangular decompositions involves various families of diagram algebras; see [CZ, §8] and §6.6 below. Note also if A is any locally unital algebra with an upper finite triangular decomposition, then the pair $(A^\sharp, A^\circ)$ is (the locally unital analog of) a left *Borelic pair* in the sense of [CZ, Definition 4.1.2]; similarly, $(A^\flat, A^\circ)$ is a right Borelic pair. Borelic pairs are related to the older notion of exact Borel subalgebra introduced by König; see [Kön, KKO]. In turn, König's work was motivated by Green's introduction of positive and negative Borel subalgebras of the classical Schur algebra $S(n,r)$ in [Gre]. However, $S(n,r)$ does not fit into the precise framework of Definition 5.24: one first has to pass to a Morita equivalent algebra whose distinguished idempotents are labelled just by partitions rather than all compositions in order to see such a structure.

The discussion so far justifies the significance of the notion of a triangular decomposition. The weaker notion of Cartan decomposition seems to be useful too. In fact, there is a tight connection between based stratified algebras and algebras with a Cartan decomposition. To pass from a based stratified algebra to an algebra with a Cartan decomposition, one needs the based stratified structure to be *split* in the following sense.

Lemma 5.27. *Let A be an upper finite (resp., essentially finite) based stratified algebra as in Definition 5.17. Recall in the context of that definition that $e_\lambda = \sum_{b \in \mathbf{B}_\lambda} e_b \in A$. Assume that the based stratified structure is* split, *by which we mean that the following properties hold for all $\lambda \in \Lambda$:*

(i) The subspace A_λ of $e_\lambda A e_\lambda$ spanned by $\bigcup_{a,b \in \mathbf{B}_\lambda} H(a,b)$ is a subalgebra.

(ii) The subspace $A^\flat e_\lambda$ of $A e_\lambda$ spanned by $\{yh \mid a,b \in \mathbf{B}_\lambda, y \in Y(a), h \in H(a,b)\}$ satisfies $A_\lambda A^\flat e_\lambda = e_\lambda A^\flat e_\lambda$.

(iii) The subspace $e_\lambda A^\sharp$ of $e_\lambda A$ spanned by $\{hx \mid a,b \in \mathbf{B}_\lambda, h \in H(a,b), x \in X(b)\}$ satisfies $e_\lambda A^\sharp A_\lambda = e_\lambda A^\sharp e_\lambda$.

Then A admits an upper finite (resp., an essentially finite) Cartan decomposition in the sense of Definition 5.23 with $\Gamma := \Lambda$, $A^\circ := \bigoplus_{\lambda \in \Lambda} A_\lambda$, $A^\flat := \bigoplus_{\lambda\in\Lambda} A^\flat e_\lambda$ and $A^\sharp := \bigoplus_{\lambda\in\Lambda} e_\lambda A^\sharp$. Moreover, the Cartan subalgebra A° is basic and $\{e_b \mid b \in \mathbf{B}\}$ is its complete set of primitive idempotents.

Remark 5.28. When the condition (i) from Lemma 5.27 holds, the natural quotient map $A \twoheadrightarrow A_{\leqslant\lambda}$ maps the subalgebra A_λ of A from the lemma isomorphically onto the subalgebra $\bar A_\lambda = \bar e_\lambda A_{\leqslant\lambda} \bar e_\lambda$ of $A_{\leqslant\lambda}$. Thus, under the splitting hypothesis, the subalgebra $\bar A_\lambda$ of $A_{\leqslant\lambda}$ is lifted to a subalgebra of A.

Proof of Lemma 5.27. It is clear that we have the required data and A° is basic as claimed since each $\bar A_\lambda \cong A_\lambda$ is basic according to (BS8); this uses Remark 5.28. It just remains to check the axioms (CD1)–(CD5). It is clear from (i) that $A^\flat A^\circ = A^\flat$, while the fact that $A^\circ A^\flat \subseteq A^\flat$ follows by (ii). Making similar arguments with $A^\flat$ replaced by $A^\sharp$ completes the check of (CD1). By considering the explicit bases from (BS5), we see that $A^\flat$ (resp. $A^\sharp$) is projective as a right (resp., left) A°-module as required for (CD3). Then (CD2) holds due to (BS5) again and the isomorphism $e_a A_\lambda \otimes_{A_\lambda} A_\lambda e_b \cong e_a A_\lambda e_b$. Finally, (CD4)–(CD5) follow from (BS6)–(BS7). $\square$

Corollary 5.29. *Suppose that A is an upper finite (resp., essentially finite) based quasi-hereditary algebra in the sense of Definition 5.1. Then A admits an upper finite (resp., essentially finite) Cartan decomposition in the sense of Definition 5.23 with Cartan subalgebra $A^\circ = \bigoplus_{\lambda \in \Lambda} \Bbbk e_\lambda$ that is basic and semisimple.*

It remains to explain how to pass from a locally unital algebra with an upper finite or essentially finite Cartan decomposition to a based statified algebra. This depends on an elementary general construction which involves making some additional choices in order to replace the given Cartan subalgebra A° by a basic algebra. How explicitly these choices can be made in practice of course depends on the particular situation. The situation is particularly straightforward if $A^\circ = \bigoplus_{\gamma \in \Gamma} \Bbbk e_\gamma$; cf. Corollary 5.29. For A as in Definition 5.23, the required choices are as follows:

- Let $\mathbf{B}$ be a set which parametrizes a complete set of pairwise inequivalent irreducible left A°-modules $\{L^\circ(b) \mid b \in \mathbf{B}\}$. Since $A^\circ = \bigoplus_{\gamma \in \Gamma} e_\gamma A^\circ e_\gamma$, the set $\mathbf{B}$ decomposes as $\mathbf{B} = \bigsqcup_{\gamma \in \Gamma} \mathbf{B}[\gamma]$ for finite sets $\mathbf{B}[\gamma]$ such that $L^\circ(b)$ is an irreducible $e_\gamma A^\circ e_\gamma$-module for each $b \in B[\gamma]$.
- Let Λ be a set which parametrizes the blocks of A°. This means that there is a function $\rho : \mathbf{B} \twoheadrightarrow \Lambda$ such that the irreducible modules $L^\circ(a), L^\circ(b)$ $(a, b \in \mathbf{B})$ belong to the same block of A° if and only if $\rho(a) = \rho(b)$. The set Λ decomposes as $\Lambda = \bigsqcup_{\gamma \in \Gamma} \Lambda[\gamma]$ where $\Lambda[\gamma] := \rho(\mathbf{B}[\gamma])$.
- Let $\leqslant$ be the partial order on Λ defined so that $\lambda \leqslant \mu$ if either $\lambda = \mu$ or $\lambda \in \Lambda[\beta], \mu \in \Lambda[\gamma]$ for $\beta, \gamma \in \Gamma$ with $\beta < \gamma$. We now have in hand a poset $(\Lambda, \leqslant)$, upper finite or interval finite according to the case, and a stratification function $\rho : \mathbf{B} \to \Lambda$. Set $\mathbf{B}_\lambda := \rho^{-1}(\lambda)$ as usual, so that $\mathbf{B}[\gamma] = \bigsqcup_{\lambda \in \Lambda[\gamma]} \mathbf{B}_\lambda$.
- Choose a set $\left\{e_{(b,r)} \;\middle|\; b \in \mathbf{B}, 1 \leqslant r \leqslant \dim L^\circ(b)\right\}$ of mutually orthogonal primitive idempotents in A° such that $e_{(b,r)} L^\circ(b) \neq 0$ for each r. In other words, for $b \in \mathbf{B}$, $e_b := e_{(b,1)}$ is an idempotent such that $P^\circ(b) := A^\circ e_b$ is a projective cover of $L^\circ(b)$, and $\{e_{(b,r)} \mid 1 \leqslant r \leqslant \dim L^\circ(b)\}$ are mutually orthogonal conjugates of e_b. For each $\gamma \in \Gamma$, these idempotents give a decomposition of e_γ, the identity element in the finite-dimensional algebra $e_\gamma A^\circ e_\gamma$, as a sum of mutually orthogonal primitive idempotents:
$$e_\gamma = \sum_{b \in \mathbf{B}[\gamma]} \sum_{r=1}^{\dim L^\circ(b)} e_{(b,r)}. \tag{5.13}$$
We now have new idempotents e_i defined for all elements i of the set
$$\widehat{I} := (I \backslash \Gamma) \sqcup \mathbf{B} \sqcup \{(b,r) \mid b \in \mathbf{B}, 2 \leqslant r \leqslant \dim L^\circ(b)\}.$$
In view of (5.13), we have that $A = \bigoplus_{i,j \in \widehat{I}} e_i A e_j$ and this is an idempotent expansion of the original decomposition $A = \bigoplus_{i,j \in I} e_i A e_j$.
- For $\lambda \in \Lambda$, let $e_\lambda := \sum_{b \in \mathbf{B}_\lambda} e_b$. The algebra $A_\lambda := e_\lambda A^\circ e_\lambda$ is basic with primitive idempotents $\{e_b \mid b \in \mathbf{B}_\lambda\}$, and A° is Morita equivalent to $\bigoplus_{\lambda \in \Lambda} A_\lambda$. An explicit equivalence $A^\circ\text{-mod} \to \bigoplus_{\lambda \in \Lambda} A_\lambda\text{-mod}$ is given by sending $V \mapsto \bigoplus_{\lambda \in \Lambda} e_\lambda V$, with quasi-inverse defined by the functor $\bigoplus_{\lambda \in \Lambda} A^\circ e_\lambda \otimes_{A_\lambda} -$. Similarly, one can write explicit equivalences between the categories of right modules.
- For $a, b \in \mathbf{B}$, let $H(a,b) := \varnothing$ if $\rho(a) \neq \rho(b)$, and let $H(a,b)$ be a basis for $e_a A_\lambda e_b$ if $\lambda := \rho(a) = \rho(b)$. The set $\bigcup_{a,b \in \mathbf{B}_\lambda} H(a,b)$ is a basis for A_λ.
- For $i \in \widehat{I}$ and any $\gamma \in \Gamma$, $e_i A^\flat e_\gamma$ makes sense and is a summand of $A^\flat e_\gamma$ thanks to (CD1). It is a projective right $e_\gamma A^\circ e_\gamma$-module by (CD3). Hence, for $\lambda \in \Lambda[\gamma]$, $e_i A^\flat e_\lambda$ is a finitely generated projective right A_λ-module. Therefore, for each $\lambda \in \Lambda$, it is possible to choose finite subsets $Y(i,a) \subset e_i A^\flat e_a$ for each $i \in \widehat{I}, a \in \mathbf{B}_\lambda$ such that
$$e_i A^\flat e_\lambda = \bigoplus_{a \in \mathbf{B}_\lambda, y \in Y(i,a)} y A_\lambda \tag{5.14}$$

and $yA_\lambda \cong e_a A_\lambda$ for each $y \in Y(i,a)$. Since $e_\gamma A^\flat e_\gamma = e_\gamma A^\circ e_\gamma$ thanks to (CD4), we have that $e_a A^\flat e_\lambda = e_a A_\lambda$, so we may assume that $Y(a,a) = \{e_a\}$ and $Y(a,b) = \varnothing$ for $a \neq b$ with $\rho(a) = \rho(b)$.

- Similarly to the previous point, we can choose finite subsets $X(b,j) \subset e_b A^\sharp e_j$ for each $j \in \widehat{I}, b \in \mathbf{B}_\lambda$ such that

$$e_\lambda A^\flat e_j = \bigoplus_{b \in \mathbf{B}_\lambda, x \in X(b,j)} A_\lambda x \tag{5.15}$$

and $A_\lambda x \cong A_\lambda e_b$ as left A_λ-modules for each $x \in X(b,j)$. Again, we may assume that $X(a,a) = \{e_a\}$ and $X(a,b) = \varnothing$ for $a \neq b$ with $\rho(a) = \rho(b)$.

Theorem 5.30. *Suppose that A is a locally unital algebra with an upper finite (resp., essentially finite) Cartan decomposition as in Definition 5.23. Apply the construction just explained to obtain the idempotent expansion $A = \bigoplus_{i,j \in \widehat{I}} e_i A e_j$, the subset $\Lambda \subseteq \widehat{I}$, and all of the other data required by (BS1)–(BS4) from Definition 5.17. This data satisfies the axioms (BS5)–(BS8), hence, we have given A the structure of an upper finite (resp., essentially finite) based stratified algebra which is split in the general sense of Lemma 5.27. Moreover:*

(1) It is an upper finite (resp., essentially finite) based quasi-hereditary algebra in the sense of Definition 5.1 if A° is semisimple.

(2) It is an upper finite (resp., essentially finite) fibered quasi-hereditary algebra in the sense of Definition 5.18 if A° is quasi-local, i.e., there are no extensions between non-isomorphic irreducible A°-modules.

