HOMOLOGICAL PROPERTIES OF ADE KHOVANOV-LAUDA-ROUQUIER ALGEBRAS

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Abstract. Building on recent works of Kato and McNamara, we give an algebraic construction of *standard modules*—infinite dimensional modules categorifying the PBW basis of the underlying quantized enveloping algebra—for Khovanov-Lauda-Rouquier algebras in finite ADE types. This allows us to prove in an elementary way that these algebras satisfy the homological properties of an "affine quasi-hereditary algebra." We also construct some Koszul-like projective resolutions of standard modules corresponding to multiplicity-free positive roots.

1. Introduction

Working over $\mathbb{Q}(q)$ for an indeterminate q, let $\mathbf{f}$ be the quantized enveloping algebra associated to a maximal nilpotent subalgebra of a simple Lie algebra $\mathfrak{g}$ of finite type A_r, D_r or E_r. It is naturally Q^+-graded

$$\mathbf{f} = \bigoplus_{\alpha \in Q^+} \mathbf{f}_\alpha,$$

where Q^+ denotes $\mathbb{N}$-linear combinations of the simple roots $\alpha_1, \dots, \alpha_r$. Moreover $\mathbf{f}$ is equipped with several distinguished bases, including Lusztig's *canonical basis* (Kashiwara's lower global crystal base) and various *PBW bases*, one for each choice $\prec$ of convex ordering of the set R^+ of positive roots. Passing to dual bases with respect to Lusztig's form $(\cdot,\cdot)$ on $\mathbf{f}$, we obtain the *dual canonical basis* (Kashiwara's upper global crystal base) and some *dual PBW bases*. See [L1], [L2] and [K]. Lusztig's approach gives a *categorification* of $\mathbf{f}$ in terms of certain categories of direct sums of degree-shifted perverse sheaves on a quiver variety. Multiplication on $\mathbf{f}$ comes from Lusztig's induction functor, and the canonical basis arises from the irreducible perverse sheaves in these categories.

In 2008 Khovanov and Lauda [KL] and Rouquier [R1] introduced for any field $\mathbb{K}$ a (locally unital) graded $\mathbb{K}$-algebra

$$H = \bigoplus_{\alpha \in Q^+} H_\alpha,$$

known as the *Khovanov-Lauda-Rouquier algebra* (KLR for short). Let $\mathrm{Proj}(H)$ be the additive category of (locally unital) finitely generated graded projective left H-modules. We make the split Grothendieck group $[\mathrm{Proj}(H)]$ of this category into a $\mathbb{Z}[q,q^{-1}]$-algebra, with multiplication arising from the induction

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product $\circ$ on modules over the KLR algebra and action of q induced by upwards degree shift. Khovanov and Lauda showed that $\mathrm{Proj}(H)$ also provides a categorification of $\mathbf{f}$: there is a unique algebra isomorphism

$$\gamma : \mathbf{f} \xrightarrow{\sim} \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} [\mathrm{Proj}(H)], \qquad \theta_i \mapsto [H_{\alpha_i}],$$

where θ_i is the generator of $\mathbf{f}$ corresponding to simple root α_i. Assuming the ground field $\mathbb{K}$ is of characteristic zero, Rouquier [R2] and Varagnolo and Vasserot [VV] have shown further that this algebraic categorification of $\mathbf{f}$ is equivalent to Lusztig's geometric one. In particular γ maps the canonical basis of $\mathbf{f}$ to the basis for $[\mathrm{Proj}(H)]$ arising from the isomorphism classes of graded self-dual indecomposable projective modules.

The setup can also be dualized. Let $\mathrm{Rep}(H)$ be the abelian category of all (locally unital) finite dimensional graded left H-modules. Its Grothendieck group $[\mathrm{Rep}(H)]$ is again a $\mathbb{Z}[q,q^{-1}]$-algebra. Taking a dual map to γ yields another algebra isomorphism

$$\gamma^* : \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} [\mathrm{Rep}(H)] \xrightarrow{\sim} \mathbf{f}.$$

When $\mathrm{char}\,\mathbb{K} = 0$, this sends the basis for $[\mathrm{Rep}(H)]$ arising from isomorphism classes of graded self-dual irreducible H-modules to the dual canonical basis for $\mathbf{f}$. In positive characteristic, the graded self-dual irreducible H-modules still give rise to a basis for $\mathbf{f}$. Although this basis definitely does *not* coincide in general with the dual canonical basis (see [Wi] and also Example 2.19 below), several people (e.g. [KOH]) have observed that it is always a *perfect basis* in the sense of Berenstein and Kazhdan [BK]. This implies for any ground field that the irreducible H-modules are parametrized in a canonical way by Kashiwara's crystal $B(\infty)$ associated to $\mathbf{f}$, a result established originally by Lauda and Vazirani [LV] without using the theory of perfect bases.

Using the geometric approach of Varagnolo and Vasserot (hence for fields of characteristic zero only), Kato [Ka] has explained further how to lift the PBW and dual PBW bases of $\mathbf{f}$ to certain graded modules $\{\widetilde{E}_b \mid b \in B(\infty)\}$ and $\{E_b \,|\, b \in B(\infty)\}$ over KLR algebras. We refer to these modules as *standard* and *proper standard* modules, respectively, motivated by the similarity to the theory of properly stratified algebras [D]. Kato establishes that each proper standard module E_b has irreducible head L_b, and the modules $\{L_b \mid b \in B(\infty)\}$ give a complete set of graded self-dual irreducible H-modules. The standard module $\widetilde{E}_b$ is infinite dimensional, and should be viewed informally as a "maximal self-extension" of the corresponding proper standard module E_b. Kato's work establishes in particular that each of the algebras H_α has finite global dimension.

More recently, McNamara [M] has found a purely algebraic way to introduce proper standard modules, similar in spirit to the approach via dominant Lyndon words developed in [KR2] but substantially more general as it makes sense for an arbitrary choice for the convex ordering $\prec$. It produces in the end the same collection of proper standard modules as above but indexed instead by the set KP of *Kostant partitions*, i.e. non-increasing sequences $\lambda = (\lambda_1 \succeq \cdots \succeq \lambda_l)$ of positive roots. Switching to this notation, we henceforth denote the proper standard module corresponding to λ by $\bar{\Delta}(\lambda)$. McNamara shows directly that $\bar{\Delta}(\lambda)$ has irreducible head $L(\lambda)$, and the modules $\{L(\lambda) \,|\, \lambda \in \mathrm{KP}\}$

give a complete set of graded self-dual irreducible H-modules. Moreover there is a "bilexicographic" partial ordering $\preceq$ on KP with respect to which the *decomposition matrix* $([\bar{\Delta}(\lambda) : L(\mu)])_{\lambda,\mu\in\mathrm{KP}}$ is unitriangular, i.e.

$$[\bar{\Delta}(\lambda) : L(\lambda)] = 1, \qquad [\bar{\Delta}(\lambda) : L(\mu)] = 0 \text{ for } \mu \not\preceq \lambda.$$

Although not relevant here, we note that McNamara's arguments work in arbitrary finite type (not just ADE), and moreover he shows that the algebras H_α have finite global dimension in non-simply-laced cases too.

Letting $P(\lambda)$ denote the projective cover of $L(\lambda)$, the standard module $\Delta(\lambda)$ corresponding to λ may be defined as

$$\Delta(\lambda) := P(\lambda) \Big/ \sum_{\mu \not\preceq \lambda} \; \sum_{f:P(\mu)\to P(\lambda)} \operatorname{im} f.$$

Taking graded duals, we also have the *costandard module* $\nabla(\lambda) := \Delta(\lambda)^{\circledast}$ and the *proper costandard module* $\bar{\nabla}(\lambda) := \bar{\Delta}(\lambda)^{\circledast}$. For $\mathbb{K}$ of characteristic zero, Kato has shown that these modules satisfy various homological properties familiar from the theory of quasi-hereditary algebras. Perhaps the most important of these is the following:

$$\operatorname{Ext}^d_H(\Delta(\lambda), \bar{\nabla}(\mu)) \cong \begin{cases} \mathbb{K} & \text{if } d = 0 \text{ and } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

There are many pleasant consequences. For example, one can deduce that the projective module $P(\lambda)$ has a finite filtration with sections of the form $\Delta(\mu)$ and multiplicities satisfying *BGG reciprocity*:

$$(P(\lambda) : \Delta(\mu)) = [\bar{\Delta}(\mu) : L(\lambda)].$$

The purpose of this article is to explain an elementary approach to the proof of these homological properties starting from the results of [M]; in particular the arguments work for $\mathbb{K}$ of arbitrary characteristic. For various reasons, we are not yet able to treat non-simply-laced finite types, but hope to address these in a sequel.

Our argument relies on another definition of the standard module $\Delta(\lambda)$. In simply-laced types, there is an easy definition of *root modules* $\Delta(\alpha)$ for each $\alpha \in R^+$ categorifying Lusztig's root vectors. The endomorphism algebra of a product $\Delta(\alpha)^{\circ m}$ of m copies of $\Delta(\alpha)$ turns out to be isomorphic to the *nil Hecke algebra* NH_m. Hence we can define the *divided power module* $\Delta(\alpha^m)$ by using a primitive idempotent in the nil Hecke algebra to project to an indecomposable direct summand of $\Delta(\alpha)^{\circ m}$. For $\lambda = (\gamma_1^{m_1}, \dots, \gamma_s^{m_s})$ with $\gamma_1 \succ \cdots \succ \gamma_s$ we show that

$$\Delta(\lambda) \cong \Delta(\gamma_1^{m_1}) \circ \cdots \circ \Delta(\gamma_s^{m_s}).$$

We then derive the homological properties by some straightforward arguments involving generalized Frobenius reciprocity; see Theorem 3.13 for the final result.

The catch with this is that it relies on a computation of Ext^1 for root modules. Fortunately McNamara has already explained *two* approaches to this computation. The first of these depends on knowing in advance that H_α has finite global dimension. The second approach is inductive in nature and depends on the existence of a *minimal pair* (β, γ) for α satisfying the *length two property* formulated precisely in §2.6. We use this argument to perform the Ext^1 calculation for a special choice of the convex ordering $\prec$; see Corollary 5.8. Then we deduce that H_α has finite global dimension (Corollary 4.6), so can complete the Ext^1 calculation for arbitrary convex orderings using the first approach; see Theorem 3.4.

