Double coset density in classical algebraic groups

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Abstract

We classify all pairs of reductive maximal connected subgroups of a classical algebraic group G that have a dense double coset in G. Using this, we show that for an arbitrary pair (H, K) of reductive subgroups of a reductive group G satisfying a certain mild technical condition, there is a dense H, K-double coset in G precisely when $G = HK$ is a factorization.

Introduction

In this paper, we consider the problem of classifying certain orbits of algebraic groups – *double cosets*. Let H and K be closed subgroups of a reductive algebraic group G, defined over an algebraically closed field of characteristic $p \geq 0$. Then, $H \times K$ acts on G by $(x, y).g = xgy^{-1}$, for $(x, y) \in H \times K, g \in G$, and the orbits are the H, K-*double cosets in* G. We shall be concerned with the following properties:

(D1) $G = HK$ is a factorization of G.

(D2) There are finitely many H, K-double cosets in G.

(D3) There is a dense H, K-double coset in G.

Notice that (D1)⇒(D2)⇒(D3). A primary aim motivating our work is to classify all triples (G, H, K) satisfying (D2) or (D3). Factorizations – property (D1) – of simple algebraic groups, with H and K either reductive or parabolic, are classified in [LSS].

For example, take H to be semisimple and $G = GL(V)$, where V is some rational irreducible H-module. Let K be the stabilizer in G of a 1-subspace of V. Then, the H, K-double cosets in G correspond naturally to the orbits of H on 1-subspaces of V. This special case of the problem has been studied in some detail by a number of authors, both for $p = 0$ and $p > 0$. The irreducible modules on which H has finitely many orbits have been classified [Kac, GLMS]. Similarly, the irreducible modules on which H has a dense orbit have been classified [SK, Ch1, Ch2]. More generally, if K is the stabilizer in G of an i-dimensional subspace of V, the H, K-double cosets correspond to H-orbits on i-subspaces of V, also studied in [GLMS]; or if H preserves a non-degenerate bilinear form on V, Guralnick and Seitz [GS] consider orbits of H on degenerate i-subspaces of V, and this problem can also easily be reformulated as a problem about double cosets.

In all the cases just mentioned, H is reductive and the subgroup K in the double coset formulation of the problem is a parabolic subgroup of G. In this paper, we are concerned instead with the case that H and K are *both* reductive subgroups of G. For example, this includes as a special case the study of orbits of a semisimple group H on non-degenerate i-subspaces of a rational irreducible H-module possessing an H-invariant bilinear form. We now state the main results of this paper.

If G is a connected reductive algebraic group, define $\mathcal{M}(G)$ to be the set of all maximal connected reductive subgroups of G that are either Levi factors or maximal connected subgroups of G. Let $\mathcal{R}(G)$ be the set of reductive subgroups $H \leq G$ for which there is a chain of connected subgroups $H^0 = H_0 < H_1 < \cdots < H_n = G$ such that for all $0 \leq i < n$, $H_i \in \mathcal{M}(H_{i+1})$. So in particular, $\mathcal{R}(G)$ contains all reductive maximal subgroups of G and all Levi factors, and if $p = 0$, $\mathcal{R}(G)$ contains *all* reductive subgroups of G. Our main result is as follows:

Theorem A. *Let G be a connected reductive algebraic group, and take $H, K \in \mathcal{R}(G)$. Then, either $G = HK$ or there is no dense H,K-double coset in G.*

This shows that, under the hypothesis of Theorem A, properties (D1)–(D3) are equivalent. This contrasts with the case when we allow one of H or K to be parabolic, when there are examples (even if $p = 0$) where each of the implications (D2)⇒(D1) or – more unexpectedly – (D3)⇒(D2) fails.

In [Lu], Luna shows that over algebraically closed fields of characteristic 0, the union of the closed H,K-double cosets in G is dense in G, for arbitrary reductive subgroups H, K of a connected reductive group G. In particular, this implies that a dense H,K-double coset in G must be closed, so that $G = HK$ is a factorization. Thus, Luna's stronger result implies Theorem A *in characteristic* 0 *only.* Luna's inductive proof depends on the construction of étale slices, which is always possible in characteristic zero thanks to complete reducibility of representations but can often fail in small positive characteristic. It is possible to prove a partial version of Theorem A using Luna's methods – see [B1, chapter 1].

The approach here is quite different, based on knowledge of the maximal subgroups of simple algebraic groups. In the case that G is simple of exceptional type, Theorem A follows from [B2]. The bulk of the work in this paper is in proving Theorem A in the case that G is simple of classical type, which will follow from the next result:

Theorem B. *Let G be a simple classical algebraic group and $H, K \in \mathcal{M}(G)$. Then, either $G = HK$ is a factorization or there is no dense H,K-double coset in G. In the former case, the triple (G, H, K) is listed in table 1.*

The existence of every factorization in table 1 follows from [LSS, Section 1]. Notation in the table will be explained in section 1. The layout of the remainder of the paper is as follows. In section 1, we review some results on maximal subgroups of simple algebraic groups and the techniques developed in [B2]. The proof of Theorem B is given in sections 2, 3 and 4, and this is applied to prove Theorem A in section 5.

1 Preliminaries

Throughout, k will be an algebraically closed field of characteristic $p \geq 0$ and G will denote an affine algebraic group defined over k. Subgroups of G will always be assumed to be closed

Table 1: Maximal reductive factorizations of simple algebraic groups

$G = Cl(V)$	n, p	H	K
SL_{2n}	$n \geq 2$	Sp_{2n}	L_1 or L_{2n-1}
SO_{2n}	$n \geq 4$	N_1	L_{n-1} or L_n
SO_{4n}	$(n,p) \neq (2,2)$	N_1	$Sp_2 \otimes Sp_{2n}$
Sp_{2n}	$p = 2$	N_i	SO_{2n}
Sp_6	$p = 2$	N_2, SO_6	$G_2, V\downarrow_K = L_K(\omega_1)$
SO_7	$p \neq 2$	L_1, N_1	$G_2, V\downarrow_K = L_K(\omega_1)$
SO_8		B_3	L_1, L_4, N_1 or ${}^\tau B_3$
SO_8		${}^\tau B_3$	L_1, L_3, N_1 or B_3
SO_8	$p \neq 2$	B_3 or ${}^\tau B_3$	N_3 or $Sp_2 \otimes Sp_4$
SO_{13}	$p = 3$	N_1	$C_3, V\downarrow_K = L_K(\omega_2)$
SO_{16}		N_1	$B_4, V\downarrow_K = L_K(\omega_4)$
SO_{20}	$p = 2$	N_1	$A_5, V\downarrow_K = L_K(\omega_3)$
SO_{25}	$p = 3$	N_1	$F_4, V\downarrow_K = L_K(\omega_4)$
SO_{32}	$p = 2$	N_1	$D_6, V\downarrow_K = L_K(\omega_5)$ or $L_K(\omega_6)$
SO_{56}	$p = 2$	N_1	$E_7, V\downarrow_K = L_K(\omega_7)$

without further notice. By a *G-module*, we mean a rational kG-module, and by a *G-variety* we mean an algebraic variety X defined over k on which G acts morphically.

By a *reductive* algebraic group, we mean a (not necessarily connected) algebraic group G with trivial unipotent radical. If G is a connected reductive algebraic group, we define a *root system* of G to be a quadruple $(T, B; \Sigma, \Pi)$ where T is a maximal torus of G and B is a Borel subgroup containing T. Let $X(T) = \mathrm{Hom}(T, k^\times)$ be the character group of T. The choice of T determines a set of roots $\Sigma \subset X(T)$. For $\alpha \in \Sigma$, we shall write U_α for the corresponding T-root subgroup of G. The choice of B determines a set of positive roots $\Sigma^+ = \{\alpha \in \Sigma \mid U_\alpha < B\}$ and hence a base Π for Σ.

Given a fixed root system $(T, B; \Sigma, \Pi)$, we will adopt the following conventions.

Write $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ where $n = \mathrm{rank}\,\Sigma$ and if G is simple, label the simple roots α_i as in [H2]. Let $W = N_G(T)/T$ be the Weyl group of G and choose a W-invariant inner product $\langle ., . \rangle$ on $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$. Then, W is generated by $s_1, \ldots, s_n$, where s_i is the simple reflection of $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$ in the hyperplane orthogonal to α_i. Fix fundamental dominant weights $\omega_1, \ldots, \omega_n \in \mathbb{R} \otimes_{\mathbb{Z}} X(T)$ such that $\frac{2\langle \omega_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}$. Let U be the unipotent radical of B, so $B = TU$. Let $B^- = TU^-$ be the opposite Borel subgroup to B. For $w \in W$, let $U_w^- < U$ be the subgroup generated by root subgroups U_α such that α is a positive root sent to a negative root by w.

If V is an arbitrary G-module and $\mu \in X(T)$, we shall write V_μ for the weight space $\{v \in V \mid t.v = \mu(t)v \text{ for all } t \in T\}$. When G is semisimple and λ is a dominant weight relative to some fixed root system of G, we shall write $L_G(\lambda)$ for the irreducible G-module of highest weight λ, and $\Delta_G(\lambda)$ for the corresponding Weyl module. Finally, we use T_i (resp.

U_i) to denote a torus (resp. a connected unipotent subgroup) of dimension i.

For the first lemma, recall the properties (D1)–(D3) introduced in the introduction.

1.1. **Lemma**. *Let H and K be subgroups of a connected algebraic group G. Let* (D) *be one of the properties* (D1)*,* (D2) *or* (D3)*.*

(i) (D) *holds for (G, H, K) if and only if it holds for (G, H^0, K^0).*

(ii) (D) *holds for (G, H, K) if and only if it holds for (G, H^g, K^h) for any $g, h \in G$.*

(iii) *Let $\theta : \tilde{G} \to G$ be a surjective morphism of algebraic groups and set $\tilde{H} = \theta^{-1}H, \tilde{K} = \theta^{-1}K$. Then* (D) *holds for (G, H, K) if and only if it holds for $(\tilde{G}, \tilde{H}, \tilde{K})$.*

Proof. This is proved for (D1) in [LSS, Lemma 1.1]. The proof for (D2) and (i), (ii) for (D3) are straightforward. So consider (iii) for (D3). Morphisms of algebraic groups are open maps so any closed subset of $\tilde{G}$ which is a union of $\ker\theta$-cosets has closed image. Now, the closure of an $\tilde{H}, \tilde{K}$-double coset is a union of double cosets, hence a union of $\ker\theta$-cosets since $\ker\theta \leq \tilde{H}$. Hence, its image is also closed, and (iii) follows easily from this observation. □

Now we review some results from invariant theory. Let G be an arbitrary algebraic group. If X is a G-variety, the algebra of G-invariants on X is defined to be

$$k[X]^G := \{f \in k[X] \mid g.f = f \text{ for all } g \in G\},$$

where $k[X]$ is the algebra of regular functions on X. Here, the action of G on $k[X]$ is defined by $(g.f)(x) := f(g^{-1}x)$ for $g \in G, f \in k[X], x \in X$. If in addition X is irreducible, we write $k(X)$ for the algebra of rational functions on X. The action of G on X also induces an action on $k(X)$, and we shall write $k(X)^G$ for the corresponding algebra of rational invariants.

In the case that G is reductive and X is an affine G-variety, the Mumford conjecture, proved in [Hab], plays a crucial role. For us, the most important consequence of the Mumford conjecture is the following lemma. We shall frequently use it to verify that there is no dense double coset in a given case in the proof of Theorem B.

1.2. **Lemma ([B2, Lemma 2.1])**. *Suppose G is reductive and X is an affine G-variety. If A and B are disjoint closed G-stable subsets of X, then there exists an invariant $f \in k[X]^G$ with $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$. In particular, if G has at least two disjoint closed orbits in X, then there is no dense G-orbit in X.*

To apply Lemma 1.2, we need to be able to prove that certain orbits are closed. Our main technique for this is the next elegant lemma, which is an easy consequence of the definition of a complete variety.

1.3. **Lemma ([S2, p68, Lemma 2])**. *Let G act on a variety X, and let $P < G$ be a subgroup of G such that G/P is complete. If $U \subset X$ is closed and P-stable, then $G.U$ is also closed.*

Combining this with Lemma 1.2, it is easy to obtain the following results. See [B2, Section 2] for details.

1.4. **Lemma**. *Let T be a maximal torus of G. Let X be an affine G-variety, and suppose that $x \in X$ is fixed by T. Then, $G.x$ is closed in X.*

1.5. **Lemma**. *Let H and K be reductive subgroups of G with maximal tori S, T respectively, such that $S \leq T$. Then, HnK is closed in G for all $n \in N_G(T)$.*

1.6. **Theorem**. *Let H be a proper reductive subgroup of a connected reductive algebraic group G. Then, there is no dense H, H-double coset in G.*

Given a reductive group G with root system $(T, B; \Sigma, \Pi)$, there is a well-defined action of W on the zero weight space V_0 of any G-module V. For later use, we record two useful lemmas:

1.7. **Lemma ([B2, Lemma 4.1])**. *Let V be a G-module and let $v, v' \in V_0$. Then, v and v' are conjugate under G if and only if they are conjugate under W.*

1.8. **Lemma**. *Let V be an irreducible G-module. Suppose that G preserves a non-degenerate bilinear form on V. If μ, ν are weights of V, then the weight spaces V_μ and V_ν are orthogonal unless $\mu = -\nu$. Hence, the restriction of the bilinear form to V_0 is non-degenerate.*

Proof. Take $u \in V_\mu, v \in V_\nu$ such that $(u, v) \neq 0$. Then, for all $t \in T$, $(u, v) = (tu, tv) = \mu(t)\nu(t)(u, v)$. Hence, $\mu(t)\nu(t) = 1$, so $\mu = -\nu$ as required for the first part of the lemma. In particular, this shows that $V_0^\perp$ contains all non-zero weight spaces, and hence $V_0 + V_0^\perp = V$. So, V_0 is indeed non-degenerate. □

Next, we need some results of Seitz [Se] and Liebeck [L] on maximal subgroups of classical algebraic groups. Recall the notation $\mathcal{M}(G)$ from the introduction. We highlight at this point the difference between *maximal connected reductive subgroups* of G and *reductive maximal connected subgroups* of G; the latter are maximal connected subgroups of G that are also reductive, whereas the former may lie in some proper parabolic subgroup of G. If G is connected and $p = 0$ *every* maximal connected reductive subgroup of G lies in $\mathcal{M}$, as a consequence of complete reducibility of representations. However, this need not be the case in arbitrary characteristic: there may be reductive subgroups of some parabolic P of G that lie in no Levi factor of P. This complication explains the need for the technical restriction that subgroups lie in $\mathcal{R}(G)$ or $\mathcal{M}(G)$ in Theorems A and B (though we believe that in fact no restriction is necessary).

We use the notation $G = Cl(V)$ to indicate that G is a connected classical algebraic group with natural module V. If $(G, p) = (B_n, 2)$, take V to be the associated $2n$-dimensional symplectic module. When $G = SO(V)$ or $Sp(V)$, let N_i denote the connected stabilizer in G of a non-degenerate subspace of V of dimension i with $i \leq \frac{1}{2} \dim V$; and when $(G, p) = (D_n, 2)$, let N_1 denote the connected stabilizer of a non-singular 1-space.

1.9. **Theorem ([Se, Theorem 3])**. *Let $G = Cl(V)$, and suppose that H is a reductive maximal connected subgroup of G. Then one of the following holds:*

(i) *$H = N_i$ for some i;*

(ii) $V = U \otimes W$ *and* $H = Cl(U) \otimes Cl(W)$*;*
(iii) $(G,H) = (SL(V), Sp(V)), (SL(V), SO(V))(p \neq 2)$ *or* $(Sp(V), SO(V))(p = 2)$*;*
(iv) H *is simple, and* $V\downarrow_H$ *is irreducible and tensor indecomposable, with* $H \neq Cl(V)$*.*

When $G = Cl(V)$, we let $\mathcal{S} = \mathcal{S}(G)$ be the set of all reductive maximal connected subgroups $H < G$ given by part (iv) of the theorem; so if $H \in \mathcal{S}$, H is a simple maximal connected subgroup of $G = Cl(V)$ and $V\downarrow_H$ is irreducible and tensor indecomposable.

We introduce the notation P_i to denote the maximal parabolic subgroup obtained by deleting the ith node from the Dynkin diagram of G, and L_i to denote a Levi factor of P_i. Explicitly, if $(T, B; \Sigma, \Pi)$ is a root system for G, then we may take $P_i = \langle B, U_{-\alpha_j} \mid j \neq i \rangle$ and $L_i = \langle T, U_{\pm\alpha_j} \mid j \neq i \rangle$.

