Some Remarks on Branching Rules and Tensor Products for Algebraic Groups*

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1 Introduction and Preliminaries

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$. In [BK1, BK2], we have revealed and exploited various relations between the branching rules from $GL_n(\mathbb{F})$ to its Levi subgroups on one hand, and decompositions of tensor products over $GL_n(\mathbb{F})$ itself on the other. For example, if L is some irreducible rational $GL_n(\mathbb{F})$-module and V is the natural $GL_n(\mathbb{F})$-module, there is a close relationship between the highest weight vectors (relative to $GL_{n-1}(\mathbb{F})$) in the restriction $L\downarrow_{GL_{n-1}(\mathbb{F})}$ and the highest weight vectors (relative to $GL_n(\mathbb{F})$) in the tensor product $L \otimes V^*$. In this paper we obtain more results in this direction, some of which are valid for an arbitrary type.

To describe our main results, we adopt standard Lie theoretic notation. Let G be a (connected) reductive algebraic group over $\mathbb{F}$. As in [J], R denotes the root system of G with respect to a fixed maximal torus T, $R^+ \subset R$ denotes the set of positive roots determined by a choice of Borel subgroup B^+ containing T, and $\{\alpha_1, \dots, \alpha_\ell\} \subset R^+$ is the corresponding base for R. Denote the highest (long) root of R by α_0 and the longest element of the Weyl group $W = N_G(T)/T$ by w_0. We write $X(T)$ for the character group $\mathrm{Hom}(T, \mathbb{F}^\times)$, $Y(T)$ for the cocharacter group $\mathrm{Hom}(\mathbb{F}^\times, T)$ and let $\langle\cdot,\cdot\rangle$ be the natural pairing $X(T) \times Y(T) \to \mathbb{Z}$. For $\alpha \in R$, $\alpha^\vee$ denotes the corresponding coroot in $Y(T)$, and $X^+(T)$ denotes the set $\{\lambda \in X(T) \mid \langle\lambda, \alpha_i^\vee\rangle \geq 0, i = 1, \dots, \ell\}$ of dominant weights.

All G-modules are assumed to be rational. For $\lambda \in X^+(T)$, we have the G-modules $L(\lambda)$, $\Delta(\lambda)$ and $\nabla(\lambda)$, which are the irreducible, the standard (or Weyl), and the costandard G-modules with highest weight λ. Let $\mathrm{Dist}(G)$ be the algebra of distributions of G as in [J, I.7], which is generated by $\mathrm{Dist}(T)$ and the 'divided power' root generators $X_\alpha^{(n)}, Y_\alpha^{(n)}$ for $\alpha \in R^+, n \geq 1$. Write $X_i^{(n)} = X_{\alpha_i}^{(n)}, Y_i = Y_{\alpha_i}^{(n)}$ for $i = 1, \dots, \ell$. If G is semisimple and simply connected (which we may assume for the proofs), $\mathrm{Dist}(G)$ coincides with the hyperalgebra of G arising from the Chevalley construction. We note that any G-module is a $\mathrm{Dist}(G)$-module in a natural way; see [J, I.7.11, II.1.20].

Given a weight $\nu \in X(T)$ and a G-module M, M_ν will denote the ν-weight space of M. If in addition $\mu \in X^+(T)$ is a dominant weight, we define

$$M^\mu := \{v \in M \mid X_i^{(b_i)} v = 0 \text{ for all } b_i > \langle\mu, \alpha_i^\vee\rangle \text{ and } i = 1, 2, \dots, \ell\}$$

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and let $M^\mu_\nu := M^\mu \cap M_\nu$ denote its ν-weight space. Our first result generalizes a well known fact in characteristic 0 which goes back to Kostant, see [PRV, Theorem 2.1] for a proof in that case. The proof in characteristic p is essentially the same.

Theorem A. *Let $\lambda, \mu \in X^+(T)$, and M be any G-module. Then*

$$\mathrm{Hom}_G(\Delta(\lambda), M \otimes \nabla(\mu)) \cong M^\mu_{\lambda-\mu}.$$

To explain our interest in the theorem, suppose that $M = L(\nu)$ is an irreducible module for some fixed $\nu \in X^+(T)$. Then, for μ large relative to ν, we see that $M^\mu_{\lambda-\mu} = M_{\lambda-\mu}$, so by the theorem, $L(\nu)_{\lambda-\mu} \cong \mathrm{Hom}_G(\Delta(\lambda), L(\nu) \otimes \nabla(\mu))$. So to compute the formal character of $L(\nu)$ it suffices to describe the Hom space in Theorem A for λ, μ large. In view of the universality of standard modules, this is equivalent to describing the highest weight vectors of weight λ in $L(\nu) \otimes \nabla(\mu)$.

We note that $\mathrm{Hom}_G(\Delta(\lambda), L(\nu) \otimes \nabla(\mu)) \cong \mathrm{Hom}_G(L(\nu^*), \nabla(\lambda^*) \otimes \nabla(\mu))$ where ν^*, λ^* are the dual dominant weights; its dimension is precisely the multiplicity of $L(\nu^*)$ in the socle of $\nabla(\lambda^*) \otimes \nabla(\mu)$. Our next result reveals some extra structure related to restricted weights of the socle of such tensor products. Recall that a dominant weight λ is called p^r*-restricted* if $\langle \lambda, \alpha_i^\vee \rangle < p^r$ for all $i = 1, 2, \dots, \ell$. A semisimple module will be called p^r-restricted if all of its composition factors have p^r-restricted highest weights.

Theorem B. *Let $\mu, \nu \in X^+(T)$ and $\alpha_0 \in R$ be the highest root. If μ is p^r-restricted and $\langle \nu, \alpha_0^\vee \rangle < p^r$ then the socle of $\nabla(\mu) \otimes \nabla(\nu)$ is p^r-restricted.*

In particular, we note that any *miniscule weight* ν satisfies the condition in Theorem B for all r. Theorem B is false if we weaken the assumption $\langle \nu, \alpha_0^\vee \rangle < p^r$ to assume only that ν is p^r-restricted; see Remark 3.5 for a counterexample in this case.

Now we specialize to the case that $G = GL(n) = GL_n(\mathbb{F})$. As usual, take T to be all diagonal matrices in $GL(n)$ and B^+ to be all upper triangular matrices. We identify the weight lattice $X(T)$ with the set $X(n)$ of all n-tuples $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of integers, λ corresponding to the character $\mathrm{diag}(t_1, \dots, t_n) \mapsto t_1^{\lambda_1} \dots t_n^{\lambda_n}$, and $X^+(T)$ with the set $X^+(n) = \{\lambda \in X(n) \mid \lambda_1 \geq \dots \geq \lambda_n\}$. We write $L_n(\lambda)$, $\Delta_n(\lambda)$, $\nabla_n(\lambda)$ for the irreducible, standard and costandard modules, and ε_i denotes the weight $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the ith position.