Proof. By (CD2), multiplication defines an isomorphism $A^\flat \otimes_{A^\circ} A^\sharp \xrightarrow{\sim} A$. Note also for $i,j \in \widehat{I}$ that left multiplication by e_i (resp., right multiplication by e_j) leaves $A^\flat$ (resp., $A^\sharp$) invariant thanks to (CD1). Hence, we have that $e_i A^\flat \otimes_{A^\circ} A^\sharp e_j \cong e_i A e_j$. By the Morita equivalences discussed above, we have that $\bigoplus_{\lambda \in \Lambda} e_i A^\flat e_\lambda \otimes_{A_\lambda} e_\lambda A^\circ \cong e_i A^\flat$ as right A°-modules and $\bigoplus_{\mu \in \Lambda} A^\circ e_\mu \otimes_{A_\mu} e_\mu A^\sharp e_j \cong A^\sharp e_j$ as left A°-modules. Tensoring these isomorphisms together, we deduce that

$$\bigoplus_{\lambda,\mu \in \Lambda} e_i A^\flat e_\lambda \otimes_{A_\lambda} e_\lambda A^\circ \otimes_{A^\circ} A^\circ e_\mu \otimes_{A^\mu} e_\mu A^\sharp e_j \cong e_i A^\flat \otimes_{A^\circ} A^\sharp e_j \cong e_i A e_j.$$

Since $e_\lambda A^\circ \otimes_{A^\circ} A^\circ e_\mu \cong e_\lambda A^\circ e_\mu$, which is zero if $\lambda \neq \mu$ and A_λ if $\lambda = \mu$, this shows that the natural multiplication map gives an isomorphism

$$\bigoplus_{\lambda \in \Lambda} e_i A^\flat e_\lambda \otimes_{A_\lambda} e_\lambda A^\sharp e_j \xrightarrow{\sim} e_i A e_j.$$

From this and (5.14)–(5.15), we deduce that multiplication gives an isomorphism

$$\bigoplus_{\lambda \in \Lambda} \bigoplus_{a \in \mathbf{B}_\lambda, y \in Y(i,a)} \bigoplus_{b \in \mathbf{B}_\lambda, x \in X(b,j)} yA_\lambda \otimes_{A_\lambda} A_\lambda x \xrightarrow{\sim} e_i A e_j.$$

For $y \in Y(i,a)$ and $x \in X(b,j)$, we have that $yA_\lambda \otimes_{A_\lambda} A_\lambda x \cong e_a A_\lambda \otimes_{A_\lambda} A_\lambda e_b \cong e_a A_\lambda e_b$, which has basis $H(a,b)$. We deduce that the products yhx for all $(y,h,x) \in \bigcup_{\lambda \in \Lambda} \bigcup_{a,b \in \mathbf{B}_\lambda} Y(i,a) \times H(a,b) \times X(b,j)$ give a basis for $e_i A e_j$. This checks (BS5).

Consider (BS6)–(BS7); we just explain for $Y(b,a)$. Suppose that $Y(b,a) \neq \varnothing$ for $a \in \mathbf{B}[\beta]$ and $b \in \mathbf{B}[\gamma]$. Let $\lambda := \rho(b)$. From (5.14), we deduce that $e_a A^\flat e_b = e_a(e_\beta A^\flat e_\gamma)e_\lambda$ is non-zero. Hence, $e_\beta A^\flat e_\gamma \neq 0$ and $\beta \leqslant \gamma$ by (CD5). Since we know already from the construction that $Y(a,a) = \{e_a\}$ and $Y(a,b) = \varnothing$ for $a \neq b$ with $\rho(a) = \rho(b)$, this proves (BS6)–(BS7).

Consider (BS8). As noted in Remark 5.28, the algebra $\bar{A}_\lambda$ from Definition 5.9 is isomorphic to the subalgebra A_λ of A in the present setup. As we noted during the construction, A_λ is basic and $\{e_b \mid b \in \mathbf{B}_\lambda\}$ is its set of its primitive idempotents. This checks (BS8). Also the based stratified structure is split thanks to (CD1).

Finally, to check (1)–(2), note by the definition of Λ that the stratification function ρ is a bijection if and only if each block A_λ has a unique irreducible module (up to isomorphism), i.e., A° is quasi-local. Moreover, we have that $A_\lambda \cong \Bbbk$ for all $\lambda \in \Lambda$ if and only if A° is semisimple. $\square$

Corollary 5.31. *Let A be a locally unital algebra with an upper finite (resp., essentially finite) Cartan decomposition and fix the choices made before Theorem 5.30. Then A-$\mathrm{mod}_{\mathrm{lfd}}$ (resp., A-$\mathrm{mod}_{\mathrm{fd}}$) is an upper finite (resp., essentially finite) fully stratified category with stratification $\rho : \mathbf{B} \to \Lambda$. Moreover:*

(1) This category is highest weight if A° is semisimple.
(2) This category is signed highest weight if A° is quasi-local.

Proof. Apply Theorems 5.20 and 5.6. $\square$

Remark 5.32. When A has a triangular decomposition rather than merely a Cartan decomposition, Corollary 5.31 can be proved more directly, starting from the construction of standard modules as modules induced from irreducible $A^\sharp$-modules. This is the standard argument in the framework of Borelic pairs.

6. Examples

In this section, we explain several examples. For the ones in §§6.4–6.7 we give very few details but have tried to indicate the relevant ingredients from the existing literature.

6.1. **A finite-dimensional example via quiver and relations.** Let A and B be the basic finite-dimensional algebras defined by the following quivers:

$$A\ (1<2):\ \begin{tikzcd} 1 \arrow[loop left, "z"] \arrow[r, bend left, "u"] & 2 \arrow[l, bend left, "v"] \end{tikzcd} \qquad \text{with relations } z^2=0, uv=0, vuzv=0,$$

$$B\ (1>2):\ \begin{tikzcd} 1 \arrow[loop left, "s"] \arrow[r, "y"] & 2 \arrow[loop right, "t"] \end{tikzcd} \qquad \text{with relations } s^2=0, t^2=0, ty=0.$$

The algebra A has basis $\{e_1, z, vu, vuz, zvu, zvuz; e_2, uzv; v, zv; u, uz, uzvu, uzvuz\}$ and B has basis $\{e_1, s; e_2, t; y, ys\}$. The irreducible A- and B-modules are indexed by the set $\{1,2\}$. We are going to consider A-$\mathrm{mod}_{\mathrm{fd}}$ and B-$\mathrm{mod}_{\mathrm{fd}}$ with the stratifications defined by the orders $1<2$ and $1>2$, respectively.

We first look at B-$\mathrm{mod}_{\mathrm{fd}}$. As usual, we denote its irreducibles by $L(i)$, indecomposable projectives by $P(i)$, standards by $\Delta(i)$, etc.. The indecomposable projectives and injectives look as follows (where we abbreviate the simple module $L(i)$ just by i):

$$P(1) = \begin{tikzcd}[column sep=small, row sep=small] & 1 \arrow[dl, dash, "s"'] \arrow[dr, dash, "y"] & \\ 1 \arrow[d, dash, "y"] & & 2 \\ 2 & & \end{tikzcd}, \quad P(2) = \begin{tikzcd}[row sep=small] 2 \arrow[d, dash, "t"] \\ 2 \end{tikzcd}, \quad I(1) = \begin{tikzcd}[row sep=small] 1 \arrow[d, dash, "s"] \\ 1 \end{tikzcd}, \quad I(2) = \begin{tikzcd}[column sep=small, row sep=small] & & 1 \arrow[d, dash, "s"] \\ 2 \arrow[dr, dash, "t"'] & & 1 \arrow[dl, dash, "y"] \\ & 2 & \end{tikzcd}. \tag{6.1}$$

It follows easily that B-$\mathrm{mod}_{\mathrm{fd}}$ is a signed highest weight category in the sense of Definition 3.9 with the structure of the standards and costandards as follows:

$$\begin{aligned} &\Delta(1) = P(1), \quad \bar\Delta(1) = \begin{tikzcd}[row sep=small] 1 \arrow[d, dash, "y"] \\ 2 \end{tikzcd}, \quad \Delta(2) = P(2), \quad \bar\Delta(2) = L(2), \\ &\nabla(1) = I(1), \quad \bar\nabla(1) = L(1), \quad \nabla(2) = \begin{tikzcd}[row sep=small] 2 \arrow[d, dash, "t"] \\ 2 \end{tikzcd}, \quad \bar\nabla(2) = L(2). \end{aligned} \tag{6.2}$$

This can also be seen from Theorem 5.20 on noting that B is a (split) fibered quasi-hereditary algebra in the sense of Definition 5.18 with $Y(2,1) = \{y\}, X(1,2) = \varnothing$ and

$H(1) = \{e_1, s\}, H(2) = \{e_2, t\}$. The basic local algebras realizing the strata are $\Bbbk[s]/(s^2)$ and $\Bbbk[t]/(t^2)$. Next we look at the tilting modules in $B\text{-mod}_{\text{fd}}$. If one takes the sign function $\varepsilon = (\varepsilon_1, \varepsilon_2)$ to be either $(+,+)$ or $(-,+)$ then one finds that the indecomposable ε-tilting modules are $P(1)$ and $P(2)$ with filtrations

$$P(1) = \begin{tikzcd}[row sep=small] \bar\Delta(1) \arrow[d, dash] \\ \bar\Delta(1) \end{tikzcd} = \begin{tikzcd}[row sep=small, column sep=small] & \nabla(1) \arrow[dl, dash] \arrow[dr, dash] & \\ \bar\nabla(2) & & \bar\nabla(2) \end{tikzcd}, \qquad P(2) = \Delta(2) = \begin{tikzcd}[row sep=small] \bar\nabla(2) \arrow[d, dash] \\ \bar\nabla(2) \end{tikzcd}. \tag{6.3}$$

These cases are not very interesting since the Ringel dual category is just $B\text{-mod}_{\text{fd}}$ again. Assume henceforth that $\varepsilon = (-,-)$ or $(+,-)$. Then the indecomposable ε-tilting modules have the following structure:

$$T(1) = \begin{tikzcd}[row sep=small, column sep=small] 1 \arrow[d, "s"] \arrow[dr, "y"] & & 2 \arrow[dl, "t"'] \\ 1 \arrow[dr, "y"] & 2 & \\ & 2 & \end{tikzcd} = \begin{tikzcd}[row sep=small, column sep=small] \bar\Delta(2) \arrow[dr, dash] & & \bar\Delta(2) \arrow[dl, dash] \\ & \Delta(1) & \end{tikzcd} = \begin{tikzcd}[row sep=small, column sep=small] & \nabla(1) \arrow[dl, dash] \arrow[dr, dash] & \\ \nabla(2) & & \nabla(2) \end{tikzcd}, \quad T(2) = P(2). \tag{6.4}$$

To see this, one just has to check that these modules are indecomposable with the appropriate Δ_ε- and ∇_ε-flags. This analysis reveals that $B\text{-mod}_{\text{fd}}$ is *not* tilting-rigid.

The minimal projective resolution of $T(1)$ takes the form

$$\cdots \longrightarrow P(2) \oplus P(2) \longrightarrow P(2) \oplus P(2) \longrightarrow P(1) \oplus P(2) \oplus P(2) \longrightarrow T(1) \longrightarrow 0,$$

in particular, it is not of finite projective dimension. Observe also that there is a non-split short exact sequence $0 \to X \to T(1) \to X \to 0$ where

$$X = \begin{tikzcd}[row sep=small, column sep=small] 1 \arrow[dr, "y"] & & 2 \arrow[dl, "t"'] \\ & 2 & \end{tikzcd}.$$

Now let $T := T(1) \oplus T(2)$. We claim that $\operatorname{End}_B(T)^{\text{op}}$ is the algebra A defined above. To prove this, one takes $z : T(1) \to T(1)$ to be an endomorphism whose image and kernel is the submodule X of $T(1)$, $u : T(2) \to T(1)$ to be a homomorphism which includes $T(2)$ as a submodule of $X \subseteq T(1)$, and $v : T(1) \to T(2)$ to be a homomorphism with kernel containing X and image $L(2) \subseteq T(2)$. Hence, $A\text{-mod}_{\text{fd}}$ is the Ringel dual of $B\text{-mod}_{\text{fd}}$ relative to T. Note also that the algebra A is based $(+,+)$- and $(-,+)$-quasi-hereditary but it is *not* based $(+,-)$- or $(-,-)$-quasi-hereditary (cf. Definition 5.10).

One can also analyze $A\text{-mod}_{\text{fd}}$ directly. Its projective modules have the following structure:

$$\widetilde{P}(1) = \begin{tikzcd}[row sep=small, column sep=small] & 1 \arrow[dl, "u"'] \arrow[dr, "z"] & \\ 2 \arrow[d, "v"] & & 1 \arrow[d, "u"] \\ 1 \arrow[d, "z"] & & 2 \arrow[d, "v"] \\ 1 \arrow[d, "u"] & & 1 \arrow[d, "z"] \\ 2 & & 1 \arrow[d, "u"] \\ & & 2 \end{tikzcd}, \qquad \widetilde{P}(2) = \begin{tikzcd}[row sep=small] 2 \arrow[d, "y"] \\ 1 \arrow[d, "z"] \\ 1 \arrow[d, "u"] \\ 2 \end{tikzcd}, \tag{6.5}$$

Continuing with $\varepsilon = (-,-)$ or $\varepsilon = (+,-)$, it is easy to check directly from this that $A\text{-mod}_{\text{fd}}$ is $(-\varepsilon)$-highest weight, as we knew already due to Theorem 4.11. However, it is not ε-highest weight for either choice of ε, so it is *not* signed highest weight.

We leave it to the reader to compute explicitly the indecomposable $(-\varepsilon)$-tilting modules $\widetilde{T}(1)$ and $\widetilde{T}(2)$ in $B\text{-mod}_{\mathrm{fd}}$. Their structure reflects the structure of the injectives $I(1)$ and $I(2)$ in $A\text{-mod}_{\mathrm{fd}}$. Let $\widetilde{T} := \widetilde{T}(1) \oplus \widetilde{T}(2) \cong T^*$. By the double centralizer property from Corollary 4.12, we have that $B = \mathrm{End}_A(\widetilde{T})^{\mathrm{op}}$, as may also be checked directly. By Theorem 4.15, the functor $\mathbb{R}\mathrm{Hom}_A(\widetilde{T}, -) : D^b(A\text{-mod}_{\mathrm{fd}}) \to D^b(B\text{-mod}_{\mathrm{fd}})$ is an equivalence. Note though that $\mathbb{R}\mathrm{Hom}_B(T, -) : D^b(B\text{-mod}_{\mathrm{fd}}) \to D^b(A\text{-mod}_{\mathrm{fd}})$ is *not* one; this follows using [Kel, Theorem 4.1] since $T(1)$ does not have finite projective dimension.