We have now explained most of the notation in table 1. There are some finer points still to be explained. In $G = SO_8$, there are three classes of subgroup of type B_3. We denote representatives of these classes by $B_3, {}^{\tau}B_3$ and N_1 in table 1. Here, B_3 denotes a subgroup of G of type B_3 such that $L_G(\omega_1)\downarrow_{B_3} = L_G(\omega_3)\downarrow_{B_3} = L_{B_3}(\omega_3)$, whilst ${}^{\tau}B_3$ denotes a subgroup of G of type B_3 such that $L_G(\omega_1)\downarrow_{{}^{\tau}B_3} = L_G(\omega_4)\downarrow_{{}^{\tau}B_3} = L_{B_3}(\omega_3)$. If in PSO_8, we let τ be a triality automorphism that induces the permutation $(\omega_1\,\omega_3\,\omega_4)$ on the fundamental weights $\{\omega_1, \ldots, \omega_4\}$ then τ induces the permutation $(B_3\ {}^{\tau}B_3\ N_1)$ on the images of $B_3, {}^{\tau}B_3$ and N_1 in PSO_8. Later, we shall refer to $B_3 < SO_8 = G$ such that $L_G(\omega_1)\downarrow_{B_3} = L_{B_3}(\omega_3)$. This is ambiguous and should be taken to mean either B_3 or ${}^{\tau}B_3$. In a similar fashion, we shall refer to $GL_n < SO_{2n}$. By this we mean either of the two classes L_{n-1} or L_n of subgroups of SO_{2n} of type GL_n. Likewise $Sp_2 \otimes Sp_{2n} < SO_{4n}$.

To list the subgroups in $\mathcal{S}(G)$ for G classical, of large dimension relative to $\dim G$, we require some known information on modules for simple algebraic groups of small dimension relative to the dimension of the group. Let G denote a simple algebraic group with fixed root system $(T, B; \Sigma, \Pi)$. If $p = 2$, we assume in addition that G is not of type B_n; we may make this assumption without loss of generality because of the existence of bijective morphisms (which are *not* isomorphisms of algebraic groups) $B_n \to C_n$ and $C_n \to B_n$ in characteristic 2. We define numbers e_G to be

G	A_n	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
e_G	$m+3$	$m+n+1$	$m+2$	$m+n$	18	96	80	192	1024

where $m = \dim G$.

If V is an irreducible G-module, we define $Cl(V)$ to be the smallest classical group on V containing the image of G in $GL(V)$ (this is well-defined as V is irreducible).

1.10. **Theorem**. *Let* $V = L_G(\lambda)$ *be an irreducible, tensor indecomposable* G*-module such that* $1 < \dim V < e_G$*. Then, up to duals and field twists,* $(G, \lambda, \dim V, Cl(V))$ *is listed in table 2.*

We remark that some care is needed interpreting table 2 if $G = C_n$ and $p = 2$. In this case, the image of G in $GL(V)$ according to the representations in table 2 need not be of type C_n. For example, the image of C_n in a spin representation in characteristic 2 is isomorphic to B_n as an algebraic group.

Table 2: Modules of small dimension

G		λ	$d = \dim V$	$Cl(V)$
A_n	$n \geq 1$	ω_1	$n+1$	$G = SL_d$
A_3		ω_2	6	$G = SO_6$
A_n	$n \geq 4$	ω_2	$\frac{1}{2}n(n+1)$	SL_d
A_5		ω_3	20	$\begin{cases} Sp_{20} & p \neq 2 \\ SO_{20} & p = 2 \end{cases}$
A_n	$n = 6, 7$	ω_3	$\frac{1}{6}(n-1)n(n+1)$	SL_d
A_n	$n \geq 1, p \neq 2$	$2\omega_1$	$\frac{1}{2}(n+1)(n+2)$	SL_d
A_n	$n \geq 1$	$\omega_1 + \omega_n$	$\begin{cases} n^2+2n & p \nmid n+1 \\ n^2+2n-1 & p \mid n+1 \end{cases}$	$\begin{cases} Sp_d & p = 2, n \equiv 1(4) \\ SO_d & \text{otherwise} \end{cases}$
A_1	$p \geq 5$	$3\omega_1$	4	Sp_4
A_3	$p = 3$	$\omega_1 + \omega_2$	16	SL_{16}
B_n	$n \geq 3, p \neq 2$	ω_1	$2n+1$	$G = SO_d$
B_n	$n \geq 3, p \neq 2$	ω_2	$2n^2+n$	SO_d
B_n	$3 \leq n \leq 6$, $p \neq 2$	ω_n	2^n	$\begin{cases} SO_d & n = 3, 4 \\ Sp_d & n = 5, 6 \end{cases}$
C_n	$n \geq 2$	ω_1	$2n$	$G = Sp_d$
C_n	$n \geq 2$	ω_2	$\begin{cases} 2n^2-n-1 & p \nmid n \\ 2n^2-n-2 & p \mid n \end{cases}$	$\begin{cases} Sp_d & p = 2, n \equiv 2(4) \\ SO_d & \text{otherwise} \end{cases}$
C_n	$n \geq 2, p \neq 2$	$2\omega_1$	$2n^2+n$	SO_d
C_n	$3 \leq n \leq 6$, $p = 2$	ω_n	2^n	SO_d
C_3	$p \neq 2$	ω_3	14	Sp_{14}
D_n	$n \geq 4$	ω_1	$2n$	$G = SO_d$
D_n	$n \geq 4$	ω_2	$\begin{cases} 2n^2-n & p \neq 2 \\ 2n^2-n-1 & p = 2, p \nmid n \\ 2n^2-n-2 & p = 2, p \mid n \end{cases}$	$\begin{cases} Sp_d & p = 2, n \equiv 2(4) \\ SO_d & \text{otherwise} \end{cases}$
D_4		ω_3, ω_4	2^3	$G = SO_8$
D_5		ω_5	2^4	SL_{16}
D_6		ω_5, ω_6	2^5	$\begin{cases} Sp_{32} & p \neq 2 \\ SO_{32} & p = 2 \end{cases}$
D_7		ω_7	2^6	SL_{64}

Table 2: Modules of small dimension (continued)

G	λ	$d = \dim V$	$Cl(V)$
G_2	ω_1	$7 - \delta_{2,p}$	$\begin{cases} SO_7 & p \neq 2 \\ Sp_6 & p = 2 \end{cases}$
G_2	ω_2	$7(2 - \delta_{p,3})$	SO_d
F_4	ω_1	$26(2 - \delta_{p,2})$	SO_d
F_4	ω_4	$26 - \delta_{3,p}$	SO_d
E_6	ω_1	27	SL_{27}
E_6	ω_2	$78 - \delta_{3,p}$	SO_d
E_7	ω_1	$133 - \delta_{2,p}$	$\begin{cases} SO_{133} & p \neq 2 \\ Sp_{132} & p = 2 \end{cases}$
E_7	ω_7	56	$\begin{cases} Sp_{56} & p \neq 2 \\ SO_{56} & p = 2 \end{cases}$
E_8	ω_8	248	SO_d

Apart from the information on $Cl(V)$, this theorem follows immediately from [L, Section 2]. To compute $Cl(V)$, the methods of [LSS, Section 2] suffice unless $p = 2$ and V is a self-dual composition factor of Lie(G). The result for this final possibility follows from results of Gow and Willems [GW] (or see [B1, chapter 2]).

We now give a first application of the information in table 2.

1.11. **Lemma**. *Let $G = Cl(V)$ be a classical algebraic group and $H < G$ be a reductive maximal connected subgroup with $\dim H \geq \frac{1}{2}\dim G$. Then (G, H) are listed below:*

G	H	*Conditions*
$Cl(V)$	N_i	
$SL(V)$	$Sp(V)$	
$Sp(V)$	$SO(V)$	$p = 2$
SO_8	B_3	$V\downarrow_H = L_H(\omega_3)$
$SO_7(p \neq 2), Sp_6(p = 2)$	G_2	$V\downarrow_H = L_H(\omega_1)$

Proof. We apply Theorem 1.9, to see that either (G, H) is as in the conclusion or $H \in \mathcal{S}(G)$. In the latter case, note that $\dim H \geq \frac{1}{2}\dim G \geq \dim V$ so the pair $(H, V\downarrow_H)$ is listed in table 2. Also, G is the group $Cl(V)$ listed in the table, since H is maximal in G. Thus, checking dimensions for each possibility in table 2 gives the conclusion. □

2 Proof of Theorem B: maximal reductive subgroups

We are now ready to prove Theorem B. The strategy is as follows. We first use the information on maximal subgroups and modules of small dimension in section 1 to list all pairs (H, K) of subgroups in $\mathcal{M}(G)$ satisfying the dimension bound $\dim H + \dim K \geq \dim G$.

We then verify each case in turn using Lemma 1.2. We divide the case analysis into two halves. In this and the next section, we consider the possibilities when both H and K are maximal reductive connected subgroups, and in section 4 we consider the remaining possibilities when one of H or K is a Levi factor.

2.1. **Proposition**. *Let G be simple and $H, K \in \mathcal{M}(G)$ with both H, K reductive maximal connected subgroups. To prove Theorem B for the triple (G, H, K) it is sufficient to show that it holds for (G, H, K) in table 3. (In the table we reference the lemma in which we treat these subgroups.)*

Proof. Let $G = Cl(V)$ and H, K be reductive maximal connected subgroups of G. If H, K are conjugate, then there is no dense H, K-double coset in G by Theorem 1.6, and if $\dim H + \dim K < \dim G$ then there can be no dense double coset by dimension. Thus, we may assume H, K are not conjugate and that $\dim H + \dim K \geq \dim G$. Moreover, if $(G, p) = (B_n, 2)$, then we can apply a bijective morphism $B_n \to C_n$ to deduce the result for B_n from the corresponding result for C_n. Hence, we will assume $(G, p) \neq (B_n, 2)$. We now apply Theorem 1.9 to list all possibilities meeting these conditions. This is easy, but rather lengthy, so we sketch the argument.

We may assume $\dim H \geq \dim K$, so $\dim H \geq \frac{1}{2} \dim G$. Hence, (G, H) is given by Lemma 1.11; we consider the possibilities one by one.

(a) $(G, H) = (Cl(V), N_i)$. List the possibilities for K using Theorem 1.9 to see one of the following holds: (i) $(G, K) = (Cl(V), N_j)$; (ii) $(G, K) = (Cl(V), Cl(U) \otimes Cl(W))$ where $V = U \otimes W$; (iii) $(G, K) = (Sp(V), SO(V))(p = 2)$; (iv) $K \in \mathcal{S}(G)$. Cases (i)-(iii) are all listed in table 3. So, consider case (iv). First, suppose $i = 1$, so $(G, H) = (SO(V), N_1)$ and $K \in \mathcal{S}(G)$ with $\dim K \geq \dim G - \dim H = \dim V - 1$. So, $(K, V\downarrow_K, \dim V, G)$ is in table 2. Hence, either $V\downarrow_K$ is a composition factor of Lie(K) or $(K, \lambda) = (A_5, \omega_3)(p = 2)$, $(B_n, \omega_n)(n = 3, 4)$, $(B_n, \omega_n)(p = 2, n = 5, 6)$, $(D_6, \omega_6)(p = 2)$, $(G_2, \omega_1)(p \neq 2)$, $(F_4, \omega_4)(p \neq 2)$ or (E_7, ω_7). All of these are included in table 3 except for B_5 in $p = 2$ which lies in D_6 so is not maximal.

Now suppose $i \geq 2$. Note if $G = SO(V)$, then $i \neq 2$ as N_2 is not maximal in $SO(V)$. Hence, $\dim K \geq \dim G - \dim H$ implies either $\dim K \geq 2 \dim V - 4$ if $G = Sp(V)$ or $\dim K \geq 3 \dim V - 9$ if $G = SO(V)$. In particular, $(K, V\downarrow_K, \dim V, G)$ is in table 2. Listing the possibilities that meet the dimension bound on $\dim K$, we deduce $(K, \lambda) = (G_2, \omega_1)$ (or (G_2, ω_2) if $p = 3$ which yields the same embedding), (B_3, ω_3), $(D_6, \omega_6)(p \neq 2)$, or $(E_7, \omega_7)(p \neq 2)$. Now for each of these cases, one computes the values of i permissible, to obtain the entries in table 3: $(G, H, K) = (SO_7, N_3, G_2)(p \neq 2)$, $(Sp_6, N_2, G_2)(p = 2)$, (SO_8, N_3, B_3), $(Sp_{32}, N_2, D_6)(p \neq 2)$ and $(Sp_{56}, N_2, E_7)(p \neq 2)$.

(b) $(G, H) = (SL(V), Sp(V))$. List the possibilities for K to obtain: (i) $K = SO(V)$; (ii) $K = Cl(U) \otimes Cl(W)$ where $V = U \otimes W$; (iii) $K \in \mathcal{S}(G)$. Case (i) is listed in the table, and case (ii) does satisfy the dimension bound $\dim H + \dim K \geq \dim G$. In case (iii), $\dim K \geq \dim G - \dim H$ implies $\dim K \geq \frac{1}{2}d(d-1) - 1$ where $d = \dim V$, which is even. So, $(K, V\downarrow_K, d, G)$ is as in table 2. Considering the cases in the table one by one, none satisfy the requirements.

Table 3: Case list involving maximal subgroups

G	p	H	K	Ref
$Cl(V)$		N_i	N_j	(2.2)
$Cl(V)$		N_i	$Cl(U) \otimes Cl(W), V = U \otimes W$	(3.1)
$SL(V)$	$p \neq 2$	$Sp(V)$	$SO(V)$	(2.4)
$Sp(V)$	$p = 2$	N_i	$SO(V)$	[LSS]
$SO(V)$		N_1	$K \in \mathcal{S}(G), V\downarrow_K$ a composition factor of Lie(K)	(2.5)
$SO(V)$	$p \neq 2$	N_1	$C_n, V\downarrow_K = L_K(\omega_2)$	(2.5)
$Sp(V)$	$p = 2$	$SO(V)$	$C_n, V\downarrow_K = L_K(\omega_2)$ and $n \equiv 2(4)$	(2.6)
$Sp(V)$	$p = 2$	$SO(V)$	$A_n, V\downarrow_K = L_K(\omega_1 + \omega_n)$ and $n \equiv 1(4)$	(2.6)
Sp_6	$p = 2$	SO_6 or N_2	$G_2, V\downarrow_K = L_K(\omega_1)$	[LSS]
SO_7	$p \neq 2$	N_1 or N_3	$G_2, V\downarrow_K = L_K(\omega_1)$	(2.3)
SO_8		N_1 or N_3	$B_3, V\downarrow_K = L_K(\omega_3)$	[LSS]
SO_{16}		N_1	$B_4, V\downarrow_K = L_K(\omega_4)$	[LSS]
SO_{20}	$p = 2$	N_1	$A_5, V\downarrow_K = L_K(\omega_3)$	[LSS]
$SO_{26-\delta_{p,3}}$		N_1	$F_4, V\downarrow_K = L_K(\omega_4)$	(2.5)
Sp_{32}	$p \neq 2$	N_2	$D_6, V\downarrow_K = L_K(\omega_6)$	(2.7)
SO_{32}	$p = 2$	N_1	$D_6, V\downarrow_K = L_K(\omega_6)$	[LSS]
Sp_{56}	$p \neq 2$	N_2	$E_7, V\downarrow_K = L_K(\omega_7)$	(2.7)
SO_{56}	$p = 2$	N_1	$E_7, V\downarrow_K = L_K(\omega_7)$	[LSS]
SO_{64}	$p = 2$	N_1	$B_6, V\downarrow_K = L_K(\omega_6)$	(2.8)
Sp_{132}	$p = 2$	SO_{132}	$E_7, V\downarrow_K = L_K(\omega_1)$	(2.6)

(c) $(G,H) = (Sp(V), SO(V))(p=2)$. By (a), we may assume $K \neq N_i$. Listing the remaining possibilities, we see: (i) $K = Cl(U) \otimes Cl(W)$ where $V = U \otimes W$; (ii) $K \in \mathcal{S}(G)$. Case (i) does not occur as $Cl(U) \otimes Cl(W)$ preserves a quadratic form on V if $p=2$, so is not maximal. In case (ii), $\dim K \geq \dim G - \dim H$ implies $\dim K \geq \dim V$, so $(K, V\downarrow_K, \dim V, G)$ is in table 2. Listing the possibilities, we see $V\downarrow_K = L_K(\lambda)$ where $(K,\lambda) = (A_n, \omega_1 + \omega_n)(n \equiv 1(4))$, $(C_n/D_n, \omega_2)(n \equiv 2(4))$, (G_2, ω_1) or (E_7, ω_1). All of these are included in table 3 except for D_n which is not maximal.

(d) $(G,H) = (SO_8, B_3)$ where $V\downarrow_H = L_{B_3}(\omega_3)$, $(SO_7, G_2)(p \neq 2)$ or $(Sp_6, G_2)(p=2)$. If $H = B_3$, we can deduce the result from the case $H = N_1$ by applying a triality automorphism to send $B_3 \to N_1$; one needs to work in PSO_8 here since triality is not defined in SO_8 if $p \neq 2$. Otherwise, $H = G_2$ and $K \in \mathcal{S}(G)$, and listing the possibilities for K using table 2, one concludes K is conjugate to a subgroup of H in all cases. □

Recall the definition of *transporter*: if V is a G-variety and $A, B \subset V$ are subsets with B closed, then $\mathrm{Tran}_G(A,B) = \{g \in G \mid g.A \subset B\}$ is a closed subset of G.