The connection between Theorem A and our earlier results [BK1, BK2] arises as follows. Embed $GL(n-1)$ into the top left hand corner of $GL(n)$. If $\mu = -\ell\varepsilon_n$ for $\ell \geq 0$, the space $M^\mu_{\lambda-\mu}$ appearing in Theorem A is precisely the space of vectors in $M_{\lambda-\mu}$ which are highest weight vectors with respect to the subgroup $GL(n-1)$, satisfying in addition $X^{(b)}_{n-1} v = 0$ for any $b > \ell$. By directly constructing the isomorphism appearing in Theorem A, we obtain the following extension of Theorem A to irreducible modules in one important special case.

Theorem C. *Fix $\lambda, \mu \in X^+(n)$ with $\lambda_n = \mu_n$. For any submodule M of $\nabla_n(\lambda)$,*

$$\mathrm{Hom}_{GL(n)}(L_n(\mu), M \otimes \nabla_n(-\ell\varepsilon_n)) \cong \mathrm{Hom}_{GL(n-1)}(L_{n-1}(\bar{\mu}), M \downarrow_{GL(n-1)})$$

where $\ell = \sum_{i=1}^n (\lambda_i - \mu_i)$ and $\bar{\mu} = (\mu_1, \dots, \mu_{n-1})$ denotes the restriction of μ to $T \cap GL(n-1)$.

We believe it is an important problem to describe the socle of $L_n(\lambda) \downarrow_{GL(n-1)}$ (which appears in Theorem C if $M = L_n(\lambda)$), for any $\lambda \in X^+(n)$. We refer to this problem as the *modular branching problem for the general linear group.* A complete answer only exists in some special cases, namely, the 'first level' and when $L_n(\lambda) \downarrow_{GL(n-1)}$ is semisimple; see [K4, B1, BKS]. By the known characteristic 0 branching rule, together with basic properties of good filtrations, the space $\mathrm{Hom}_{GL(n-1)}(\Delta_{n-1}(\mu), \nabla_n(\lambda) \downarrow_{GL(n-1)})$ is 0 unless $\lambda_{i+1} \leq \mu_i \leq \lambda_i$ for $i = 1, \dots, n-1$, when it is 1-dimensional. Hence, each of the three spaces

$$\mathrm{Hom}_{GL(n-1)}(\Delta_{n-1}(\mu), L_n(\lambda) \downarrow_{GL(n-1)}),$$
$$\mathrm{Hom}_{GL(n-1)}(L_{n-1}(\mu), \nabla_n(\lambda) \downarrow_{GL(n-1)}),$$
$$\mathrm{Hom}_{GL(n-1)}(L_{n-1}(\mu), L_n(\lambda) \downarrow_{GL(n-1)})$$

are at most 1-dimensional, the last of which computes the socle. Moreover, the last Hom space is non-zero if and only if both of the first two are non-zero.

Our final result, which is a consequence of Theorem C, reduces the problem of calculating any of the three Hom spaces to just the first. We are not aware of a direct proof of Theorem D working only with branching rules.

Theorem D. *Fix $\lambda \in X^+(n)$ and $\mu \in X^+(n-1)$ such that $\lambda_{i+1} \leq \mu_i \leq \lambda_i$ for $1 \leq i \leq n-1$. Let $\overline{\lambda^*} = (-\lambda_n, -\lambda_{n-1}, \dots, -\lambda_2) \in X^+(n-1)$ and $\tilde{\mu}^* = (-\lambda_n, -\mu_{n-1}, \dots, -\mu_1) \in X^+(n)$. Then,*

$$\mathrm{Hom}_{GL(n-1)}(L_{n-1}(\mu), \nabla_n(\lambda) \downarrow_{GL(n-1)}) \cong \mathrm{Hom}_{GL(n-1)}(\Delta_{n-1}(\overline{\lambda^*}), L_n(\tilde{\mu}^*) \downarrow_{GL(n-1)}).$$

Consequently, $L_{n-1}(\mu)$ lies in the socle of $L_n(\lambda) \downarrow_{GL(n-1)}$ if and only if both

$$\mathrm{Hom}_{GL(n-1)}(\Delta_{n-1}(\mu), L_n(\lambda) \downarrow_{GL(n-1)}) \quad \text{and} \quad \mathrm{Hom}_{GL(n-1)}(\Delta_{n-1}(\overline{\lambda^*}), L_n(\tilde{\mu}^*) \downarrow_{GL(n-1)})$$

are non-zero.

In particular, Theorem D means that to calculate the socle of $L_n(\lambda) \downarrow_{GL(n-1)}$ for all λ, it is sufficient to calculate the space of $GL(n-1)$-highest weight vectors in $L_n(\lambda)$ for all λ, or equivalently, the socle of $L_n(\lambda) \downarrow_{B^+ \cap GL(n-1)}$. In [B2, §5.3], the first author described an algorithm for calculating the space of highest weight vectors in $L_n(\lambda) \downarrow_{GL(n-1)}$. This is computationally intensive, depending on first calculating the Gram matrix for the contravariant form on certain weight spaces of Weyl modules, so is viable only for partitions of size $|\lambda| < 12$. Combining this with Theorem D means it is now possible to compute explicitly the socle of $L_n(\lambda) \downarrow_{GL(n-1)}$ for small λ.

Finally, we remark that there is an analogue of Theorem B for the branching problem: if $\lambda \in X^+(n)$ is p^r-restricted, the socle of $\nabla_n(\lambda) \downarrow_{GL(n-1)}$ is also p^r-restricted. This is a generalization of [K1, Theorem B] (for type A), where this was proved with $\nabla_n(\lambda)$ replaced by $L_n(\lambda)$. In fact, the proof of the more general version is identical to the proof in [K1], combined with Lemma 3.2 from this paper.

2 Proof of Theorem A

We will assume throughout the section that G is semisimple and simply connected. Theorem A (and Theorem B) as stated in the introduction reduce to this case by standard arguments.

The point is that then, the algebra of distributions $\mathrm{Dist}(G)$ can be identified with the *hyperalgebra* U of G, so can be constructed explicitly by first choosing a Chevalley system $(x_\alpha)_{\alpha\in R}, (h_i)_{1\le i\le \ell}$ in the corresponding semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, then taking the $\mathbb{Z}$-subalgebra $U_\mathbb{Z}$ of the universal enveloping algebra of $\mathfrak{g}$ generated by all $x_\alpha^k/k!$, and finally setting $U = U_\mathbb{Z} \otimes F$; see [J, II.1.12] and [S]. The elements $X_\alpha^{(n)}, Y_\alpha^{(n)} \in \mathrm{Dist}(G)$ coincide with $(x_\alpha^n/n!) \otimes 1, (x_{-\alpha}^n/n!) \otimes 1 \in U$ respectively, for $\alpha \in R^+$.

By [J, II.1.20], there is an equivalence of categories between the category of all G-modules and the category of locally finite U-modules. We denote by U^+ (resp. U^-) the subalgebra of U generated by all $X_\alpha^{(k)}$ (resp. $Y_\alpha^{(k)}$) for $\alpha \in R^+, k \ge 0$. Also, let U^0 be the subalgebra generated by all

$$\binom{H_i}{k} := \frac{h_i(h_i-1)\dots(h_i-k+1)}{k!} \otimes 1$$

for $1 \le i \le \ell$ and $k \ge 0$. Kostant's $\mathbb{Z}$-form for $U_\mathbb{Z}$ [S, Theorem 2] gives a PBW type basis for each of U, U^-, U^0 and U^+, on tensoring with $\mathbb{F}$.