6.2. **An explicit semi-infinite example.** In this subsection, we give a baby example involving a lower finite highest weight category. Let C be the coalgebra with basis

$$\left\{c^{(\ell)}_{i,j} \,\middle|\, i,j,\ell \in \mathbb{Z}, \ell \geqslant i,j \geqslant 0\right\},$$

counit defined by $\epsilon(c^{(\ell)}_{i,j}) := \delta_{i,\ell}\delta_{j,\ell}$, and comultiplication $\delta : C \to C \otimes C$ defined by

$$c^{(i)}_{i,j} \mapsto \sum_{\substack{k=0\\ i\not\equiv j(2)}}^{j} c^{(i)}_{i,k} \otimes c^{(j)}_{k,j} + \sum_{\substack{k=j\\ k\equiv i(2)}}^{i} c^{(i)}_{i,k} \otimes c^{(k)}_{k,j}, \qquad c^{(j)}_{i,j} \mapsto \sum_{\substack{k=0\\ i\not\equiv j(2)}}^{i} c^{(i)}_{i,k} \otimes c^{(j)}_{k,j} + \sum_{\substack{k=i\\ k\equiv j(2)}}^{j} c^{(k)}_{i,k} \otimes c^{(j)}_{k,j},$$

$$c^{(\ell)}_{i,j} \mapsto c^{(\ell)}_{i,\ell} \otimes c^{(\ell)}_{\ell,j} + \sum_{\substack{k=0\\ i\not\equiv \ell(2)}}^{i} c^{(i)}_{i,k} \otimes c^{(\ell)}_{k,j} + \sum_{\substack{k=i\\ k\equiv \ell(2)}}^{\ell-1} c^{(k)}_{i,k} \otimes c^{(\ell)}_{k,j} + \sum_{\substack{k=0\\ j\not\equiv \ell(2)}}^{j} c^{(\ell)}_{i,k} \otimes c^{(j)}_{k,j} + \sum_{\substack{k=j\\ k\equiv \ell(2)}}^{\ell-1} c^{(\ell)}_{i,k} \otimes c^{(k)}_{k,j}$$

for $i,j \geqslant 0$, $\ell > \max(i,j)$. We will show that $\mathcal{R} := \mathrm{comod}_{\mathrm{fd}}\text{-}C$ is a lower finite highest weight category with weight poset $\Lambda := \mathbb{N}$ ordered in the natural way. Then we will determine the costandard, standard and indecomposable injective and tilting objects explicitly, and describe the Ringel dual category $\widetilde{\mathcal{R}}$. To do this, we mimic some arguments for reductive groups which we learnt from [Jan1].

We will need comodule induction functors, which we review briefly. For any coalgebra C with comultiplication δ, a right C-comodule V with structure map $\eta_R : V \to V \otimes C$, and a left C-comodule W with structure map $\eta_L : W \to C \otimes W$, the *cotensor product* $V \,\square_C\, W$ is the subspace of the vector space $V \otimes W$ that is the equalizer of the diagram

$$V \otimes W \underset{\mathrm{id}\otimes\eta_L}{\overset{\eta_R\otimes\mathrm{id}}{\rightrightarrows}} V \otimes C \otimes W.$$

In particular, $\eta_R : V \to V \otimes C$ is an isomorphism from V to the subspace $V \,\square_C\, C$, and similarly $\eta_L : W \xrightarrow{\sim} C \,\square_C\, W$. Now suppose that $\pi : C \to C'$ is a coalgebra homomorphism and V is a right C'-comodule. Viewing C as a left C'-comodule with structure map $\delta_L := (\pi \otimes \mathrm{id}) \circ \delta : C \to C' \otimes C$, we define the induced comodule to be

$$\mathrm{ind}^C_{C'} V := V \,\square_{C'}\, C.$$

This is a subcomodule of the right C-comodule $V \otimes C$ (with structure map $\mathrm{id} \otimes \delta$). In fact, $\mathrm{ind}^C_{C'} : \mathrm{comod}\text{-}C' \to \mathrm{comod}\text{-}C$ defines a functor which is right adjoint to the exact restriction functor $\mathrm{res}^C_{C'}$, so it is left exact and sends injectives to injectives.

Now let C be the coalgebra defined above, and consider the natural quotient maps $\pi^\flat : C \twoheadrightarrow C^\flat$ and $\pi^\sharp : C \twoheadrightarrow C^\sharp$, where $C^\flat$ and $C^\sharp$ are the quotients of C by the coideals spanned by $\left\{c^{(\ell)}_{i,j} \,\middle|\, \ell > j\right\}$ or $\left\{c^{(\ell)}_{i,j} \,\middle|\, \ell > i\right\}$, respectively. These coalgebras have bases denoted $\left\{c_{i,j} := \pi^\flat(c^{(j)}_{i,j}) \,\middle|\, 0 \leqslant i \leqslant j\right\}$ and $\left\{c_{i,j} := \pi^\sharp(c^{(i)}_{i,j}) \,\middle|\, i \geqslant j \geqslant 0\right\}$, and comultiplications $\delta^\flat$ and $\delta^\sharp$ satisfying

$$\delta^\flat(c_{i,j}) = c_{i,i} \otimes c_{i,j} + \sum_{\substack{k=i+1\\ k\equiv j(2)}}^{j} c_{i,k} \otimes c_{k,j}, \quad \delta^\sharp(c_{i,j}) = c_{i,j} \otimes c_{j,j} + \sum_{\substack{k=j+1\\ k\equiv i(2)}}^{i} c_{i,k} \otimes c_{k,j}, \tag{6.6}$$

respectively. Also let $C^\circ \cong \bigoplus_{i\geqslant 0} \Bbbk$ be the semisimple coalgebra with basis $\{c_i \mid i \geqslant 0\}$ and comultiplication $\delta^\circ : c_i \mapsto c_i \otimes c_i$. Note C° is a quotient of both $C^\flat$ and $C^\sharp$ via the obvious maps sending $c_{i,j} \mapsto \delta_{i,j} c_i$; hence, it is also a quotient of C. It may also be identified with a subcoalgebra of both $C^\flat$ and $C^\sharp$ via the maps sending $c_i \mapsto c_{i,i}$.

Let $L^\circ(i)$ be the one-dimensional irreducible right C°-comodule spanned by $c_{i,i}$. Since C° is semisimple with these as its irreducible comodules, any irreducible right C°-comodule V decomposes as $V = \bigoplus_{i\in I} V_i$ with the "weight spaces" V_i being a direct sum of copies of $L^\circ(i)$. Similarly, any left C°-comodule V decomposes as $V = \bigoplus_{i\in I} {}_iV$. This applies in particular to left and right $C^\flat, C^\sharp$ or C-comodules, since these may be viewed as C°-comoodules by restriction.

Since C° is a subcoalgebra of $C^\flat$, the irreducible comodule $L^\circ(i)$ may also be viewed as an irreducible right $C^\flat$-comodule. We denote this instead by $L^\flat(i)$; it is the subcomodule of $C^\flat$ spanned by the vector $c_{i,i}$. For $i \geqslant 0$, let $I(i) := {}_iC \cong \operatorname{ind}_{C^\circ}^C L^\circ(i)$, let $\nabla(i)$ be the subcomodule of $I(i)$ spanned by the vectors $\{c^{(i)}_{i,j} \mid 0 \leqslant j \leqslant i\}$, and let $L(i)$ be the one-dimensional irreducible subcomodule of $\nabla(i)$ spanned by the vector $c^{(i)}_{i,i}$. Now we proceed in several steps.

Claim 1: *Viewed as a functor to vector spaces, the induction functor* $\operatorname{ind}_{C^\flat}^C$ *is isomorphic to the functor* $V \mapsto V \,\square_{C^\circ}\, C^\sharp \cong \bigoplus_{i\geqslant 0} V_i \otimes {}_iC^\sharp$. *Hence, this functor is exact.* To prove this, let $\delta_{LR} := (\pi^\flat \bar{\otimes} \pi^\sharp) \circ \delta : C \to C^\flat \,\square_{C^\circ}\, C^\sharp$. As $\delta_{LR}(c^{(\ell)}_{i,j}) = c_{i,\ell} \otimes c_{\ell,j}$ and these vectors for all $\ell \geqslant \max(i,j)$ give a basis for $C^\flat \,\square_{C^\circ}\, C^\sharp$, this map is a linear isomorphism. Moreover, the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\delta_L"] \arrow[d, "\delta_{LR}"'] & C^\flat \otimes C \arrow[d, "\mathrm{id}\otimes\delta_{LR}"] \\ C^\flat \,\square_{C^\circ}\, C^\sharp \arrow[r, "\delta^\flat\otimes\mathrm{id}"'] & C^\flat \otimes C^\flat \,\square_{C^\circ}\, C^\sharp. \end{tikzcd}$$

The vertical maps are isomorphisms. Using the definition of $\operatorname{ind}_{C^\flat}^C$, it follows for any right $C^\flat$-comodule V with structure map η that the induced module $\operatorname{ind}_{C^\flat}^C V$ is isomorphic as a vector space (indeed, as a right $C^\sharp$-comodule) to the equalizer of the diagram

$$V \otimes C^\flat \,\square_{C^\circ}\, C^\sharp \overset{\eta\otimes\mathrm{id}\otimes\mathrm{id}}{\underset{\mathrm{id}\otimes\delta^\flat\otimes\mathrm{id}}{\rightrightarrows}} V \otimes C^\flat \otimes C^\flat \,\square_{C^\circ}\, C^\sharp.$$

Since $\operatorname{ind}_{C^\flat}^{C^\flat} V \cong V$, this is naturally isomorphic to $V \,\square_{C^\circ}\, C^\sharp$. As $C^\sharp \cong \bigoplus_{i\geqslant 0} {}_iC^\sharp$, we get finally that $V \,\square_{C^\circ}\, C^\sharp \cong \bigoplus_{i\geqslant 0} V_i \otimes {}_iC^\sharp$.

Claim 2: *For* $i \geqslant 0$*, the right* $C^\flat$*-comodule* ${}_iC^\flat \cong \operatorname{ind}_{C^\circ}^{C^\flat} L^\circ(i)$ *has an exhaustive ascending filtration* $0 < V_0 < V_1 < \cdots$ *such that* $V_0 \cong L^\flat(i)$ *and* $V_r/V_{r-1} \cong L^\flat(i+2r-1) \oplus L^\flat(i+2r)$ *for* $r \geqslant 1$. *Also, the modules* $\{L^\flat(i) \mid i \geqslant 0\}$ *give a complete set of pairwise inequivalent irreducible* $C^\flat$*-comodules.* The first statement follows from (6.6), defining V_0 to be the subspace spanned by $c_{i,i}$, and V_r is spanned by $c_{i,i+2r-1}, c_{i,i+2r}$. To prove the second statement, take any irreducible $C^\flat$-comodule L. Take a non-zero homomorphism $\operatorname{res}_{C^\circ}^{C^\flat} L \to L^\circ(i)$ for some i. Then use adjointness of $\operatorname{res}_{C^\circ}^{C^\flat}$ and $\operatorname{ind}_{C^\circ}^{C^\flat}$ to obtain an embedding $L \hookrightarrow {}_iC^\flat$. Hence, $L \cong L^\flat(i)$ as a $C^\flat$-comodule.

Claim 3: *We have that* $\nabla(i) \cong \operatorname{ind}_{C^\flat}^C L^\flat(i)$ *and it is uniserial with composition factors* $L(i), L(i-2), L(i-4), \dots, L(a), L(b), \cdots L(i-3), L(i-1)$ *(for* $(a,b) \in \{(0,1),(1,0)\}$

depending on parity of i) in order from bottom to top:

$$\nabla(i) = \begin{array}{c} i-1 \\ | \\ i-3 \\ \vdots \\ i-2 \\ | \\ i \end{array} \tag{6.7}$$

The restriction of $\delta_L : C \to C^\flat \otimes C$ to $\nabla(i)$ gives an embedding of $\nabla(i)$ into $\mathrm{ind}_{C^\flat}^C L^\flat(i)$. This embedding is an isomorphism since we know $\mathrm{ind}_{C^\flat}^C L^\flat(i)$ has the same dimension $(i+1)$ as $\nabla(i)$ thanks to Claim 1. The determinaton of the subcomodule structure is straightforward using the definition of $\delta(c_{i,j}^{(i)})$ for $0 \leqslant j \leqslant i$.

Claim 4: *The injective C-comodule $I(i)$ has an exhaustive filtration $0 < I_0 < I_1 < \cdots$ such that $I_0 \cong \nabla(i)$ and $I_r/I_{r-1} \cong \nabla(i+2r-1) \oplus \nabla(i+2r)$ for $r \geqslant 1$:*

$$I(i) = \begin{array}{ccccc} \vdots & & & & \vdots \\ \nabla(i+3) & & & & \nabla(i+4) \\ | & & & & | \\ \nabla(i+1) & & & & \nabla(i+2) \\ & \diagdown & & \diagup & \\ & & \nabla(i) & & \end{array} \tag{6.8}$$

This follows from Claims 1, 2 and 3.

Claim 5: *The irreducible C-comodules $\{L(i) \mid i \geqslant 0\}$ give a complete set of pairwise inequivalent irreducibles. Moreover, $I(i)$ is the injective hull of $L(i)$, and the natural order on $\mathbb{N}$ defines a stratification of $\mathcal{R} := \mathrm{comod}_{\mathrm{fd}}\text{-}C$ in the sense of Definition 3.1.* By Claim 3, the last part of Claim 2, and an adjunction argument, any irreducible C-comodule embeds into $\nabla(i)$ for some i, hence, it is isomorphic to $L(i)$. The module $I(i)$ is injective, and it has irreducible socle $L(i)$ by another adjunction argument. Hence, it is the injective hull of $L(i)$. To see that we have a stratification, it remains to check that for each $k \geqslant 0$ that the largest submodule of $I(i)$ all of whose composition factors are of the form $L(j)$ for $j \leqslant k$ is finite-dimensional. This follows on considering from the filtration from Claim 4 using the fact that $\nabla(j)$ has irreducible socle $L(j)$.