Based on extensive computations with Lyndon orderings, we actually believe that any minimal pair (β, γ) for any $\alpha \in R^+$ with respect to any convex ordering satisfies the length two property; see Conjecture 2.16. This is interesting because we show in Theorem 4.4 that the length two property implies the existence of a short exact sequence

$$0 \to q\Delta(\beta) \circ \Delta(\gamma) \to \Delta(\gamma) \circ \Delta(\beta) \to \Delta(\alpha) \to 0.$$

From this one can inductively construct projective resolutions of the root modules. For multiplicity-free roots (including all roots in type A) these resolutions are very explicit and should be viewed as a variation on the classical Koszul resolution from commutative algebra; see Theorem 4.7. The first non-trivial example comes from the highest root $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ in type A_3. Adopting the notation of Example 5.1 and choosing minimal pairs as in Remark 5.4, our resolution of $\Delta(\alpha)$ in this special case is

$$0 \longrightarrow q^2 H_\alpha 1_{321} \xrightarrow{(-\tau_1\tau_2\ \tau_2)} qH_\alpha 1_{213} \oplus qH_\alpha 1_{312} \xrightarrow{\left(\begin{smallmatrix}\tau_1\\ \tau_1\tau_2\end{smallmatrix}\right)} H_\alpha 1_{123} \longrightarrow 0,$$

where we view elements of the direct sum as row vectors and the maps are defined by right multiplication by the given matrices.

Conventions. By a module V over a $\mathbb{Z}$-graded algebra H, we *always* mean a graded left H-module. We write $\operatorname{rad} V$ (resp. $\operatorname{soc} V$) for the intersection of all maximal submodules (resp. the sum of all irreducible submodules) of V. We write q for the upwards degree shift functor: if $V = \bigoplus_{n\in\mathbb{Z}} V_n$ then qV has $(qV)_n := V_{n-1}$. More generally, given a formal Laurent series $f(q) = \sum_{n\in\mathbb{Z}} f_n q^n$ with coefficients $f_n \in \mathbb{N}$, $f(q)V$ denotes $\bigoplus_{n\in\mathbb{Z}} q^n V^{\oplus f_n}$. For modules U and V, we write $\hom_H(U, V)$ for homogeneous H-module homomorphisms, reserving $\mathrm{Hom}_H(U, V)$ for the graded vector space $\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_H(U, V)_n$ where

$$\mathrm{Hom}_H(U, V)_n := \hom_H(q^n U, V) = \hom_H(U, q^{-n} V).$$

We define $\mathrm{ext}^d_H(U, V)$ and $\mathrm{Ext}^d_H(U, V)$ similarly. Finally, if V is a locally finite dimensional graded vector space, its graded dimension is

$$\operatorname{Dim} V := \sum_{n\in\mathbb{Z}} (\dim V_n) q^n.$$

2. KLR algebras

We begin with some basic facts about the representation theory of KLR algebras. To simplify the exposition, we restrict ourselves from the outset to finite ADE types. On making the usual adjustments to the definitions, the results stated in this section are true in any finite type. The discussion of the contravariant form in §2.7 is new.

2.1. **The quantum group f.** Fix once and for all a quiver of finite ADE type. Let I be the set indexing the vertices of the quiver. For $i, j \in I$, we write $i \to j$ if there is an arrow from i to j, and we write $i \not- j$ if $i \neq j$ are not connected by an edge. Let $C = (c_{i,j})_{i,j \in I}$ be the underlying *Cartan matrix*, so $c_{i,i} := 2$, $c_{i,j} := 0$ if $i \not- j$, and $c_{i,j} := -1$ for all other $i, j \in I$. Let P and Q be the corresponding *weight* and *root lattices*, respectively. Thus P is the free abelian group on basis $(\varpi_i)_{i\in I}$, and Q is the subgroup of P spanned by the *simple roots* $(\alpha_i)_{i \in I}$ defined from $\alpha_j := \sum_{i\in I} c_{i,j}\varpi_i$. There is a unique symmetric bilinear form $P \times P \to \mathbb{Q}, (\alpha, \beta) \mapsto \alpha \cdot \beta$ such that $\varpi_i \cdot \alpha_j = \delta_{i,j}$ for all $i, j \in I$. Let $Q^+ := \bigoplus_{i\in I} \mathbb{N}\alpha_i \subset Q$ and define the *height* of $\alpha \in Q^+$ from $\mathrm{ht}(\alpha) := \sum_{i\in I} \varpi_i \cdot \alpha$.

Now let q be an indeterminate and $\mathbb{A} := \mathbb{Q}(q)$. Let $[n] := (q^n - q^{-n})/(q - q^{-1})$ be the quantum integer and $[n]! := [n][n-1]\cdots[1]$ be the quantum factorial. Let $\mathbf{f}$ be the free associative $\mathbb{A}$-algebra on generators $\{\theta_i \mid i \in I\}$ subject to the *quantum Serre relations*

$$\sum_{r+s=1-c_{i,j}} (-1)^r \theta_i^{(r)} \theta_j \theta_i^{(s)} = 0$$

for all $i, j \in I$ and $r \geq 1$, where $\theta_i^{(r)}$ denotes the *divided power* $\theta_i^r/[r]!$.

Let $\langle I \rangle$ be the free monoid on I, that is, the set of all *words* $\boldsymbol{i} = i_1 \cdots i_n$ for $n \geq 0$ and $i_1, \ldots, i_n \in I$ with multiplication given by concatenation of words. For a word $\boldsymbol{i} = i_1 \cdots i_n$ of length n and a permutation $w \in S_n$, we let

$$|\boldsymbol{i}| := \alpha_{i_1} + \cdots + \alpha_{i_n}, \qquad w(\boldsymbol{i}) := i_{w^{-1}(1)} \cdots i_{w^{-1}(n)},$$

$$\theta_{\boldsymbol{i}} := \theta_{i_1} \cdots \theta_{i_n}, \qquad \deg(w; \boldsymbol{i}) := -\sum_{\substack{1 \leq j < k \leq n \\ w(j) > w(k)}} c_{i_j, i_k}.$$

Setting $\langle I \rangle_\alpha := \{\boldsymbol{i} \in \langle I \rangle \mid |\boldsymbol{i}| = \alpha\}$, the words $\{\theta_{\boldsymbol{i}} \mid \boldsymbol{i} \in \langle I \rangle_\alpha\}$ span the α-weight space $\mathbf{f}_\alpha$ of $\mathbf{f}$. In his book [L2, §1.2.5, §33.1.2], Lusztig shows that there is a well-defined non-degenerate symmetric bilinear form $(\cdot, \cdot)$ on $\mathbf{f}$ defined by properties equivalent to the formula

$$(\theta_{\boldsymbol{i}}, \theta_{\boldsymbol{j}}) = \frac{1}{(1-q^2)^n} \sum_{\substack{w \in S_n \\ w(\boldsymbol{i}) = \boldsymbol{j}}} q^{\deg(w; \boldsymbol{i})}$$

for all words $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle$ with $\boldsymbol{i}$ of length n.

The field $\mathbb{A}$ possesses a unique automorphism called the *bar involution* such that $\overline{q} = q^{-1}$. With respect to this involution, let $\mathtt{b} : \mathbf{f} \to \mathbf{f}$ be the anti-linear algebra automorphism such that $\mathtt{b}(\theta_i) = \theta_i$ for all $i \in I$. Also let $\mathtt{b}^* : \mathbf{f} \to \mathbf{f}$ be the adjoint anti-linear map to $\mathtt{b}$ with respect to Lusztig's form, so $\mathtt{b}^*$ is defined from $(x, \mathtt{b}^*(y)) = \overline{(\mathtt{b}(x), y)}$ for any $x, y \in \mathbf{f}$. It is well known that

$$\mathtt{b}^*(xy) = q^{\beta\cdot\gamma} \mathtt{b}^*(y) \mathtt{b}^*(x) \tag{2.1}$$

for $x \in \mathbf{f}_\beta$ and $y \in \mathbf{f}_\gamma$.

Let $\mathcal{A} := \mathbb{Z}[q, q^{-1}] \subset \mathbb{A}$. Lusztig's $\mathcal{A}$-form $\mathbf{f}_\mathcal{A}$ for $\mathbf{f}$ is the $\mathcal{A}$-subalgebra of $\mathbf{f}$ generated by all $\theta_i^{(r)}$. Also let $\mathbf{f}^*_\mathcal{A}$ be the dual of $\mathbf{f}_\mathcal{A}$ with respect to the form $(\cdot, \cdot)$, i.e. $\mathbf{f}^*_\mathcal{A} := \{y \in \mathbf{f} \mid (x, y) \in \mathcal{A} \text{ for all } x \in \mathbf{f}_\mathcal{A}\}$. Both $\mathbf{f}_\mathcal{A}$ and $\mathbf{f}^*_\mathcal{A}$ are free

$\mathcal{A}$-modules and we can identify $\mathbf{f} = \mathbb{A} \otimes_{\mathcal{A}} \mathbf{f}_{\mathcal{A}} = \mathbb{A} \otimes_{\mathcal{A}} \mathbf{f}^*_{\mathcal{A}}$. The maps $\mathtt{b}$ and $\mathtt{b}^*$ preserve $\mathbf{f}_{\mathcal{A}}$ and $\mathbf{f}^*_{\mathcal{A}}$, respectively.