We call a weight vector v in a G-module a *highest weight vector* if it is annihilated by all $X_\alpha^{(k)}$ for $\alpha \in R^+, k \ge 1$. The following fundamental result can be found in [J, II.2.13b)].

2.1. **(Universality of standard modules)** *The module $\Delta(\mu)$ is generated by any highest weight vector v_μ of weight μ, and, moreover, any G-module generated by a highest weight vector of weight μ is a quotient of $\Delta(\mu)$.*

We will often regard elements of $X(T)$ as homomorphisms $U^0 \to \mathbb{F}$. For a dominant weight μ let

$$X(\mu) := \{X_i^{(b_i)} \mid 1 \le i \le \ell, b_i > \langle \mu, \alpha_i^\vee\rangle\},$$
$$Y(\mu) := \{Y_i^{(b_i)} \mid 1 \le i \le \ell, b_i > \langle \mu, \alpha_i^\vee\rangle\},$$
$$\Omega(\mu) := \{X_\alpha^{(b_\alpha)}, H - \mu(H) \mid \alpha \in R^+, b_\alpha \ge 1, H \in U^0\}.$$

The next lemma is well known. We prove it for completeness as we could not find a proof in the literature.

2.2. **Lemma.** *For $\mu \in X^+(T)$, let $I(\mu)$ be the left ideal of U generated by $Y(\mu) \cup \Omega(\mu)$. Then, $\Delta(\mu) \cong U/I(\mu)$.*

Proof. Let v_μ be a highest weight vector in $\Delta(\mu)$ of weight μ. Consider the U-module homomorphism $U \to \Delta(\mu),\ u \mapsto uv_\mu$. As $Uv_\mu = \Delta(\mu)$ and $I(\mu)v_\mu = 0$, this homomorphism yields a surjection $U/I(\mu) \to \Delta(\mu)$. By the universality of standard modules and the equivalence of categories between locally finite U-modules and G-modules, it suffices to prove that $V(\mu) := U/I(\mu)$ is finite dimensional.

We prove this as in [H2, 21.4] by showing that the weights of $V(\mu)$ are permuted by the Weyl group W associated to the root system R. Let $s_i \in W$ be the simple reflection corresponding to α_i. Since W is generated by its simple reflections, we just need to prove that $s_i\nu$ is a weight of $V(\mu)$ whenever ν is a weight of $V(\mu)$.

Take $0 \ne v \in V(\mu)_\nu$. Our goal is to establish that $X_i^{(k)}v = Y_i^{(k)}v = 0$ for $k \gg 0$. Then the vector $\exp(X_i)\exp(-Y_i)\exp(X_i)v$ will be a well-defined non-zero vector of weight

$s_i\nu$. Note that $\nu + k\alpha_i$ is not a weight of $V(\mu)$ for k large enough, so $X_i^{(k)}v = 0$ for such k. To prove the claim for Y_i we may assume, using the PBW type basis for U^-, that $v = Y_{\beta_1}^{(b_1)} \dots Y_{\beta_m}^{(b_m)} + I(\mu)$ where $\{\beta_1, \dots, \beta_m\}$ are the positive roots.

By induction on $b_1 + \dots + b_m$ we now show that $Y_i^{(k)} Y_{\beta_1}^{(b_1)} \dots Y_{\beta_m}^{(b_m)} \in I(\mu)$ if $k > 3(b_1 + \dots + b_m) + \mu_i$. If $b_1 + \dots + b_m = 0$ this is clear as $Y_i^{(k)} \in I(\mu)$ for $k > \mu_i$. For the inductive step, let $r = \min\{s \mid b_s > 0\}$. To apply the inductive hypothesis it suffices to note that $Y_i^{(k)} Y_{\beta_r}^{(b_r)}$ is a linear combination of elements of the form $u_j Y_i^{(k-j)}$ where $j \le 3b_r$, which follows for example by [K1, 1.8(ii)]. □

Lemma 2.2 gives generators and relations for $\Delta(\mu)$ as a U-module. However, to prove Theorem A, we need generators and relations for $\Delta(\mu)$ as a U^+-module.

2.3. **Lemma.** *For $\mu \in X^+(T)$, let $I^-(\mu)$ be the left ideal of U^- generated by $Y(\mu)$. Then, $\Delta(\mu) \downarrow_{U^-} \cong U^-/I^-(\mu)$.*

Proof. Let $J(\mu)$ denote the left ideal of U generated by $\Omega(\mu)$. Then, $Z(\mu) := U/J(\mu)$ is the *Verma module* of highest weight μ. Using the PBW type bases, the map $\theta : U^- \to Z(\mu)$, $Y \mapsto Y + J(\mu)$ is an isomorphism of U^--modules. Lemma 2.2 implies that $\Delta(\mu) \cong Z(\mu)/F(\mu)$ where $F(\mu)$ is the image of $I(\mu)$ in $Z(\mu)$. So it suffices to show that θ maps $I^-(\mu)$ onto $F(\mu)$, or equivalently, that $UY(\mu) \subseteq U^-Y(\mu) + J(\mu)$.

We can write $U = U^-U^+U^0$ by [S, Theorem 2]. Clearly, elements of U^0 applied to the elements of $Y(\mu)$ change them to proportional ones. So we just need to prove that for any $X \in U^+$, the element $XY_i^{(b)}$ belongs to $U^-Y(\mu) + J(\mu)$ providing $b > \langle \mu, \alpha_i^\vee \rangle$. We may assume that $X = X_\alpha^{(a)}$ for some $\alpha \in R^+, a \ge 1$. If $\alpha \ne \alpha_i$, the weight $a\alpha - b\alpha_i$ is not a sum of negative roots, so $X_\alpha^{(a)} Y_i^{(b)}$ lies in $J(\mu)$. So we may assume that $\alpha = \alpha_i$ is a simple root, and moreover, by weights, that $a \le b$. Then, using [S, Lemma 5], we get

$$X_i^{(a)} Y_i^{(b)} + J(\mu) = Y_i^{(b-a)} \binom{H_i - b - a + 2a}{a} + J(\mu) = Y_i^{(b-a)} \binom{\langle \mu, \alpha_i^\vee \rangle - (b-a)}{a} + J(\mu).$$

If $b - a > \langle \mu, \alpha_i^\vee \rangle$ we have $Y_i^{(b-a)} \in Y(\mu)$. Otherwise $\langle \mu, \alpha_i^\vee \rangle - (b - a)$ is a non-negative integer strictly less than a, so $\binom{\langle \mu, \alpha_i^\vee \rangle - (b-a)}{a} = 0$. □

2.4. **Corollary.** *For $\mu \in X^+(T)$, let $I^+(\mu)$ be the left ideal of U^+ generated by $X(\mu)$. Then, $\Delta(\mu) \downarrow_{U^+} \cong U^+/I^+(-w_0\mu)$.*

Proof. Let $n_0 \in N_G(T)$ be any representative of $w_0 \in W = N_G(T)/T$. This acts on U by the adjoint action Ad. Moreover, $\mathrm{Ad}\, n_0$ sends U^- isomorphically onto U^+ and $I^-(\mu)$ isomorphically onto $I^+(-w_0\mu)$. Using these observations, the result follows immediately from Lemma 2.3. □

Recall the definition of $M_{\lambda-\mu}^\mu$ from the introduction. Now we can prove Theorem A.