Claim 6: *The category $\mathcal{R} := \mathrm{comod}_{\mathrm{fd}}\text{-}C$ is a lower finite highest weight category with costandard objects $\nabla(i)\,(i \geqslant 0)$. It also possesses a duality interchanging $\nabla(i)$ and $\Delta(i)$.* It is clear that $\nabla(i)$ is the largest subcomodule of $I(i)$ all of whose composition factors are of the form $L(j)$ for $j \leqslant i$, hence, this is the costandard object. Then to show that $\mathcal{R}$ is a lower finite highest weight category, it just remains to check the property $(I\nabla)$, which follows from Claim 4. The duality is defined using the evident coalgebra antiautomorphism of C which maps $c_{i,j}^{(\ell)} \mapsto c_{j,i}^{(\ell)}$.

Claim 7: *The indecomposable tilting comodule $T(i)$ is equal to $L(i) = \Delta(i) = \nabla(i)$ if $i = 0$, and there are non-split short exact sequences $0 \to \Delta(i) \to T(i) \to \Delta(i-1) \to 0$ and $0 \to \nabla(i-1) \to T(i) \to \nabla(i) \to 0$ if $i > 0$.*
This is immediate in the case $i = 0$. Now for $i > 0$, let $T(i)$ be the non-split extension of $\nabla(i-1)$ by $\nabla(i)$ that is the subcomodule of $I(i-1)$ spanned by the vectors $\{c_{i-1,j}^{(i-1)}, c_{i-1,k}^{(i)} \mid 0 \leqslant j \leqslant i-1, 0 \leqslant k \leqslant i\}$. Then one checks that this submodule is self-dual. Since it has a ∇-flag it therefore also has a Δ-flag, so it must be the desired tilting object by Theorem 4.2.

Claim 8: *The Ringel dual category $\widetilde{\mathcal{R}}$ is the category A-$\mathrm{mod}_{\mathrm{lfd}}$ of locally finite-dimensional left modules over the locally unital algebra A defined by the following quiver:*

$$
A\,:\,\begin{tikzcd} 0 \arrow[r, bend left, "y_0"] & 1 \arrow[l, bend left, "x_0"] \arrow[r, bend left, "y_1"] & 2 \arrow[l, bend left, "x_1"] & \cdots \end{tikzcd} \qquad \textit{with relations } y_{i+1}y_i = x_i x_{i+1} = x_i y_i = 0.
$$

We need to find an isomorphism of algebras $A \xrightarrow{\sim} \left(\bigoplus_{i,j\geqslant 0} \mathrm{Hom}_C(T(i),T(j))\right)^{\mathrm{op}}$. For this, we consider the $T(i)$ ($i = 0,1,2,3,\dots$) with their ∇-flags:

$$
\begin{array}{ccccccccc}
 & & & & & & 2 & & \\
 & & & & & & | & & \\
 & & & & 1 & & 0 & & \\
 & & & & | & & | & & \\
 & & 0 & & 0 & & 1 & & \\
 & & | & & | & & | & & \\
0 & \underset{\xrightarrow{x_0}}{\xleftarrow{y_0}} & 1 & \underset{\xrightarrow{x_1}}{\xleftarrow{y_1}} & 2 & \underset{\xrightarrow{x_2}}{\xleftarrow{y_2}} & 3 & \underset{\xrightarrow{x_3}}{\xleftarrow{y_3}} & \cdots \\
 & & | & & | & & | & & \\
 & & 0 & & 0 & & 1 & & \\
 & & & & | & & | & & \\
 & & & & 1 & & 0 & & \\
 & & & & & & | & & \\
 & & & & & & 2 & &
\end{array}
\tag{6.9}
$$

We will describe the images, also called e_i, x_i, y_i, of the generators of A. We send e_i to the identity endomorphism of $T(i)$, x_i to the morphism $T(i) \to T(i+1)$ sending the quotient $\nabla(i)$ of $T(i)$ to the subcomodule $\nabla(i)$ of $T(i+1)$ and y_i to the morphism $T(i+1) \to T(i)$ sending the quotient $\Delta(i)$ of $T(i+1)$ to the submodule $\Delta(i)$ of $T(i)$. The relations are easy to check (remembering the op, e.g., one must verify that $y_2 \circ x_2 = 0 \neq x_2 \circ y_2$). Since the algebra A is very easy to understand, one also sees that this homomorphism is injective, then it is an isomorphism by dimension considerations.

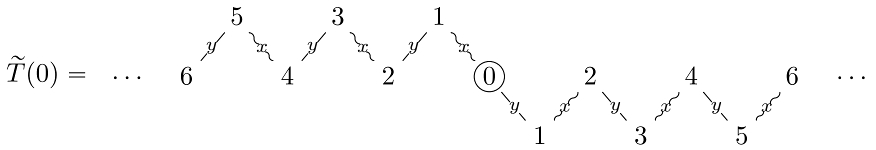
Remark 6.1. The above analysis of $\mathrm{comod}_{\mathrm{fd}}$-$C$ relies ultimately on the observation that the coalgebra C has a triangular decomposition in a precise sense which is the analog for coalgebras of Definition 5.24. There are also coalgebra analogs of the other definitions from the previous section, e.g., the coalgebra analog of Definition 5.1 is the notion of a (lower finite) based quasi-hereditary coalgebra. This will be developed in a sequel to this article.

One can argue in the opposite direction too, starting from the algebra A just defined, and computing its Ringel dual to recover the coalgebra C (in fact, this is how we discovered the coalgebra C in the first place). Note for this that A has an upper finite triangular decomposition in the sense of Definition 5.24 with $A^\circ = \bigoplus_{i\in\mathbb{N}} \Bbbk e_i$, $A^\flat = \{e_i, y_i \mid i \in \mathbb{N}\}$ and $A^\sharp = \{e_i, x_i \mid i \in \mathbb{N}\}$. By Lemma 5.26 and Corollary 5.31, A-$\mathrm{mod}_{\mathrm{lfd}}$ is an upper finite highest weight category. Its standard and costandard modules have the structure

$$
\widetilde{\Delta}(i) = \begin{array}{c} i \\ \downarrow{\scriptstyle y_i} \\ i+1 \end{array}, \qquad\qquad \widetilde{\nabla}(i) = \begin{array}{c} i+1 \\ \downarrow{\scriptstyle x_i} \\ i \end{array}. \tag{6.10}
$$

Using the characterization from Theorem 4.17(i), it follows that the indecomposable tilting modules for A have a similar structure to $\widetilde{T}(0)$, which is as follows (to get $\widetilde{T}(i)$

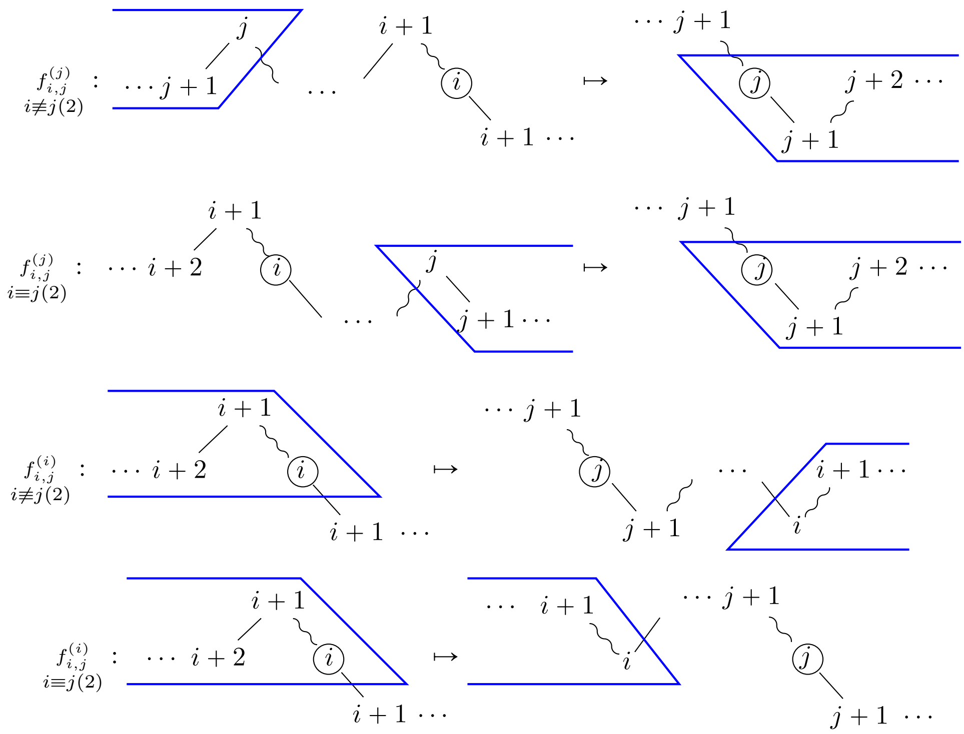
in general one just has to add i to all of the labels):



(6.11)

This diagram demonstrates that $\widetilde{T}(0)$ has both an infinite ascending Δ-flag with $\widetilde{\Delta}(0)$ at the bottom and subquotients as indicated by the straight lines, and an infinite descending ∇-flag with $\widetilde{\nabla}(0)$ at the top and subquotients indicated by the wiggly lines; cf. Claim 4 above. Given the indecomposable tilting modules $\widetilde{T}(i)$ for A, one can now compute the coalgebra C arising from the tilting generator $\widetilde{T} := \bigoplus_{i\geqslant 0}\widetilde{T}(i)$ according to the general recipe from Definition 4.21. We leave this to the reader, but display below the homomorphisms $f^{(\ell)}_{i,j} : \widetilde{T}(i) \to \widetilde{T}(j)$ in the endomorphism algebra $B := \operatorname{End}_A(\widetilde{T})^{\mathrm{op}}$ which are "dual" to the basis elements $c^{(\ell)}_{i,j}$ of the coalgebra $C = B^*$ as above.

The map $f^{(i)}_{i,i} : \widetilde{T}(i) \to \widetilde{T}(i)$ is the identity endomorphism, and $f^{(\ell)}_{i,j} : \widetilde{T}(i) \to \widetilde{T}(j)$ for $\ell > \max(i,j)$ has irreducible image and coimage isomorphic to $\widetilde{L}(\ell)$, i.e., it sends the (unique) irreducible copy of $\widetilde{L}(\ell)$ in the head of $\widetilde{T}(i)$ to the irreducible $\widetilde{L}(\ell)$ in the socle of $\widetilde{T}(j)$. The remaining maps $f^{(i)}_{i,j}, f^{(j)}_{i,j} : \widetilde{T}(i) \to \widetilde{T}(j)$ for $i \neq j$ are depicted below:



Remark 6.2. The above example can be changed slightly to obtain an essentially finite example with poset $\Lambda := \mathbb{Z}$ with the ordering reversed to the natural ordering: Let D be the essentially finite-dimensional locally unital algebra defined by the following quiver:

$$D\ :\ \begin{tikzcd} \cdots -1 \arrow[r, bend left, "y_{-1}"] & 0 \arrow[l, bend left, "x_{-1}"] \arrow[r, bend left, "y_0"] & 1 \arrow[l, bend left, "x_0"] \arrow[r, bend left, "y_1"] & 2 \cdots \arrow[l, bend left, "x_1"] \end{tikzcd} \quad \text{with relations } y_{i+1}y_i = x_i x_{i+1} = x_i y_i = 0,$$

Like for A, this algebra has a triangular decomposition, so $D\text{-mod}_{\text{fd}}$ is an essentially finite highest weight category. Recalling that the construction of tilting modules in the essentially finite case explained in §4.5 is by passing to an upper finite truncation, the indecomposable tilting module $T(0)$ for D has the same structure as for A; see (6.11). This module is infinite-dimensional; thus $D\text{-mod}_{\text{fd}}$ is *not* tilting-bounded. Note also that this algebra D can be obtained from the general construction from Remark 5.25, starting from the obvious triangular decomposition of the $\mathbb{Z}$-graded algebra $\bar{A} = \Bbbk\langle x, y \mid x^2 = y^2 = 0, xy = 0\rangle$ with x in degree 1 and y in degree -1; cf. [BT, Example 5.12].

6.3. **Category $\mathcal{O}$ for affine Lie algebras.** Perhaps the first naturally-occurring examples of finite highest weight categories came from the blocks of the BGG category $\mathcal{O}$ for a semisimple Lie algebra. This context also provides natural examples of finite ε-highest weight categories; see [Maz1] for a survey. To get examples of *semi-infinite* highest weight categories, one can consider instead blocks of the category $\mathcal{O}$ for an affine Kac-Moody Lie algebra. We briefly recall the setup referring to [Kac], [Car] for more details.

Let $\mathring{\mathfrak{g}}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ and

$$\mathfrak{g} := \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

be the corresponding affine Kac-Moody algebra. Fix also a Cartan subalgebra $\mathring{\mathfrak{h}}$ contained in a Borel subalgebra $\mathring{\mathfrak{b}}$ of $\mathring{\mathfrak{g}}$. There are corresponding subalgebras $\mathfrak{h}$ and $\mathfrak{b}$ of $\mathfrak{g}$, namely,

$$\mathfrak{h} := \mathring{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d, \qquad \mathfrak{b} := \left(\mathring{\mathfrak{b}} \otimes_{\mathbb{C}} \mathbb{C}[t] + \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} t\mathbb{C}[t]\right) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Let $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ and $\{h_i \mid i \in I\} \subset \mathfrak{h}$ be the simple roots and coroots of $\mathfrak{g}$ and $(\cdot|\cdot)$ be the normalized invariant form on $\mathfrak{h}^*$, all as in [Kac, Ch. 7–8]. The *basic imaginary root* $\delta \in \mathfrak{h}^*$ is the positive root corresponding to the canonical central element $c \in \mathfrak{h}$ under $(\cdot|\cdot)$. The linear automorphisms of $\mathfrak{h}^*$ defined by $s_i : \lambda \mapsto \lambda - \lambda(h_i)\alpha_i$ generate the Weyl group W of $\mathfrak{g}$. Let $\rho \in \mathfrak{h}^*$ be the element satisfying $\rho(h_i) = 1$ for all $i \in I$ and $\rho(d) = 0$. Then define the shifted action of W on $\mathfrak{h}^*$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$ for $w \in W$, $\lambda \in \mathfrak{h}^*$.