2.2. **The KLR algebra.** Fix now a field $\mathbb{K}$. For $\alpha \in Q^+$ of height n, the *KLR algebra* H_α is the associative, unital $\mathbb{K}$-algebra defined by generators

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in \langle I \rangle_\alpha\} \cup \{x_1, \dots, x_n\} \cup \{\tau_1, \dots, \tau_{n-1}\}$$

subject only to the following relations:

- the elements $\{1_{\boldsymbol{i}} \,|\, \boldsymbol{i} \in \langle I \rangle_\alpha\}$ are mutually orthogonal idempotents whose sum is the identity $1_\alpha \in H_\alpha$;
- $x_k 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} x_k$ and $x_k x_l = x_l x_k$;
- $\tau_k 1_{\boldsymbol{i}} = 1_{(k\,k{+}1)(\boldsymbol{i})} \tau_k$ and $\tau_k \tau_l = \tau_l \tau_k$ if $|k-l| > 1$;
- $\tau_k^2 1_{\boldsymbol{i}} = \begin{cases} 0 & \text{if } i_k = i_{k+1}, \\ (x_k - x_{k+1}) 1_{\boldsymbol{i}} & \text{if } i_k \leftarrow i_{k+1}, \\ (x_{k+1} - x_k) 1_{\boldsymbol{i}} & \text{if } i_k \rightarrow i_{k+1}, \\ 1_{\boldsymbol{i}} & \text{if } i_k \not\!- i_{k+1}; \end{cases}$
- $(\tau_k x_l - x_{(k\,k{+}1)(l)} \tau_k) 1_{\boldsymbol{i}} = \begin{cases} 1_{\boldsymbol{i}} & \text{if } l = k+1 \text{ and } i_k = i_{k+1}, \\ -1_{\boldsymbol{i}} & \text{if } l = k \text{ and } i_k = i_{k+1}, \\ 0 & \text{otherwise;} \end{cases}$
- $(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) 1_{\boldsymbol{i}} = \begin{cases} 1_{\boldsymbol{i}} & \text{if } i_k = i_{k+2} \text{ and } i_k \leftarrow i_{k+1}, \\ -1_{\boldsymbol{i}} & \text{if } i_k = i_{k+2} \text{ and } i_k \rightarrow i_{k+1}, \\ 0 & \text{otherwise.} \end{cases}$

The algebra H_α is naturally $\mathbb{Z}$-graded with each $1_{\boldsymbol{i}}$ in degree zero, each x_k in degree two, and $\tau_k 1_{\boldsymbol{i}}$ in degree $-c_{i_k, i_{k+1}}$. There's also an anti-automorphism $T : H_\alpha \to H_\alpha$ which fixes all the generators. Here are a few other basic facts about the structure of these algebras established in [KL] or [R1]:

- Fix once and for all a reduced expression for each $w \in S_n$ and let τ_w be the corresponding product of the τ-generators of H_α. Note that $\tau_w 1_{\boldsymbol{i}}$ is of degree $\deg(w; \boldsymbol{i})$. The monomials
$$\{x_1^{k_1} \cdots x_n^{k_n} \tau_w 1_{\boldsymbol{i}} \mid w \in S_n, k_1, \dots, k_n \geq 0, \boldsymbol{i} \in \langle I \rangle_\alpha\} \tag{2.2}$$
give a basis for H_α. In particular, H_α is locally finite dimensional and bounded below.
- Pick $\boldsymbol{i} \in \langle I \rangle_\alpha$ so that $S_{\boldsymbol{i}} := \mathrm{Stab}_{S_n}(\boldsymbol{i})$ is a standard parabolic subgroup of S_n. For $j = 1, \dots, n$, let
$$y_j := \sum_{w \in S_n / S_{\boldsymbol{i}}} x_{w(j)} 1_{w(\boldsymbol{i})}, \tag{2.3}$$
where $S_n / S_{\boldsymbol{i}}$ denotes the set of minimal length left coset representatives. These elements generate a free polynomial algebra $\mathbb{K}[y_1, \dots, y_n]$ inside H_α. Moreover, letting S_n act on $\mathbb{K}[y_1, \dots, y_n]$ by permuting the generators, the invariant subalgebra $\mathbb{K}[y_1, \dots, y_n]^{S_{\boldsymbol{i}}}$ is exactly the center $Z(H_\alpha)$ of the KLR algebra. Combined with the basis theorem, it follows that H_α is free of finite rank as a module over its center (forgetting the grading the rank is $(n!)^2$).
- For $m \geq 1$ and $i \in I$, the KLR algebra $H_{m\alpha_i}$ is identified with the *nil Hecke algebra* NH_m, that is, the algebra with generators $x_1, \dots, x_m$ and

$\tau_1, \dots, \tau_{m-1}$ subject to the following relations: $x_i x_j = x_j x_i$; $\tau_i x_j = x_j \tau_i$ for $j \neq i, i+1$; $\tau_i x_{i+1} = x_i \tau_i + 1$; $x_{i+1}\tau_i = \tau_i x_i + 1$; $\tau_i^2 = 0$; and the usual type A braid relations amongst $\tau_1, \dots, \tau_{m-1}$. It is well known that the nil Hecke algebra is a matrix algebra over its center; see e.g. [R2, §2] for a recent exposition. Moreover, writing $w_{[1,m]}$ for the longest element of S_m, the degree zero element

$$e_m := x_2 x_3^2 \cdots x_m^{m-1} \tau_{w_{[1,m]}} \tag{2.4}$$

is a primitive idempotent, hence $P(\alpha_i^m) := q^{\frac{1}{2}m(m-1)} H_{m\alpha_i} e_m$ is an indecomposable projective $H_{m\alpha_i}$-module. The degree shift here has been chosen so that irreducible head $L(\alpha_i^m)$ of $P(\alpha_i^m)$ has graded dimension $[m]^!$. Thus $H_{m\alpha_i} \cong [m]^! P(\alpha_i^m)$ as a left module.

For $\beta, \gamma \in Q^+$, there is an evident non-unital algebra embedding $H_\beta \otimes H_\gamma \hookrightarrow H_{\beta+\gamma}$. We denote the image of the identity $1_\beta \otimes 1_\gamma \in H_\beta \otimes H_\gamma$ by $1_{\beta,\gamma} \in H_{\beta+\gamma}$. Then for an $H_{\beta+\gamma}$-module U and an $H_\beta \otimes H_\gamma$-module V, we set

$$\mathrm{res}^{\beta+\gamma}_{\beta,\gamma} U := 1_{\beta,\gamma} U, \qquad \mathrm{ind}^{\beta+\gamma}_{\beta,\gamma} V := H_{\beta+\gamma} 1_{\beta,\gamma} \otimes_{H_\beta \otimes H_\gamma} V,$$

which are naturally $H_\beta \otimes H_\gamma$- and $H_{\beta+\gamma}$-modules, respectively. These definitions extend in an obvious way to situations where there are more than two tensor factors. The following Mackey-type theorem is of crucial importance.

Theorem 2.1. *Suppose we are given $\beta, \gamma, \beta', \gamma' \in Q^+$ of heights m, n, m', n', respectively, such that $\beta + \gamma = \beta' + \gamma'$. Setting $k := \min(m, n, m', n')$, let $\{1 = w_0 < \cdots < w_k\}$ be the set of minimal length $S_{m'} \times S_{n'} \backslash S_{m+n} / S_m \times S_n$-double coset representatives ordered via the Bruhat order. For any $H_\beta \otimes H_\gamma$-module V, there is a filtration*

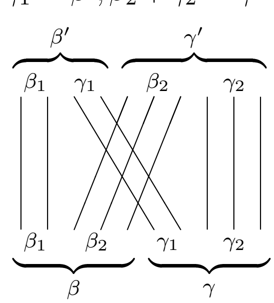
$$0 = V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k = \mathrm{res}^{\beta'+\gamma'}_{\beta',\gamma'} \circ \mathrm{ind}^{\beta+\gamma}_{\beta,\gamma}(V)$$

defined by $V_j := \sum_{i=0}^{j} \sum_{w \in (S_{m'} \times S_{n'}) w_i (S_m \times S_n)} 1_{\beta',\gamma'} \tau_w 1_{\beta,\gamma} \otimes V$. Moreover there is a unique isomorphism of $H_{\beta'} \otimes H_{\gamma'}$-modules

$$V_j / V_{j-1} \xrightarrow{\sim} \bigoplus_{\beta_1,\beta_2,\gamma_1,\gamma_2} q^{-\beta_2 \cdot \gamma_1} \mathrm{ind}^{\beta',\gamma'}_{\beta_1,\gamma_1,\beta_2,\gamma_2} \circ I^* \circ \mathrm{res}^{\beta,\gamma}_{\beta_1,\beta_2,\gamma_1,\gamma_2}(V),$$

$$1_{\beta',\gamma'} \tau_{w_j} 1_{\beta,\gamma} \otimes v + V_{j-1} \mapsto \sum_{\beta_1,\beta_2,\gamma_1,\gamma_2} 1_{\beta_1,\gamma_1,\beta_2,\gamma_2} \otimes 1_{\beta_1,\beta_2,\gamma_1,\gamma_2} v,$$

where $I : H_{\beta_1} \otimes H_{\gamma_1} \otimes H_{\beta_2} \otimes H_{\gamma_2} \xrightarrow{\sim} H_{\beta_1} \otimes H_{\beta_2} \otimes H_{\gamma_1} \otimes H_{\gamma_2}$ is the obvious isomorphism, and the sums are taken over all $\beta_1, \beta_2, \gamma_1, \gamma_2 \in Q^+$ such that $\beta_1 + \beta_2 = \beta, \gamma_1 + \gamma_2 = \gamma, \beta_1 + \gamma_1 = \beta', \beta_2 + \gamma_2 = \gamma'$ and $\min(\mathrm{ht}(\beta_2), \mathrm{ht}(\gamma_1)) = j$*:*