2.5. **Theorem.** *Let $\lambda, \mu \in X^+(T)$, and M be any G-module. Then*

$$\mathrm{Hom}_G(\Delta(\lambda), M \otimes \nabla(\mu)) \cong M_{\lambda-\mu}^\mu.$$

Proof. Let $\mathbb{F}_\lambda$ be the 1-dimensional B^+-module of weight λ, and let $A \lhd B^+$ be the unipotent radical of B^+. Using the universality of standard modules we get

$$\begin{aligned} \mathrm{Hom}_G(\Delta(\lambda), M \otimes \nabla(\mu)) &\cong \mathrm{Hom}_{B^+}(\mathbb{F}_\lambda, M \otimes \nabla(\mu)) \\ &\cong \left((M \otimes \nabla(\mu))^A\right)_\lambda \\ &\cong \mathrm{Hom}_A(\nabla(\mu)^*, M)_\lambda \end{aligned}$$

where the last λ-weight space is taken with respect to the action $(t \cdot \varphi)(f) = t\varphi(t^{-1}f)$ for $\varphi \in \mathrm{Hom}_A(\nabla(\mu)^*, M)$, $f \in \nabla(\mu)^*$. Moreover, since $\nabla(\mu)^* \cong \Delta(-w_0\mu)$ and $U^+ \cong \mathrm{Dist}(A)$, [J, I.7.16] implies

$$\mathrm{Hom}_A(\nabla(\mu)^*, M)_\lambda \cong \mathrm{Hom}_{U^+}(\Delta(-w_0\mu), M)_\lambda.$$

The natural isomorphism $\mathrm{Hom}_{U^+}(U^+, M) \to M$ combined with Corollary 2.4 induces an isomorphism

$$F : \mathrm{Hom}_{U^+}(\Delta(-w_0\mu), M) \to M^\mu, \ \varphi \mapsto \varphi(v_{-\mu})$$

where $v_{-\mu}$ is a lowest weight vector in $\Delta(-w_0\mu)$ of weight $-\mu$. For $t \in T$ and a weight vector $\varphi \in \mathrm{Hom}_{U^+}(\Delta(-w_0\mu), M)_\lambda$,

$$t(\varphi(v_{-\mu})) = t(\varphi(t^{-1}tv_{-\mu})) = (t \cdot \varphi)(tv_{-\mu}) = (\lambda - \mu)(t)\varphi(v_{-\mu}).$$

Hence, F sends the λ-weight space of $\mathrm{Hom}_{U^+}(\Delta(-w_0\mu), M)$ isomorphically onto the $(\lambda-\mu)$-weight space of M^μ. □

3 Proof of Theorem B

Now we turn to the proof of Theorem B, which will ultimately be deduced as a consequence of Steinberg's tensor product theorem. We continue with the notation and assumptions from section 2; in particular, G is semisimple and simply connected.

3.1. **Lemma.** *For $\nu \in X^+(T)$ and $m \geq 0$, $\nu - w_0\nu \not\geq m\alpha_0$ if and only if $\langle \nu, \alpha_0^\vee \rangle < m$.*

Proof. If $\langle \nu, \alpha_0^\vee \rangle \geq m$ then $\nu - m\alpha_0$ is a weight of $\Delta(\nu)$, hence $\nu - m\alpha_0 \geq w_0\nu$, which is the lowest weight of $\Delta(\nu)$. Hence, $\nu - w_0\nu \geq m\alpha_0$. Conversely, suppose $\nu - w_0\nu \geq m\alpha_0$. Then $\nu - w_0\nu = m\alpha_0 + \kappa$ where κ is a sum of positive roots. Now, $\langle \nu - w_0\nu, \alpha_0^\vee \rangle = m\langle \alpha_0, \alpha_0^\vee \rangle + \langle \kappa, \alpha_0^\vee \rangle \geq m\langle \alpha_0, \alpha_0^\vee \rangle = 2m$. On the other hand, $\langle \nu - w_0\nu, \alpha_0^\vee \rangle = \langle \nu, \alpha_0^\vee \rangle - \langle w_0\nu, \alpha_0^\vee \rangle = 2\langle \nu, \alpha_0^\vee \rangle$, since $\langle w_0\nu, \alpha_0^\vee \rangle = \langle \nu, (w_0^{-1}\alpha_0)^\vee \rangle = \langle \nu, (w_0\alpha_0)^\vee \rangle$ and $w_0\alpha_0 = -\alpha_0$. Thus, $\langle \nu, \alpha_0^\vee \rangle \geq m$. □

3.2. **Lemma.** *Let $\lambda \in X^+(T)$ be p^r-restricted,and $v_\mu \in \nabla_n(\lambda)$ be a non-zero weight vector of weight μ. If v_μ is annihilated by all $X_\alpha^{(k)}$ for all $1 \leq k < p^r$ and all $\alpha \in R^+$, then $\mu = \lambda$.*

Proof. We let U_r denote the subalgebra of U generated by $\{X_\alpha^{(k)}, Y_\alpha^{(k)}\}_{\alpha \in R^+, k < p^r}$, which is the algebra of distributions of G_r, the rth Frobenius kernel of G (see [J, II.3]). The assumptions imply that the U_r-module M generated by v_μ is non-zero and has all weights less than or equal to μ. Pick $L(\nu)$ lying in the socle of M, so that ν is p^r-restricted with $\nu \leq \mu$. Certainly, $\mu \leq \lambda$, so the result will follow if we can show that $\nu = \lambda$.

For this, we claim that $\nabla(\lambda)$ has simple socle $L(\lambda)$ as a U_r-module. By the argument of [H1, Proposition 1.1] (which proves the special case $r = 1$), $\Delta(\lambda)$ is generated as a U_r-module by any highest weight vector of weight λ. This easily implies that $\Delta(\lambda)$ has simple head as a U_r-module, hence proving the claim on dualizing. □

3.3. **Theorem.** *Fix $\mu, \nu \in X^+(T)$ where μ is p^r-restricted and $\langle \nu, \alpha_0^\vee \rangle < p^r$. The socle of $\nabla(\mu) \otimes \nabla(\nu)$ is p^r-restricted.*