We define the *level* of $\lambda \in \mathfrak{h}^*$ to be $(\lambda + \rho)(c) \in \mathbb{C}$. It is *critical* if it equals the level of $\lambda = -\rho$, i.e., it is zero[7]. We usually restrict our attention to integral weights λ, that is, weights $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_i) \in \mathbb{Z}$ for all $i \in I$. The level of an integral weight is either *positive*, *negative* or *critical* (= zero). For any $\lambda \in \mathfrak{h}^*$, we define

$$\tilde{\lambda} := -\lambda - 2\rho. \tag{6.12}$$

Since $w \cdot (-\lambda - 2\rho) = -w \cdot \lambda - 2\rho$, weights λ and μ are in the same orbit under the shifted action of W if and only if so are $\tilde{\lambda}$ and $\tilde{\mu}$. Note also that the level of λ is positive (resp., critical) if and only if the level of $\tilde{\lambda}$ is negative (resp., critical). A crucial fact is that the orbit $W \cdot \lambda$ of an integral weight λ of positive level contains a unique weight $\lambda_{\max}$ such that $\lambda_{\max} + \rho$ is dominant; e.g., see [Kum, Exercise 13.1.E8a and Proposition 1.4.2]. By [Kum, Corollary 1.3.22], this weight is maximal in its orbit with respect to the usual dominance ordering $\leqslant$ on weights, i.e., $\mu \leqslant \lambda$ if $\lambda - \mu \in \bigoplus_{i \in I} \mathbb{N}\alpha_i$. If λ is integral of negative level, we deduce from this discussion that its orbit contains a unique minimal weight $\lambda_{\min}$.

For $\lambda \in \mathfrak{h}^*$, let $\Delta(\lambda)$ be the Verma module with highest weight λ and $L(\lambda)$ be its unique irreducible quotient. Although Verma modules need not be of finite length, the composition multiplicities $[\Delta(\lambda) : L(\mu)]$ are always finite. There is also the *dual Verma module* $\nabla(\lambda)$ which is the restricted dual $\Delta(\lambda)^{\#}$ of $\Delta(\lambda)$, i.e., the sum of the duals of the weight spaces of $\Delta(\lambda)$ with the $\hat{\mathfrak{g}}$-action twisted by the Chevalley antiautomorphism. All of the modules just introduced are objects in the category $\mathcal{O}$ consisting of all $\mathfrak{g}$-modules M which are semisimple over $\mathfrak{h}$ with finite-dimensional weight spaces and such that the

[7]Many authors define the level to be $\lambda(c)$, in which case the critical level is $-\check{h}$, where $\check{h}$ is the dual Coxeter number.

set of weights of M is contained in the lower set generated by a finite subset of $\mathfrak{h}^*$; see [Kum, Section 2.1]. There is also a larger category $\widehat{\mathcal{O}}$ consisting of the $\mathfrak{g}$-modules M which are semisimple over $\mathfrak{h}$ and locally finite-dimensional over $\mathfrak{b}$.

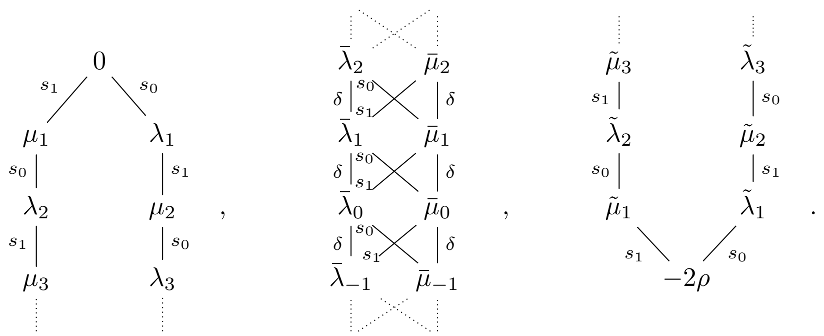
Let $\sim$ be the equivalence relation on $\mathfrak{h}^*$ generated by $\lambda \sim \mu$ if there exists a positive root γ and $n \in \mathbb{Z}$ such that $2(\lambda+\rho|\gamma) = n(\gamma|\gamma)$ and $\lambda-\mu = n\gamma$. For a $\sim$-equivalence class Λ, let $\mathcal{O}_\Lambda$ (resp., $\widehat{\mathcal{O}}_\Lambda$) be the full subcategory of $\mathcal{O}$ (resp., $\widehat{\mathcal{O}}$) consisting of all $M \in \mathcal{O}$ (resp., $M \in \widehat{\mathcal{O}}$) such that $[M : L(\lambda)] \neq 0 \Rightarrow \lambda \in \Lambda$. In view of the linkage principle from [KK, Theorem 2], these subcategories may be called the *blocks* of $\mathcal{O}$ and of $\widehat{\mathcal{O}}$, respectively. In particular, by [DGK, Theorem 4.2], any $M \in \mathcal{O}$ decomposes uniquely as a direct sum $M = \bigoplus_{\Lambda \in \mathfrak{h}^*/\sim} M_\Lambda$ with $M_\Lambda \in \mathcal{O}_\Lambda$. Note though that $\mathcal{O}$ is not the coproduct of its blocks in the strict sense since it is possible to find $M \in \mathcal{O}$ such that M_Λ is non-zero for infinitely many different Λ. The situation is more satisfactory for $\widehat{\mathcal{O}}$: $\widehat{\mathcal{O}}$ is the product of its blocks since by [Soe, Theorem 6.1] the functor

$$\prod_{\Lambda \in \mathfrak{h}^*/\sim} \widehat{\mathcal{O}}_\Lambda \to \widehat{\mathcal{O}}, \qquad (M_\Lambda)_{\Lambda \in \mathfrak{h}^*/\sim} \mapsto \bigoplus_{\Lambda \in \mathfrak{h}^*/\sim} M_\Lambda \tag{6.13}$$

is an equivalence of categories. Note also that $[\Delta(\lambda) : L(\mu)] \neq 0$ implies that the level of λ equals that of μ, since the scalars by which c acts on $L(\lambda)$ and $L(\mu)$ must agree. Consequently, we can talk simply about the level of a block.

A general combinatorial description of the $\sim$-equivalence classses Λ can be found for instance in [Fie3, Lemma 3.9]. For simplicity, we restrict ourselves from now on to integral blocks. In non-critical levels, one gets exactly the W-orbits $W \cdot \lambda$ of the integral weights of non-critical level. In critical level, one needs to incorporate also the translates by $\mathbb{Z}\delta$. From this description, it follows that the poset $(\Lambda, \leqslant)$ underlying an integral block $\mathcal{O}_\Lambda$ is upper finite with unique maximal element $\lambda_{\max}$ if $\mathcal{O}_\Lambda$ is of positive level, and lower finite with unique minimal element $\lambda_{\min}$ if $\mathcal{O}_\Lambda$ is of negative level. In case of the critical level, the poset is neither upper finite nor lower finite, but it is always interval finite.

Example 6.3. Here we give some explicit examples of posets which can occur for $\mathfrak{g} = \hat{\mathfrak{sl}}_2$, the Kac-Moody algebra for the Cartan matrix $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. The labelling set for the principal block is $W \cdot 0 = \{\lambda_k, \mu_k \mid k \geqslant 0\}$ where $\lambda_k := -\frac{1}{2}k(k+1)\alpha_0 - \frac{1}{2}k(k-1)\alpha_1$ and $\mu_k := -\frac{1}{2}k(k-1)\alpha_0 - \frac{1}{2}k(k+1)\alpha_1$. This is a block of positive level with maximal element $\lambda_0 = \mu_0 = 0$. Applying the map (6.12), we deduce that $W \cdot (-2\rho) = \{\tilde{\lambda}_k, \tilde{\mu}_k \mid k \geqslant 0\}$. This is the labelling set for a block of negative level with minimal element $\tilde{\lambda}_0 = \tilde{\mu}_0 = -2\rho$. Finally, we have that $W \cdot (\alpha_0 - \rho) \sqcup W \cdot (\alpha_1 - \rho) = \{\bar{\lambda}_k, \bar{\mu}_k \mid k \in \mathbb{Z}\}$ where $\bar{\lambda}_k := (k+1)\alpha_0 + k\alpha_1 - \rho$ and $\bar{\mu}_k := k\alpha_0 + (k+1)\alpha_1 - \rho$. This is the labelling set for a block of critical level, and it is neither upper nor lower finite. The following pictures illustrate these three situations:



Recall the definitions of upper finite and lower finite highest weight categories from Definitions 3.36 and 3.53, respectively.

Theorem 6.4. *Let $\mathcal{O}_\Lambda$ be an integral block of $\mathcal{O}$ of non-critical level. Then it is an upper finite or lower finite highest weight category according to whether the level is positive or negative, respectively. In both cases, the standard and costandard objects are the Verma modules $\Delta(\lambda)$ and the dual Verma modules $\nabla(\lambda)$, respectively, for $\lambda \in \Lambda$. The partial order $\leqslant$ on Λ is the dominance order.*

Proof. First, we prove the result for an integral block $\mathcal{O}_\Lambda$ of positive level. As explained above, the poset Λ is upper finite in this case. Let $\lambda_{\max}$ be its unique maximal weight.

Claim 1: *In the positive level case, $\mathcal{O}_\Lambda$ is the full subcategory of $\widehat{\mathcal{O}}_\Lambda$ consisting of all modules M such that $[M : L(\lambda)] < \infty$ for all $\lambda \in \Lambda$.* To prove this, given $M \in \mathcal{O}_\Lambda$, it is obvious that all of its composition multiplicities are finite since M has finite-dimensional weight spaces. Conversely, suppose that all of the composition multiplicities of $M \in \widehat{\mathcal{O}}_\Lambda$ are finite. All weights of M lie in the lower set generated by $\lambda_{\max}$. Moreover, for $\lambda \leqslant \lambda_{\max}$, the dimension of the λ-weight space of M is

$$\dim M_\lambda = \sum_{\mu \in \Lambda} [M : L(\mu)] \dim L(\mu)_\lambda.$$

Since the poset is upper finite, there are only finitely many $\mu \in \Lambda$ such that the λ-weight space $L(\mu)_\lambda$ is non-zero, and these weight spaces are finite-dimensional, so we deduce that $\dim M_\lambda < \infty$. This proves the claim.

Now we observe that the Verma module $M(\lambda_{\max})$ with maximal possible highest weight is projective in $\widehat{\mathcal{O}}_\Lambda$. From this and a standard argument involving translation functors through walls (see e.g. [Nei]) and the combinatorics from [Fie1, §4] (see also the introduction of [Fie2]), it follows that each of the irreducible modules $L(\lambda)$ ($\lambda \in \Lambda$) has a projective cover $P(\lambda)$ in $\widehat{\mathcal{O}}_\Lambda$. Moreover, these projective covers have (finite) Δ-flags as in the axiom ($P\Delta$). In particular, this shows that each $P(\lambda)$ actually belongs to $\mathcal{O}_\Lambda$. All that is left to complete the proof of the theorem in the positive level case is to show that $\mathcal{O}_\Lambda$ is a Schurian category. Let $A := \left(\bigoplus_{\lambda,\mu \in \Lambda} \operatorname{Hom}_{\mathfrak{g}}(P(\lambda), P(\mu))\right)^{\mathrm{op}}$. Since the multiplicities $[P(\mu) : L(\lambda)]$ are finite, A is a locally finite-dimensional locally unital algebra. Using Lemma 2.4, we deduce that $\widehat{\mathcal{O}}_\Lambda$ is equivalent to the category A-mod of all left A-modules. As explained in the discussion after (2.18), A-mod$_{\mathrm{lfd}}$ is the full subcategory of A-mod consisting of all modules with finite composition multiplicities. Combining this with Claim 1, we deduce that the equivalence between $\widehat{\mathcal{O}}_\Lambda$ and A-mod restricts to an equivalence between $\mathcal{O}_\Lambda$ and A-mod$_{\mathrm{lfd}}$. Hence, $\mathcal{O}_\Lambda$ is a Schurian category.

We turn our attention to an integral block $\mathcal{O}_\Lambda$ of negative level. In this case, we know already that the poset Λ is lower finite with a unique minimal element $\lambda_{\min}$.

Claim 2: *In the negative level case, the category $\mathcal{O}_\Lambda$ is the full subcategory of $\widehat{\mathcal{O}}_\Lambda$ consisting of all modules of finite length.* For this, it is obvious that any module in $\widehat{\mathcal{O}}_\Lambda$ of finite length belongs to $\mathcal{O}_\Lambda$. Conversely, any object in $\mathcal{O}_\Lambda$ is of finite length thanks to the formula [Kum, 2.1.11 (1)], taking λ therein to be $\lambda_{\min}$.

From Claim 2 and Lemma 2.1, it follows that $\mathcal{R} := \mathcal{O}_\Lambda$ is a locally finite Abelian category. By [Fie1, Theorem 2.7] the Serre subcategory $\mathcal{R}^\downarrow$ of $\mathcal{R}$ associated to $\Lambda^\downarrow$ is a finite highest weight category for each finite lower set $\Lambda^\downarrow$ of Λ. We deduce that $\mathcal{R}$ is a lower finite highest weight category using Theorem 3.63. □

Let $\mathcal{O}_\Lambda$ be an integral block of non-critical level. The following assertions about projective and injective modules follow from Theorem 6.4 and the general theory from §§2.1–2.3; see also [Soe, Remark 6.5].