$w_j =$ 

Proof. This follows from [KL, Proposition 2.18]. $\square$

2.3. **The categorification theorem.** Let $\mathrm{Rep}(H_\alpha)$ denote the abelian category of finite dimensional H_α-modules and set

$$\mathrm{Rep}(H) := \bigoplus_{\alpha \in Q^+} \mathrm{Rep}(H_\alpha).$$

This is a graded $\mathbb{K}$-linear monoidal category with respect to the induction product $U \circ V := \mathrm{ind}_{\beta,\gamma}^{\beta+\gamma}(U \boxtimes V)$ for $U \in \mathrm{Rep}(H_\beta)$ and $V \in \mathrm{Rep}(H_\gamma)$. Let $[\mathrm{Rep}(H)] = \bigoplus_{\alpha \in Q^+}[\mathrm{Rep}(H_\alpha)]$ denote its Grothendieck ring, which we make into an $\mathcal{A}$-algebra so that $q[V] = [qV]$. Dually, we have the additive category $\mathrm{Proj}(H_\alpha)$ of finitely generated projective H_α-modules and set

$$\mathrm{Proj}(H) := \bigoplus_{\alpha \in Q^+} \mathrm{Proj}(H_\alpha).$$

Again this is a graded $\mathbb{K}$-linear monoidal category with respect to the induction product, and again the split Grothendieck group $[\mathrm{Proj}(H)] = \bigoplus_{\alpha \in Q^+}[\mathrm{Proj}(H_\alpha)]$ is naturally an $\mathcal{A}$-algebra. Moreover there is a non-degenerate pairing

$$(\cdot,\cdot) : [\mathrm{Proj}(H)] \times [\mathrm{Rep}(H)] \to \mathcal{A}$$

defined on $P \in \mathrm{Proj}(H_\alpha)$ and $V \in \mathrm{Rep}(H_\beta)$ by declaring that $([P],[V]) := \mathrm{Dim}\, T^*(P) \otimes_{H_\alpha} V$ if $\beta = \alpha$, $([P],[V]) := 0$ otherwise. Finally there are dualities $\circledast$ on $\mathrm{Rep}(H_\alpha)$ and $\#$ on $\mathrm{Proj}(H_\alpha)$ inducing antilinear involutions on the Grothendieck groups. These are defined from $V^\circledast := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ and $P^\# := \mathrm{Hom}_{H_\alpha}(P, H_\alpha)$, respectively, both viewed as left modules via the anti-automorphism T. By [KR2, Lemma 3.2], we have that $([P^\#],[V]) = \overline{([P],[V^\circledast])}$.

Theorem 2.2 (Khovanov-Lauda)**.** *There is a unique adjoint pair of $\mathcal{A}$-algebra isomorphisms*

$$\gamma : \mathbf{f}_\mathcal{A} \xrightarrow{\sim} [\mathrm{Proj}(H)], \qquad \gamma^* : [\mathrm{Rep}(H)] \xrightarrow{\sim} \mathbf{f}_\mathcal{A}^*$$

such that $\gamma(\theta_i^{(n)}) = [P(\alpha_i^n)]$. Under these isomorphisms, the antilinear involutions $\mathtt{b}$ and $\mathtt{b}^*$ on $\mathbf{f}_\mathcal{A}$ and $\mathbf{f}_\mathcal{A}^*$ correspond to the dualities $\#$ and $\circledast$, respectively.

Proof. See [KL, §3] for the statement about γ. The dual statement is implicit in [KL]; see also [KR2, Theorem 4.4] for an expanded account. $\square$

Henceforth we will *identify* $\mathbf{f}_\mathcal{A}$ with $[\mathrm{Proj}(H)]$ and $\mathbf{f}_\mathcal{A}^*$ with $[\mathrm{Rep}(H)]$ according to Theorem 2.2. We also note the following lemma, which is the module-theoretic analogue of (2.1).

Lemma 2.3. *For $U \in \mathrm{Rep}(H_\beta)$ and $V \in \mathrm{Rep}(H_\gamma)$, there is a natural isomorphism $(U \circ V)^\circledast \cong q^{\beta\cdot\gamma}(V^\circledast \circ U^\circledast)$.*

Proof. This is [LV, Theorem 2.2(2)]. $\square$

Any H_α-module V admits a decomposition into *word spaces* $V = \bigoplus_{\boldsymbol{i} \in \langle I\rangle_\alpha} 1_{\boldsymbol{i}} V$. Then the *character* of $V \in \mathrm{Rep}(H_\alpha)$ is the formal sum

$$\mathrm{Ch}\, V = \sum_{\boldsymbol{i} \in \langle I\rangle_\alpha} (\mathrm{Dim}\, 1_{\boldsymbol{i}} V)\boldsymbol{i} \in \mathcal{A}\langle I\rangle_\alpha,$$

where $\mathcal{A}\langle I\rangle_\alpha$ is the free $\mathcal{A}$-module on basis $\langle I\rangle_\alpha$. Letting $\mathcal{A}\langle I\rangle := \bigoplus_{\alpha\in Q^+}\mathcal{A}\langle I\rangle_\alpha$, we obtain from this a map $\mathrm{Ch} : [\mathrm{Rep}(H)] \to \mathcal{A}\langle I\rangle$. Viewing $\mathcal{A}\langle I\rangle$ as an $\mathcal{A}$-algebra with respect to the *shuffle product* $\circ$ defined on words $\boldsymbol{i}$ and $\boldsymbol{j}$ of lengths m and n, respectively, by

$$\boldsymbol{i}\circ\boldsymbol{j} := \sum_{\substack{w\in S_{m+n}\\ w(1)<\cdots<w(m)\\ w(m+1)<\cdots<w(m+n)}} q^{\deg(w;\boldsymbol{ij})} w(\boldsymbol{ij}), \tag{2.5}$$

it is known that Ch is an injective algebra homomorphism; see [KL, §3]. Obviously, Ch intertwines the duality $\circledast$ on $[\mathrm{Rep}(H)]$ with the antilinear involution of $\mathcal{A}\langle I\rangle$ which fixes all the basis vectors $\boldsymbol{i}\in\langle I\rangle$.

2.4. **PBW and dual PBW bases.** Returning to the discussion of the quantum group $\mathbf{f}$, let W be the *Weyl group* associated to our fixed Cartan matrix. So W is the subgroup of $GL(P)$ generated by the *simple reflections* $\{s_i \mid i \in I\}$ defined from $s_i(\beta) := \beta - (\alpha_i\cdot\beta)\alpha_i$. Let $R := \bigcup_{i\in I} W\alpha_i$ be the root system and $R^+ := R \cap Q^+$ denote the *positive roots.*

Lemma 2.4. *Given positive roots $\beta_1,\dots,\beta_l$ ($l \geq 2$) whose sum is a positive root, there exists a partition $J \sqcup K$ of the set $\{1,\dots,l\}$ such that $\sum_{j\in J}\beta_j$ and $\sum_{k\in K}\beta_k$ are positive roots.*

Proof. See [M, Lemma 2.1]. □

Recall that a *convex ordering* on R^+ is a total ordering $\prec$ such that

$$\beta,\gamma,\beta+\gamma\in R^+, \beta\prec\gamma \quad\Rightarrow\quad \beta\prec\beta+\gamma\prec\gamma.$$

By [P], there is a bijection between convex orderings of R^+ and reduced expressions for the longest element w_0 of W: given a reduced expression $w_0 = s_{i_1}\cdots s_{i_N}$ the corresponding convex ordering on R^+ is given by

$$\alpha_{i_1} \prec s_{i_1}(\alpha_{i_2}) \prec s_{i_1}s_{i_2}(\alpha_{i_3}) \prec \cdots \prec s_{i_1}\cdots s_{i_{N-1}}(\alpha_{i_N}).$$

We assume henceforth that such a convex ordering/reduced expression has been specified.

Lemma 2.5. *Suppose we are given positive roots $\alpha,\beta_1,\dots,\beta_k,\gamma_1,\dots,\gamma_l$ such that $\beta_i \preceq \alpha \preceq \gamma_j$ for all i and j. We have that $\beta_1+\cdots+\beta_k = \gamma_1+\cdots+\gamma_l$ if and only if $k = l$ and $\beta_1 = \cdots = \beta_k = \gamma_1 = \cdots = \gamma_l = \alpha$.*

Proof. Suppose that $\beta_1+\cdots+\beta_k = \gamma_1+\cdots+\gamma_l$. We may assume for suitable $0 \leq k' \leq k$ and $0 \leq l' \leq l$ that $\beta_i = \alpha$ for $1 \leq i \leq k'$, $\beta_i \prec \alpha$ for $k'+1 \leq i \leq k$ and $\gamma_i = \alpha$ for $1 \leq i \leq l'$, $\gamma_i \succ \alpha$ for $l'+1 \leq i \leq l$. Then we need to show that $k = k' = l' = l$. Assume the convex ordering corresponds to reduced expression $w_0 = s_{i_1}\cdots s_{i_N}$ as above. Then $\alpha = s_{i_1}\cdots s_{i_{j-1}}(\alpha_{i_j})$ for a unique $1 \leq j \leq N$. If $k' \geq l'$, let $w := s_{i_j}\cdots s_{i_1}$. From $\beta_1+\cdots+\beta_k = \gamma_1+\cdots+\gamma_l$, we deduce that

$$(k'-l')w(\alpha) + w(\beta_{k'+1}) + \cdots + w(\beta_k) = w(\gamma_{l'+1}) + \cdots + w(\gamma_l).$$

By [B, Ch. VI, §6, Cor. 2], the set of positive roots sent to negative roots by w is the set $\{\alpha' \in R^+ \mid \alpha' \preceq \alpha\}$. Hence the left hand side of the above equation is a sum of negative roots and the right hand side is a sum of positive roots. So both sides are zero and we deduce that $k = k' = l' = l$. For the case $k' \leq l'$, argue in a similar way with $w := s_{i_{j-1}} \cdots s_{i_1}$, so that the set of positive roots sent to negative by w is $\{\alpha' \in R^+ \mid \alpha' \prec \alpha\}$. $\square$

Corresponding to the chosen convex ordering/reduced expression, Lusztig has introduced *root vectors* $\{r_\alpha \mid \alpha \in R^+\}$ in $\mathbf{f}$ via a certain braid group action. The definition uses the embedding of $\mathbf{f}$ into the full quantum group so we only summarize it briefly: we take the positive embedding (so $\theta_i \mapsto E_i$) and use the braid group generators $T_i := T''_{i,+}$ from [L2, §37.1.3] (with $v = q^{-1}$); then for $\alpha \in R^+$ we have that

$$r_\alpha := T_{i_1} \cdots T_{i_{j-1}}(\theta_{i_j})$$

if $\alpha = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$. For example, in type A_2 with $I = \{1,2\}$ and fixed reduced expression $w_0 = s_1 s_2 s_1$, so that $\alpha_1 \prec \alpha_1 + \alpha_2 \prec \alpha_2$, we have that $r_{\alpha_1} = \theta_1, r_{\alpha_1+\alpha_2} = \theta_1\theta_2 - q\theta_2\theta_1, r_{\alpha_2} = \theta_2$. Also introduce the *dual root vector*

$$r^*_\alpha := (1 - q^2) r_\alpha. \tag{2.6}$$

This is invariant under $\mathtt{b}^*$, as can be checked directly using (2.1) and the formulae in [L2, §37.2.4].