Proof. We say a vector $v \in \nabla(\mu) \otimes \nabla(\nu)$ is *weakly primitive* if $X_\alpha^{(k)} v = 0$ for all $\alpha \in R^+$ and all k with $0 < k < p^r$. Fix a weakly primitive weight vector $v \in \nabla(\mu) \otimes \nabla(\nu)$ of weight δ. Write $\delta = \mu + \nu - \kappa$ for some $\kappa \in X(T)$. We first claim that $\kappa \leq \nu - w_0\nu$. Write

$$v = \sum_{\gamma, i, j} x^i_{\delta-\gamma} \otimes y^j_\gamma$$

summing over $\gamma \in X(T)$ and i, j over index sets I_γ, J_γ respectively. In this expression, $\{x^i_\beta\}_{i \in I_\beta}$ and $\{y^j_\gamma\}_{j \in J_\gamma}$ denote linearly independent vectors of the weight spaces $\nabla(\mu)_\beta$ and $\nabla(\nu)_\gamma$ respectively. Let γ_0 be a minimal weight such that J_{γ_0} is non-empty. Then for any $\alpha \in R^+$ and any k with $0 < k < p^r$ we have

$$0 = X_\alpha^{(k)} v = \sum_{i,j} \left(X_\alpha^{(k)} x^i_{\delta-\gamma_0} \right) \otimes y^j_{\gamma_0}$$
$$+ [\text{a linear combination of vectors of the form } x^i_\beta \otimes y^j_\gamma \text{ with } \gamma \not\leq \gamma_0].$$

We conclude by linear independence of $\{y^j_\gamma\}_{j \in J_\gamma}$ that $X_\alpha^{(k)} x^i_{\delta-\gamma_0} = 0$ for any $\alpha \in R^+$ and $0 < k < p^r$. Since μ is p^r-restricted, it follows from Lemma 3.2 that $x^i_{\delta-\gamma_0}$ is a high weight vector in $\nabla(\mu)$. Thus, $\delta - \gamma_0 = \mu$, hence $\gamma_0 = \delta - \mu = \nu - \kappa$. This shows that $\nu - \kappa$ is a weight of $\nabla(\nu)$, so $\nu - \kappa \geq w_0\nu$, which implies the claim.

Now, assume for a contradiction that the Steinberg tensor product $L(\lambda) \otimes L(\lambda')^{[r]}$ is a submodule of $\nabla(\mu) \otimes \nabla(\nu)$ for some p^r-restricted λ and some $\lambda' \neq 0$. Let v_λ and $v^+_{\lambda'}$ be high weight vectors of $L(\lambda)$ and $L(\lambda')^{[r]} = L(p^r\lambda')$, respectively. Also let $v^-_{\lambda'}$ be the lowest weight vector of $L(\lambda')^{[r]}$. Then both $v_\lambda \otimes v^+_{\lambda'}$ and $v_\lambda \otimes v^-_{\lambda'}$ are weakly primitive (in the latter case, this follows by the definition of the action of U on Frobenius twists). The weights of these two vectors are $\lambda + p^r\lambda'$ and $\lambda + p^r w_0\lambda'$ respectively. Set

$$\lambda + p^r\lambda' = \mu + \nu - \kappa_1, \quad \lambda + p^r w_0\lambda' = \mu + \nu - \kappa_2.$$

By the claim, we have $\kappa_2 \leq \nu - w_0\nu$. On the other hand, $\kappa_2 - \kappa_1 = p^r(\lambda' - w_0\lambda') \geq p^r\alpha_0$, the last inequality being true by [K3, Lemma 1.5]. It follows that $\kappa_2 \geq \kappa_1 + p^r\alpha_0 \geq p^r\alpha_0$, whence $\nu - w_0\nu \geq p^r\tilde{\alpha}$. This contradicts the assumption on ν because of Lemma 3.1. □

3.4. **Corollary.** *Let μ be a dominant p^r-restricted weight, and ν be any miniscule weight. If M is any submodule of $\nabla(\mu)$ then the socle of $M \otimes L(\nu)$ is p^r-restricted. In particular, the socle of $L(\mu) \otimes L(\nu)$ is p^r-restricted.*

Proof. This follows immediately from Theorem 3.3, since if ν is miniscule then $\langle \nu, \alpha^\vee \rangle$ is 0 or 1 for all $\alpha \in R^+$. □

3.5. **Remark.** One might ask whether Theorem 3.3 is true more generally, namely, is it true that the socle of $\nabla(\mu) \otimes \nabla(\nu)$ is p-restricted as long as both $\mu, \nu \in X^+(n)$ are p-restricted. We give a counterexample which shows that this is false in general. Consider the 2-restricted dominant weights $\mu = \nu = 3\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$ for $GL(4)$. Put $\lambda = 6\varepsilon_1 + 3\varepsilon_2 + 2\varepsilon_3 + \varepsilon_4$. By the Littlewood-Richardson rule and [W], the module $\nabla(\mu) \otimes \nabla(\nu)$ has a ∇-filtration, with $\nabla(\lambda)$ as one of its quotients. Hence there exists a non-zero homomorphism from $\Delta(\lambda)$ to $\nabla(\mu) \otimes \nabla(\nu)$. However, $\nabla(\lambda)$ is irreducible in characteristic 2, as follows e.g. from [K2, 2.2(iv)]. So we get a non-2-restricted irreducible module in the socle of $\nabla(\mu) \otimes \nabla(\nu)$.

4 Proof of Theorems C and D

From now on, we assume that $G = GL(n)$. In the notation from the introduction, the root system $R \subset X(T)$ is the set $\{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n\}$. For $i < j$, we denote the root $\varepsilon_i - \varepsilon_j$ by $\alpha(i, j)$. Write $E_{i,j}^{(k)}$ for $X_{\alpha(i,j)}^{(k)}$ and $F_{i,j}^{(k)}$ for $Y_{\alpha(i,j)}^{(k)}$. We fix an integer $\ell \geq 0$ throughout the section.

Let $P = LY$ be the standard parabolic subgroup of $GL(n)$, where $L \cong GL(n-1)GL(1)$ (embedded diagonally) and Y is the unipotent radical generated by the root subgroups corresponding to the roots $\alpha(i, n)$, for $i = 1, 2, \dots n-1$. Note that for any $GL(n)$-module N, the Y-fixed points N^Y of N are L-invariant, so we can regard N^Y as a $GL(n-1)$-module in a natural way. Also, for $\lambda = (\lambda_1, \dots, \lambda_n) \in X^+(n), j \in \mathbb{Z}$ and any submodule M of $\nabla_n(\lambda)$, the *jth level* of M is defined by

$$M^j := \bigoplus_{\substack{\nu \in X(n), \\ \nu_n = \lambda_n + j}} M_\nu.$$

This is a weight space for the 1-dimensional torus $GL(1)$ that centralizes $GL(n-1)$ in $GL(n)$, so

$$M \downarrow_{GL(n-1)} \cong \sum_{j \geq 0} M^j.$$

4.1. **Lemma.** *Let $\lambda, \mu \in X^+(n)$. The dimension of* $\mathrm{Hom}_{GL(n)}(\Delta_n(\mu), \nabla_n(\lambda) \otimes \nabla(-\ell\varepsilon_n))$ *is* 1 *if $\mu_n \leq \lambda_n$ and $\lambda_{i+1} \leq \mu_i \leq \lambda_i$ for $i = 1, \dots, n-1$, and is* 0 *otherwise.*

Proof. By [W], $\nabla_n(\lambda) \otimes \nabla(-\ell\varepsilon_n)$ has a good filtration, so [J, II.4.16a)] implies that the Hom dimension is equal to the multiplicity of $\nabla_n(\mu)$ in a good filtration of $\nabla_n(\lambda) \otimes \nabla(-\ell\varepsilon_n)$. Now the result follows for example from the Littlewood-Richardson rule. □