We now prove Theorem B for all cases in table 2 except $(Cl(V), N_i, Cl(U) \otimes Cl(W))$, which we postpone to section 3.

2.2. **Lemma**. *There is no dense H,K-double coset in G if*

$$(G,H,K) = (Cl(V), N_i, N_j).$$

Proof. We may assume $j \neq i$ since otherwise H and K are conjugate and the result holds by Theorem 1.6. So, let $\dim V = n$ and $1 \leq j < i \leq \frac{1}{2}n$. Then, $G = Sp(V)$ or $SO(V)$, H is the stabilizer of a non-degenerate subspace V_H of dimension i, and K is the stabilizer of a non-degenerate subspace V_K of dimension j (or if $G = SO(V)$, $p=2$ and $j=1$, K is the stabilizer of a non-singular line V_K). Conjugating, we may assume that $V_K < V_H$.

We claim $HK = \mathrm{Tran}_G(V_K, V_H)$. One inclusion is obvious, so take g to be an element of $\mathrm{Tran}_G(V_K, V_H)$. Then, gV_K is a non-degenerate (or non-singular) subspace of V_H of dimension j; H acts transitively on these, so we can find $x \in H$ such that $xgV_K = V_K$. Hence, $xg \in K$ and $g \in HK$. Thus, $HK = \mathrm{Tran}_G(V_K, V_H)$, and in particular, HK is closed.

Now, pick $h \in G$ such that $hV_K < V_H^{\perp}$. Then, hKh^{-1} is the stabilizer of the subspace hV_K, and an identical argument shows $HhKh^{-1} = \mathrm{Tran}_G(hV_K, V_H^{\perp})$, so $HhK = \mathrm{Tran}_G(V_K, V_H^{\perp})$, which is closed. Also, $\mathrm{Tran}_G(V_K, V_H)$ and $\mathrm{Tran}_G(V_K, V_H^{\perp})$ are clearly disjoint. So, HK and HhK are disjoint closed H,K-double cosets in G, and the result follows by Lemma 1.2. □

2.3. **Lemma**. *Theorem B holds if*

$$(G,H,K) = (SO_7, N_1, G_2),$$
$$(G,H,K) = (SO_7, N_3, G_2)$$

where $V\downarrow_{G_2} = L_{G_2}(\omega_1)$ and $p \neq 2$.

Proof. The case $(G, H, K) = (SO_7, N_1, G_2)$ is a factorization by [LSS]. So, we just need to consider $(G, H, K) = (SO_7, N_3, G_2)$.

Let V be the spin module for $G = B_3$, so $V = L_{B_3}(\omega_3)$. Fixing a maximal torus T of G, we will write a base for G relative to T as $\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3\}$ in the usual way. Then, the non-zero weight spaces in V correspond to weights $\{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3)\}$; we abbreviate $\frac{1}{2}(+\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ as $+++$ etc .... Let $H = N_3$, a maximal rank subgroup of G which may be chosen to be generated by root subgroups $U_{\pm(\varepsilon_1-\varepsilon_2)}, U_{\pm(\varepsilon_1+\varepsilon_2)}$ and $U_{\pm\varepsilon_3}$. Let V_H be the span of weight spaces corresponding to weights $+++, ++-, ---, --+$, which is H-stable since each of the root group generators of H stabilizes V_H. The restriction of the form on V to V_H is non-degenerate by Lemma 1.8 so $V = V_H \oplus V_H^\perp$ is a direct sum of H-modules. Let V_K be a non-degenerate line in V_H, and set $K = \mathrm{stab}_G(V_K)$, a subgroup of type G_2 as $G_2 = B_3 \cap N_1$ in $SO(V)$.

We know $SO(V) = B_3N_1$, so G acts transitively on non-degenerate lines in V and we may pick $h \in G$ such that $hV_K < V_H^\perp$. Then, as in Lemma 2.2, $HK \subset \mathrm{Tran}_G(V_K, V_H)$ whilst $HhK \subset \mathrm{Tran}_G(V_K, V_H^\perp)$. These transporters are closed and disjoint, so contain disjoint closed H, K-double cosets to complete the proof by Lemma 1.2. □

2.4. **Lemma**. *There is no dense H, K-double coset in G if*

$$(G, H, K) = (SL(V), Sp(V), SO(V))(p \neq 2).$$

Proof. The proof of this case is identical to the proof for $(G, H, K) = (E_6, F_4, C_4)$ in [B2, Lemma 6.1]. □

2.5. **Lemma**. *Theorem B holds if $G = SO(V), H = N_1$ and $K \in \mathcal{S}$, where $V\downarrow_K = L_K(\lambda)$ is either a composition factor of the adjoint module of K or $(K, \lambda) = (F_4, \omega_4)$ or (C_n, ω_2).*

Proof. Notice here we are excluding $K = A_n (n \equiv 1(4)), C_n (n \equiv 2(4))$ and E_7 when $p = 2$, as then K is not a subgroup of G by [GW]. Let T be a maximal torus of K. If $(G, H, K, p) = (SO_{25}, N_1, F_4, 3)$ or $(SO_{13}, N_1, C_3, 3)$, then $G = HK$ is a factorization by [LSS]. Otherwise, $\dim V_0 \geq 2$. Hence, since by Lemma 1.8, the restriction of the bilinear form to V_0 is non-degenerate, there are infinitely many vectors of length 1 in V_0. By Lemma 1.7, there must therefore be infinitely many non-conjugate vectors of length 1 in V_0. Furthermore, by Lemma 1.4, these all have closed K-orbits. Thus, under the usual identification between H, K-double cosets in G and K-orbits on vectors of length 1 in V, we have found infinitely many closed H, K-double cosets and the result follows by Lemma 1.2 again. □

2.6. **Lemma**. *There is no dense H, K-double coset in G if $p = 2$ and*

$$(G, H, K, \lambda) = (Sp(V), SO(V), A_n, \omega_1 + \omega_n)(n \equiv 1(4)),$$
$$(G, H, K, \lambda) = (Sp(V), SO(V), C_n, \omega_2)(n \equiv 2(4)),$$
$$(G, H, K, \lambda) = (Sp(V), SO(V), E_7, \omega_1)$$

where $V\downarrow_K = L_K(\lambda)$.

PROOF. Let $d = \dim V$. Let W be a $(d+2)$-dimensional orthogonal space with quadratic form Q, and fix a non-singular line $\langle w \rangle$ where $Q(w) = 1$. Let $\overline{G} \cong SO_{d+1}$ be the stabilizer of $\langle w \rangle$ in $SO(W)$. Then, $V = \langle w \rangle^{\perp}/\langle w \rangle$ is a d-dimensional symplectic space, and the map $\langle w \rangle^{\perp} \to V$ induces a bijective morphism $\overline{G} \to G \cong Sp_d$. Let $\overline{H}, \overline{K}$ be the pre-images of H, K respectively. It will be sufficient to show that there are disjoint closed $\overline{H}, \overline{K}$-double cosets in $\overline{G}$, because of Lemma 1.1(iii). As $\overline{H} \cong SO_d$, it stabilizes some complement U to $\langle w \rangle$ in $\langle w \rangle^{\perp}$. Let $Z = (\langle w \rangle^{\perp})^*$ and note that $\overline{H}$ is the stabilizer in $\overline{G}$ of the vector $f_0 \in Z$ where $f_0(w) = 1, f_0(U) = 0$. Hence, it is sufficient to show that $\overline{K}$ has disjoint closed orbits in $\overline{G}.f_0$.

We claim $\overline{G}.f_0 = \{f \in Z \mid f(w) = 1\}$. To prove this, it is sufficient to show $\overline{G}$ acts transitively on complements to $\langle w \rangle$ in $\langle w \rangle^{\perp}$, or equivalently that $\overline{G}$ acts transitively on non-degenerate 2-spaces in W containing $\langle w \rangle$. This follows easily by Witt's lemma.

Let T be a maximal torus of $\overline{H}$. Then, $\dim Z_0 = \dim(\langle w \rangle^{\perp})_0 = \dim V_0 + 1$. Hence, since $K \neq A_1$ or C_2 (when $K = G$), $\dim Z_0 \geq 2$. So there are infinitely many vectors of weight 0 in $\overline{G}.f_0$, and the result follows by Lemma 1.4 and Lemma 1.7. □

The proof of the next lemma is based on the proof that A_1D_6 (resp. A_1E_7) has infinitely many orbits on $L_{A_1}(\omega_1) \otimes L_{D_6}(\omega_6)$ (resp. $L_{A_1}(\omega_1) \otimes L_{E_7}(\omega_7)$) in [GLMS]. We also treat the case N_2 in $SO(V)$ here for later use, even though it is not maximal.

2.7. **Lemma**. *There is no dense H, K-double coset in G if*

$$(G,H,K) = (Sp(V), N_2, D_6) \text{ or } (Sp(V), N_2, E_7) \text{ and } p \neq 2,$$
$$(G,H,K) = (SO(V), N_2, D_6) \text{ or } (SO(V), N_2, E_7) \text{ and } p = 2$$

where $V\downarrow_K = L_K(\omega_6)$ if $K = D_6$ or $L_K(\omega_7)$ if $K = E_7$.

PROOF. Let $d = \dim V$. Let U be a 2-dimensional symplectic space, and set $W = U \otimes V$, a $2d$-dimensional orthogonal space. Write $(.,.)$ for the bilinear forms on U, V, W. Let $L = Sp(U) \otimes K \cong A_1D_6$ or A_1E_7, a subgroup of $SO(W)$. Let A be the A_1-factor of L. Let $\Omega_1 = \{w \in W \mid (w,w) = 1\}$ and $\Omega_2 = \{v_1 \wedge v_2 \in \bigwedge^2 V \mid (v_1, v_2) = 1\}$. To prove the lemma, it will be sufficient to show that K has at least two disjoint closed orbits in Ω_2.

Let e, f be a basis for U with $(e,f) = 1$. Any vector $w \in W$ with $(w,w) = 1$ can be written uniquely as $e \otimes v_1 + f \otimes v_2$ where $v_1, v_2 \in V$ and $(v_1, v_2) = 1$. So, we can define a surjective morphism $\theta : \Omega_1 \to \Omega_2$ by $e \otimes v_1 + f \otimes v_2 \mapsto v_1 \wedge v_2$. Letting A act on Ω_2 trivially, θ is L-equivariant. It is easy to check that the fibres of θ are A-orbits so that θ is an orbit map for the action of A on Ω_1. Moreover, θ is separable so by [Borel, 6.6], (Ω_2, θ) is the quotient of Ω_1 by A in the sense of [Borel, 6.3]. Hence, as θ is L-equivariant and $K = L/A$, there is a bijection between closed L-orbits in Ω_1 and closed K-orbits in Ω_2. Thus, it is sufficient to show that L has at least two disjoint closed orbits in Ω_1.

Let G be a simply connected group of type E_7 (resp. E_8) if $K = D_6$ (resp. E_7). Let $(T, B; \Sigma, \Pi)$ be a root system for G. The maximal rank subgroup obtained by deleting α_1 (resp. α_8) from the extended Dynkin diagram of G is of type A_1D_6 (resp. A_1E_7), and we identify this with L. Let $\{H_\alpha, X_\beta \mid \alpha \in \Pi, \beta \in \Sigma\}$ be a Chevalley basis for $\mathfrak{g} = \mathrm{Lie}(G)$. We now claim that we may identify W with the L-submodule of $\mathfrak{g}$ spanned by $X_{\pm\alpha}$ such that $\alpha = a_1\alpha_1 + \cdots + a_r\alpha_r$, $a_1, \ldots, a_r \in \mathbb{Z}$ and $a_1 = \pm 1$ (resp. $a_8 = \pm 1$). To prove this, observe that L certainly stabilizes this subspace since each of the root group generators of L stabilizes it. Also, $-\alpha_1$ (resp. $-\alpha_8$) is equal to the weight $\omega_1 \otimes \omega_6$ (resp. $\omega_1 \otimes \omega_7$) when written in terms of fundamental dominant weights for a root system of L, so this subspace must be isomorphic to W as an L-module.

Now, there is a well defined bilinear form on $\mathfrak{g}$ (defined by reduction modulo p from a scalar multiple of the Killing form on the corresponding Lie algebra over $\mathbb{C}$) satisfying $(X_\alpha, X_{-\alpha}) = 1$ for $\alpha \in \Sigma$. Under the identification, this is precisely the bilinear form preserved by L on W. Choose $\alpha, \beta \in \Sigma^+$ such that $X_\alpha, X_\beta \in W$ and $\alpha + \beta \notin \Sigma$. Let $X_{a,b} = a(X_\alpha + X_{-\alpha}) + b(X_\beta + X_{-\beta})$ for $a, b \in k$. Consider the infinite set $\Omega_0 = \{X_{a,b} \mid a, b \in k, (X_{a,b}, X_{a,b}) = 1\} \subset \Omega_1$. Now, as a, b vary, we obtain elements in the fundamental $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ generated by $X_{\pm\alpha}, X_{\pm\beta}$ with infinitely many different eigenvalues. Hence, if $p \neq 2$ (when $X_{a,b}$ is semisimple) there are infinitely many closed G-orbits intersecting Ω_0. If $p = 2$, there are infinitely many elements in Ω_0 with non-conjugate semisimple parts, so again there are infinitely many G-orbits intersecting Ω_0 with disjoint closures. Hence, there are infinitely many closed L-orbits in Ω_1 completing the proof. □

2.8. **Lemma**. *There is no dense H, K-double coset in G if*

$$(G, H, K, p) = (SO(V), B_6, N_1, 2)$$

where $V \downarrow_{B_6} = L_{B_6}(\omega_6)$.

Proof. Let $L = D_7$ and $V = L_{D_7}(\omega_7)$ with $p = 2$. Then, B_6 is a subgroup of L and $V \downarrow_{B_6} = L_{B_6}(\omega_6)$. Note that B_6 preserves a quadratic form Q on V, but L does not preserve Q. Let L_1 be the product of L and a one dimensional torus which acts on V by scalars.

We first show that L_1 has a dense orbit in V with generic stabilizer G_2G_2. To prove this, it is sufficient by dimensions to show that there is $v \in V$ with $\mathrm{stab}_L(v) = G_2G_2$. Now, $V \downarrow_{C_3C_3} = L_{C_3}(\omega_3) \otimes L_{C_3}(\omega_3)$. As $G_2G_2 < C_3C_3$ and $L_{C_3}(\omega_3) \downarrow_{G_2}$ has a fixed 1-space, it follows that there is some vector $v \in V$ with stabilizer containing G_2G_2. But if the connected stabilizer of v is any larger than G_2G_2 it must contain G_2C_3, which is impossible as C_3 fixes no vector in V.

Now we show that $k[V]^L \neq k$. Since $k[V]$ is a unique factorization domain and L has no rational characters, $k(V)^L$ is the field of fractions of $k[V]^L$. Hence, it is sufficient to show $k(V)^L \neq k$, which by Rosenlicht's theorem [Ros] is equivalent to L having no dense orbit in V. But if L has a dense orbit in V, then it lies in the dense L_1-orbit in V and so the generic stabilizer is G_2G_2 by the previous paragraph, which is a contradiction as $\dim D_7/G_2G_2 = 63 \neq \dim V$.

So, we can find a homogeneous invariant $f \in k[V]^L - k$ of degree d say. Since $B_6 < L$, we just need to show that f is not constant on $\Omega = \{v \in V \mid Q(v) = 1\}$, for then f will separate infinitely many closed B_6-orbits in $\Omega \cong SO(V)/N_1$. So, suppose by way of contradiction that f is constant on this set. Consider the regular function $\theta \in k[L]$ defined by $h \mapsto Q(h.v_0)^d$ for $h \in L$, where $v_0 \in V$ is some fixed vector with $Q(v_0) = 1$. Since L does not preserve Q, we can choose v_0 so that θ is not constant on L and $f(v_0) \neq 0$.

Suppose h lies in the dense subset of L defined by the non-vanishing of θ. Then, $\lambda^2 = Q(h.v_0)$ is non-zero and so $w_0 = \frac{1}{\lambda}h.v_0$ satisfies $Q(w_0) = 1$, hence $f(w_0) = f(v_0)$ by assumption. But then $f(v_0) = f(h.v_0) = f(\lambda w_0) = \lambda^d f(w_0) = \lambda^d f(v_0)$, so $\lambda^d = 1$ and $\theta(h) = Q(h.v_0)^d = Q(\lambda w_0)^d = \lambda^{2d} = 1$. This implies that θ is constant on a dense subset of L, hence constant on all of L, which is a contradiction. □

3 Proof of Theorem B: tensor embeddings

In this section, we verify Theorem B for the remaining case in table 3. We fix some notation throughout the section. Let U and W be non-degenerate symplectic or orthogonal spaces with $\dim U = a, \dim W = b$. Let H be the group $Cl(U) \otimes Cl(W)$, a central product of the classical groups on U and W. Writing $(.,.)$ for the bilinear forms on U and W, H preserves the non-degenerate form on $V = U \otimes W$ defined by $(u \otimes w, u' \otimes w') = (u, u')(w, w')$ for $u, u' \in U, w, w' \in W$, so embeds into the corresponding classical group $G = Cl(V)$. In characteristic 2, we assume that $H = Sp(U), K = Sp(W)$; here, $H \otimes K$ embeds into $G = SO(V)$. Fix $1 \leq i \leq \frac{1}{2}ab$, where we assume that i is even if $G = Sp(V)$ and that i is even or equal to 1 if $(G, p) = (SO(V), 2)$. Let $K = N_i < G$.