A *Kostant partition* of $\alpha \in Q^+$ is a sequence $\lambda = (\lambda_1, \dots, \lambda_l)$ of positive roots such that $\lambda_1 \succeq \cdots \succeq \lambda_l$ and $\lambda_1 + \cdots + \lambda_l = \alpha$. Denote the set of all Kostant partitions of α by $\mathrm{KP}(\alpha)$. For $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}(\alpha)$, let $m_\beta(\lambda)$ denote the multiplicity of $\beta \in R^+$ as a part of λ. Also set $\lambda'_k := \lambda_{l+1-k}$ for $k = 1, \dots, l$. Then define a partial ordering $\preceq$ on $\mathrm{KP}(\alpha)$ so that $\lambda \prec \mu$ if and only if both of the following hold:

- $\lambda_1 = \mu_1, \dots, \lambda_{k-1} = \mu_{k-1}$ and $\lambda_k \prec \mu_k$ for some k such that λ_k and μ_k both make sense;
- $\lambda'_1 = \mu'_1, \dots, \lambda'_{k-1} = \mu'_{k-1}$ and $\lambda'_k \succ \mu'_k$ for some k such that λ'_k and μ'_k both make sense.

This ordering was introduced in [M, §3], and the following useful lemmas were noted already there (at least implicitly).

Lemma 2.6. *For $\alpha \in R^+$ and $m \geq 1$, the Kostant partition (α^m) is the unique smallest element of $\mathrm{KP}(m\alpha)$.*

Proof. Suppose that $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}(\alpha)$ satisfies $\lambda \not\succ (\alpha^m)$. Then we either have that $\lambda_1 \preceq \alpha$ or that $\lambda'_1 \succeq \alpha$. In the former case, $\lambda_k \preceq \alpha$ for all k, while in the latter $\lambda_k \succeq \alpha$ for all k. Either way, applying Lemma 2.5 to the equality $\lambda_1 + \cdots + \lambda_l = \alpha + \cdots + \alpha$ (m times), we deduce that $\lambda = (\alpha^m)$. $\square$

Lemma 2.7. *For $\alpha \in R^+$, suppose that $\lambda \in \mathrm{KP}(\alpha)$ is minimal such that $\lambda \succ \alpha$. Then λ has two parts, i.e. $\lambda = (\beta, \gamma)$ for positive roots $\beta \succ \alpha \succ \gamma$.*

Proof. Suppose for a contradiction that $\lambda = (\lambda_1, \dots, \lambda_l)$ with $l \geq 3$. By Lemma 2.4, we can partition the set $\{1, \dots, l\}$ as $J \sqcup K$ so that $\beta := \sum_{j \in J} \lambda_j$ and $\gamma := \sum_{k \in K} \lambda_k$ are positive roots with $\beta \succ \gamma$. Each λ_j is $\preceq \lambda_1$ hence by Lemma 2.5 we have that $\beta \preceq \lambda_1$. Moreover if it happens that $\beta = \lambda_1$ then $\gamma = \alpha - \beta = \lambda_2 + \cdots + \lambda_l$ and we see similarly that $\gamma \preceq \lambda_2$. As $l \geq 3$, this shows that either $\beta \prec \lambda_1$, or $\beta = \lambda_1$ and $\gamma \prec \lambda_2$. A similar argument shows that either $\gamma \succ \lambda'_1$, or $\gamma = \lambda'_1$ and $\beta \succ \lambda'_2$. Hence $(\beta, \gamma) \prec \lambda$. But also we know that $(\beta, \gamma) \succ (\alpha)$ by Lemma 2.6. So this contradicts the minimality of λ. $\square$

Let $\mathrm{KP} := \bigcup_{\alpha \in Q^+} \mathrm{KP}(\alpha)$. For $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}$, we set

$$r_\lambda := r_{\lambda_1} \cdots r_{\lambda_l} / [\lambda]!, \qquad r^*_\lambda := q^{s_\lambda} r^*_{\lambda_1} \cdots r^*_{\lambda_l}, \tag{2.7}$$

where

$$[\lambda]! := \prod_{\beta \in R^+} [m_\beta(\lambda)]!, \qquad s_\lambda := \sum_{\beta \in R^+} m_\beta(\lambda)(m_\beta(\lambda) - 1)/2.$$

The following key result is due to Lusztig; it gives us the *PBW* and *dual PBW bases* for $\mathbf{f}$ arising from the given convex ordering $\prec$.

Theorem 2.8 (Lusztig)**.** *The monomials* $\{r_\lambda \mid \lambda \in \mathrm{KP}\}$ *and* $\{r^*_\lambda \mid \lambda \in \mathrm{KP}\}$ *give a pair of dual bases for the free* $\mathcal{A}$*-modules* $\mathbf{f}_\mathcal{A}$ *and* $\mathbf{f}^*_\mathcal{A}$*, respectively.*

Proof. This follows from [L2, Corollary 41.1.4(b)], [L2, Proposition 41.1.7] and [L2, Proposition 38.2.3]. □

2.5. **Proper standard modules.** The next important results were obtained by McNamara in [M, §3]. Note that we have slightly different conventions to [M]; we have reformulated the statements below accordingly. The modules $L(\alpha)$ defined in the next theorem are called *cuspidal modules*; this language originated in [KR2].

Theorem 2.9 (McNamara)**.** *For* $\alpha \in R^+$ *there is a unique (up to isomorphism) irreducible* H_α*-module* $L(\alpha)$ *such that* $[L(\alpha)] = r^*_\alpha$*. Moreover, for any* $m \geq 1$*, the module* $L(\alpha^m) := q^{\frac{1}{2}m(m-1)} L(\alpha)^{\circ m}$ *is irreducible.*

Proof. The existence of $L(\alpha)$ is the first part of [M, Theorem 3.1]. The second part is [M, Lemma 3.4]. □

Suppose we are given $\alpha \in Q^+$ and $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}(\alpha)$. Define the *proper standard module*

$$\bar{\Delta}(\lambda) := q^{s_\lambda} L(\lambda_1) \circ \cdots \circ L(\lambda_l). \tag{2.8}$$

It is immediate from Theorem 2.9 and the definition (2.7) that $[\bar{\Delta}(\lambda)] = r^*_\lambda$, i.e. proper standard modules categorify the dual PBW basis. Let

$$L(\lambda) := \bar{\Delta}(\lambda) / \mathrm{rad}\, \bar{\Delta}(\lambda).$$

The following theorem asserts in particular that this is a self-dual irreducible module.

Theorem 2.10 (McNamara)**.** *For* $\alpha \in Q^+$*, the modules* $\{L(\lambda) \mid \lambda \in \mathrm{KP}(\alpha)\}$ *give a complete set of pairwise inequivalent* $\circledast$*-self-dual irreducible* H_α*-modules. Moreover, for any* $\lambda \in \mathrm{KP}(\alpha)$*, all composition factors of* $\mathrm{rad}\, \bar{\Delta}(\lambda)$ *are of the form* $q^n L(\mu)$ *for* $\mu \prec \lambda$ *and* $n \in \mathbb{Z}$*.*

Proof. This is [M, Theorem 3.1]. □

For $\lambda \in \mathrm{KP}$, we denote the projective cover of $L(\lambda)$ by $P(\lambda)$. Also introduce the *proper costandard module*

$$\bar{\nabla}(\lambda) := \bar{\Delta}(\lambda)^\circledast. \tag{2.9}$$

It is immediate from Theorem 2.10 that $\bar{\nabla}(\lambda)$ has irreducible socle $L(\lambda)^\circledast \cong L(\lambda)$. Let us also record "McNamara's Lemma" which is at the heart of the proof of both of the above theorems.

Lemma 2.11. *Suppose we are given $\alpha \in R^+$ and $\beta, \gamma \in Q^+$ with $\beta + \gamma = \alpha$. If $\mathrm{res}^\alpha_{\beta,\gamma} L(\alpha) \neq 0$ then β is a sum of positive roots $\preceq \alpha$ and γ is a sum of positive roots $\succeq \alpha$.*

Proof. This is [M, Lemma 3.2]. $\square$

Here are some further consequences.