Let V be the natural $GL(n)$-module, and let $\{f_1, \dots, f_n\}$ be the basis of V^* dual to the canonical basis of V. By [J, I.2.16(4)], the module $\nabla(-\ell\varepsilon_n)$ is precisely the ℓth symmetric power $S^\ell(V^*)$. Let $\Lambda(n, \ell)$ be the set of all n-tuples $(\lambda_1, \dots, \lambda_n)$ of non-negative integers with $\lambda_1 + \dots + \lambda_n = \ell$. For $\beta = (\beta_1, \dots, \beta_n) \in \Lambda(n, \ell)$, we set

$$\begin{aligned} f_\beta &= f_1^{\beta_1} \dots f_n^{\beta_n} && \in S^\ell(V^*), \\ E(\beta) &= E_{1,n}^{(\beta_1)} E_{2,n}^{(\beta_2)} \dots E_{n-1,n}^{(\beta_{n-1})} && \in \mathrm{Dist}(GL(n)). \end{aligned}$$

Then $\{f_\beta \mid \beta \in \Lambda(n, \ell)\}$ is a basis for $S^\ell(V^*)$, and in particular, the set of weights of $S^\ell(V^*)$ is precisely the set $-\Lambda(n, \ell)$, all with multiplicity one. Also if $\beta = (\beta_1, \dots, \beta_n)$, $\gamma =$

$(\gamma_1, \dots, \gamma_n) \in \Lambda(n, \ell)$, we write

$$|\bar{\beta}| = \beta_1 + \cdots + \beta_{n-1}, \qquad \binom{\bar{\gamma}}{\bar{\beta}} = \binom{\gamma_1}{\beta_1} \cdots \binom{\gamma_{n-1}}{\beta_{n-1}}.$$

Then for $1 \le r < s \le n, t \ge 1$, we have

$$E_{r,s}^{(t)} f_\beta = (-1)^t \binom{\beta_r}{t} f_{\beta - t\alpha(r,s)} \quad \text{and} \quad F_{r,s}^{(t)} f_\beta = (-1)^t \binom{\beta_s}{t} f_{\beta + t\alpha(r,s)}$$

(note that $\binom{\beta_i}{t} = 0$ if $\beta_i < t$, so the right hand sides above are interpreted as 0 if $\beta_r < t$ or $\beta_s < t$, respectively).

Let M be an arbitrary $GL(n)$-module. As $\{f_\beta \mid \beta \in \Lambda(n, \ell)\}$ is a basis of $S^\ell(V^*)$, any element $w \in M \otimes S^\ell(V^*)$ can be written uniquely in the form

$$w = \sum_{\beta \in \Lambda(n,r)} w_\beta \otimes f_\beta.$$

We refer to w_β as the *β-component* of w. Define a linear map

$$e : M \to M \otimes S^\ell(V^*), \ v \mapsto \sum_{\beta \in \Lambda(n,\ell)} (E(\beta)v) \otimes f_\beta.$$

4.2. **Lemma.** *For any $GL(n)$-module M, the map e is an injective $GL(n-1)$-module homomorphism.*

Proof. Clearly, e linear. It is injective since the $\ell\varepsilon_n$-component of $e(v)$ is v. Let B' be the subgroup $B^+ \cap GL(n-1)$ of all upper triangular matrices in $GL(n-1)$, and W' be the subgroup of all permutation matrices in $GL(n-1)$. As B' and W' generate $GL(n-1)$, it suffices to prove that e is both a B'-homomorphism and a W'-homomorphism.

To prove that e is a W'-homomorphism take σ to be a permutation of $\{1, \dots, n-1\}$ and denote by the same letter σ the corresponding permutation matrix. Then

$$\begin{aligned} \sigma e(v) &= \sigma \sum_{\beta \in \Lambda(n,\ell)} E(\beta)v \otimes f_\beta = \sum_{\beta \in \Lambda(n,\ell)} (\sigma E(\beta)v) \otimes (\sigma f_\beta) \\ &= \sum_{\beta \in \Lambda(n,\ell)} (\sigma E(\beta)\sigma^{-1}\sigma v) \otimes f_{\sigma\beta} = \sum_{\beta \in \Lambda(n,\ell)} (E(\sigma\beta)\sigma v) \otimes f_{\sigma\beta} = e(\sigma v). \end{aligned}$$

To prove that e is a B'-homomorphism, we first note that it is a T-homomorphism, which follows from the fact that for $v \in M_\mu$, the restriction of the weight of any $(E(\beta)v) \otimes f_\beta$ to T is the same as the restriction of μ. So, it suffices to prove that $e(E_{r,s}^{(t)}v) = E_{r,s}^{(t)}e(v)$ for any $v \in M$, $1 \le r < s \le n-1$ and $t > 0$. First, we note that for $m \ge 0$, the commutator formula from [S, Lemma 15] implies that

$$\begin{aligned} E_{r,s}^{(m)} E(\beta) &= E_{r,s}^{(m)} E_{s,n}^{(\beta_s)} E(\beta - \beta_s\alpha(s,n)) = \sum_{i=0}^{\min(m,\beta_s)} E_{r,n}^{(i)} E_{s,n}^{(\beta_s - i)} E_{r,s}^{(m-i)} E(\beta - \beta_s\alpha(s,n)) \\ &= \sum_{i=0}^{\min(m,\beta_s)} E_{r,n}^{(i)} E(\beta - i\alpha(s,n)) E_{r,s}^{(m-i)} = \sum_{i=0}^{\min(m,\beta_s)} \binom{i+\beta_r}{i} E(\beta + i\alpha(r,s)) E_{r,s}^{(m-i)}. \end{aligned}$$

Hence,

$$E_{r,s}^{(t)}e(v) = E_{r,s}^{(t)} \sum_{\beta\in\Lambda(n,\ell)} (E(\beta)v)\otimes f_\beta = \sum_{\substack{\beta\in\Lambda(n,\ell),\\ 0\le m\le t}} \left(E_{r,s}^{(m)}E(\beta)v\right)\otimes\left(E_{r,s}^{(t-m)}f_\beta\right)$$

$$= \sum_{\substack{\beta\in\Lambda(n,\ell),\\ 0\le m\le t}} \sum_{i=0}^{\min(m,\beta_s)} (-1)^{t-m}\binom{\beta_r}{t-m}\binom{i+\beta_r}{i}\left(E(\beta+i\alpha(r,s))E_{r,s}^{(m-i)}v\right)\otimes f_{\beta-(t-m)\alpha(r,s)}.$$

Fix any $\gamma\in\Lambda(n,\ell)$. Then the γ-component of the above expression is

$$\sum_{m=\max(0,t-\gamma_s)}^{t} \sum_{i=0}^{\min(m,\gamma_s-(t-m))} (-1)^{t-m}\binom{\gamma_r+t-m}{t-m}\binom{i+\gamma_r+t-m}{i} E(\gamma+(t-m+i)\alpha(r,s))E_{r,s}^{(m-i)}v$$

which equals

$$\sum_{j=\max(0,t-\gamma_s)}^{t} c_j E(\gamma+(t-j)\alpha(r,s))E_{rs}^{(j)}v$$

where $c_j = \sum_{m=j}^{t}(-1)^{t-m}\binom{\gamma_r+t-m}{t-m}\binom{\gamma_r+t-j}{m-j}$. An elementary substitution now shows that $c_j=0$ if $j<t$. Hence, the γ-component is $E(\gamma)E_{r,s}^{(t)}v$, proving that $E_{r,s}^{(t)}e(v)=e(E_{r,s}^{(t)}v)$. □