3.1. **Proposition**. *With above notation, Theorem B holds for the triple (G, H, K).*

We prove Proposition 3.1 with a series of lemmas. Recall that by Lemma 1.2, we just need to show that there are two disjoint closed H, K-double cosets in G.

3.2. **Lemma**. *Theorem B holds for the triple (G, H, K) if $i = 1$.*

Proof. Here, $G = SO(V)$. We need to consider the following cases.
(i) $H = Sp(U) \otimes Sp(W), 2 \leq a \leq b$;
(ii) $H = SO(U) \otimes SO(W), 3 \leq a \leq b$ and $p \neq 2$.

Pick bases $u_1, \dots, u_a$ and $w_1, \dots, w_b$ for U, W respectively with respect to which the bilinear forms on U, W correspond to $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ if the form is symplectic (where $n = \frac{a}{2}$ or $\frac{b}{2}$) or to the identity matrix if the form is orthogonal (possible as $p \neq 2$). Denote the matrices corresponding to the forms on U, W by J_1, J_2 respectively. We realise V as the set of $a \times b$ matrices over k, with the matrix m corresponding to the vector $\sum m_{st}\ u_s \otimes w_t$ in V. If $x = x_1 \otimes x_2$ is an element of H written as a pair of matrices in terms of the chosen bases (so $x_s^T J_s x_s = J_s$ for $s = 1, 2$), the action of x on V is given by $x : m \mapsto x_1 m x_2^T$. Writing $M_a(k)$ for the set of $a \times a$ matrices over k, define $\theta : V \to M_a(k)$ by $\theta : m \mapsto J_1 m J_2 m^T$. Then, $\theta : x_1 m x_2^T \mapsto x_1^{-T}(J_1 m J_2 m^T) x_1^T$, so H-orbits in V are send to H-conjugacy classes in $M_a(k)$.

We first claim that the set Ω of matrices of the form mJ_2m^T for some $a \times b$ matrix m is just the set of all symmetric matrices in case (ii) or the set of all alternating matrices in case (i). As $b \geq a$, it is sufficient to prove the claim in the case $b = a$. Any symmetric (resp. alternating) matrix F can be regarded as the matrix for a symmetric (resp. alternating) bilinear form on V. Any such form can be reduced to a canonical form by change of basis, which corresponds to the matrix operation $F \mapsto mFm^T$ for some matrix m. Since Ω is closed under this operation, one just needs to check that the matrices corresponding to one's favourite canonical form for symmetric (resp. alternating) bilinear forms lie in Ω, which is straightforward. Hence, the image of θ is the set of all symmetric matrices in case (ii) or

$$\left\{ \left(\begin{array}{cc} m_1 & m_2 \\ m_3 & m_1^T \end{array} \right) \middle| \text{for all } m_1, m_2, m_3 \in M_a(k) \text{ with } m_2, m_3 \text{ alternating} \right\}$$

in case (i).

Now, if $p \neq 2$, we can realise K as the stabilizer of a vector $v \in V$ with $(v, v) = 1$. Then, $\theta(G.v)$ is just the set of all matrices in $\theta(V)$ of trace 1. We now claim that there are at least two disjoint closed H-orbits in $\theta(G.v)$ if $a > 2$ (the case $a = 2$ is a factorization by [LSS]). The result will then follow with an application of Lemma 1.2, for on taking pre-images, we see that there are at least two disjoint closed H-orbits in $G.v \cong G/K$. To prove the claim, note that the action of H on $M_a(k)$ is conjugation. There are clearly non-conjugate diagonal matrices in $\theta(G.v)$ for the action of all of $GL_a(k)$ on $M_a(k)$ providing $a > 2$. Each $GL_a(k)$-conjugacy class of a diagonal matrix is closed, hence always contains at least one closed H-conjugacy class. Hence, H has disjoint closed orbits in $\theta(G.v)$ as required.

If $p = 2$, one needs to argue further. The quadratic form preserved by G on V is given explicitly, in terms of the basis, by $Q(u_s \otimes w_t) = 0$ for all s, t. Given this, it is not hard to exhibit vectors $v, v' \in V$ with $Q(v) = 1 = Q(v')$ such that $\theta(v)$ and $\theta(v')$ are non-conjugate diagonal matrices, and then the preceeding argument completes the proof. □

Now assume $i > 1$. Some of the arguments here break down for the case of N_2 in $SO(V)$ (which is a Levi subgroup), so we shall divide into five cases:

(i) $H = SO(U) \otimes SO(W), p \neq 2$ and $3 \leq a \leq b$;
(ii) $H = Sp(U) \otimes SO(W), p \neq 2$ and *either* $2 \leq a \leq b, 3 \leq b$ *or* $i = 2, 3 \leq b < a$;
(iii) $H = SO(U) \otimes Sp(W), p \neq 2$ and $3 \leq a < b$ with $i \neq 2$;
(iv) $H = Sp(U) \otimes Sp(W)$, $2 \leq a \leq b, 4 \leq b$ and $i \neq 2$;
(v) $H = Sp(U) \otimes Sp(W)$, $2 \leq a \leq b, 4 \leq b$ and $i = 2$.

3.3. **Lemma**. *There is no dense H, K-double coset in G if $i > b$.*

PROOF. We show that in this case, $\dim H + \dim K < \dim G$. This is just an elementary (but messy) manipulation of inequalities. Note $a \leq b$ (the second possibility in (ii) does not occur as $i > b$). The inequality $\dim H + \dim K \geq \dim G$ is then equivalent to

$$i \leq \begin{cases} \frac{1}{2}ab - \frac{1}{2}\sqrt{(a^2b^2 - 2a^2 - 2b^2 + 2a + 2b)} & \text{case (i)} \\ \frac{1}{2}ab - \frac{1}{2}\sqrt{(a^2b^2 - 2a^2 - 2b^2 - 2a + 2b)} & \text{case (ii)} \\ \frac{1}{2}ab - \frac{1}{2}\sqrt{(a^2b^2 - 2a^2 - 2b^2 + 2a - 2b)} & \text{case (iii)} \\ \frac{1}{2}ab - \frac{1}{2}\sqrt{(a^2b^2 - 2a^2 - 2b^2 - 2a - 2b)} & \text{case (iv).} \end{cases}$$

Hence, in all cases, it is sufficient to show that

$$(\dagger) \qquad b \geq \frac{1}{2}ab - \frac{1}{2}\sqrt{(a^2b^2 - 2a^2 - 2b^2 - 2a - 2b)}.$$

First, if $a = b$, then this inequality reduces to $b \geq \frac{1}{2}b^2 - \frac{1}{2}\sqrt{(b^4 - 4b^2 - 4b)}$, which is easily seen to be true providing $b \geq 3$. Otherwise, $a < b$. Then, $a(a+1) \leq b(b-1)$. So, $0 \leq 2b^2 - 2a^2 - 2b - 2a$. Hence, $a^2b^2 - 4b^2 \leq a^2b^2 - 2b^2 - 2a^2 - 2b - 2a$. Making this substitution in (†), the inequality is then easy to verify. □

Thus, from now on, we assume that we are in one of cases (i)-(iv) – we will treat case (v) separately in Lemma 3.7 – and that $1 < i \leq b$. We now fix standard bases for U, W. Let $A = [\frac{a}{2}], B = [\frac{b}{2}]$. Let $e_1, \ldots, e_A, f_1, \ldots, f_A$, together with d if a is odd, be a basis for U such that

$$\begin{array}{lllllll}(e_s, e_t) & = & (f_s, f_t) & = & 0, & (e_s, f_t) & = & \delta_{st}, \\ (d, e_s) & = & (d, f_s) & = & 0, & (d, d) & = & 1\end{array}$$

for all s, t. Similarly, let $e'_1, \ldots, e'_B, f'_1, \ldots, f'_B$, together with d' if b is odd, be a basis for W.

Let $V_1 = \langle v_1, \ldots, v_i \rangle$ be a non-degenerate i-subspace of V, where $v_1, \ldots, v_i$ are chosen so that the determinant of the $i \times i$ matrix with st-entry equal to (v_s, v_t) is 1. Let $K = \mathrm{stab}_G(V_1)$; equivalently, $K = \mathrm{stab}_G(v_1 \wedge \cdots \wedge v_i)$, the stabilizer of a vector in $\bigwedge^i V$. The G-orbit containing $v_1 \wedge \cdots \wedge v_i$ is just $\{w_1 \wedge \cdots \wedge w_i \mid w_s \in V, \det(w_s, w_t) = 1\}$. Define a linear map $\theta : \bigwedge^i V \to S^iU \otimes \bigwedge^i W$ by $\theta : (u_1 \otimes w_1) \wedge \cdots \wedge (u_i \otimes w_i) \mapsto (u_1. \ldots .u_i) \otimes (w_1 \wedge \cdots \wedge w_i)$ for $u_s \in U, w_t \in W$. Then, θ is H-equivariant, so the morphism $\bar{\theta} : G \to S^iU \otimes \bigwedge^i W$, defined by $\bar{\theta} : g \mapsto \theta(g.v_1 \wedge \cdots \wedge v_i)$ for $g \in G$, sends H, K-double cosets in G to H-orbits in $S^iU \otimes \bigwedge^i W$. It is sufficient to show that there are two H-orbits in $\bar{\theta}(G)$ with disjoint closures, since their pre-images then contain two disjoint closed H, K-double cosets in G and Lemma 1.2 then implies Theorem B holds in this case. The first closed orbit is easy to find (this is the reason we chose the conditions in cases (i)-(iv) with some care).

3.4. **Lemma**. *In cases (i)–(iv),* $0 \in \bar{\theta}(G)$.

Proof. If $i \geq 4$, we may choose V_1 such that $v_1 = e_1 \otimes e'_1, v_2 = f_1 \otimes f'_1, v_3 = e_1 \otimes f'_1, v_4 = f_1 \otimes e'_1$. Then, $\bar{\theta}(1) = \theta(v_1 \wedge \cdots \wedge v_i)$ which is zero as every term in the expression will contain $e'_1 \wedge e'_1$. If $i = 3$, so $p \neq 2$, we may choose V_1 so that $v_1 = e_1 \otimes e'_1, v_2 = f_1 \otimes f'_1, v_3 = \frac{1}{\sqrt{2}}(e_1 \otimes f'_1 + f_1 \otimes e'_1)$, whence it is easily checked that $\bar{\theta}(1) = 0$. Finally, if $i = 2$ then $Cl(W) = SO(W)$ and $p \neq 2$. Let w be a non-singular vector in W. Then, we may choose V_1 so that $v_1 = e_1 \otimes w, v_2 = f_1 \otimes w$, and $\bar{\theta}(1) = 0$. □

Thus, it is sufficient in cases (i)–(iv) to show that there is $v \in \bar{\theta}(G)$ with $0 \notin \overline{H.v}$ (except for case (v)). Let T be the maximal torus of H that acts diagonally on the standard bases for U, W chosen above.

3.5. **Lemma**. *In cases (i)–(iv) with i even, there is no dense H, K-double coset in G.*

Proof. Let V_2 be the non-degenerate k-space spanned by $e_1 \otimes e'_1, f_1 \otimes f'_1, \ldots, e_1 \otimes e'_j, f_1 \otimes f'_j$, where $j = \frac{i}{2}$. Then, the image of this in $\bar{\theta}(G)$ is $v = (e_1.f_1)^j \otimes (e'_1 \wedge f'_1 \wedge \cdots \wedge e'_j \wedge f'_j)$. This is a non-zero vector of $S^iU \otimes \bigwedge^i W$ which is fixed by T. Hence, by Lemma 1.4, $H.v$ is closed, hence $\overline{H.v}$ is disjoint from 0 as required. □

To treat the remaining cases, we have to do rather more work. Suppose that U_1 and W_1 are H_1-modules for some algebraic group H_1, and that H_1 preserves a bilinear form $\langle ., . \rangle$ on

W_1. We now define an H_1-equivariant morphism $U_1 \otimes W_1 \to U_1 \otimes U_1$, which we shall use a number of times in the remainder of the section. Fix bases $u_1, \ldots, u_N$ and $w_1, \ldots, w_M$ for U_1 and W_1 respectively. Define the morphism $\phi : U_1 \otimes W_1 \to U_1 \otimes U_1$ by

$$(*) \qquad \phi : \sum_{s,t} a_{st} u_s \otimes w_t \mapsto \sum_{s,t,s',t'} a_{st} a_{s't'} \langle w_t, w_{t'} \rangle u_s \otimes u_{s'}$$

where s, s' sum between 1 and N and t, t' sum between 1 and M. This is certainly a morphism. A routine (if slightly gruesome) check shows that the definition of ϕ does not depend on the choice of basis. Using this observation, it is easy to check that ϕ is H-equivariant.

3.6. **Lemma**. *In cases (i)–(iv) with i odd, there is no dense H, K-double coset in G.*

PROOF. As i is odd and $i \neq 1$, either $H = SO(U) \otimes SO(W)$ or $H = Sp(U) \otimes Sp(W)$, and $p \neq 2$. Let V_2 be the non-degenerate i-space spanned by $e_1 \otimes e'_1, f_1 \otimes f'_1, \ldots, e_1 \otimes e'_j, f_1 \otimes f'_j, e_1 \otimes e'_{j+1} + f_1 \otimes f'_{j+1}$ where $j = [\frac{i}{2}]$. Its image in $S^iU \otimes \bigwedge^i W$ is $v = e_1^j . f_1^{j+1} \otimes e'_1 \wedge f'_1 \wedge \cdots \wedge e'_j \wedge f'_j \wedge f'_{j+1} + e_1^{j+1} . f_1^j \otimes e'_1 \wedge f'_1 \wedge \cdots \wedge e'_j \wedge f'_j \wedge e'_{j+1}$. We show $0 \notin \overline{H.v}$ which will complete the proof.

Let $N = \dim S^iU$, $M = \dim \bigwedge^i W$ and let $u_1, \ldots, u_N$ (resp. $w_1, \ldots, w_M$) be bases for S^iU (resp. $\bigwedge^i W$) such that $u_1 = e_1^{j+1}.f_1^j, u_2 = e_1^j.f_1^{j+1}$ and $w_1 = e'_1 \wedge f'_1 \wedge \cdots \wedge e'_j \wedge f'_j \wedge e'_{j+1}, w_2 = e'_1 \wedge f'_1 \wedge \cdots \wedge e'_j \wedge f'_j \wedge f'_{j+1}$.

Now, H preserves the canonical form on $\bigwedge^i W$, defined by

$$\langle a_1 \wedge \cdots \wedge a_i, b_1 \wedge \cdots \wedge b_i \rangle = \sum_{\sigma \in S_i} \varepsilon(\sigma)(a_1, b_{\sigma 1}). \ldots .(a_i, b_{\sigma i})$$

for $a_s, b_t \in W$. We now construct an H-equivariant morphism $\phi : S^iU \otimes \bigwedge^i W \to S^iU \otimes S^iU$ as in $(*)$ (taking $(H_1, U_1, W_1) = (H, S^iU, \bigwedge^i W)$). Explicitly,

$$\phi : \sum_{s,t} a_{st} u_s \otimes w_t \mapsto \sum_{s,t,s',t'} a_{st} a_{s't'} \langle w_t, w_{t'} \rangle u_s \otimes u_{s'}.$$

Now, one computes the image of $v = u_1 \otimes w_1 + u_2 \otimes w_2$, to show $\phi(v) = (-1)^j (u_1 \otimes u_2 + u_2 \otimes u_1)$ or $u_1 \otimes u_2 - u_2 \otimes u_1$ according to whether $H = SO(U) \otimes SO(W)$ or $Sp(U) \otimes Sp(W)$ respectively. Now, $u_1 \otimes u_2 = e_1^{j+1}.f_1^j \otimes e_1^j.f_1^{j+1}$ which is fixed by T, and similarly for $u_2 \otimes u_1$. Hence, $\phi(v)$ is fixed by T, so its H-orbit is closed by Lemma 1.4, and disjoint from 0. Hence, the pre-image of $H.\phi(v) = \phi(H.v)$ will be closed and disjoint from zero, and contains $H.v$. But this shows $0 \notin \overline{H.v}$ as required. □

It remains to treat case (v). Here, we have to proceed slightly differently.