Lemma 2.12. *For $\alpha \in R^+$ and $m \geq 1$, we have that*

$$\left[\mathrm{res}^{m\alpha}_{\alpha,\dots,\alpha} L(\alpha^m)\right] = [m]! \left[L(\alpha)^{\boxtimes m}\right].$$

Proof. It suffices to show for $m \geq 2$ that

$$\left[\mathrm{res}^{m\alpha}_{\alpha,(m-1)\alpha} L(\alpha^m)\right] = [m] \left[L(\alpha) \boxtimes L(\alpha^{m-1})\right].$$

For this we apply Theorem 2.1, noting that $L(\alpha^m) = q^{m-1} L(\alpha) \circ L(\alpha^{m-1})$. To understand the non-zero layers in the Mackey filtration, we need to find all quadruples $(\beta_1, \beta_2, \gamma_1, \gamma_2)$ such that $\beta_1 + \beta_2 = \beta_1 + \gamma_1 = \alpha, \gamma_1 + \gamma_2 = \beta_2 + \gamma_2 = (m-1)\alpha$, $\mathrm{res}^\alpha_{\beta_1,\beta_2} L(\alpha) \neq 0$ and $\mathrm{res}^{(m-1)\alpha}_{\gamma_1,\gamma_2} L(\alpha^{m-1}) \neq 0$. By Lemma 2.11 and Mackey, both β_1 and γ_1 are sums of positive roots $\preceq \alpha$. Since $\beta_1 + \gamma_1 = \alpha$, we deduce using Lemma 2.5 that either $\beta_1 = 0$ or $\gamma_1 = 0$. This analysis shows that there are just two non-zero layers in the Mackey filtration. The bottom non-zero layer is obviously equal in the Grothendieck group to $q^{m-1}\left[L(\alpha) \circ L(\alpha^{m-1})\right]$. Also using some induction on m, the top non-zero layer contributes $q^{-1}[m-1]\left[L(\alpha) \circ L(\alpha^{m-1})\right]$. Finally observe that $q^{m-1} + q^{-1}[m-1] = [m]$. $\square$

Lemma 2.13. *Suppose we are given $\alpha \in Q^+$ and $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}(\alpha)$. Let $\mathrm{res}^\alpha_\lambda$ denote the functor $\mathrm{res}^\alpha_{\lambda_1,\dots,\lambda_l}$. Then*

$$\left[\mathrm{res}^\alpha_\lambda \bar{\Delta}(\lambda)\right] = [\lambda]! \left[L(\lambda_1) \boxtimes \cdots \boxtimes L(\lambda_l)\right],$$

Moreover for any $\mu \not\preceq \lambda$ we have that $\mathrm{res}^\alpha_\mu \bar{\Delta}(\lambda) = 0$.

Proof. This follows from [M, Lemma 3.3] and Lemma 2.12. $\square$

2.6. **Minimal pairs and the length two conjecture.** Still following [M], we refer to the pairs $\lambda = (\beta, \gamma)$ from the statement of Lemma 2.7 as the *minimal pairs* for α. Let $\mathrm{MP}(\alpha)$ denote the set of all minimal pairs for $\alpha \in R^+$.

For $\lambda = (\beta, \gamma) \in \mathrm{MP}(\alpha)$, it is immediate from Theorem 2.10 that all composition factors of $\mathrm{rad}\, \bar{\Delta}(\lambda)$ are isomorphic to $L(\alpha)$ (up to degree shift). Since $\bar{\Delta}(\lambda) = L(\beta) \circ L(\gamma)$ and $(L(\beta) \circ L(\gamma))^\circledast \cong q^{-1} L(\gamma) \circ L(\beta)$ by Lemma 2.3, we deduce that there are short exact sequences

$$0 \longrightarrow X \longrightarrow L(\beta) \circ L(\gamma) \longrightarrow L(\lambda) \longrightarrow 0, \tag{2.10}$$

$$0 \longrightarrow qL(\lambda) \longrightarrow L(\gamma) \circ L(\beta) \longrightarrow Y \longrightarrow 0, \tag{2.11}$$

for certain finite dimensional modules X and Y, both of which only have composition factors isomorphic to $L(\alpha)$ (up to degree shift). We say that λ has the *length two property* if $[\bar{\Delta}(\lambda)] = [L(\lambda)] + q[L(\alpha)]$. In that case, (2.10)–(2.11) simplify to

$$0 \longrightarrow qL(\alpha) \longrightarrow L(\beta) \circ L(\gamma) \longrightarrow L(\lambda) \longrightarrow 0, \tag{2.12}$$

$$0 \longrightarrow qL(\lambda) \longrightarrow L(\gamma) \circ L(\beta) \longrightarrow L(\alpha) \longrightarrow 0. \tag{2.13}$$

Here are some more of the important consequences of the length two property.

Lemma 2.14. *Let $\lambda = (\beta, \gamma)$ be a minimal pair for $\alpha \in R^+$ with the length two property. Then we have that*

$$r_\alpha = r_\gamma r_\beta - q r_\beta r_\gamma, \qquad\qquad r_\alpha^* = (r_\gamma^* r_\beta^* - q r_\beta^* r_\gamma^*)/(1-q^2).$$

Proof. On passing to the Grothendieck group, the short exact sequences (2.12)–(2.13) imply that

$$[L(\gamma) \circ L(\beta)] - q[L(\beta) \circ L(\gamma)] = (1-q^2)[L(\alpha)].$$

The second equality follows from this combined with Theorems 2.2 and 2.9. Then divide both sides by $(1-q^2)$ and use (2.6) to deduce the first equality. □

Lemma 2.15. *Suppose that $\lambda = (\beta, \gamma)$ is a minimal pair for $\alpha \in R^+$. Then:*

1. $\mathrm{res}^\alpha_{\beta,\gamma} L(\alpha) = 0$.
2. $\mathrm{res}^\alpha_{\beta,\gamma} L(\beta) \circ L(\gamma) \cong \mathrm{res}^\alpha_{\beta,\gamma} L(\lambda) \cong L(\beta) \boxtimes L(\gamma)$.
3. $\mathrm{res}^\alpha_{\gamma,\beta} L(\alpha)$ *is non-zero and all of its composition factors are of the form* $L(\gamma) \boxtimes L(\beta)$ *(up to degree shift).*
4. $\mathrm{res}^\alpha_{\gamma,\beta} L(\alpha) \cong L(\gamma) \boxtimes L(\beta)$ *providing λ has the length two property.*

Proof. A special case of Lemma 2.13 shows both that $\mathrm{res}^\alpha_{\beta,\gamma} L(\alpha) = 0$ and that $\mathrm{res}^\alpha_{\beta,\gamma} L(\beta) \circ L(\gamma) \cong L(\beta) \boxtimes L(\gamma)$. Then apply the exact functor $\mathrm{res}^\alpha_{\beta,\gamma}$ to (2.10) to deduce that $\mathrm{res}^\alpha_{\beta,\gamma} L(\lambda) \cong L(\beta) \boxtimes L(\gamma)$. The fact that $\mathrm{res}^\alpha_{\gamma,\beta} L(\alpha)$ is non-zero is justified in the course of the proof of [M, Theorem 3.1(1)]. Also [M, Lemma 4.1] shows that $\left[\mathrm{res}^\alpha_{\gamma,\beta} L(\alpha)\right]$ is a scalar multiple of $[L(\gamma) \boxtimes L(\beta)]$. To see that the scalar is one when the length two property holds, we make a computation using Lusztig's form and Lemma 2.14:

$$\begin{aligned}(r_\gamma \otimes r_\beta, \Delta(r_\alpha^*)) &= (r_\gamma r_\beta, r_\alpha^*) = (r_\gamma r_\beta - q r_\beta r_\gamma, r_\alpha^*) + q(r_\beta r_\gamma, r_\alpha^*)\\ &= (r_\alpha, r_\alpha^*) + q(r_\beta \otimes r_\gamma, \Delta(r_\alpha^*)) = 1.\end{aligned}$$

(Here Δ denotes the coproduct on $\mathbf{f}$ which corresponds to restriction under the categorification theorem.) □

We say that the convex ordering $\prec$ has the *length two property* if all minimal pairs for all $\alpha \in R^+$ have the length two property; see §5.2 for examples.

Conjecture 2.16. *All convex orderings have the length two property.*

2.7. **The contravariant form and Williamson's counterexample.** The results in this subsection are not needed in the remainder of the article but are of independent interest.

Theorem 2.17. *For $\lambda \in \mathrm{KP}(\alpha)$, there is a unique (up to scalars) non-zero bilinear form $\langle .,. \rangle$ on $\bar{\Delta}(\lambda)$ such that*

$$\langle hv, v' \rangle = \langle v, T(h)v' \rangle \tag{2.14}$$

for all $v, v' \in \bar{\Delta}(\lambda)$ and $h \in H_\alpha$. Moreover:

1. *The radical of the form $\langle .,. \rangle$ coincides with the unique maximal submodule of $\bar{\Delta}(\lambda)$.*

(2) *For $\boldsymbol{i}, \boldsymbol{i}' \in \langle I \rangle_\alpha$ and $m, m' \in \mathbb{Z}$, we have that $\langle 1_{\boldsymbol{i}} \bar{\Delta}(\lambda)_m, 1_{\boldsymbol{i}'} \bar{\Delta}(\lambda)_{m'} \rangle = 0$ unless $\boldsymbol{i} = \boldsymbol{i}'$ and $m + m' = 0$.*

(3) *The form $\langle .,. \rangle$ is symmetric.*

Proof. There is a bijection between the space of bilinear forms on $\bar{\Delta}(\lambda)$ with the property (2.14) and the vector space $\mathrm{Hom}_{H_\alpha}(\bar{\Delta}(\lambda), \bar{\nabla}(\lambda))$, under which $f : \bar{\Delta}(\lambda) \to \bar{\nabla}(\lambda)$ corresponds to the form with $\langle v, v' \rangle := f(v)(v')$. Moreover $\mathrm{Hom}_{H_\alpha}(\bar{\Delta}(\lambda), \bar{\nabla}(\lambda))$ is one-dimensional, indeed, it is spanned by a homogeneous homomorphism that sends the head of $\bar{\Delta}(\lambda)$ onto the socle of $\bar{\nabla}(\lambda)$. The existence and uniqueness of the form $\langle .,. \rangle$ follow at once, and (1) and (2) are immediate consequences too.

To prove (3), it suffices to show that the induced form on the irreducible quotient $L(\lambda)$ of $\bar{\Delta}(\lambda)$ is symmetric. Of course this induced form is also characterized uniquely (up to scalars) by the contravariant property (2.14) and it satisfies (2); moreover it is non-degenerate thanks to (1). We proceed by induction on height. To start with, consider the case that $\lambda = (\alpha_i^n)$ for some $i \in I$ and $n \geq 1$. Then $L(\lambda)$ has basis $\{\tau_w \otimes v \mid w \in S_n\}$ where v spans the one-dimensional vector space $L(\alpha_i)^{\circ n}$, and $\tau_w \otimes v$ is of degree $q^{\frac{1}{2}n(n-1) - \ell(w)}$. By (2) we may choose the scalar so that $\langle \tau_{w_{[1,n]}} \otimes v, 1 \otimes v \rangle = 1$. Then for arbitrary $x, y \in S_n$ we have by contravariance that $\langle \tau_x \otimes v, \tau_y \otimes v \rangle = \delta_{x^{-1}y, w_{[1,n]}}$, hence the form is symmetric.