4.3. **Lemma.** *For any $GL(n)$-module M we have $(M\otimes S^\ell(V^*))^Y\subseteq e(M)$.*

Proof. Let $w=\sum_{\beta\in\Lambda(n,\ell)} w_\beta\otimes f_\beta\in(M\otimes S^\ell(V^*))^Y$. We have to show that

$$w_\gamma = E(\gamma)w_{\ell\varepsilon_n}$$

for any $\gamma\in\Lambda(n,\ell)$. We prove this by induction on $|\bar\gamma|$. If $|\bar\gamma|=0$, the result is clear. Let $|\bar\gamma|>0$. Considering the $\ell\varepsilon_n$-component of the equation $E(\gamma)w=0$ gives

$$\sum_{\beta\preceq\gamma}(-1)^{|\bar\beta|}E(\gamma-\beta)w_\beta=0$$

where $\beta\preceq\gamma$ means $\beta_1\le\gamma_1,\dots,\beta_{n-1}\le\alpha_{n-1}$. Now the induction hypothesis gives us

$$0=\sum_{\beta\prec\gamma}(-1)^{|\bar\beta|}E(\gamma-\beta)E(\beta)w_{\ell\varepsilon_n}+(-1)^{|\bar\gamma|}w_\gamma=\sum_{\beta\prec\gamma}(-1)^{|\bar\beta|}\binom{\bar\gamma}{\bar\beta}E(\gamma)w_{\ell\varepsilon_n}+(-1)^{|\bar\gamma|}w_\gamma.$$

The lemma now follows from the identity $\sum_{\beta\preceq\gamma}(-1)^{|\bar\beta|}\binom{\bar\gamma}{\bar\beta}=0$. □

4.4. **Lemma.** *Let M be a submodule of $\nabla_n(\lambda)$. Let $v\in M^0\oplus\cdots\oplus M^\ell$. Then $e(v)\in\left(M\otimes S^\ell(V^*)\right)^Y$.*

Proof. Let $1 \le r < n$ and $t > 0$. Then

$$E_{r,n}^{(t)} e(v) = E_{r,n}^{(t)} \sum_{\beta \in \Lambda(n,\ell)} (E(\beta)v) \otimes f_\beta = \sum_{\beta \in \Lambda(n,\ell)} \sum_{m=0}^{t} \left(E_{r,n}^{(t-m)} E(\beta) v \right) \otimes \left(E_{r,n}^{(m)} f_\beta \right)$$

$$= \sum_{\beta \in \Lambda(n,\ell)} \sum_{m=0}^{\min(t,\beta_r)} \left(\binom{t-m+\beta_r}{t-m} E(\beta + (t-m)\alpha(r,n)) v \right) \otimes \left((-1)^m \binom{\beta_r}{m} f_{\beta - m\alpha(r,n)} \right).$$

Let $\gamma \in \Lambda(n,\ell)$. The γ-component of the expression above is

$$\sum_{m=0}^{\min(t,\gamma_n)} (-1)^m \binom{t+\gamma_r}{t-m} \binom{\gamma_r+m}{m} E(\gamma + t\alpha(r,n))v.$$

If $t > \gamma_n$ then $\gamma_1 + \cdots + \gamma_{n-1} + t > \ell$, so $E(\gamma + t\alpha(r,n))v = 0$ as v belongs to the top ℓ levels. Otherwise,

$$\sum_{m=0}^{t} (-1)^m \binom{t+\gamma_r}{t-m} \binom{\gamma_r+m}{m} = 0$$

which completes the proof. □

4.5. **Theorem.** *Let M be any submodule of $\nabla_n(\lambda)$ and let $\bar{e}$ denote the restriction of e to $M^0 \oplus \cdots \oplus M^\ell$. Then $\bar{e}$ is an isomorphism between $M^0 \oplus \cdots \oplus M^\ell$ and $\left(M \otimes S^\ell(V^*)\right)^Y$ as $GL(n-1)$-modules.*

Proof. Lemma 4.2 and Lemma 4.4 imply that $\bar{e}$ is a well-defined injective homomorphism of $GL(n-1)$-modules. To prove that it is surjective, take $v \in \left(M \otimes S^\ell(V^*)\right)^Y$ of weight $\nu = (\nu_1, \ldots, \nu_n)$. By Lemma 4.3, $v = e(v_{\ell\varepsilon_n})$, so it remains to show that $v_{\ell\varepsilon_n}$ lies in the first ℓ levels of M. Suppose for a contradiction this is false, and choose v, ν so that ν is maximal in the dominance order subject to the condition $v_{\ell\varepsilon_n} \notin M^0 \oplus \cdots \oplus M^\ell$.

If $v_{\ell\varepsilon_n}$ is a $GL(n-1)$-highest weight vector, then by Lemma 4.2, $v = e(v_{\ell\varepsilon_n})$ is a $GL(n-1)$-highest weight vector, which by assumption is Y-invariant. Hence, v is a $GL(n)$-highest weight vector so, using the universality of standard modules, Lemma 4.1 implies that $\nu_n \le \lambda_n$. But the weight of $v_{\ell\varepsilon_n}$ is $\nu + \ell\varepsilon_n$, so this contradicts the assumption $v_{\ell\varepsilon_n} \notin M^0 \oplus \cdots \oplus M^\ell$.

So, $v_{\ell\varepsilon_n}$ is not a $GL(n-1)$-highest weight vector, so we can find some $1 \le i < j < n, k > 0$ such that $E_{i,j}^{(k)} v_{\ell\varepsilon_n} \ne 0$. Applying the injective map e, this implies that $E_{i,j}^{(k)} v \ne 0$. As $E_{i,j}^{(k)} v$ is Y-invariant, the maximality of ν now implies that $E_{i,j}^{(k)} v_{\ell\varepsilon_n} \in M^0 \oplus \cdots \oplus M^\ell$. But again this implies that $v_{\ell\varepsilon_n} \in M^0 \oplus \cdots \oplus M^\ell$, giving the desired contradiction. □

4.6. **Corollary.** *Fix $\lambda, \mu \in X^+(n)$ and a submodule $M \le \nabla_n(\lambda)$. The restriction of the map $\bar{e}$ from Theorem 4.5 gives a bijection between the $GL(n-1)$-highest weight vectors in $M^0 \oplus \cdots \oplus M^\ell$ of weight $\mu + \ell\varepsilon_n$ and the $GL(n)$-highest weight vectors in $M \otimes S^\ell(V^*)$ of weight μ.*

Proof. This follows from Theorem 4.5 since a vector $v \in M \otimes S^\ell(V^*)$ is $GL(n)$-primitive if and only if it is $GL(n-1)$-primitive and lies in $(M \otimes S^\ell(V^*))^Y$. □

4.7. **Theorem.** *Fix* $\lambda, \mu \in X^+(n)$ *and a submodule* $M \leq \nabla_n(\lambda)$. *Let* $\bar{\mu} = (\mu_1, \dots, \mu_{n-1})$.