3.7. **Lemma**. *There is no dense H, K-double coset in G in case (v).*

Proof. Let Q be the quadratic form associated to $(.,.)$ on V preserved by G. Let $\Omega = \{v_1 \otimes v_2 \in V \otimes V \mid (v_1, v_2) = 1, Q(v_1) = 0 = Q(v_2)\}$. Then, G acts transitively on Ω and K is the stabilizer of a vector in Ω, so we just need to show H has at least two disjoint closed orbits in Ω. Let $\langle .,. \rangle$ be the H-invariant bilinear form on $W \otimes W$ defined by $\langle w_1 \otimes w_2, w_1' \otimes w_2' \rangle = (w_1, w_2')(w_2, w_1')$. Now, we define an H-equivariant morphism $V \otimes V \to S^2(U \otimes U)$ by composition

$$\theta : V \otimes V \longrightarrow U \otimes U \otimes W \otimes W \xrightarrow{\phi} U \otimes U \otimes U \otimes U \longrightarrow S^2(U \otimes U)$$

where the first and last maps are canonical and ϕ is as defined in $(*)$ with $(H_1, U_1, W_1) = (H, U \otimes U, W \otimes W)$. Compute the image of $(\eta e_1 \otimes e_1' + (1-\eta) f_1 \otimes f_2') \otimes (f_1 \otimes f_1' + e_1 \otimes e_2') \in \Omega$. It is $-\eta^2 (e_1 \otimes f_1)^2 + 2\eta(1-\eta) e_1 \otimes e_1 . f_1 \otimes f_1 - (1-\eta)^2 (f_1 \otimes e_1)^2$. In particular, the image consists of infinitely many elements of weight zero as η varies. Hence, there are infinitely many closed H-orbits in $\theta(\Omega)$ by Lemma 1.7 and Lemma 1.4, and the conclusion follows by Lemma 1.2. □

This completes the proof of Proposition 3.1.

Finally, we consider a case in $SL(V)$, using the same argument as Lemma 3.7.

3.8. **Lemma**. *There is no dense H, K-double coset in G if*

$$(G, H, K) = (SL_{ab}, SL_a \otimes SL_b, GL_{ab-1}).$$

Proof. Let $V = U \otimes W$ with $\dim V = ab, \dim U = a, \dim W = b$, and let $G = SL(V), H = SL(U) \otimes SL(W)$. Fix bases $u_1, \ldots, u_a$ for U and $w_1, \ldots, w_b$ for W and let $\bar{u}_1, \ldots, \bar{u}_a$ (resp. $\bar{w}_1, \ldots, \bar{w}_b$) be the corresponding dual basis for U^* (resp. W^*).

Note that K is the stabilizer of an element $v_0 \otimes \bar{v}_0 \in V \otimes V^*$ where $\bar{v}_0(v_0) = 1$. Hence, define $\Omega = \{v \otimes \bar{v} \in V \otimes V^* \mid \bar{v}(v) = 1\}$, the G-orbit of $v_0 \otimes \bar{v}_0$. We need to prove that H has at least two disjoint closed orbits in Ω, by Lemma 1.2.

Let $\langle .,. \rangle$ be the H-equivariant bilinear form on $W \otimes W^*$ defined by $\langle w_s \otimes \bar{w}_t, w_{s'} \otimes \bar{w}_{t'} \rangle = \bar{w}_t(w_{s'}) \bar{w}_{t'}(w_s)$ for $1 \leq s, s', t, t' \leq b$. Now, $V \otimes V^*$ is just $U \otimes W \otimes U^* \otimes W^*$. Define an H-equivariant morphism $\theta : V \otimes V^* \to S^2(U \otimes U^*)$ by the composition

$$V \otimes V^* \longrightarrow U \otimes U^* \otimes W \otimes W^* \xrightarrow{\phi} U \otimes U^* \otimes U \otimes U^* \longrightarrow S^2(U \otimes U^*)$$

where the first and last maps are canonical, and ϕ is as defined in $(*)$ with $(H_1, U_1, W_1) = (H, U \otimes U^*, W \otimes W^*)$. Compute the image of $(\eta u_1 \otimes w_1 + (1-\eta) u_2 \otimes w_2) \otimes (\bar{u}_1 \otimes \bar{w}_1 + \bar{u}_2 \otimes \bar{w}_2) \in \Omega$. It is $\eta^2 (u_1 \otimes \bar{u}_1)^2 + 2\eta(1-\eta)(u_1 \otimes \bar{u}_2 \otimes u_2 \otimes \bar{u}_1) + (1-\eta)^2 (u_2 \otimes \bar{u}_2)^2$. The important thing is that the image is of weight zero with respect to the maximal torus T of H that acts diagonally on the basis, and as η varies we obtain infinitely many distinct elements. Hence, there are infinitely many closed H-orbits in $\theta(\Omega)$ by Lemma 1.7 and Lemma 1.4. □

4 Proof of Theorem B: Levi factors

We now consider Theorem B when one or both of H or K is a Levi factor. So, let G be a connected reductive algebraic group. We fix a root system $(T, B; \Sigma, \Pi)$ for G for the remainder of this section. Let $P = P_J$ be the standard parabolic subgroup of G corresponding to the subset J of $I = \{1, \ldots, n\}$. So, $P = \langle B, U_{-\alpha_j} \,|\, j \in J \rangle$ and $L = \langle T, U_{\pm\alpha_j} \,|\, j \in J \rangle$ is a Levi factor of P. Let $W_L = N_L(T)/T$ be the Weyl group of L, a subgroup of W.

4.1. **Lemma ([B2, Lemma 3.2])**. *Let H be a connected reductive subgroup of G with maximal torus $S \le T$; let $W_H = N_H(S)/S$ be the Weyl group of H. Suppose $N_H(S) = N_H(T)$, so that W_H can be identified with a subgroup of W. If there is a dense H, L-double coset in G, then $W = W_H W_L$ is a factorization of W.*

The condition $N_H(S) = N_H(T)$ in Lemma 4.1 is obviously satisfied if H is of maximal rank, for then we can take $S = T$. As an immediate application, we have the following:

4.2. **Corollary**. *Let L_J and $L_{J'}$ be standard Levi factors corresponding to proper subsets $J, J' \subset I$. Then, there is no dense $L_J, L_{J'}$-double coset in G.*

Proof. Let $W_J, W_{J'}$ be the corresponding parabolic subgroups of W. By Lemma 4.1, we just need to show that $W \neq W_J W_{J'}$, which is well known. □

To apply Lemma 4.1 to subgroups H which are not of maximal rank, we need to verify the condition $N_H(S) = N_H(T)$. We now consider this problem. We write $\mathfrak{t}$ and $\mathfrak{g}$ for the Lie algebras of T and G respectively. We shall need the next known lemma:

4.3. **Lemma ([B2, Lemma 3.6])**. *Suppose G is simple, simply connected and (G, p) is not $(C_n, 2)$ for any $n \ge 1$. Then, $N_G(\mathfrak{t}) = N_G(T)$.*

4.4. **Corollary**. *Let G be as in the lemma and H be a connected reductive subgroup with maximal torus $S < T$. Suppose the zero weight space of $\mathfrak{g}$ relative to S is equal to $\mathfrak{t}$. Then, $N_H(S) = N_H(T)$.*

Proof. By assumption, $N_H(S)$ normalizes $\mathfrak{g}$. Hence, by Lemma 4.3, $N_H(S) \le H \cap N_G(\mathfrak{t}) = H \cap N_G(T) = N_H(T)$. Conversely, let $n \in N_H(T)$ and let $s \in S = T \cap H$. Then, $nsn^{-1} \in T \cap H = S$, so n normalizes S. □

4.5. **Lemma**. *Let $G = Cl(V)$ and $H < G$ be a connected reductive subgroup with maximal torus $S < T$. Then, the zero weight space of $\mathfrak{g}$ relative to S equals $\mathfrak{t}$ if and only if the following hold:*

(i) *for every $\mu \in X(S)$, $\dim(V\downarrow_S)_\mu \le 1$;*

(ii) *if $G = C_n$ or $B_n(p = 2)$, then $(V\downarrow_S)_0 = 0$.*

Proof. We prove this for the case $G = Sp(V)$; other types of G are similar. Let $e_1, \ldots, e_n, f_n, \ldots, f_1$ be a symplectic basis for V, where $(e_i, e_j) = 0 = (f_i, f_j)$ and $(e_i, f_j) = \delta_{ij}$. Let T be the maximal torus of G which acts diagonally on this basis, so elements of T have the form $t = \mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1})$ when written with respect to the basis. Let ε_i be the character of T sending $t \mapsto t_i$ for $1 \le i \le n$. Then, the weights of e_i, f_i are $\varepsilon_i, -\varepsilon_i$ respectively. Let μ_i be the restriction of ε_i to S. The weights of $\mathfrak{g}$ are $\{0^n, \pm 2\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \,|\, 1 \le i < j \le n\}$. We therefore require the weights $2\mu_i$ and $\mu_i \pm \mu_j$ to S to

Table 4: Case list involving a Levi factor

G	p	H	K	Ref
$SL(V)$		$SL(U)\otimes SL(W), V=U\otimes W$	L_i	(4.7), (3.8)
$SL(V)$		$Sp(V)$	L_i	(4.7)
$SL(V)$	$p\neq 2$	$SO(V)$	L_i	(4.7)(4.8)
$SO(V)$		$Cl(U)\otimes Cl(W), V=U\otimes W$	N_2	(3.1)
Sp_{2n}	$p\neq 2$	N_i	GL_n	(4.9)
SO_{2n}		N_i	GL_n	(4.7), (4.9)
SO_7	$p\neq 2$	$G_2, V\downarrow_H = L_H(\omega_1)$	N_2	[LSS]
SO_8		$B_3, V\downarrow_H = L_H(\omega_1)$	N_2	[LSS]
SO_8		N_1	N_2	(2.2)
SO_{16}		B_4	N_2	(4.11)
$SO_{26-\delta p,3}$		$F_4, V\downarrow_H = L_H(\omega_4)$	N_2	(4.14)
SO_{32}	$p=2$	$D_6, V\downarrow_H = L_H(\omega_6)$	N_2	(2.7)
SO_{56}	$p=2$	$E_7, V\downarrow_H = L_H(\omega_7)$	N_2	(2.7)

be non-zero for each $1\leq i<j\leq n$. This is clearly if and only if no $\mu_i=0$, so $(V\downarrow_S)_0=0$, and no $\mu_i=\pm\mu_j$, so $\dim(V\downarrow_S)_\mu\leq 1$ for all $\mu\in X(S)$. □

Now we apply these results to the remaining cases in the proof of Theorem B, when one of H, K is a Levi factor. We begin by obtaining a case list as before.

4.6. **Proposition**. *Let G be simple and $H, K\in\mathcal{M}(G)$ with K a Levi factor. To prove Theorem B for the triple (G,H,K) it is sufficient to show that it holds for for (G,H,K) in table 4. (In the table we reference the lemma in which we treat these subgroups.)*

Proof. The possibilities for (G,K) are easy to compute. Moreover, it is sufficient to list the possibilities for H, K up to graph automorphisms of G. Hence, $(G,K)=(A_n,L_i)(1\leq i\leq\frac{n}{2})$, (B_n,L_1), (C_n,L_n), (D_n,L_1) or $(D_n,L_n)(n>4)$. Next, note that we can exclude $(G,K,p)=(C_n,L_n,2)$ as then L_n lies in a subgroup D_n so that $K\notin\mathcal{M}(G)$. Then, by Corollary 4.2, H is not a Levi factor, so H is a reductive maximal connected subgroup of G with $\dim H\geq\dim G/K$. Now the possibilities can be listed, copying the proof of Proposition 2.1. □

We begin the proof of these cases by considering factorizations in the Weyl group, applying Lemma 4.1 when possibile.

4.7. **Lemma**. *Theorem B holds if*

$$
\begin{aligned}
(G,H,K)&=(SL(V),SL(U)\otimes SL(W),L_i)(i\neq 1),\\
(G,H,K)&=(SO_{2n},N_i,GL_n)(i\ \textit{even}),\\
(G,H,K)&=(SL(V),Sp(V),L_i),\\
(G,H,K)&=(SL(V),SO(V),L_i)(p\neq 2).
\end{aligned}
$$

PROOF. First observe that for each case the condition in Lemma 4.5 is satisfied. Thus, we may apply Corollary 4.4 to deduce that the condition $N_H(S) = N_H(T)$ is satisfied in every case. Thus, it suffices to show $W \neq W_H W_L$, where W_H and W_L are the Weyl groups of H and $L = K$ respectively by Lemma 4.1. We consider this for each case in turn.

(i) $(G, H, K) = (SL(V), SL(U) \otimes SL(W), L_i)(i \neq 1)$. Let $n = \dim G$, $r = n - 1$. By applying a graph automorphism, we may assume $i \leq \frac{r}{2}$. We shall show that $W \neq W_H W_L$.

Choose bases $u_1, \ldots, u_a$ and $w_1, \ldots, w_b$ for U, W respectively, and assume $a \leq b$. An easy dimension argument shows that we may assume $i \leq b$. Let $T_U \otimes T_W$ be the corresponding maximal torus of H that acts diagonally on these basis elements. Let $v_1, \ldots, v_{ab}$ be the basis for $U \otimes W$ such that $v_{(i-1)b+j} = u_i \otimes w_j$ where $1 \leq i \leq a, 1 \leq j \leq b$. Let T be the maximal torus of G that acts diagonally on this basis, so $T_U \otimes T_W < T$. Writing ε_i for the element of $X(T)$ that sends $t \mapsto t_i$ when $t \in T$ is written as $\mathrm{diag}(t_1, \ldots, t_{ab})$ with respect to the basis $v_1, \ldots, v_{ab}$. Let μ_{ij} be the element of $X(T_U \otimes T_W)$ obtained by restricting $\varepsilon_{(i-1)b+j}$.

Now, by the proof of [B2, Lemma 3.2], W_L is the stabilizer of ω_i in $X(T)$, and $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$. Hence, $W.\omega_i = \{\varepsilon_{j_1} + \cdots + \varepsilon_{j_i} \mid 1 \leq j_1 < \cdots < j_i \leq ab\}$ of order $\binom{ab}{i}$. On the other hand, we are assuming $i \leq b$, so $\omega_i \downarrow_{T_U \otimes T_W} = \mu_{11} + \cdots + \mu_{1i}$. So, $W_H.\omega_i = \{\mu_{jk_1} + \cdots + \mu_{jk_i} \mid 1 \leq j \leq a, 1 \leq k_1 < \cdots < k_i \leq b\}$ of order $a\binom{b}{i}$. Hence, $W_H.\omega_i \neq W.\omega_i$ unless $i = 1$, and the result follows.

(ii) $(G, H, K) = (SO_{2n}, N_{2i}, GL_n)$. Again $W \neq W_H W_L$. Let $e_1, \ldots, e_n, f_n, \ldots, f_1$ be a basis for V such that $(e_s, e_t) = 0 = (f_s, f_t), (e_s, f_t) = \delta_{st}$ and $Q(e_s) = Q(f_t) = 0$ where Q is the associated quadratic form if $p = 2$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be the corresponding weights of the maximal torus T that acts diagonally on this basis. Then, W_L is the stabilizer of $\omega_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$ in $X(T)$. Also, W acts on $\pm\varepsilon_1, \ldots, \pm\varepsilon_n$ as all permutations of $1, \ldots, n$ and all sign changes of even signature. Hence, $|W.\omega_n| = 2^{n-1}$. On the other hand, $|W_H.\omega_n| = 2^{i-1}2^{n-i-1} = 2^{n-2}$. Hence, $W \neq W_H W_L$.

(iii) $(G, H, K) = (SL_{2n}, Sp_{2n}, L_i)(i \neq 1)$. If $i = 1$, this is a factorization by [LSS]. So, suppose $1 < i \leq n$. Let T be a maximal torus of G and $\varepsilon_1, \ldots, \varepsilon_{2n}$ be the weights of T as in (i). Then, we may choose T_H and weights $\mu_1, \ldots, \mu_n$ as usual such that $\varepsilon_i \downarrow_{T_H} = -\varepsilon_{n+i} \downarrow_{T_H} = \mu_i$ for $1 \leq i \leq n$. Then, L_i is the stabilizer of $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$. Hence, $|W.\omega_i| = \binom{2n}{i}$ as in (i). Now, W_H acts as all permutations and sign changes on $\pm\mu_1, \ldots, \pm\mu_n$. Hence, $|W_H.\omega_i| = 2^i\binom{n}{i}$. Hence, $W \neq W_H W_L$ providing $i \neq 1$.