- In the positive level case, when $\mathcal{O}_\Lambda$ is a Schurian category, $\widehat{\mathcal{O}}_\Lambda$ has enough projectives and injectives. Moreover, the projective covers of the irreducible modules are the modules $\{P(\lambda) \mid \lambda \in \Lambda\}$ constructed in the proof of Theorem 6.4, and these belong to $\mathcal{O}_\Lambda$. Their restricted duals $I(\lambda) := P(\lambda)^\#$ are the indecomposable injective modules in $\widehat{\mathcal{O}}_\Lambda$, and also belong to $\mathcal{O}_\Lambda$.

- The situation is completely different in the negative level case, as we need to pass to $\widehat{\mathcal{O}}_\Lambda$, which is the ind-completion of the finite Abelian category $\mathcal{O}_\Lambda$, before we can talk about injective modules. In $\widehat{\mathcal{O}}_\Lambda$, the irreducible module $L(\lambda)$ ($\lambda \in \Lambda$) has an injective hull $I(\lambda)$ in $\widehat{\mathcal{O}}_\Lambda$, which possesses a (possibly infinite) ascending ∇-flag in the sense of Definition 3.52. However, $\widehat{\mathcal{O}}_\Lambda$ usually does not have any projectives at all (although one could construct such modules in the pro-completion of $\mathcal{O}_\Lambda$ as done e.g. in [Fie2]).

The following results about tilting modules are consequences of the general theory developed in §4.1 and §4.3. They already appeared in an equivalent form in [Soe].

- In the negative level case, tilting modules are objects in $\mathcal{O}_\Lambda$ admitting both a (finite) Δ-flag and a (finite) ∇-flag. The isomorphism classes of indecomposable tilting modules in $\mathcal{O}_\Lambda$ are parametrized by their highest weights. They may also be constructed by applying translation functors to the Verma module $\Delta(\lambda_{\min})$.
- In the positive level case, tilting modules are objects in $\mathcal{O}_\Lambda$ which admit both a (possibly infinite) ascending Δ-flag and a (possibly infinite) descending ∇-flag in the sense of Definition 3.37. Again, the isomorphism classes of indecomposable tilting modules are parametrized by their highest weights.

In both cases, our characterization of the indecomposable tilting module $T(\lambda)$ of highest weight λ is slightly different from the one given in [Soe, Definition 6.3]. From our definition, one sees immediately that $T(\lambda)^\# \cong T(\lambda)$.

Remark 6.5. Elsewhere in the literature dealing with positive level, it is common to pass to a different category of modules, e.g., to the Whittaker category in [BY] or to truncated versions of $\mathcal{O}$ in [SVV, Section 3], before contemplating tilting modules.

Our next result is concerned with the Ringel duality between integral blocks of positive and negative level. This depends crucially on a special case of the *Arkhipov-Soergel equivalence* from [Ark], [Soe]. Let S be Arkhipov's semi-regular bimodule, which is the bimodule S_γ of [Soe] with $\gamma := 2\rho$ as in [Soe, Lemma 7.1]. For $\lambda \in \mathfrak{h}^*$, let $T(\lambda)$ be the indecomposable tilting module from [Soe, Definition 6.3]. Also let $P(\lambda)$ be a projective cover of $L(\lambda)$ in $\widehat{\mathcal{O}}$ whenever such an object exists; cf. [Soe, Remark 6.5(2)].

Theorem 6.6 (Arkhipov, Soergel)**.** *Tensoring with the semi-regular bimodule defines an equivalence $S \otimes_{U(\mathfrak{g})} - : \Delta(\mathcal{O}) \to \nabla(\mathcal{O})$ between the exact subcategories of $\mathcal{O}$ consisting of objects with (finite) Δ- and ∇-flags, respectively, such that*

(1) $S \otimes_{U(\mathfrak{g})} \Delta(\lambda) \cong \nabla(\tilde{\lambda})$;

(2) $S \otimes_{U(\mathfrak{g})} P(\lambda) \cong T(\tilde{\lambda})$ (assuming $P(\lambda)$ exists).

Corollary 6.7. *Assume that $\mathcal{O}_\Lambda$ is an integral block of negative level. Let $\widetilde{\mathcal{R}}$ be the Ringel dual of $\mathcal{R} := \mathcal{O}_\Lambda$ relative to some choice of $T = \bigoplus_{i \in I} T_i$ as in Definition 4.19, and let F be the Ringel duality functor from (4.8). Also let $\widetilde{\Lambda} := \{\tilde{\lambda} \mid \lambda \in \Lambda\}$. Then there is an equivalence of categories $E : \widetilde{\mathcal{R}} \to \mathcal{O}_{\widetilde{\Lambda}}$ such that $E \circ F : \nabla(\mathcal{O}_\Lambda) \to \Delta(\mathcal{O}_{\widetilde{\Lambda}})$ is a quasi-inverse to the Arkhipov-Soergel equivalence $S \otimes_{U(\mathfrak{g})} - : \Delta(\mathcal{O}_{\widetilde{\Lambda}}) \to \nabla(\mathcal{O}_\Lambda)$.*

Proof. Note to start with that $\mathcal{O}_{\widetilde{\Lambda}}$ is an integral block of positive level. Moreover, the map $\Lambda^{\mathrm{op}} \to \widetilde{\Lambda}, \lambda \mapsto \tilde{\lambda}$ is an order isomorphism.

Choose a quasi-inverse D to $S \otimes_{U(\mathfrak{g})} - : \Delta(\mathcal{O}_{\widetilde{\Lambda}}) \to \nabla(\mathcal{O}_\Lambda)$, and set $P_i := DT_i$. By Theorem 6.6(2), $(P_i)_{i \in I}$ is a projective generating family for $\widehat{\mathcal{O}}_{\widetilde{\Lambda}}$. Moreover, recalling that $\widetilde{\mathcal{R}}$ is the category A-mod$_{\mathrm{lfd}}$ where $A := \bigoplus_{i,j \in I} \operatorname{Hom}_{\mathcal{O}_\Lambda}(T_i, T_j)$, the functor D induces an isomorphism via which we can identify A with $\bigoplus_{i,j \in I} \operatorname{Hom}_{\mathcal{O}_{\widetilde{\Lambda}}}(P_i, P_j)$.

As explained in the proof of Theorem 6.4, the functor

$$H := \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{O}_{\tilde{\Lambda}}}(P_i, -) : \mathcal{O}_{\tilde{\Lambda}} \to A\text{-mod}_{\mathrm{lfd}}$$

is an equivalence of categories. Moreover, we have that

$$H \circ D = \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{O}_{\tilde{\Lambda}}}(P_i, D(-)) \cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{O}_{\Lambda}}(S \otimes_{U(\mathfrak{g})} P_i, -) \cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{O}_{\Lambda}}(T_i, -) = F.$$

Letting E be a quasi-inverse equivalence to H, it follows that $E \circ F \cong D$. □

Remark 6.8. In the setup of Corollary 6.7, the Arkhipov-Soergel equivalence extends to an equivalence $S \otimes_{U(\mathfrak{g})} - : \Delta^{\mathrm{asc}}(\mathcal{O}_{\tilde{\Lambda}}) \to \nabla^{\mathrm{asc}}(\mathcal{O}_{\Lambda})$, which is a quasi-inverse to $E \circ F : \nabla^{\mathrm{asc}}(\mathcal{O}_{\Lambda}) \to \Delta^{\mathrm{asc}}(\mathcal{O}_{\tilde{\Lambda}})$. These functors interchange the indecomposable injectives in $\widehat{\mathcal{O}}_{\Lambda}$ with the indecomposable tiltings in $\mathcal{O}_{\tilde{\Lambda}}$.

Finally we discuss the situation for an integral critical block $\mathcal{O}_{\Lambda}$. As we have already explained, in this case the poset Λ is neither upper nor lower finite. In fact, these blocks do not fit into the framework of this article at all, since the Verma modules have infinite length and there are no projectives. One sees this already for the Verma module $\Delta(-\rho)$ for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, which has composition factors $L(-\rho - m\delta)$ for $m \geqslant 0$, each appearing with multiplicity equal to the number of partitions of m; see e.g. [AF1, Theorem 4.9(1)]. However, there is an auto-equivalence $\Sigma := L(\delta) \otimes - : \widehat{\mathcal{O}}_{\Lambda} \to \widehat{\mathcal{O}}_{\Lambda}$, which makes it possible to pass to the *restricted category* $\widehat{\mathcal{O}}^{\mathrm{res}}_{\Lambda}$, which we define next.

Let A_n be the vector space of natural transformations $\Sigma^n \to \mathrm{Id}$. This gives rise to a graded algebra $A := \bigoplus_{n\in\mathbb{Z}} A_n$. Then the restricted category $\widehat{\mathcal{O}}^{\mathrm{res}}_{\Lambda}$ is the full subcategory of $\widehat{\mathcal{O}}_{\Lambda}$ consisting of all modules which are annihilated by the induced action of A_n for $n \neq 0$; cf. [AF1, §4.3]. The irreducible modules in the restricted category are the same as in $\widehat{\mathcal{O}}_{\Lambda}$ itself. There are also the *restricted Verma modules*

$$\Delta(\lambda)^{\mathrm{res}} := \Delta(\lambda) \Big/ \sum_{\eta\in A_{\neq 0}} \mathrm{im}\left(\eta_{\Delta(\lambda)} : \Sigma^n \Delta(\lambda) \to \Delta(\lambda)\right) \tag{6.14}$$

from [AF1, §4.4]. In other words, $\Delta(\lambda)^{\mathrm{res}}$ is the largest quotient of $\Delta(\lambda)$ that belongs to the restricted category. Similarly, the *restricted dual Verma module* $\nabla(\lambda)^{\mathrm{res}}$ is the largest submodule of $\nabla(\lambda)$ that belongs to the restricted category.

The restricted category $\widehat{\mathcal{O}}^{\mathrm{res}}_{\Lambda}$ is no longer indecomposable: by [AF2, Theorem 5.1] it decomposes further as

$$\widehat{\mathcal{O}}^{\mathrm{res}}_{\Lambda} = \prod_{\overline{\Lambda}\in\Lambda/W} \widehat{\mathcal{O}}^{\mathrm{res}}_{\overline{\Lambda}} \tag{6.15}$$

where Λ/W denotes the orbits of W under the dot action. For instance, the poset Λ for the critical level displayed in Example 6.3 splits into two orbits $W \cdot (\alpha_0 - \rho)$ and $W \cdot (\alpha_1 - \rho)$ (i.e., one removes the edges labelled by δ). In the most singular case, $\widehat{\mathcal{O}}^{\mathrm{res}}_{-\rho}$ is a product of simple blocks; in particular, $\Delta^{\mathrm{res}}(-\rho) = L(-\rho) = \nabla^{\mathrm{res}}(-\rho)$.

Conjecture 6.9. Let $\widehat{\mathcal{O}}^{\mathrm{res}}_{\overline{\Lambda}}$ be a regular (in the sense of [AF2]) integral critical block. Let $\mathcal{O}^{\mathrm{res}}_{\overline{\Lambda}} := \mathrm{Fin}\left(\widehat{\mathcal{O}}^{\mathrm{res}}_{\overline{\Lambda}}\right)$ be the full subcategory consisting of all modules of finite length. Then $\mathcal{O}^{\mathrm{res}}_{\overline{\Lambda}}$ is an essentially finite highest weight category with standard and costandard objects $\Delta(\lambda)^{\mathrm{res}}$ and $\nabla(\lambda)^{\mathrm{res}}$ for $\lambda \in \overline{\Lambda}$. Moreover, the indecomposable projective modules in $\mathcal{O}^{\mathrm{res}}_{\overline{\Lambda}}$ are also its indecomposable tilting modules, and therefore $\mathcal{O}^{\mathrm{res}}_{\overline{\Lambda}}$ is tilting-bounded and Ringel self-dual.

This conjecture is true for the basic example of a critical block from Example 6.3 thanks to [Fie3, Theorem 6.6]; the same category arises as the principal block of category $\mathcal{O}$ for $\mathfrak{gl}_{1|1}(\mathbb{C})$ discussed in §6.7 below. The conjecture is also consistent with the so-called *Feigin-Frenkel conjecture* [AF1, Conjecture 4.7], which says that composition multiplicities of restricted Verma modules are related to the periodic Kazhdan-Lusztig polynomials from [Lus] (and Jantzen's generic decomposition patterns from [Jan2]). These polynomials depend on the relative position of the given pair of weights and, when not too close to walls, they vanish for weights that are far apart. This is consistent with the conjectured existence of indecomposable projectives of finite length in regular blocks of the restricted category.

Remark 6.10. It seems to us that the Feigin-Frenkel conjecture might have a geometric explanation in terms of a sequence of equivalences of categories similar to [FG, (7)]. Ultimately this should connect $\mathcal{O}_{\Lambda}^{\mathrm{res}}$ with representations of the quantum group analog of Jantzen's thickened Frobenius kernel G_1T. The latter are already known by [AJS, §17] to be essentially finite highest weight categories controlled[8] by the periodic Kazhdan-Lusztig polynomials. Also, in these categories, tilting modules are projective, hence, the Ringel self-duality would be an obvious consequence.

6.4. **Rational representations.** As we noted in Remark 3.66, the definition of lower finite highest weight category originated in the work of Cline, Parshall and Scott [CPS1]. As well as the BGG category $\mathcal{O}$ already mentioned, their work was motivated by the representation theory of a connected reductive algebraic group G in positive characteristic, as developed for example in [Jan1]: the rigid symmetric monoidal category $\mathcal{R}ep(G)$ of finite-dimensional rational representations of G is a lower finite highest weight category. Tilting modules for G were studied in [Don3], although our formulation of semi-infinite Ringel duality from §4.4 is not mentioned explicitly there: Donkin instead took the approach pioneered in [Don2] of truncating to a finite lower set before taking Ringel duals.