(i) *If* $\mu_n \leq \lambda_n$ *then* $\mathrm{Hom}_{GL(n)}(\Delta_n(\mu), M \otimes S^\ell(V^*)) \cong \mathrm{Hom}_{GL(n-1)}(\Delta_{n-1}(\bar{\mu}), M^{\ell+\mu_n-\lambda_n})$.

(ii) *If* $\mu_n = \lambda_n$ *then* $\mathrm{Hom}_{GL(n)}(L_n(\mu), M \otimes S^\ell(V^*)) \cong \mathrm{Hom}_{GL(n-1)}(L_{n-1}(\bar{\mu}), M^\ell)$.

Proof. (i) A vector $v \in M$ has $GL(n)$-weight $\mu + \ell\varepsilon_n$ if and only if it has $GL(n-1)$-weight $\bar{\mu}$ and lies in $M^{\ell+\mu_n-\lambda_n}$. Since $\mu_n \leq \lambda_n$, such vectors lie in the first ℓ levels of M. So Corollary 4.6 now implies that there is a bijection between the $GL(n-1)$-highest weight vectors of $(T \cap GL(n-1))$-weight $\bar{\mu}$ in $M^{\ell+\mu_n-\lambda_n}$ and the $GL(n)$-highest weight vectors in $M \otimes S^\ell(V^*)$ of weight μ. The result now follows using the universality of standard modules.

(ii) Suppose now that $\mu_n = \lambda_n$. In view of (i), it suffices to show that a $GL(n-1)$-highest weight vector $v \in M^\ell$ of weight $\mu + \ell\varepsilon_n$ generates an irreducible $GL(n-1)$-module if and only if $e(v) \in M_n(\lambda) \otimes S^\ell(V^*)$ generates an irreducible $GL(n)$-module. Equivalently, applying Theorem 4.5, we need to prove that a $GL(n-1)$-highest weight vector w in $(M \otimes S^\ell(V^*))^Y$ of weight μ generates an irreducible $GL(n-1)$-module if and only if it generates an irreducible $GL(n)$-module. If the $GL(n)$-highest weight vector w generates an irreducible $GL(n)$-module then it certainly generates an irreducible $GL(n-1)$-module by [J, II.2.11].

Conversely, let $w \in M \otimes S^\ell(V^*)$ be a $GL(n)$-highest weight vector of weight μ that generates a reducible $GL(n)$-module. Then, we can find an operator Y in the negative part of $\mathrm{Dist}(GL(n))$ generated by all $F_{i,j}^{(k)}$ such that $Yw \in M$ is a non-zero $GL(n)$-highest weight vector of weight $\nu < \mu$. Observe that $\nu_n \geq \mu_n = \lambda_n$, while by Lemma 4.1, $\nu_n \leq \lambda_n$. Thus, $\nu_n = \lambda_n$, so by weights Y lies in $\mathrm{Dist}(GL(n-1))$. But this implies that the $GL(n-1)$-module generated by w is also reducible as required. □

We remark that Theorem 4.7(i) can also be deduced from [BK1, Corollary 2.10]. Part (ii) is certainly false if we try to weaken the assumption to $\mu_n \leq \lambda_n$. Theorem C follows from Theorem 4.7(ii):

4.8. **Corollary.** *Fix* $\lambda, \mu \in X^+(n)$ *with* $\lambda_n = \mu_n$. *Let* $\ell = \sum_{i=1}^n (\lambda_i - \mu_i)$ *and* $\bar{\mu} = (\mu_1, \dots, \mu_{n-1})$. *For any submodule* M *of* $\nabla_n(\lambda)$,

$$\mathrm{Hom}_{GL(n)}(L_n(\mu), M \otimes S^\ell(V^*)) \cong \mathrm{Hom}_{GL(n-1)}(L_{n-1}(\bar{\mu}), M \downarrow_{GL(n-1)}).$$

Proof. This follows immediately from Theorem 4.7(ii) since the jth level M^j is the sum of all weight spaces M_ν with $\nu \in X^+(n)$ satisfying $\nu_1 + \cdots + \nu_{n-1} = \lambda_1 + \cdots + \lambda_{n-1} - j$. □

Finally, we deduce Theorem D. For any $\lambda = (\lambda_1, \dots, \lambda_n) \in X^+(n)$, we write λ^* for the dominant weight $-w_0\lambda = (-\lambda_n, -\lambda_{n-1}, \dots, -\lambda_1) \in X^+(n)$.

4.9. **Theorem.** *Fix any* $\lambda \in X^+(n)$ *and* $\mu \in X^+(n-1)$ *with* $\lambda_{i+1} \leq \mu_i \leq \lambda_i$ *for* $i = 1, \dots, n-1$. *Let* $\tilde{\mu} = (\mu_1, \dots, \mu_{n-1}, \lambda_n) \in X^+(n)$ *and* $\bar{\lambda} = (\lambda_1, \dots, \lambda_{n-1}) \in X^+(n-1)$. *Then,*

$$\mathrm{Hom}_{GL(n-1)}(L_{n-1}(\mu), \nabla_n(\lambda) \downarrow_{GL(n-1)}) \cong \mathrm{Hom}_{GL(n-1)}(\Delta_{n-1}(\overline{\lambda^*}), L_n(\tilde{\mu}^*) \downarrow GL(n-1)).$$

Proof. We note that $\overline{\lambda^*} = (-\lambda_n, \dots, -\lambda_2)$ and $\tilde{\mu}^* = (-\lambda_n, -\mu_{n-1}, \dots, -\mu_1)$. Let $\gamma = \tilde{\mu}, \bar{\gamma} = \mu$ and $\ell = \sum_{i=1}^n (\lambda_i - \gamma_i)$. Arguing as in Corollary 4.8, it suffices to prove that

$$\mathrm{Hom}_{GL(n-1)}(L_{n-1}(\bar{\gamma}), \nabla_n(\lambda)^\ell) \cong \mathrm{Hom}_{GL(n-1)}(\Delta_{n-1}(\overline{\lambda^*}), L_n(\gamma^*)^{\ell+\mu_1-\lambda_1}).$$

Using Theorem 4.7 and the fact that $\mu_1 \leq \lambda_1$, we have that

$$\begin{aligned}\mathrm{Hom}_{GL(n-1)}(L_{n-1}(\bar{\gamma}), \nabla_n(\lambda)^\ell) &\cong \mathrm{Hom}_{GL(n)}(L_n(\gamma), \nabla_n(\lambda) \otimes S^\ell(V^*)) \\ &\cong \mathrm{Hom}_{GL(n)}(\Delta_n(\lambda^*), L_n(\gamma^*) \otimes S^\ell(V^*)) \\ &\cong \mathrm{Hom}_{GL(n-1)}(\Delta_{n-1}(\overline{\lambda^*}), L_n(\gamma^*)^{\ell+\mu_1-\lambda_1})\end{aligned}$$

as claimed. □

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