(iv) $(G, H, K) = (SL(V), SO(V), L_i)(p \neq 2)$. If $\dim V$ is even and $i > 1$, the argument of (iii) shows $W \neq W_H W_L$ to complete the proof. We shall prove the result if $\dim V$ is even and $i = 1$ in Lemma 4.8. So, we may assume $\dim V = 2n + 1$ is odd and $1 \leq i \leq n$. Let $T_H < T$ be maximal tori of H, G respectively and ε_i the usual weights of T. We can ensure that $\varepsilon_i \downarrow_{T_H} = -\varepsilon_{n+i} \downarrow_{T_H} = \mu_i$ for $1 \leq i \leq n$ and $\varepsilon_{2n+1} \downarrow_{T_H} = 0$. Then, W_L is the stabilizer of $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ and $|W.\omega_i| = \binom{2n+1}{i}$. Now, W_H acts as all permutations and sign changes on $\pm\mu_1, \ldots, \pm\mu_n$. Hence, $|W_H.\omega_i| = 2^i\binom{n}{i}$. Hence, $W \neq W_H W_L$ for all i. □

However, in all remaining cases in table 4 (except $(G, H, K) = (SO_{26}, F_4, N_2)$ when the condition of Lemma 4.5 fails) one can show that $W = W_H W_L$ is a factorization. So, Lemma 4.1 is of no use here and we must treat them explicitly.

4.8. **Lemma**. *There is no dense H, K-double coset in G if*

$$(G, H, K) = (SL_{2n}, SO_{2n}, L_1)(p \neq 2).$$

Proof. Let $G = GL(V)$, $H = SO(V)$ and $v_1, \dots, v_{2n}$ be an orthonormal basis for V with respect to the bilinear form $(.,.)$ preserved by H on V. Let $f_1, \dots, f_{2n}$ be the corresponding dual basis for V^*. Then, the homomorphism defined by $\phi : f_i \mapsto v_i$ is H-equivariant. Let K be the stabilizer of $v_1 \otimes f_1$. Then, $G.(v_1 \otimes f_1) = \{v \otimes f \in V \otimes V^* \mid f(v) = 1\}$. Define $\theta : V \otimes V^* \to S^2V$ by $v \otimes f \mapsto v.\phi(f)$, an H-equivariant morphism. Let $\bar{\theta}(g) = \theta(g.(v_1 \otimes f_1))$ so that $\bar{\theta} : G \to S^2V$ is constant on H, K-double cosets. Also, $\bar{\theta}(G) = \{v.w \in S^2V \mid (v, w) = 1\}$. Now, we can construct S^2V as symmetric matrices, the matrix M corresponding to the vector $\sum_{i,j} M_{ij} v_i.v_j$. The action of H on S^2V is then just $x : M \mapsto xMx^T = xMx^{-1}$ as $x^T = x^{-1}$ for $x \in H$ when written as a matrix in terms of the orthonormal basis. So, the action is just conjugation. Now, one checks that as η varies, the elements $(v_1+\eta v_2).v_1 \in \bar{\theta}(G)$ give infinitely many non-conjugate semisimple matrices, and the proof is completed as usual applying Lemma 1.2. □

4.9. **Lemma**. *Theorem B holds if*

$$(G, H, K) = (SO_{2n}, N_{2i+1}, GL_n),$$
$$(G, H, K) = (Sp_{2n}, N_{2i}, GL_n)(p \neq 2).$$

Proof. If $(G, H, K) = (SO_{2n}, GL_n, N_1)$ then $G = HK$ is a factorization by [LSS]. Excluding this case, we may assume that $i \geq 1$ and $p \neq 2$.

Let $e_1, \dots, e_n, f_1, \dots, f_n$ be a basis for the natural G-module V with $(e_i, f_j) = \delta_{ij}$, $(e_i, e_j) = 0 = (f_i, f_j)$ for all i, j. Let K be the connected stabilizer of the direct sum decomposition decomposition $V = \langle e_1, \dots, e_n \rangle \oplus \langle f_1, \dots, f_n \rangle$. Let H be the connected stabilizer of the non-degenerate subspace $U = \langle e_1, \dots, e_i, f_1, \dots, f_i \rangle$ if $G = Sp(V)$ or $U = \langle e_1, \dots, e_i, f_1, \dots, f_i, e_n + f_n \rangle$ if $G = SO(V)$. Let $L = V \otimes V$ and $z = e_1 \otimes f_1 + \dots + e_n \otimes f_n$; observe that in either case, K fixes z. Moreover, $L\downarrow_H \cong U \otimes U \oplus U \otimes U^\perp \oplus U^\perp \otimes U \oplus U^\perp \otimes U^\perp$. Let $\pi : L \to U \otimes U = M$ be projection along this direct sum decomposition, an H-equivariant morphism.

Now, the morphism $\theta : G \to M$ defined by $g \mapsto \pi(gz)$ sends H, K-double cosets in G to H-orbits in M, so it suffices to show that there exist two closed H-orbits in $\pi(Gz)$. Let $g_\lambda \in G$ be the map sending

$$e_1 \mapsto e_1 \pm \lambda f_{i+1}, e_{i+1} \mapsto \lambda f_1 + e_{i+1}, f_1 \mapsto e_{i+1} + (\lambda + 1) f_1, f_{i+1} \mapsto (\lambda + 1) f_{i+1} \pm e_1$$

where the ambiguous sign is chosen to be $+$ if $G = Sp(V)$, $-$ if $G = SO(V)$. Now note that $\pi(g_\lambda z) = \pi(z) + \lambda(e_1 \otimes f_1 \pm f_1 \otimes e_1)$. So as λ varies, we obtain infinitely many elements of weight zero with respect to the maximal torus of H which acts diagonally on the given basis. So Lemma 1.7 and Lemma 1.4 imply that there are infinitely many closed H-orbits in $\pi(Gz)$ to complete the proof. □

The next case will follow easily from the next general lemma.

4.10. **Lemma**. *Suppose $H, K, L < G$ are all connected algebraic groups. Suppose HL and KL are dense in G. Then, there is a dense $H \cap K, H \cap L$-double coset in H if and only if there is a dense $H \cap K, K \cap L$-double coset in K.*

PROOF. Suppose $(H \cap K)x(H \cap L)$ is dense in H for some $x \in H$. Then, $(H \cap K)xL$ is dense in HL by Lemma 5.1, hence in G as HL is dense in G. So, it meets KL as G is an irreducible variety and KL is dense in G. Therefore, it actually lies in KL as KL is stable under the action of $(H \cap K) \times L$ and $(H \cap K)xL$ is a single $(H \cap K) \times L$ orbit. So, $(H \cap K)xL = (H \cap K)yL$ for some $y \in K$, and this is dense in KL. Hence by Lemma 5.1 again $(H \cap K)y(K \cap L)$ is dense in K. The converse is the same. □

4.11. **Corollary**. *There is no dense $\bar{H}, \bar{K}$-double coset in $\bar{G}$ if*

$$(\bar{G}, \bar{H}, \bar{K}) = (SO(V), N_2, B_4)$$

where $V\downarrow_K = L_K(\omega_4)$.

PROOF. Let V be a spin module for D_5 and B_4; V possesses a non-degenerate B_4-invariant bilinear form. To apply Lemma 4.10, let $G = SL(V)$, $H = D_5$, $K = SO(V)$, so $H \cap K = B_4$ as D_5 does not preserve the bilinear form and B_4 is maximal in D_5. We show that there is a non-degenerate 2-space with D_5-stabilizer A_1G_2. For, let $A_1G_2 < SO_3 \times SO_7 < SO_{10} = D_5$ be the usual subgroup of D_5. Then, $V\downarrow_{A_1G_2} = L_{A_1}(\omega_1) \otimes \Delta_7$, where Δ_7 is the 7-dimensional Weyl module for G_2. Now, G_2 fixes a 1-space in Δ_7, hence A_1G_2 fixes a 2-space in V. But, the only connected subgroup of D_5 containing A_1G_2 is N_3, and $V\downarrow_{N_3} = L_{A_1}(\omega_1) \otimes L_{B_3}(\omega_3)$, so this fixes no 2-space. Hence, there is some 2-space ω in $\mathbb{P}^2(V)$ with D_5-stabilizer A_1G_2. Hence by dimension, $D_5.\omega$ is dense in $\mathbb{P}^2(V)$. But the non-degenerate 2-spaces are also dense in $\mathbb{P}^2(V)$, hence there is some non-degenerate 2-space ω_1 in $D_5.\omega$; this will have D_5-stabilizer A_1G_2 as required.

Let $L = \mathrm{stab}_G(\omega_1)$, a parabolic subgroup of G. Then, $K \cap L$ is the group N_2 in $SO(V)$, and $H \cap L$ is A_1G_2. Hence, by dimensions, HL and KL are dense in G. Now, there is no dense $H \cap K, H \cap L$-double coset in H by Lemma 2.2 $H \cap K$ is N_1 in $H = SO_{10}$ and $H \cap L$ lies in N_3). Hence, by Lemma 4.10 there is no dense $H \cap K, K \cap L$-double coset in K which is the required result. □

It just remains to verify *the last case*

$$(G, H, K) = (SO_{26-\delta_{p,3}}, F_4, N_2)$$

to complete the proof of Theorem B. This seems to be rather harder than the other cases, especially if $p = 3$. For the remainder of the section, we work with the 27-dimensional module V for $E = E_6$. The construction of V described here is taken originally from Cohen and Cooperstein's paper [CC]. We shall use the notation defined in [B2, Section 4], and refer the reader to [B2] for fuller details.

Let V be a 27-dimensional vector space over k whose elements are triples $x = [x_1, x_2, x_3]$ with $x_i \in M_3(k)$. We set

$$\tilde{E} = \{g \in GL(V) \mid \text{there is } \lambda \in k^\times \text{ such that, for all } x \in V,\ \mathcal{D}(g.x) = \lambda \mathcal{D}(x)\},$$

where $\mathcal{D} : V \to k$ is the cubic form $\mathcal{D}(x) = \det x_1 + \det x_2 + \det x_3 - \mathrm{tr}(x_1 x_2 x_3)$. Then, $E = \tilde{E}'$ is a simply connected simple algebraic group of type E_6, and $\tilde{E}$ is an extension of E by a 1-dimensional torus. Let e^i_{jk} be the element $[x_1, x_2, x_3]$ of V all of whose entries are 0 except the jk-entry of x_i which is 1. Let $e_i = e^1_{ii}$ for $i = 1, 2, 3$ and $e = e_1 + e_2 + e_3$. Let $G = E_e$, a simple algebraic group of type F_4 (by [CC]).

Note G preserves the non-degenerate symmetric bilinear form $(.,.)$ given by $(x, y) = \mathrm{tr}(x_1 y_1 + x_2 y_3 + x_3 y_2)$ for $x = [x_1, x_2, x_3], y = [y_1, y_2, y_3] \in V$. Finally, the G-equivariant map $\# : V \to V$, $x \mapsto x^\#$ is defined by the identity

$$\mathcal{D}(x + ty) = \mathcal{D}(x) + (x^\#, y)t + (x, y^\#)t^2 + \mathcal{D}(y)t^3,$$

for $x, y \in V$ and t an indeterminate. Explicitly, the map $\#$ is given by

$$x^\# = [x_1^\# - x_2 x_3, x_3^\# - x_1 x_2, x_2^\# - x_3 x_1]$$

where for $c \in M_3(k)$, $c^\#$ is the adjoint of c (the matrix whose ij-entry is the ji-cofactor of c).

We begin by defining a certain subgroup of G of type $T_1 G_2 U_{14}$. Here, we exclude $p = 2$ so that there are no degeneracies in the commutator relations. Let P be the B_3-parabolic of G obtained by deleting β_4 from the Dynkin diagram of G, labelling the simple roots of G by $\beta_1, \ldots, \beta_4$ as in [B2, 4.6]. So $P = \langle B_1, K_{-\beta_j} \mid j = 1, 2, 3\rangle$. Let $P = LQ$ where $L = \langle T_1, K_{\pm\beta_j} \mid j = 1, 2, 3\rangle$ is a Levi factor and Q is the unipotent radical. Let $L = L'R$ where R is the 1-dimensional radical. Observe that the subgroup $Z \cong G_2$ constructed in [B2, Lemma 4.10] is a subgroup of L'.

By [ABS], Q has an L-composition series $Q = Q_1 > Q_2 > Q_3 = 0$ such that the factors $V_i = Q_i / Q_{i+1}$ are the L'-modules $L_{B_3}(\omega_3)$ and $L_{B_3}(\omega_1)$ for $i = 1, 2$ respectively. As $L_{B_3}(\omega_3)\downarrow_{G_2} = L_{G_2}(\omega_1) \oplus k$, there is a unique Z-invariant subgroup $Q_0 \lhd Q$ of dimension 14. We have thus defined a subgroup $H = RZQ_0$ of type $T_1 G_2 U_{14}$. Let

$$(y_{\beta_1}(s), y_{\beta_2+2\beta_3+\beta_4}(t)) = y_{\beta_1+\beta_2+2\beta_3+\beta_4}(Ast)$$
$$(y_{\beta_3}(s), y_{\beta_1+\beta_2+\beta_3+\beta_4}(t)) = y_{\beta_1+\beta_2+2\beta_3+\beta_4}(Bst)$$

for some $A, B \in k^\times$, applying Chevalley's commutator formula. Let

$$v(t) = y_{\beta_2+2\beta_3+\beta_4}(Bt) y_{\beta_1+\beta_2+\beta_3+\beta_4}(-At).$$

Then a routine check using the commutator formula and the known action of root subgroups K_{β_i} in [B2, 4.6] shows that each of the generators of Z defined in [B2, Lemma 4.10] centralizes $V = \{v(t) \mid t \in k\}$. In particular, the image VQ_2/Q_2 in V_1 must be the 1-space fixed by Z. So, $Q = Q_0 V$.

4.12. **Lemma**. *Let H be the subgroup of $G = F_4$ of type $T_1 G_2 U_{14}$ defined above. Then, there is no dense H, H-double coset in G.*

PROOF. Recall $H < P$ where P is a B_3-parabolic. Let n_0 be a coset representative for the longest element of the Weyl group $N_G(T_1)/T_1$. Suppose HgH is dense in G for some $g \in G$; in particular, $\dim H \cap gHg^{-1} = 6$. Now, Pn_0P is dense in G so it must intersect HgH. Hence, as $H < P$, $HgH < Pn_0P$ and $g = p_1n_0p_2$ for some $p_1, p_2 \in P$. Hence, $\dim {}^{p_1}H \cap {}^{p_2^-}H^- = 6$ where $p_2^- = {}^{n_0}p_2 \in P^-$, the opposite parabolic, and $H^- = {}^{n_0}H < P^-$. However, ${}^{p_1}H \cap {}^{p_2^-}H^- < P \cap P^- = L$ so we consider ${}^{p_1}H \cap L$. Write $p_1 = lvx$ for $l \in L, x \in Q_0 < H, v \in V$. Then, we showed above that v centralizes Z, so ${}^{p_1}H \cap L >{}^{l}Z$. Arguing similarly for ${}^{p_2^-}H^- \cap L$, we deduce that there are conjugates of Z in L with intersection of dimension ≤ 6. But this implies that there is a dense G_2, G_2-double coset in T_1B_3 which is not the case by Theorem 1.6. □

We can now describe the orbits of G on $e^\perp/\langle e\rangle$ when $p = 3$, in terms of our basis $\{e^i_{jk} \mid 1 \leq i, j, k \leq 3\}$ for V.

4.13. **Lemma**. *Suppose $p = 3$ and $W = e^\perp/\langle e\rangle$ is the 25-dimensional module $L_{F_4}(\omega_4)$. Then, $G = F_4$ has 3 orbits on the 1-subspaces of W with orbit representatives l_1, l_2, l_3 where $l_i = \langle u_i + \langle e\rangle\rangle$ and $u_1 = e^2_{11}, u_2 = e^3_{22} + e^3_{33}, u_3 = e_1 - e_2$. Moreover, $G_{l_1} = G_{\langle u_1\rangle} = P$, a B_3-parabolic, and $G_{l_2} = G_{\langle u_2\rangle} = H$ of type $T_1G_2U_{14}$.*

PROOF. We first prove the result on stabilizers. Observe l_1 and $\langle u_1\rangle$ are spanned by highest weight vectors of weight ω_4 in both V, W from the computation of weights, hence both stabilizers equal $P = LQ$, a B_3-parabolic with Levi factor L and unipotent radical Q. A direct calculation using the definition in [B2] shows that $\langle e, u_2\rangle^\# \subset \langle e, u_1, u_2\rangle$. Now, G_{l_2} stabilizes $\langle e, u_2\rangle$ hence $\langle e, u_2\rangle^\#$ hence $\langle e, u_1, u_2\rangle$. But $\langle u_1\rangle$ is the only line L in $\langle e, u_1, u_2\rangle$ with $L^\# = \{0\}$. As $\#$ is G-equivariant, G_{l_2} therefore stabilizes $\langle u_1\rangle$, so lies in P. Now, suppose $g \in G_{l_2}$. Then, $g.u_2 = \mu u_2 + \nu e$ for some μ, ν. The weights of e^3_{22} and e^3_{33} are $\beta_1 + \beta_2 + \beta_3 + \beta_4$ and $\beta_2 + 2\beta_3 + \beta_4$ in terms of the roots $\beta_1, \ldots, \beta_4$ of G, by restricting weights in [B2, Table 3]. In particular, they both involve β_4, whilst the weight of e is 0 so does not. But then, $g \in P$ forces $\nu = 0$. This shows $G_{l_2} = G_{\langle u_2\rangle}$.