To give more details, we fix a maximal torus T contained in a Borel subgroup B of G. Then the weight poset Λ is the set $X^+(T)$ of dominant characters of T with respect to B; Jantzen's convention is that the roots of B are negative. The costandard objects $\nabla(\lambda)$ are the induced modules $H^0(G/B, \mathcal{L}_\lambda)$. For the partial order $\leqslant$, one can use the usual dominance ordering on $X^+(T)$, or the more refined Bruhat order of [Jan1, §II.6.4]. This makes $\mathcal{R}ep(G)$ into a lower finite highest weight category by [Jan1, Proposition II.4.18] and [Jan1, Proposition II.6.13]. In fact, in the case of $\mathcal{R}ep(G)$, all of the general results about ascending ∇-flags found in §3.5 were known already before the time of [CPS1], e.g., they are discussed in Donkin's book [Don1] (and called there *good filtrations*).

Let $\mathcal{T}ilt(G)$ be the full subcategory of $\mathcal{R}ep(G)$ consisting of all tilting modules. A key theorem in this setting is that tensor products of tilting modules are tilting; this is the Donkin-Mathieu-Wang theorem [Don1], [Mat], [Wan]. Thus, $\mathcal{T}ilt(G)$ is a Karoubian rigid symmetric monoidal category. Let $(T_i)_{i\in I_0}$ be a *monoidal tilting generating family* for $\mathcal{T}ilt(G)$, that is, a family of tilting modules such that every indecomposable tilting module is isomorphic to a summand of a tensor product $T_{\boldsymbol{i}} := T_{i_1}\otimes\cdots\otimes T_{i_r}$ for some finite (possibly empty) sequence $\boldsymbol{i} = (i_1, \dots, i_r)$ of elements of I_0. Let I denote the set of all such sequences. Then define $\mathcal{A}$ to be the category with objects I, morphisms defined from $\mathrm{Hom}_{\mathcal{A}}(\boldsymbol{i}, \boldsymbol{j}) := \mathrm{Hom}_G(T_{\boldsymbol{i}}, T_{\boldsymbol{j}})$, composition being induced by composition in $\mathcal{R}ep(G)$. The category $\mathcal{A}$ is naturally a strict monoidal category, with the tensor product of objects being by concatenation of sequences. The evident monoidal functor $\mathcal{A} \to \mathcal{T}ilt(G)$ extends to the Karoubi envelope of $\mathcal{A}$, and the resulting functor $\mathrm{Kar}(\mathcal{A}) \to \mathcal{T}ilt(G)$ is a monoidal equivalence.

Forgetting the monoidal structure, one can think instead in terms of the locally finite-dimensional locally unital algebra $A = \bigoplus_{\boldsymbol{i},\boldsymbol{j}\in I} e_{\boldsymbol{i}} A e_{\boldsymbol{j}}$ that is the path algebra of $\mathcal{A}^{\mathrm{op}}$ in the sense of Remark 2.3. Then the Ringel dual of $\mathcal{R}ep(G)$ relative to $(T_{\boldsymbol{i}})_{\boldsymbol{i}\in I}$ in the general sense of Definition 4.19 is the category $A\text{-mod}_{\mathrm{lfd}}$. It is naturally an upper finite highest weight category. Theorem 5.7 shows moreover that there is an idempotent expansion of A making A into a based quasi-hereditary algebra in the sense of Definition 5.1.

[8]One needs to assume that the root of unity ℓ is odd and bigger than or equal to the Coxeter number.

The simplest interesting example comes from $G = SL_2$. For this, we may take $I_0 := \{1\}$, so that $I = \mathbb{N}$, and T_1 to be the natural two-dimensional representation. The strict monoidal category $\mathcal{A}$ in this case has an explicit diagrammatical description: it is the the *Temperley-Lieb category* $\mathcal{TL}(\delta)$ at parameter $\delta := 2$; see e.g. [GW]. The Temperley category comes with a natural diagrammatic basis, hence, we get bases for morphism spaces in $\mathcal{A}$ which turns out already to be an object-adapted cellular basis in the sense of [ELau]. Equivalently, the natural basis is an idempotent-adapted cellular basis for the path algebra A of $\mathcal{A}^{\mathrm{op}}$ making A into an upper finite based quasi-hereditary algebra in the sense of Definition 5.1. In this based quasi-hereditary structure, the upper finite poset is $I = \mathbb{N}$ ordered by the opposite of the natural ordering, and for $\lambda \in I$ the set $Y(\lambda)$ (resp., $X(\lambda)$) consists of all cap-free Temperley-Lieb diagrams with λ strings at the bottom (resp., all cup-free Temperley-Lieb diagrams with λ strings at the top). This example is somewhat misleading in its simplicity, for example, one does not need to pass to any sort of idempotent expansion.

The case that $G = GL_n$ is also quite classical. If one assumes that $p := \operatorname{char} \Bbbk$ is either 0 or satisfies $p > n$, then one can take $I_0 := \{\uparrow, \downarrow\}$, $T_\uparrow$ to be the natural G-module V, and $T_\downarrow$ to be its dual V^*. By a reformulation of Schur-Weyl duality, the resulting strict monoidal category $\mathcal{A}$ is isomorphic to a quotient of the so-called *oriented Brauer category* $\mathcal{OB}(\delta)$ at parameter $\delta := n$ by an explicitly described tensor ideal; see [Bru, Theorem 1.10]. Theorem 5.7 implies that there is an idempotent expansion of the path algebra A of $\mathcal{A}^{\mathrm{op}}$ which is an upper finite based quasi-hereditary algebra. We expect that an explicit idempotent-adapted cellular basis for A could be constructed using the methods of [Eli].

The general principles outlined so far are valid also when G is replaced by the corresponding quantum group $U_q(\mathfrak{g})$, possibly at a root of unity (taking the Lusztig form). When at a root of unity over the ground field is $\mathbb{C}$, it turns out that the indecomposable projectives and injectives in the category of rational representations of $U_q(\mathfrak{g})$ are all finite-dimensional, i.e., the category is essentially finite Abelian. Tiltings are also finite-dimensional, indeed, the category is tilting-bounded in the sense of §4.5. The first example of this nature comes from $U_q(\mathfrak{sl}_2)$ at a root of unity over $\mathbb{C}$. In this case, see e.g. [AT, Theorem 3.12, Definition 3.3], the principal block is Morita equivalent to the locally unital algebra A defined as the path algebra of the quiver

$$\begin{tikzcd} 0 \arrow[r, bend left, "x_0"] & 1 \arrow[l, bend left, "y_0"] \arrow[r, bend left, "x_1"] & 2 \arrow[l, bend left, "y_1"] \arrow[r, bend left, "x_2"] & 3 \cdots \arrow[l, bend left, "y_2"] \end{tikzcd} \qquad \text{with relations } x_{i+1}x_i = y_iy_{i+1} = x_iy_i - y_{i+1}x_{i+1} = 0.$$

The appropriate partial order on the weight poset $\mathbb{N}$ is the natural order $0 < 1 < \cdots$. The indecomposable projectives have the following structure:

$$P(0) = \begin{tikzcd}[row sep=small] 0 \arrow[d, squiggly, "x"] \\ 1 \arrow[d, squiggly, "y"] \\ 0 \end{tikzcd}, \quad P(1) = \begin{tikzcd}[row sep=small, column sep=small] & 1 \arrow[dl, squiggly, "y"'] \arrow[dr, squiggly, "x"] & \\ 0 \arrow[dr, squiggly, "x"'] & & 2 \arrow[dl, squiggly, "y"] \\ & 1 & \end{tikzcd}, \quad P(2) = \begin{tikzcd}[row sep=small, column sep=small] & 2 \arrow[dl, squiggly, "y"'] \arrow[dr, squiggly, "x"] & \\ 1 \arrow[dr, squiggly, "x"'] & & 3 \arrow[dl, squiggly, "y"] \\ & 2 & \end{tikzcd}, \quad P(3) = \begin{tikzcd}[row sep=small, column sep=small] & 3 \arrow[dl, squiggly, "y"'] \arrow[dr, squiggly, "x"] & \\ 2 \arrow[dr, squiggly, "x"'] & & 4 \arrow[dl, squiggly, "y"] \\ & 3 & \end{tikzcd}, \ldots.$$

The tilting objects are $T(0) := L(0)$ and $T(n) := P(n-1)$ for $n \geqslant 1$. From this, it is easy to see that the Ringel dual is described by the same quiver with one additional relation, namely, that $b_0a_0 = 0$ (and of course the order is reversed).

6.5. **Tensor product categorifications.** Until quite recently, most of the naturally-occurring examples were highest weight categories (like the ones described in the previous two subsections). But the work of Webster [Web1], [Web2] and Losev and Webster [LW] has brought to prominence a very general source of examples that are fully stratified but seldom highest weight.

Fundamental amongst these new examples are the categorifications of tensor products of irreducible highest weight modules of symmetrizable Kac-Moody Lie algebras. Rather than attempting to repeat the definition of these here, we refer the reader to [LW]. All of these examples are finite fully stratified categories possessing a duality as in Corollary 3.23. They are also tilting-rigid; the proof of this depends on an argument involving translation/projective functors. Consequently, the Ringel dual is again a finite fully signed category that is tilting-rigid. In fact, the Ringel dual category is always another tensor product categorification[9] (reverse the order of the tensor product). In the earlier article [Web2], Webster also wrote down explicit finite-dimensional algebras which give realization of these categories. In view of Theorem 5.22, all of Webster's algebras admit bases making them into based stratified algebras, but explicit such bases have only been found in a few examples in type A which are actually highest weight.

In [Web1], Webster also introduced some more general tensor product categorifications, including ones which categorify the tensor product of an integrable lowest weight module tensored with an integrable highest weight module; see also [BD1, Construction 4.13]. The latter are particularly important since they may be realized as generalized cyclotomic quotients of the Kac-Moody 2-category. They are upper finite fully stratified categories.

6.6. **Deligne categories.** Another source of upper finite highest weight categories comes from various Deligne categories. The definition of these categories is diagrammatic in nature. For example, in characteristic zero, the Deligne category $\underline{\mathrm{Re}}\mathrm{p}(GL_\delta)$ is the Karoubi envelope of the oriented Brauer category $\mathcal{OB}(\delta)$ mentioned earlier. This case was studied in the PhD thesis of Reynolds [Rey] based on the observation that it admits a triangular decomposition in the sense of Definition 5.24; see also [Bru]. This was also noticed independently by Coulembier and Zhang [CZ, §8], who also extended it to the other types of Deligne category (and several other families of diagram algebras).

The category of locally finite-dimensional representations of the Deligne category $\underline{\mathrm{Re}}\mathrm{p}(GL_\delta)$ can also be interpreted as a special case of the lowest weight tensored highest weight tensor product categorifications discussed in the previous subsection; see [Bru, Theorem 1.11]. The Ringel dual in this example is equivalent to the Abelian envelope $\underline{\mathrm{Re}}\mathrm{p}^{ab}(GL_\delta)$ of Deligne's category constructed by Entova, Hinich and Serganova [EHS], which is a monoidal lower finite highest weight category. In [Ent], it is shown that $\underline{\mathrm{Re}}\mathrm{p}^{ab}(GL_\delta)$ categorifies a highest weight tensored lowest weight representation, which is the dual result to that of [Bru]. This example will be discussed further in the sequel to this article, where we give an explicit description of the blocks of $\underline{\mathrm{Re}}\mathrm{p}^{ab}(GL_\delta)$ via Khovanov's arc coalgebra (an interesting explicit example of a based quasi-hereditary coalgebra), thereby proving a conjecture formulated in the introduction of [BS2].

The other classical families of Deligne categories $\underline{\mathrm{Re}}\mathrm{p}(O_\delta)$, $\underline{\mathrm{Re}}\mathrm{p}(P)$ and $\underline{\mathrm{Re}}\mathrm{p}(Q)$ are also being investigated actively along similar lines by several groups of authors, and there has been considerable recent progress. There are many interesting connections here with rational representations of the corresponding families of classical supergroups.

6.7. **Representations of Lie superalgebras.** Finally, we mention briefly an interesting source of essentially finite highest weight categories: the analogs of the BGG category $\mathcal{O}$ for classical Lie superalgebras. A detailed account in the case of the Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{C})$ can be found in [BLW]. Its category $\mathcal{O}$ gives an essentially finite highest weight category which is neither lower finite nor upper finite. Moreover, it is tilting-bounded as in §4.5, so that the Ringel dual category is also an essentially finite highest weight category.

[9]This was noted in Remark 3.10 of the `arxiv` version of [LW] but the authors removed this remark in the published version (along with Remark 2.7 which was misleading).

There is one very easy special case: the principal block of category $\mathcal{O}$ for $\mathfrak{gl}_{1|1}(\mathbb{C})$ is equivalent to the category of finite-dimensional modules over the essentially finite-dimensional locally unital algebra which is the path algebra of the following quiver:

$$\cdots -1 \underset{y_{-1}}{\overset{x_{-1}}{\rightleftarrows}} 0 \underset{y_0}{\overset{x_0}{\rightleftarrows}} 1 \underset{y_1}{\overset{x_1}{\rightleftarrows}} 2 \cdots \text{ with relations } x_{i+1}x_i = y_iy_{i+1} = x_iy_i - y_{i+1}x_{i+1} = 0,$$

see e.g. [BS1, p. 380]. This is very similar to the $U_q(\mathfrak{sl}_2)$-example from §6.4, but now the poset $\mathbb{Z}$ (ordered naturally) is neither lower nor upper finite. From the category $\mathcal{O}$ perspective, this example is rather misleading since its projective, injective and tilting objects coincide, hence, it is Ringel self-dual.

One gets similar examples from $\mathfrak{osp}_{m|2n}(\mathbb{C})$, as discussed for example in [BW] and [ES]. The simplest non-trivial case of $\mathfrak{osp}_{3|2}(\mathbb{C})$ produces the path algebra of a D_∞ quiver (replacing than the A_∞ quiver above); see [ES, §II]. The "strange" families $\mathfrak{p}_n(\mathbb{C})$ and $\mathfrak{q}_n(\mathbb{C})$ also exhibit similar structures. The former has not yet been investigated systematically (although basic aspects of the finite-dimensional finite-dimensional representations and category $\mathcal{O}$ were recently studied in [B+9] and [CC], respectively). It is an interesting example of a naturally-occurring highest weight category which does not admit a duality. For $\mathfrak{q}_n(\mathbb{C})$, we refer to [BD2] and the references therein. In fact, the integral blocks for $\mathfrak{q}_n(\mathbb{C})$ are signed highest weight categories; this observation is due to Frisk [Fri2].