Now for the induction step, suppose that λ is not of the form (α_i^n). Pick $i \in I$ and $m \geq 1$ such that $\mathrm{res}^\alpha_{\alpha - m\alpha_i, m\alpha_i} L(\lambda) \neq 0$ but $\mathrm{res}^\alpha_{\alpha-(m+1)\alpha_i,(m+1)\alpha_i} L(\lambda) = 0$ (interpreting $\mathrm{res}^\alpha_{\alpha - m\alpha_i, m\alpha_i} L(\lambda)$ as zero if $\alpha - m\alpha_i \notin Q^+$). By general theory we have that $\mathrm{res}^\alpha_{\alpha - m\alpha_i, m\alpha_i} L(\lambda) \cong L(\mu) \boxtimes L(\alpha_i^m)$ for some $\mu \in \mathrm{KP}(\alpha - m\alpha_i)$; see e.g. [KL, Lemma 3.7]. By (1)–(2), the restriction of the form $\langle .,. \rangle$ to this copy of $L(\mu) \boxtimes L(\alpha_i^m)$ is a non-degenerate contravariant form, hence it is the product of contravariant forms on $L(\mu)$ and $L(\alpha_i^m)$. The former is symmetric by induction and the latter is symmetric by the previous paragraph. This shows the form is symmetric on restriction to some non-zero word space of $L(\lambda)$. Since $L(\lambda)$ is irreducible, this implies it is symmetric on the entire module. $\square$

The next lemma is useful when trying to compute the contravariant form on $\bar{\Delta}(\lambda)$ in practice. Suppose that $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}(\alpha)$ and set $n := \mathrm{ht}(\alpha)$. Let S_λ be the parabolic subgroup $S_{\mathrm{ht}(\lambda_1)} \times \cdots \times S_{\mathrm{ht}(\lambda_l)}$ of S_n and D_λ be the set of minimal length S_n/S_λ-coset representatives. Recalling that $\bar{\Delta}(\lambda) = q^{s_\lambda} H_\alpha 1_\lambda \otimes_{H_\lambda} (L(\lambda_1) \boxtimes \cdots \boxtimes L(\lambda_l))$, where $H_\lambda := H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_l}$ with identity 1_λ, any element of $\bar{\Delta}(\lambda)$ is a sum of vectors of the form $\tau_w 1_\lambda \otimes (v_1 \otimes \cdots \otimes v_k)$ for $w \in D_\lambda$ and $v_i \in L(\lambda_i)$. Let x be the longest element of D_λ such that $\tau_x 1_\lambda = 1_\lambda \tau_x$, and let y be the longest element of S_l such that $\lambda_{y(i)} = \lambda_i$ for each $i = 1, \dots, l$.

Lemma 2.18. *In the preceding notation, the contravariant form $\langle .,. \rangle$ on $\bar{\Delta}(\lambda)$ satisfies*

$$\langle \tau_w 1_\lambda \otimes (v_1 \otimes \cdots \otimes v_l), 1_\lambda \otimes (v_1' \otimes \cdots \otimes v_l') \rangle = \delta_{x,w} \langle v_{y(1)}, v_1' \rangle_1 \cdots \langle v_{y(l)}, v_l' \rangle_l$$

for all $w \in D_\lambda$ and $v_i, v_i' \in L(\lambda_i)$. Here $\langle .,. \rangle_i$ denotes some choice of (non-degenerate) contravariant form on $L(\lambda_i)$ for each $i = 1, \dots, l$.

Proof. Let $L' := L(\lambda_1) \boxtimes \cdots \boxtimes L(\lambda_l)$ for short, and denote the product of the forms $\langle .,. \rangle_i$ for $i = 1, \dots, l$ by $\langle .,. \rangle'$, which is a non-degenerate form on L'. Recall from the proof of Theorem 2.17 that the contravariant form on $\bar{\Delta}(\lambda)$ is defined from $\langle v, v' \rangle := f(v)(v')$ where $f : \bar{\Delta}(\lambda) \to \bar{\nabla}(\lambda)$ is a non-zero homomorphism. We can identify $\bar{\nabla}(\lambda) = \mathrm{Hom}_{\mathbb{K}}(\bar{\Delta}(\lambda), \mathbb{K})$ with the "coinduced" module $q^{-s_\lambda}\mathrm{Hom}_{H_\lambda}(1_\lambda H_\alpha, L')$ so that $\theta : 1_\lambda H_\alpha \to L'$ is identified with the functional $\bar{\Delta}(\lambda) \mapsto \mathbb{K}, h1_\lambda \otimes v \mapsto \langle \theta(1_\lambda T(h)), v \rangle'$. By adjointness of tensor and hom, we have that

$$\begin{aligned}\mathrm{Hom}_{H_\alpha}(\bar{\Delta}(\lambda), \bar{\nabla}(\lambda)) &= \mathrm{Hom}_{H_\alpha}(\bar{\Delta}(\lambda), q^{-s_\lambda}\mathrm{Hom}_{H_\lambda}(1_\lambda H_\alpha, L') \\ &\cong \mathrm{Hom}_{H_\lambda}(\mathrm{res}^\alpha_\lambda \bar{\Delta}(\lambda), q^{-s_\lambda} L').\end{aligned}$$

Now we observe as in the proof of Lemma 2.13 that the top non-zero layer in the Mackey filtration of $\mathrm{res}^\alpha_\lambda \bar{\Delta}(\lambda)$ is isomorphic to $q^{-s_\lambda} L'$. In this way, we obtain an explicit homomorphism $\bar{f} : \mathrm{res}^\alpha_\lambda \bar{\Delta}(\lambda) \to q^{-s_\lambda} L'$ such that

$$\bar{f}(1_\lambda \tau_w 1_\lambda \otimes (v_1 \otimes \cdots \otimes v_l)) = \delta_{x,w} v_{y(1)} \otimes \cdots \otimes v_{y(l)}$$

for $w \in D_\lambda$ and $v_i \in L(\lambda_i)$. Then choose $f : \bar{\Delta}(\lambda) \to \bar{\nabla}(\lambda)$ so that it corresponds to $\bar{f}$ under the above isomorphism. This means that

$$\langle h1_\lambda \otimes v, h'1_\lambda \otimes v' \rangle = f(h1_\lambda \otimes v)(h'1_\lambda \otimes v') = \langle \bar{f}(1_\lambda T(h')h1_\lambda \otimes v), v' \rangle'.$$

The lemma follows from the last two displayed formulae. $\square$

Now we recall a conjecture from [KR2, Conjecture 7.3] asserting in finite type that the formal character of an irreducible H_α-module $L(\lambda)$ does not depend on the characteristic p of the ground field $\mathbb{K}$. Using geometric techniques, Williamson [Wi] has recently shown that this is false (too optimistic!), and the question of finding a satisfactory bound on p remains open. The smallest counterexample found by Williamson is as follows.

Example 2.19. Assume we are in type A_5 with quiver

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow 5.$$

The positive roots are $\alpha_{i,j} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $1 \leq i \leq j \leq 5$. Let $\prec$ be the convex ordering defined so that $\alpha_{i,j} \prec \alpha_{k,l}$ if either $i < k$, or $i = k$ and $j < l$. For this choice the cuspidal representations are trivial to construct: $L(\alpha_{i,j})$ is the one-dimensional module spanned by a degree zero vector belonging to the $i(i+1)\cdots j$-word space. Let

$$\begin{aligned}\lambda &:= (\alpha_{4,5}, \alpha_{4,5}, \alpha_3, \alpha_3, \alpha_{2,4}, \alpha_{2,4}, \alpha_{1,2}, \alpha_{1,2}), \\ \boldsymbol{i} &:= 4534234523123412.\end{aligned}$$

We claim that $1_{\boldsymbol{i}} L(\lambda)_0$ has dimension 2 if $\mathrm{char}\,\mathbb{K} = 2$ and dimension 3 in all other characteristics. To see this we compute the rank of the contravariant form on $1_{\boldsymbol{i}} \bar{\Delta}(\lambda)_0$. Let v span $L(\alpha_{4,5})^{\boxtimes 2} \boxtimes L(\alpha_3)^{\boxtimes 2} \boxtimes L(\alpha_{2,4})^{\boxtimes 2} \boxtimes L(\alpha_{1,2})^{\boxtimes 2}$. Adopting all the notation from Lemma 2.18, the vectors $\{\tau_w 1_\lambda \otimes v \mid w \in D_\lambda\}$ give a basis for $\bar{\Delta}(\lambda)$ with $1_\lambda \otimes v$ of degree 4 and $\tau_x 1_\lambda \otimes v$ of degree -4. We normalize the contravariant form on $\bar{\Delta}(\lambda)$ so that $\langle \tau_x 1_\lambda \otimes v, 1_\lambda \otimes v\rangle = -1$. Let

$$\begin{aligned}
a &:= \tau_3\tau_7\tau_6\tau_5\tau_4\tau_9\tau_8\tau_7\tau_6\tau_{12}\tau_{11}\tau_{13}\tau_{12},\\
b &:= \tau_3\tau_7\tau_6\tau_5\tau_4\tau_{12}\tau_{11}\tau_{10}\tau_9\tau_8\tau_7\tau_6\tau_{13}\tau_{12},\\
c_1 &:= \tau_2\tau_1\tau_3\tau_2,\\
c_2 &:= \tau_5,\\
c_3 &:= \tau_9\tau_8\tau_7\tau_{10}\tau_9\tau_8\tau_{11}\tau_{10}\tau_9,\\
c_4 &:= \tau_{14}\tau_{13}\tau_{15}\tau_{14}.
\end{aligned}$$

We have that $c_1c_2c_3c_4 1_\lambda \otimes v = \tau_x 1_\lambda \otimes v$ (as should be clear on drawing the appropriate diagrams), and $1_{\boldsymbol{i}}\bar{\Delta}(\lambda)_0$ is 5-dimensional with basis

$$ac_1c_2c_3c_4 1_\lambda \otimes v, bc_2c_3c_4 1_\lambda \otimes v, bc_1c_3c_4 1_\lambda \otimes v, bc_1c_2c_4 1_\lambda \otimes v, bc_1c_2c_3 1_\lambda \otimes v.$$

Using Lemma 2.18 and making some explicit but lengthy straightening calculations, one can then check that the Gram matrix of the contravariant form on $1_{\boldsymbol{i}}\bar{\Delta}(\lambda)_0$ with respect to this basis is

$$\begin{pmatrix} 0&1&1&1&1\\ 1&0&0&0&1\\ 1&0&0&0&1\\ 1&0&0&0&1\\ 1&1&1&1&0 \end{pmatrix}.$$

It remains to compute the rank of this matrix.