We now show that $H = RZQ_0 = G_{\langle u_2\rangle}$. For, Z fixes u_2 by [B2, Lemma 4.10] and R acts by scalars. A direct check shows that K_{β_4} fixes u_2; this corresponds to the lowest weight vector of $L_{G_2}(\omega_1) = Q_0/Q_2$. Moreover, this extension is not split (it is not even abelian), so K_{β_4} generates all of Q_0 as an RZ-group. So, $H < G_{\langle u_2\rangle}$. Conversely, take $lq \in G_{\langle u_2\rangle}$ where $l \in L, q \in Q$. If $l \notin RZ$, then $G_{\langle u_2\rangle}$ contains a conjugate of L as RZQ/Q is maximal in LQ/Q. But this implies that the unipotent radical of $G_{\langle u_2\rangle}$, which contains Q_0, is all of Q. An explicit computation (using the commutator formula) shows that $K_{\beta_1+\beta_2+\beta_3+\beta_4}$ does not stabilize $\langle u_2\rangle$, so this is a contradiction. Hence, $l \in T_1Z$, so $q \in G_{\langle u_2\rangle}$, so $q \in Q_0$, so $lq \in T_1ZQ_0$. Hence, $H = G_{\langle u_2\rangle}$ as required.

It is now straightforward to deduce that there are just three orbits with representatives as given. Observe l_1, l_2 are non-conjugate degenerate 1-spaces, whilst l_3 is non-degenerate. As $SO_{25} = F_4N_1(p = 3)$ by [LSS], G has just one orbit on non-degenerate lines, and l_3 is a representative. So, we need to show that there are precisely two orbits on degenerate lines. This is proved in [CC] over finite fields by a counting argument. Copying [B2, Lemma 4.7], it is easy to deduce from the finite fields case that G can therefore have no more than two orbits on degenerate lines. But we have exhibited two disjoint orbits, and the result follows. □

4.14. **Proposition**. *There is no dense G, K-double coset in L if*

$$(L, G, K) = (SO(W), F_4, N_2)$$

where $W\downarrow_G = L_G(\omega_4)$.

Proof. Let G be the group F_4 constructed above.

First, suppose $p \neq 3$. Then W is just the module $e^\perp$, and K is the stabilizer in L of a vector $a \otimes b \in V \otimes V$ where $(a, a) = 0 = (b, b), (a, b) = 1$ and $a, b \in e^\perp$. By [CC], we can define a G-equivariant map $* : V \otimes V \to V$ by $v_1 \otimes v_2 \mapsto (v_1+v_2)^\# - v_1^\# - v_2^\#$. It is sufficient to show that there are at least two disjoint closed G-orbits in $L.(a \otimes b)$. Consider the vector $(e_{22}^2 + \eta e_{33}^2) \otimes ((1-\eta)e_{22}^3 + e_{33}^3) \in L.(a \otimes b)$. Its image under $*$ is $(\eta - 1)e_3 - \eta e_2$. This is of weight 0 relative to the maximal torus T_1 of G (by [B2, Table 3]), so as η varies we obtain infinitely many 0-weight vectors. Hence, by Lemma 1.7 and Lemma 1.4 there are infinitely many closed G-orbits in $L.(a \otimes b)$ as required.

So, now suppose $p = 3$. Let H be the group of type $T_1G_2U_{14}$ constructed above. Let $W = e^\perp/\langle e\rangle$, which is isomorphic to $L_G(\omega_4)$. Choose $a, b \in W$ with $(a, a) = 0 = (b, b), (a, b) = 1$. Suppose there is a dense G, K-double coset in L. As K is the stabilizer of $a \otimes b \in W \otimes W$, this implies by dimension (conjugating if necessary) that $G_{a\otimes b} = G_{\langle a\rangle} \cap G_{\langle b\rangle}$ is of dimension 6.

Suppose one of a, b is G-conjugate to $u_1 + \langle e\rangle$ as in Lemma 4.13; let it be a. Then, $G_{\langle a\rangle} = P$, a B_3-parabolic of dimension 37, and P lies in the parabolic subgroup P_1 of L that stabilizes $\langle a\rangle$. Hence, the P-orbit of $\langle a\rangle$ is of dimension at most $\dim P_1.\langle a\rangle = 23$. So, $\mathrm{stab}_P(\langle b\rangle)$ is certainly of dimension at least $37 - 23 > 6$, a contradiction.

Hence, we may assume both a and b are G-conjugate to $u_2 + \langle e\rangle$ by Lemma 4.13. But then, $G_{\langle a\rangle}$ and $G_{\langle b\rangle}$ are conjugate to the subgroup H of type $T_1G_2U_{14}$ by Lemma 4.13, and $G_{\langle a\rangle} \cap G_{\langle b\rangle}$ of dimension 6 implies there is a dense H, H-double coset in G. This is not the case by Lemma 4.12. □

This completes the proof of Theorem B.

5 Proof of Theorem A

We now deduce Theorem A from Theorem B and [B2]. Recall the notation $\mathcal{R}(G)$ from the introduction. Observe initially that if $H \in \mathcal{R}(G)$ and $K \in \mathcal{R}(H)$ then $K \in \mathcal{R}(G)$. Of course, in characteristic 0, $\mathcal{R}(G)$ contains all reductive subgroups of G. In general, $\mathcal{R}(G)$ will contain very many but not necessarily all reductive subgroups of G. The main difficulty in deducing Theorem A is to show that $\mathcal{R}(G)$ is "closed under intersections" in the following sense: if $H, K \in \mathcal{R}(G)$ give rise to a factorization $G = HK$ then $H \cap K$ is in both $\mathcal{R}(H)$ and $\mathcal{R}(K)$. This need not be the case if HK is not a factorization of G, even if the intersection is reductive.

We begin with an elementary lemma.

5.1. **Lemma**. *Let $H, K \leq G$ and $Z \leq H$. If $x \in H$, then $Zx(H \cap K)$ is dense in H if and only if ZxK is dense in HK.*

PROOF. The bijective morphism $H/H \cap K \to HK/K$ defined by $x(H \cap K) \mapsto xK$ for $x \in H$ is a homeomorphism; for [H1, p56, ex. 4] implies that it is an open map. Hence, $Zx(H \cap K)$ is dense in H if and only if ZxK is dense in HK. □

We say a semisimple group G is of *length* n if G can be written as $G_1 \ldots . G_n$ as a commuting product of n simple factors. If $G = G_1G_2$ as a commuting product of reductive (but not necessarily simple) factors, we say a subgroup $H < G$ is *diagonally embedded* if the projections $\pi_i : H \to G_i$ are bijective for each i. Note that if H is diagonally embedded in G, then $H \in \mathcal{R}(G)$ – if the G_i are simple then H is maximal in G and in the general case it is easy to write down a chain $H = H_0 < H_1 < \cdots < H_n = G$ with H_i maximal in H_{i+1} from this observation. We now consider diagonally embedded subgroups. The first lemma is well known.

5.2. **Lemma ([B3, Lemma 4.4])**. *Let G be a semisimple algebraic group of adjoint type of length at least 2. If $H \in \mathcal{M}(G)$ then one of the following holds:*
(i) *Some simple factor $1 \neq G_1 \lhd G$ is contained in H.*
(ii) *G is of length 2 and H is diagonally embedded in G.*

We now consider double cosets of diagonally embedded subgroups as in Lemma 5.2(ii). We begin with a preliminary lemma.

5.3. **Lemma**. *Let G, G_1 and G_2 be simple algebraic groups, where G_1 and G_2 are of adjoint type. Suppose that $\pi_i : G \to G_i$ is a bijective morphism (but not necessarily an isomorphism of algebraic groups) for $i = 1, 2$. Then, one of $\pi_1 \circ {\pi_2}^{-1} : G_2 \to G_1$ or $\pi_2 \circ \pi_1^{-1} : G_1 \to G_2$ is a morphism.*

PROOF. Let $(T, B; \Sigma, \Pi)$ be a root system for G. For each i, let $B_i = \pi_i(B), T_i = \pi_i(T)$ for each i, hence defining a root system $(T_i, B_i; \Sigma_i, \Pi_i)$ for G_i. Let $E = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and $E_i = X(T_i) \otimes_{\mathbb{Z}} \mathbb{R}$, and fix inner products on E and E_i invariant under the corresponding Weyl group. Given non-zero vectors v and w in either of these spaces, let $R(v, w)$ denote the ratio of their lengths.

Denote the root subgroups of G and G_i by $U_\alpha(\alpha \in \Sigma)$ and $U_{i,\alpha}(\alpha \in \Sigma_i)$ respectively. The map $\pi_i : G \to G_i$ sends the set of root subgroups of G bijectively onto the root subgroups of G_i, hence induces a bijection $\theta_i : \Sigma \to \Sigma_i$ such that $\pi_i(U_\alpha) = U_{i,\theta_i(\alpha)}$ for all $\alpha \in \Sigma$. Let $\bar{\pi}_i : T \to T_i$ denote the restriction of π_i to T, and let $\pi_i^* : X(T_i) \to X(T)$ denote the associated comorphism. Since $\bar{\pi}_i$ is surjective, π_i^* is injective. Fix parametrizations $x_\alpha : k \to U_\alpha(\alpha \in \Sigma)$ and $x_{i,\alpha} : k \to U_{i,\alpha}(\alpha \in \Sigma_i)$ of the root subgroups. As π_i is a morphism, $\pi_i(x_\alpha(t)) = x_{i,\theta_i(\alpha)}(c_{i,\alpha}t^{q_{i,\alpha}})$ for coefficients $c_{i,\alpha} \in k^\times$ and certain powers $q_{i,\alpha}$ of p, depending on $\alpha \in \Sigma$ [if $p = 0$, $q_{i,\alpha} = 1$ always].

We claim that either $q_{1,\alpha} \geq q_{2,\alpha}$ for all $\alpha \in \Sigma$, or $q_{1,\alpha} \leq q_{2,\alpha}$ for all $\alpha \in \Sigma$. Well, otherwise, $p \neq 0$ and we can find $\alpha, \beta \in \Sigma$ such that $q_{1,\alpha} > q_{2,\alpha}$ and $q_{1,\beta} < q_{2,\beta}$. A simple calculation shows that $\pi_i^*(\theta_i(\alpha)) = q_{i,\alpha}\alpha$ for all $\alpha \in \Sigma$, so $R(\theta_i(\alpha), \theta_i(\beta)) = R(q_{i,\alpha}\alpha, q_{i,\beta}\beta) = \frac{q_{i,\alpha}}{q_{i,\beta}} R(\alpha, \beta)$ for each i. So,

$$\frac{R}{p} \geq \frac{q_{1,\beta}}{q_{2,\beta}} R(\theta_1(\alpha), \theta_1(\beta)) = \frac{q_{1,\alpha}}{q_{2,\alpha}} R(\theta_2(\alpha), \theta_2(\beta)) \geq \frac{p}{R},$$

where R is the ratio of a long root to a short root in Σ. This implies that $R^2 \geq p^2 \geq 4$ which is a contradiction as R^2 equals $1, 2$ or 3.

So, without loss of generality, assume that $q_{1,\alpha} \geq q_{2,\alpha}$ for all $\alpha \in \Sigma$. We claim in that case that $\pi_1 \circ \pi_2^{-1}$ is a morphism. Arguing as in [LS, Lemma 1.2], it suffices to check that the restriction of $\pi_1 \circ \pi_2^{-1}$ to each root subgroup and to T_2 is a morphism. On the root subgroup $U_{2,\theta_2(\alpha)}$ for $\alpha \in \Sigma$, $\pi_1 \circ \pi_2^{-1}$ is the map $x_{2,\theta_2(\alpha)}(t) \mapsto x_{1,\theta_1(\alpha)}(c_\alpha t^{q_\alpha})$ where $c_\alpha = c_{1,\alpha}/c_{2,\alpha}$ and $q_\alpha = q_{1,\alpha}/q_{2,\alpha}$, which is a morphism as $q_{1,\alpha} \geq q_{2,\alpha}$ and both are powers of p [or 1 if $p = 0$]. It remains to check that the restriction of $\pi_1 \circ \pi_2^{-1}$ to T_2 is a morphism. Now, $q_{1,\alpha}$ is an integer multiple of $q_{2,\alpha}$. So, $\pi_1^*(\Sigma_1)$ is contained in $\pi_2^*(\Sigma_2)$. As G_1 is of adjoint type, $X(T_1)$ is generated as an abelian group by Σ_1 so in fact $\pi_1^*(X(T_1))$ is a subgroup of $\pi_2^*(X(T_2))$. Hence, there is a well-defined homomorphism of abelian groups $(\pi_2^*)^{-1} \circ \pi_1^* : X(T_1) \to X(T_2)$. This induces a morphism $T_2 \to T_1$ of tori, which necessarily equals the restriction of $\pi_1 \circ \pi_2^{-1}$ to T_2. Consequently, this restriction is a morphism, completing the proof. $\square$

5.4. **Proposition**. *Suppose $G = G_1G_2$ is a semisimple, adjoint algebraic group of length 2. If $H, K < G$ are diagonally embedded subgroups, then either $G = HK$ or there is no dense H, K-double coset in G.*

Proof. We may conjugate to assume that the double coset HK is closed. Let $\pi_i : H \to G_i$ and $\delta_i : K \to G_i$ be the projections, bijective morphisms for each i. By Lemma 5.3, one of $\pi_1 \circ \pi_2^{-1}$ or $\pi_2 \circ \pi_1^{-1}$ is a morphism; assume without loss of generality that $\pi_2 \circ \pi_1^{-1}$ is a morphism. Then, either $\delta_1 \circ \delta_2^{-1}$ is a morphism, in which case $\delta_1 \circ \delta_2^{-1} \circ \pi_2 \circ \pi_1^{-1}$ is a morphism, or both $\pi_2 \circ \pi_1^{-1}$ and $\delta_2 \circ \delta_1^{-1}$ are bijective morphisms from G_1 to G_2. In the latter case, Lemma 5.3 again implies that either $\pi_2 \circ \pi_1^{-1} \circ \delta_1 \circ \delta_2^{-1}$ or $\delta_2 \circ \delta_1^{-1} \circ \pi_1 \circ \pi_2^{-1}$ is a morphism. We have shown that at least one of the maps

$$\pi_1 \circ \pi_2^{-1} \circ \delta_2 \circ \delta_1^{-1}, \pi_2 \circ \pi_1^{-1} \circ \delta_1 \circ \delta_2^{-1}, \delta_1 \circ \delta_2^{-1} \circ \pi_2 \circ \pi_1^{-1} \text{ or } \delta_2 \circ \delta_1^{-1} \circ \pi_1 \circ \pi_2^{-1}$$

is a morphism.

Now assume without loss of generality that $\pi_1 \circ \pi_2^{-1} \circ \delta_2 \circ \delta_1^{-1}$ is a morphism, the other three cases being entirely similar. Let $\theta = \pi_2 \circ \pi_1^{-1} : G_1 \to G_2$ and $\phi = \delta_2 \circ \delta_1^{-1} : G_1 \to G_2$, both isomorphisms of abstract groups. Then, $H = \{g\theta(g) \mid g \in G_1\}$ and $K = \{g\phi(g) \mid g \in G_1\}$. Hence, $H \cap K = \{g\theta(g) \mid g \in G_1, \theta(g) = \phi(g)\} \cong G_1^\sigma$ where $\sigma = \theta^{-1} \circ \phi$ is an abstract automorphism of G_1 which by assumption is a morphism of algebraic groups. So now we can apply [S1, 10.13] to deduce that there are two distinct possibilities:

(i) G_1^σ is finite. Then, $\dim HK = \dim H + \dim K - \dim H \cap K = \dim G$. So HK is dense and closed, so $G = HK$ is a factorization.

(ii) σ is an algebraic automorphism of G_1. Consider an arbitrary double coset HhK with $h \in G_1$, with stabilizer $h^{-1}Hh \cap K = \{g\theta(g) \mid g \in G_1, \theta(hgh^{-1}) = \phi(g)\}$. Let $\mathrm{Int}(h)$ be the inner automorphism of G_1 defined by $h \in G_1$. Then, $h^{-1}Hh \cap K$ is isomorphic to the fixed points of $\mathrm{Int}(h)^{-1} \circ \sigma$ in G_1. But this is an algebraic automorphism of G_1, so again by [S1, 10.13], the set of fixed points $h^{-1}Hh \cap K$ is of positive dimension. Hence, HhK is not dense in G, for all $h \in G_1$. □

5.5. **Remark**. This proof gives rise to some interesting 'diagonal' factorizations. For example, let $G = G_1G_2$ be a product of two isomorphic simple factors, where $\theta : G_1 \to G_2$ is an isomorphism. Let $\sigma : G_1 \to G_1$ be a Frobenius automorphism of G_1 ($p \neq 0$). Then, $G = HK$ is a factorization if $H = \{g\theta(g) \mid g \in G_1\}$ and $K = \{g\theta(\sigma(g)) \mid g \in G_1\}$. The existence of these factorizations can also be proved using Lang's theorem.