References

[ADL] I. Ágoston, V. Dlab and E. Lukács, Stratified algebras, *Math. Rep. Acad. Sci. Canada* **20** (1998), 22–28.

[AHLU] I. Ágoston, D. Happel, E. Lukács and L. Unger, Standardly stratified algebras and tilting, *J. Algebra* **226** (2000), 144–160.

[And] H. H. Andersen, Tilting modules and cellular categories; `arXiv:1912.00817`.

[AJS] H. H. Andersen, J. C. Jantzen and W. Soergel, Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p, *Astérisque* **220** (1994).

[AST] H. H. Andersen, C. Stroppel and D. Tubbenhauer, Cellular structures using U_q-tilting modules, *Pacific J. Math.* **292** (2018), 21–59.

[AT] H. H. Andersen and D. Tubbenhauer, Diagram categories for U_q-tilting modules at roots of unity, *Transform. Groups* **22** (2017), 29–89.

[AF1] T. Arakawa and P. Fiebig, On the restricted Verma modules at the critical level, *Trans. Amer. Math. Soc.* **364** (2012), 4683–4712.

[AF2] ———, The linkage principle for restricted critical level representations of affine Kac-Moody algebras, *Compositio Math.* **148** (2012), 1787–1810.

[Ark] S. Arkhipov, Semi-infinite cohomology of associative algebras and bar duality, *Int. Math. Res. Notices* 1997, 833–863.

[B+9] M. Balagovic, Z. Daugherty, I. Entova-Aizenbud, I. Halacheva, J. Hennig, M. S. Im, G. Letzter, E. Norton, V. Serganova and C. Stroppel, Translation functors and decomposition numbers for the periplectic Lie superalgebra $\mathfrak{p}(n)$, *Math. Res. Lett.* **26** (2019), 643–710.

[BW] H. Bao and W. Wang, A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs, *Astérisque* 2018, no. 402, vii+134 pp..

[BBD] A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers, *Astérisque* **100** (1982), 5–171.

[BT] G. Bellamy and U. Thiel, Highest weight theory for finite-dimensional graded algebras with triangular decomposition, *Advances Math.* **330** (2018), 361–419.

[BY] R. Bezrukavnikov and Z. Yun, On Koszul duality for Kac-Moody groups, *Represent. Theory* **17** (2013), 1–98.

[Bru] J. Brundan, Representations of oriented skein categories; `arXiv:1712.08953`.

[BD1] J. Brundan and N. Davidson, Categorical actions and crystals, *Contemp. Math.* **684** (2017), 116–159.

[BD2] ———, Type C blocks of super category $\mathcal{O}$, *Math. Z.* **293** (2019), 867–901.

[BLW] J. Brundan, I. Losev and B. Webster, Tensor product categorifications and the super Kazhdan-Lusztig conjecture, *Int. Math. Res. Notices* 2017, 6329–6410.

[BS1] J. Brundan and C. Stroppel, Highest weight categories arising from Khovanov's diagram algebra IV: the general linear supergroup, *J. Eur. Math. Soc.* **14** (2012), 373–419.

[BS2] ——— Gradings on walled Brauer algebras and Khovanov's arc algebra, *Advances Math.* **231** (2012), 709–773.

[Car] R. W. Carter, *Lie Algebras of Finite and Affine Type*, Cambridge Studies in Advanced Mathematics **96**, Cambridge University Press, 2005.

[CC] C.-W. Chen and K. Coulembier, The primitive spectrum and category $\mathcal{O}$ for the periplectic Lie superalgebra; `arXiv:1805.08885`.

[CPS1] E. Cline, B. Parshall and L. Scott, Finite dimensional algebras and highest weight categories, *J. Reine Angew. Math.* **391** (1988), 85–99.

[CPS2] ———, Stratifying endomorphism algebras, *Mem. Amer. Math. Soc.* **124** (1996), 1–119.

[Cou] K. Coulembier, The classification of blocks in BGG category $\mathcal{O}$, *Math. Z.*, to appear.

[CZ] K. Coulembier and R.-B. Zhang, Borelic pairs for stratified algebras, *Advances Math.* **345** (2019), 53–115.

[DGK] V. Deodhar, O. Gabber and V. Kac, Structure of some categories of representations of infinite-dimensional Lie algebras, *Advances Math.* **45** (1982), 92–116.

[Dla1] V. Dlab, Quasi-hereditary algebras revisited, *An. Şt. Univ. Ovidius Constanţa* **4** (1996), 43–54.

[Dla2] ———, Properly stratified algebras, *C. R. Acad. Sci. Paris* **331** (2000), 191–196.

[DR] V. Dlab and C. Ringel, The module theoretical approach to quasi-hereditary algebras, in: "Representations of Algebras and Related Topics (1990)," pp. 200–224, London Math. Soc. Lecture Note Ser. **168**, Cambridge Univ. Press, Cambridge, 1992.

[Don1] S. Donkin, *Rational Representations of Algebraic Groups*, LNM **1140**, Springer, 1985.

[Don2] ———, On Schur algebras and related algebras I, *J. Algebra* **104** (1986), 310–328.

[Don3] ———, On tilting modules for algebraic groups, *Math. Z.* **212** (1993), 39–60.

[Don4] ———, *The q-Schur Algebra*, London Math. Soc., 1998.

[DuR] J. Du and H. Rui, Based algebras and standard bases for quasi-hereditary algebras, *Trans. Amer. Math. Soc.* **350** (1998), 3207–3235.

[ES] M. Ehrig and C. Stroppel, On the category of finite-dimensional representations of $OSp(r|2n)$: part I, in: "Representation Theory–Current Trends and Perspectives," 109–170, *EMS Ser. Congr. Rep.*, Eur. Math. Soc., Zurich, 2017.

[Eli] B. Elias, Light ladders and clasp conjectures; `arXiv:1510.06840`.

[ELau] B. Elias and A. Lauda, Trace decategorification of the Hecke category, *J. Algebra* **449** (2016), 615–634.

[ELos] B. Elias and I. Losev, Modular representation theory in type A via Soergel bimodules; `arXiv:1701.00560`.

[Ent] I. Entova-Aizenbud, Categorical actions and multiplicities in the Deligne category $\mathrm{Rep}(GL_t)$ *J. Algebra* **504** (2018), 391–431.

[EHS] I. Entova-Aizenbud, V. Hinich and V. Serganova, Deligne categories and the limit of categories $\mathrm{Rep}(GL(m|n))$; `arXiv:1511.07699`.

[EGNO] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik, *Tensor Categories*, Amer. Math. Soc., 2015.

[Fie1] P. Fiebig, Centers and translation functors for the category $\mathcal{O}$ over Kac-Moody algebras, *Math. Z.* **243** (2003), 689–717.

[Fie2] ———, The combinatorics of category $\mathcal{O}$ over symmetrizable Kac-Moody algebras, *Transform. Groups* **11** (2006), 29–49.

[Fie3] ———, On the subgeneric restricted blocks of affine category $\mathcal{O}$ at the critical level, in: "Symmetries, Integrable Systems and Representations," pp. 65–84, *Springer Proc. Math. Stat.* **40**, 2013.

[FG] E. Frenkel and D. Gaitsgory, Localization of $\mathfrak{g}$-modules on the affine Grassmannian, *Ann. of Math.* (2) **170** (2009), no. 3, 1339–1381.

[FKS] I. Frenkel, M. Khovanov and C. Stroppel, A categorification of finite-dimensional irreducible representations of quantum $\mathfrak{sl}_2$ and their tensor products, *Selecta Math.* (N.S.) **12** (2006), 379–431.

[Fre] P. Freyd, Abelian categories, *Reprints in Theory and Applications of Categories* **3** (2003), 23–164.

[Fri1] A. Frisk, Dlab's theorem and tilting modules for stratified algebras, *J. Algebra* **314** (2007) 507–537.

[Fri2] ———, Typical blocks of the category $\mathcal{O}$ for the queer Lie superalgebra, *J. Algebra Appl.* **6** (2007), 731–778.

[Gab] P. Gabriel, Des catégories Abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.

[GHK] M. Hazewinkel, N. Gubareni and V. Kirichenko, *Algebras, Rings and Modules*, vol. 2, Mathematics and Its Applications (Springer), **586**, Springer, Dordrecht, 2007.

[GW] F. Goodman and H. Wenzl, The Temperley-Lieb algebra at roots of unity, *Pacific J. Math.* **161** (1993), 307–334.

[GL] J. Graham and G. Lehrer, Cellular algebras, *Invent. Math.* **123** (1996), 1–34.

[Gre] J. A. Green, Combinatorics and the Schur algebra, *J. Pure Appl. Algebra* **88** (1993), 89–106.

[Hap] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*, London Math. Soc., 1988.

[Jan1] J. C. Jantzen, *Representations of Algebraic Groups*, Academic Press, Orlando, 1987.

[Jan2] ——, Über das Dekompositionsverhalten gewisser modularer Darstellungen halbeinfacher Gruppen und ihrer Lie-Algebren, *J. Algebra* **49** (1977), 441–469.

[HN] R. Holmes and D. Nakano, Brauer-type reciprocity for a class of graded associative algebras, *J. Algebra* **144** (1991), 117–126.

[Kac] V. Kac, *Infinite-dimensional Lie Algebras*, Cambridge University Press, 1985.

[KK] V. Kac and D. Kazhdan, Structure of representations with highest weight of infinite dimensional Lie algebras, *Advances Math.* **34** (1979), 97–108.

[KS] M. Kashiwara and P. Shapira, *Sheaves and Categories*, Springer, 2000.

[Kel] B. Keller, Derived categories and tilting, in: "Handbook of Tilting Theory," pp. 49–104, London Math. Soc. Lecture Notes **332**, Cambridge Univ. Press, Cambridge, 2007.

[KM] A. Kleshchev and R. Muth, Based quasi-hereditary algebras; `arXiv:1810.02844`.

[Kön] S. König, Exact Borel subalgebras of quasi-hereditary algebras I, with an appendix by Leonard Scott, *Math. Z.* **220** (1995), 399–426.

[KKO] S. König, J. Külshammer and S. Ovsienko, Quasi-hereditary algebras, exact Borel subalgebras, A_∞-categories and boxes, *Advances Math.* **262** (2014), 546–592.

[Kum] S. Kumar, *Kac-Moody Groups, their Flag Varieties and Representation Theory*, Progress in Mathematics **204**, Birkhäuser Boston, 2002.

[LW] I. Losev and B. Webster, On uniqueness of tensor products of irreducible categorifications, *Selecta Math.* **21** (2015), 345–377.

[Lus] G. Lusztig, Hecke algebras and Jantzen's generic decomposition patterns, *Advances Math.* **37** (1980), 121–164.

[Mac] S. Mac Lane, *Categories for the Working Mathematician*, Springer, 1978.

[Mat] O. Mathieu, Filtrations of G-modules, *Ann. Sci. ENS* **23** (1990), 625–644.

[Maz1] V. Mazorchuk, Stratified algebras arising in Lie theory, in: "Representations of Finite Dimensional Algebras and Related Topics in Lie Theory and Geometry," pp. 245–260, *Fields Inst. Commun.* **40**, Amer. Math. Soc., Providence, RI, 2004.

[Maz2] ——, Koszul duality for stratified algebras II, *J. Aust. Math. Soc.* **89** (2010), 23-49.

[MS] V. Mazorchuk and C. Stroppel, Categorification of (induced) cell modules and the rough structure of generalised Verma modules, *Advances Math.* **219** (2008), 1363–1426.

[Mit] B. Mitchell, Rings with several objects, *Advances Math.* **8** (1972), 1–161.

[Nei] W. Neidhardt, Translation to and fro over Kac-Moody algebras, *Pacific J. Math.* **139** (1989), 107–153.

[PS] L. Positselski and J. Šťovíček, ∞-tilting theory, *Pacific J. Math.* **301** (2019), 297–334.

[Rey] A. Reynolds, *Representations of Oriented Brauer Categories*, PhD thesis, University of Oregon, 2015.

[Rin] C. Ringel, The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences, *Math. Z.* **208** (1991), 209–225.

[SVV] P. Shan, M. Varagnolo, and E. Vasserot, Koszul duality of affine Kac-Moody algebras and cyclotomic rational double affine Hecke algebras, *Advances Math.* **262** (2014), 370–435.

[Sim] D. Simson, Coalgebras, comodules, pseudocompact algebras and tame comodule type, *Colloq. Math.* **90** (2001), 101–150.

[Soe] W. Soergel, Character formulas for tilting modules over Kac-Moody algebras, *Represent. Theory* **2** (1998), 432–448.

[Tak] M. Takeuchi, Morita theorems for categories of comodules, *J. Fac. Sci. Univ. Tokyo* **24** (1977), 629–644.

[Wak] T. Wakamatsu, On modules with trivial self-extensions, *J. Algebra* **114** (1988), 106–114.

[Wan] J.-P. Wang, Sheaf cohomology on G/B and tensor products of Weyl modules, *J. Algebra* **77** (1982), 162–185.

[Web1] B. Webster, Canonical bases and higher representation theory, *Compositio Math.* **151** (2015), 121–166.

[Web2] ——, Knot invariants and higher representation theory, *Mem. Amer. Math. Soc.* **250** (2017), 1–141.

[Wei] C. Weibel, *An Introduction to Homological Algebra*, CUP, 1994.

[Wes] B. Westbury, Invariant tensors and cellular categories, *J. Algebra* **321** (2009), 3563–3567.

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