5.6. **Lemma**. *Let G be simple, and suppose $G = HK$ where $H, K \in \mathcal{M}(G)$. Then $H \cap K$ is in both $\mathcal{R}(H)$ and $\mathcal{R}(K)$.*

PROOF. We verify this explicitly for each entry in table 1. First, we claim that if $H < G$ is a connected reductive subgroup normalized by some maximal torus T of G then $H \in \mathcal{R}(G)$. For, clearly $H \in \mathcal{R}(HT)$, so we may assume $T < H$. Moreover, by induction, we may assume H is a maximal connected reductive subgroup. Then, either H has some central torus, so that H is a Levi factor, or H is a maximal connected subgroup. In either case $H \in \mathcal{M}(G)$, proving the claim. In particular, this observation proves the lemma if G is exceptional, when the possible factorizations are determined in [B2], or if $(G, H, K, p) = (Sp_{2n}, N_i, SO_{2n}, 2)$, since in either case both H, K are maximal rank, so $H \cap K$ is of maximal rank in both H, K. We now consider the remaining cases in table 1; it is sufficient to do this up to graph automorphisms of G.

(i) We first consider the first three entries in table 1. Here the intersections are given in the table below.

G	H	K	$(H \cap K)^0$
SL_{2n}	Sp_{2n}	GL_{2n-1}	T_1Sp_{2n-2}
SO_{2n}	SO_{2n-1}	GL_n	GL_{n-1}
SO_{4n}	SO_{4n-1}	$Sp_2 \otimes Sp_{2n}$	$Sp_2 \times Sp_{2n-2}$

In the first two cases, $H \cap K$ and its embedding in H and K is straightforward to compute and the result follows. In the third case, we can find a subgroup $Z \cong Sp_2 \times Sp_{2n-2}$ of G such that, if V is the natural module for G,

$$\begin{aligned} V{\downarrow_Z} &\cong L_Z(\omega_1, 0) \otimes (L_Z(\omega_1, 0) \oplus L_Z(0, \omega_1)) \\ &\cong (L_Z(\omega_1, 0) \otimes L_Z(\omega_1, 0)) \oplus (L_Z(\omega_1, 0) \otimes L_Z(0, \omega_1)). \end{aligned}$$

From the first isomorphism here, we see that Z is a subgroup of $K = Sp_2 \otimes Sp_{2n}$, and there is a chain of embeddings $Sp_2 \times Sp_{2n-2} < Sp_2 \otimes (Sp_2Sp_{2n-2}) < Sp_2 \otimes Sp_n$, each maximal in the next, proving that $Z \in \mathcal{R}(K)$. On the other hand, from the second isomorphism, we see that $Z < SO_4SO_{4n-4}$. Now, the diagonal subgroup $Sp_2 < Sp_2 \otimes Sp_2 < SO_4$ fixes a non-singular line in the 4-dimensional orthogonal space (if e, f is a symplectic basis then Sp_2 fixes $e \otimes f - f \otimes e$), and this identifies Sp_2 with the subgroup $SO_3 < SO_4$. Hence, we have a chain of embeddings $Sp_2 \times Sp_{2n-2} < Sp_2(Sp_2 \otimes Sp_{2n-2}) = SO_3(Sp_2 \otimes Sp_{2n-2}) <$

$SO_3SO_{4n-4} < SO_{4n-1}$ proving that $Z \in \mathcal{R}(H)$. Finally, dimension implies Z is indeed equal to $(H \cap K)^0$.

(ii) We next consider the factorizations involving G_2 or B_3 in table 1. By applying graph automorphisms to $G = PSO_8$, it is sufficient to prove the lemma for the cases in the table below – we have already treated the cases $(G, H, K) = (SO_8, N_1, GL_4)$ and $(SO_8, N_1, Sp_2 \otimes Sp_4)$ in (i).

G	p	H	K	$(H \cap K)^0$
Sp_6	$p = 2$	SO_6	G_2	A_2
SO_7	$p \neq 2$	N_1	G_2	A_2
Sp_6	$p = 2$	N_2	G_2	$A_1\tilde{A}_1$
SO_7	$p \neq 2$	N_2	G_2	A_1T_1
SO_8		N_1	B_3 or ${}^\tau B_3$	G_2

In the first two cases, the intersection must have dimension 8, so the only possibility is the long root subgroup A_2 of G_2 (or possibly $\tilde{A}_2$ if $p = 3$); this is of maximal rank in G_2 so certainly lies in $\mathcal{R}(K)$. The embedding $A_2 < SO_6$ is well known; it is $A_2 < GL_3 < SO_6$, so $H \cap K \in \mathcal{R}(H)$. In the final case, G_2 fixes a non-singular 1-space in $L_{B_3}(\omega_3)$ so $N_1 \cap B_3$ contains G_2, hence equals G_2 by dimension. Here, $H \cap K$ is maximal in both H and K, so the result follows.

To consider the third and fourth cases, let V be the natural module for G and compute the restriction $V\downarrow_{A_1\tilde{A}_1}$, where $A_1\tilde{A}_1$ is the maximal subgroup of G_2. By considering weights, the restriction splits as $L_{\tilde{A}_1}(2\omega_1) \oplus (L_{A_1}(\omega_1) \otimes L_{\tilde{A}_1}(\omega_1))$. Hence, if $p = 2$, $A_1\tilde{A}_1 \leq N_2 \cap G_2$, hence equals the intersection by dimension. This is of maximal rank in G_2, so we just need to show $H \cap K \in \mathcal{R}(H)$. For this, there is a chain of subgroups $A_1\tilde{A}_1 < Sp_2SO_4 < Sp_2Sp_4 = N_2$ with each maximal in the next, proving the result. Finally, if $p \neq 2$ then the subgroup $A_1T_1 < A_1\tilde{A}_1$ is of the correct dimension to be the intersection $(H \cap K)^0$, and is a maximal rank subgroup of G_2. So we just need to show that $A_1T_1 \in \mathcal{R}(H)$. Here it is clear from the structure of $V\downarrow_{A_1\tilde{A}_1}$ that $A_1T_1 < GL_2T_1 < SO_4T_1 < SO_5T_1 = N_2$ is a chain of embeddings proving $A_1T_1 \in \mathcal{R}(H)$.

(iii) For remaining cases, the intersection is computed in [LSS, Proposition 1.9].

G	p	H	K	$(H \cap K)^0$
SO_{56}	$p = 2$	E_7	N_1	E_6
SO_{32}	$p = 2$	D_6	N_1	A_5
SO_{25}	$p = 3$	F_4	N_1	D_4
SO_{20}	$p = 2$	A_5	N_1	A_2A_2
SO_{16}		B_4	N_1	B_3
SO_{13}	$p = 3$	C_3	N_1	$A_1A_1A_1$

In each case, it is shown in [LSS] that $(H \cap K)^0$ is normalized by some maximal torus of H, so it just remains to show $H \cap K \in \mathcal{R}(K)$. For this, we just exhibit a chain of subgroups proving that $H \cap K \in \mathcal{R}(K)$, leaving the details to the reader; for this, the embeddings are explained in more detail in [LSS, Proposition 1.9].

$E_6 \overset{\omega_1}{<} SL_{27} < GL_{27} < SO_{54} < SO_{55} = N_1$,

$A_5 \overset{\omega_2}{<} SL_{15} < GL_{15} < SO_{30} < SO_{31} = N_1,$
$D_4 \overset{\omega_1,\omega_3,\omega_4}{<} SO_8SO_8SO_8 < SO_8SO_{16} < SO_{24} = N_1,$
$A_2A_2 \overset{\omega_1\otimes\omega_2}{<} SL_9 < GL_9 < SO_{18} = N_1,$
$B_3 \overset{\omega_1,\omega_3}{<} SO_7SO_7 < SO_7SO_8 < SO_{15} = N_1,$
$A_1A_1A_1 < SO_4SO_4SO_4 < SO_4SO_8 < SO_{12} = N_1.$

In the last case here, the embedding $A_1A_1A_1 < SO_4SO_4SO_4$ is such that the highest weights of $A_1A_1A_1$ on the SO_4 factors are $\omega_1 \otimes \omega_1 \otimes 0, \omega_1 \otimes 0 \otimes \omega_1, 0 \otimes \omega_1 \otimes \omega_1$ respectively. □

5.7. **Lemma**. *Suppose G is a connected reductive algebraic group and $H, K \in \mathcal{M}(G)$ are such that $G = HK$ is a factorization. Then, $H \cap K$ is in both $\mathcal{R}(H)$ and $\mathcal{R}(K)$.*

Proof. We prove that $H \cap K \in \mathcal{R}(K)$ by induction on $\dim G$, the case $G = 1$ being trivial. First suppose G is semisimple. If G is of length 1, then the result is precisely Lemma 5.6. So, G is of length greater than 1. Also, we may replace G by the corresponding adjoint group $\operatorname{Ad} G$ as $\operatorname{Ad} \mathcal{R}(K) = \mathcal{R}(\operatorname{Ad} K)$. So by Lemma 5.2 there are two cases:

(i) Some simple factor $1 \neq G_1 \lhd G$ is contained in H. Then, $H/G_1 \in \mathcal{M}(G/G_1)$. Notice that either $KG_1/G_1 = G/G_1$ (which will not cause problems) or $KG_1/G_1 \in \mathcal{M}(G/G_1)$. By induction, $(H/G_1) \cap (KG_1/G_1) \in \mathcal{R}(KG_1/G_1)$. Now, $H \cap (KG_1) = (H \cap K)G_1$. Let $A = H \cap K$ and $B = AG_1/G_1$. Then as $B \in \mathcal{R}(KG_1/G_1)$, there is a chain of subgroups $B^0 = B_0 < B_1 < \cdots < B_n = KG_1/G_1$ with $B_i \in \mathcal{M}(B_{i+1})$ for each i. Let A_i be the connected pre-image of B_i under $q : K \to KG_1/G_1$. Then, $A_0 = ((AG_1) \cap K)^0 = (H \cap (KG_1) \cap K)^0 = (H \cap K)^0 = A$. So, we obtain a chain of subgroups $A = A_0 < A_1 < \cdots < A_n = K$ with each $A_i \in \mathcal{M}(A_{i+1})$. So, $H \cap K \in \mathcal{R}(K)$ as required.

(ii) G is of length 2 and H is diagonally embedded in G. Again by Lemma 5.2 there are two cases for K. First, suppose K is also diagonally embedded. Then, $\dim H \cap K = \dim H + \dim K - \dim G = 0$ so $H \cap K$ is finite and the result follows. Otherwise, some simple factor of G is contained in K. Without loss of generality, suppose $G = G_1G_2$ with G_i simple and that $G_2 \leq K$. Then, $K = K_1G_2$ with $K_1 \in \mathcal{M}(G_1)$. Let $\pi_i : H \to G_i$ be the projections, bijective morphisms. Then, $H \cap K = \pi_1^{-1}K_1$. Let $K_2 = \{\pi_1(x_1)\pi_2(x_2) \mid x_1, x_2 \in H \cap K\}$. Then, $H \cap K$ is diagonally embedded in K_2, so that $H \cap K \in \mathcal{R}(K_2)$, and $K_2 \in \mathcal{M}(K)$. Hence, $H \cap K \in \mathcal{R}(K)$.

Finally, suppose G is reductive and not semisimple. Let $R \neq 1$ be the radical of G. If $HR = G$ then $G' \leq H$ so $K' \leq H \cap K \leq K$ and it is clear from this that $H \cap K \in \mathcal{R}(K)$. So, we may assume $HR \neq G$, so that by maximality, $HR = H$. Then the argument of (i) (with $G_1 = R$) gives the result. □

We are finally in a position to prove that $\mathcal{R}(G)$ is closed under intersections in full generality.

5.8. **Proposition**. *Suppose G is a reductive algebraic group and $H, K \in \mathcal{R}(G)$ are such that $G = HK$ is a factorization. Then, $H \cap K$ is in both $\mathcal{R}(H)$ and $\mathcal{R}(K)$.*

Proof. Again we use induction on $\dim G$, the case G finite being trivial. We may assume G, H, K are connected, so let $H, K \in \mathcal{R}(G)$ be connected. We may embed $H \leq H_1, K \leq K_1$

with $H_1, K_1 \in \mathcal{M}(G)$ and $H \in \mathcal{R}(H_1), K \in \mathcal{R}(K_1)$. Then, $G = HK$ implies

$$H_1 = H(H_1 \cap K), K_1 = (H \cap K_1)K.$$

In particular, $H_1 = H(H_1 \cap K_1)$. By Lemma 5.7 $H_1 \cap K_1 \in \mathcal{R}(H_1)$, and $H \in \mathcal{R}(H_1)$ by definition, so induction implies that $H \cap K_1$ is in both $\mathcal{R}(H)$ and $\mathcal{R}(H_1 \cap K_1)$. Similarly, $H_1 \cap K$ is in both $\mathcal{R}(K)$ and $\mathcal{R}(H_1 \cap K_1)$.

Now, $H_1 = H(H_1 \cap K)$ implies $H_1 \cap K_1 = (H \cap K_1)(H_1 \cap K)$. We have shown that $H \cap K_1$ and $H_1 \cap K$ are in $\mathcal{R}(H_1 \cap K_1)$. Hence, by induction, $H \cap K$ is in both $\mathcal{R}(H \cap K_1)$ and $\mathcal{R}(H_1 \cap K)$. The result follows as $H \cap K_1$ is in $\mathcal{R}(H)$ and $H_1 \cap K$ is in $\mathcal{R}(K)$. □

Now we can prove Theorem A. For convenience, we restate the theorem:

5.9. **Theorem**. *Let G be a connected reductive algebraic group, and take $H, K \in \mathcal{R}(G)$. Then, either $G = HK$ or there is no dense H, K-double coset in G.*

Proof. We prove this by induction on $\dim G$; the induction starts with $G = 1$. So, let G be a connected reductive group of dimension d and suppose the result holds for all groups of dimension less than d.

(i) We first show that the result holds if $H, K \in \mathcal{M}(G)$ and G is semisimple. If G is of length 1, then the result holds by Theorem B or by [B2] if G is exceptional. So suppose G is of length greater than 1. By Lemma 1.1, we may assume G is adjoint, so Lemma 5.2 applies. Suppose first that some simple factor $1 \neq G_1$ of G is contained in H. Then, if $G_1 K = G$, $G = HK$ and the result follows. Otherwise, maximality implies $G_1 K = K$ and the result follows by induction from the case $(G/G_1, H/G_1, K/G_1)$. A similar argument applies if some simple factor of G is contained in K. Hence, both H, K are diagonally embedded and G is of length 2. Now the result follows by Proposition 5.4.

(ii) We now show that the result holds if $H, K \in \mathcal{M}(G)$ and G is reductive. Let R be the radical of G and conjugate to assume that HK is closed. Then, if $HKR = G$ either $HK = G$, as required, or we can find $r \in R - HK$. In this case, $HrK = HKr$ is also closed and disjoint from HK. Hence, there are at least two disjoint closed H, K-double cosets in G and the result follows by Lemma 1.2. So, we may assume $HKR \neq G$. Then, $HR \neq G$ and $KR \neq G$, so by maximality, $HR = H$, $KR = K$. Then, the result follows by induction from the case $(G/R, H/R, K/R)$.

(iii) We now show that the result holds if $H, K \in \mathcal{R}(G)$ and G is reductive. Observe that by Lemma 1.1, we may assume H, K are connected. So, let $H, K \in \mathcal{R}(G)$ be connected. Then, we may embed $H \leq H_1, K \leq K_1$ with $H_1, K_1 \in \mathcal{M}(G)$ such that $H \in \mathcal{R}(H_1), K \in \mathcal{R}(K_1)$. If $G \neq H_1 K_1$, then there is no dense H_1, K_1-double coset in G by (ii) so the result holds. So, suppose $G = H_1 K_1$; if $H = H_1, K = K_1$, then $G = HK$ and we are done. So, assume without loss of generality that $H \neq H_1$. Since $G = H_1 K_1$, Proposition 5.8 implies $H_1 \cap K_1 \in \mathcal{R}(H_1)$. So by induction either $H_1 = H(H_1 \cap K_1)$ or there is no dense $H, H_1 \cap K_1$-double coset in H_1. In the latter case Lemma 5.1 implies there is no dense H, K_1-double coset in $H_1 K_1 = G$ and the result follows. In the former case, $G = H_1 K_1 = HK_1$. Hence, by Proposition 5.8, $H \cap K_1 \in \mathcal{R}(K_1)$, and also $K \in \mathcal{R}(K_1)$ by definition. So by induction, either $K_1 = (H \cap K_1)K$ or there is no dense $H \cap K_1, K$-double coset in K_1. But the former case implies $G = HK_1 = HK$, and in the latter case there is no dense H, K-double coset in $HK_1 = G$ by Lemma 5.1. □

This completes the proof of the main results of the paper.

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