3. Standard modules I

Throughout the section, we fix a choice of convex ordering $\prec$ and adopt all the notation from the previous section. From Theorem 3.4 onwards we will assume in addition that the following holds:

(GD) For each $\alpha \in Q^+$ the algebra H_α has finite global dimension.

By [Ka, Corollary 2.3], (GD) is already known to hold for $\mathbb{K}$ of characteristic zero. In fact, it is also true in positive characteristic as may be established by mimicking McNamara's arguments from the proof of [M, Theorem 4.6]; we'll explain a slightly different argument in the next section. In this section we give an elementary definition of modules $\Delta(\lambda)$ which categorify the PBW basis elements r_λ. We show that these modules satisfy homological properties analogous to the standard modules of a quasi-hereditary algebra, hence they are isomorphic to the modules $\widetilde{E}_b$ constructed using Saito reflection functors in [Ka, §4].

3.1. **Root modules.** For any $\alpha \in Q^+$ of height n, let H'_α be the subalgebra of H_α generated by $\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in \langle I\rangle_\alpha\} \cup \{x_1 - x_2, \dots, x_{n-1} - x_n\} \cup \{\tau_w \mid w \in S_n\}$.

Lemma 3.1. *For $\alpha \in Q^+$ of height n, the monomials*

$$\left\{(x_1-x_2)^{k_1}\cdots(x_{n-1}-x_n)^{k_{n-1}}\tau_w 1_{\boldsymbol{i}} \;\middle|\; w \in S_n, k_1,\dots,k_{n-1} \geq 0, \boldsymbol{i} \in \langle I\rangle_\alpha\right\}$$

give a basis for H'_α. Assume moreover that we are given $i \in I$ such that the integer $\varpi_i \cdot \alpha$ has non-zero image in the ground field $\mathbb{K}$. Then there exists a *degree two central element $z \in Z(H_\alpha)$ such that multiplication defines an algebra isomorphism $\mathbb{K}[z] \otimes H'_\alpha \xrightarrow{\sim} H_\alpha$.*

Proof. In view of the basis (2.2), the first part follows on checking that the span of the given monomials is closed under multiplication; this is an easy application of the defining relations. For the second part, let $m := \varpi_i \cdot \alpha$, and pick $\boldsymbol{i} \in \langle I \rangle_\alpha$ so that $S_{\boldsymbol{i}}$ is a parabolic subgroup of S_n with $i_1 = \cdots = i_m = i$. Using the notation (2.3), the element

$$z := y_1 + \cdots + y_m = \sum_{w \in S_n/S_{\boldsymbol{i}}} (x_{w(1)} + \cdots + x_{w(m)}) 1_{w(\boldsymbol{i})}$$

is central of degree 2. Observe that

$$m x_n = z - \sum_{w \in S_n/S_{\boldsymbol{i}}} \big((x_{w(1)} - x_n) + \cdots + (x_{w(m)} - x_n)\big) 1_{w(\boldsymbol{i})}.$$

Hence the natural multiplication map $\mathbb{K}[z] \otimes H'_\alpha \to H_\alpha$ is surjective. It remains to observe that the two algebras have the same graded dimension. $\square$

Assume for the rest of the subsection that $\alpha \in R^+$. Let $L'(\alpha)$ denote the restriction of the cuspidal module $L(\alpha)$ to H'_α.

Lemma 3.2. *The H'_α-module $L'(\alpha)$ is irreducible. Also for $d \geq 1$ we have that*

$$\mathrm{Ext}^d_{H_\alpha}(L(\alpha), L(\alpha)) \cong \mathrm{Ext}^d_{H'_\alpha}(L'(\alpha), L'(\alpha)) \oplus q^{-2}\mathrm{Ext}^{d-1}_{H'_\alpha}(L'(\alpha), L'(\alpha)).$$

Proof. There exists $i \in I$ as in the statement of Lemma 3.1; indeed, apart from the highest root of E_8 one can choose i so that $\varpi_i \cdot \alpha = 1$; for the highest root of E_8 one can choose i so that $\varpi_i \cdot \alpha = 2$ if $\mathrm{char}\,\mathbb{K} \neq 2$ and $\varpi_i \cdot \alpha = 3$ if $\mathrm{char}\,\mathbb{K} = 2$. Hence there exists $z \in Z(H_\alpha)$ of degree 2 such that $H_\alpha = \mathbb{K}[z] \otimes H_{\alpha'}$. The irreducibility of $L'(\alpha)$ is then immediate from the irreducibility of $L(\alpha)$ combined with Schur's lemma. Under the identification $H_\alpha = \mathbb{K}[z] \otimes H_{\alpha'}$, we have that $L(\alpha) = \mathbb{K} \boxtimes L'(\alpha)$, where $\mathbb{K}$ is viewed as a $\mathbb{K}[z]$-module so that z acts as zero. So

$$\mathrm{Ext}^*_{H_\alpha}(L(\alpha), L(\alpha)) \cong \mathrm{Ext}^*_{H'_\alpha}(L'(\alpha), L'(\alpha)) \otimes \mathrm{Ext}^*_{\mathbb{K}[z]}(\mathbb{K}, \mathbb{K}).$$

Now note $\mathrm{Hom}_{\mathbb{K}[z]}(\mathbb{K}, \mathbb{K}) \cong \mathbb{K}, \mathrm{Ext}^1_{\mathbb{K}[z]}(\mathbb{K}, \mathbb{K}) \cong q^{-2}\mathbb{K}$ and $\mathrm{Ext}^d_{\mathbb{K}[z]}(\mathbb{K}, \mathbb{K}) = 0$ for $d \geq 2$. $\square$

Introduce the *root module*

$$\Delta(\alpha) := H_\alpha \otimes_{H'_\alpha} L'(\alpha). \tag{3.1}$$

Theorem 3.3. *For $\alpha \in R^+$, there is a unique (up to scalars) non-zero H_α-module endomorphism $x : \Delta(\alpha) \to \Delta(\alpha)$ of degree 2. We then have that $\mathrm{End}_{H_\alpha}(\Delta(\alpha)) = \mathbb{K}[x]$. Also there is a short exact sequence*

$$0 \to q^2\Delta(\alpha) \xrightarrow{x} \Delta(\alpha) \to L(\alpha) \to 0,$$

and the head of $\Delta(\alpha)$ is isomorphic to $L(\alpha)$.

Proof. As in the proof of Lemma 3.2, we can identify H'_α with $\mathbb{K}[z] \boxtimes H'_\alpha$ for some $z \in Z(H_\alpha)_2$, and then $L(\alpha) = \mathbb{K} \boxtimes L'(\alpha)$ and $\Delta(\alpha) = \mathbb{K}[z] \boxtimes L'(\alpha)$. The theorem follows from this and Lemma 3.2. For x we can take the endomorphism defined by multiplication by the central element z. $\square$

The short exact sequence from Theorem 3.3 shows $[L(\alpha)] = (1-q^2)[\Delta(\alpha)]$. Hence, recalling (2.6) and Theorem 2.9, we have proved that $[\Delta(\alpha)] = r_\alpha$. To say more, we need to assume the finite global dimension hypothesis (GD). The following is extracted from the second half of the proof of [M, Proposition 4.5] (where the same thing is proved for non-simply-laced types).

Theorem 3.4. *For $\alpha \in R^+$ and $d \geq 2$, we have that*

$$\mathrm{Ext}^1_{H_\alpha}(L(\alpha), L(\alpha)) \cong q^{-2}\mathbb{K}, \qquad \mathrm{Ext}^d_{H_\alpha}(L(\alpha), L(\alpha)) = 0.$$

Proof. Proceed by induction on height. For the induction step, take a minimal pair $\lambda = (\beta, \gamma) \in \mathrm{MP}(\alpha)$. Let X, Y be as in (2.10)–(2.11). By Lemma 2.15(1) and generalized Frobenius reciprocity, $\mathrm{Ext}^d_{H_\alpha}(L(\beta) \circ L(\gamma), L(\alpha)) = 0$ for all $d \geq 1$. Thus the long exact sequence obtained by applying $\mathrm{Hom}_{H_\alpha}(-, L(\alpha))$ to (2.10) tells us that

$$\mathrm{Ext}^d_{H_\alpha}(X, L(\alpha)) \cong \mathrm{Ext}^{d+1}_{H_\alpha}(L(\lambda), L(\alpha))$$

for all $d \geq 1$. Similarly, using instead Lemma 2.15(3) and the induction hypothesis, we have that $\mathrm{Ext}^d_{H_\alpha}(L(\gamma) \circ L(\beta), L(\alpha)) = 0$ for all $d \geq 3$. So the long exact sequence coming from (2.11) tells us that

$$\mathrm{Ext}^d_{H_\alpha}(qL(\lambda), L(\alpha)) \cong \mathrm{Ext}^{d+1}_{H_\alpha}(Y, L(\alpha))$$

for all $d \geq 3$.

Now pick $d \geq 0$ maximal so that $\mathrm{Ext}^d_{H_\alpha}(L(\alpha), L(\alpha)) \neq 0$; this makes sense as H_α has finite global dimension by the assumption (GD). Suppose for a contradiction that $d \geq 2$. Since all composition factors of X are of the form $L(\alpha)$ (up to shift) we must have that $\mathrm{Ext}^d_{H_\alpha}(X, L(\alpha)) \neq 0$, hence

$$\mathrm{Ext}^{d+2}_{H_\alpha}(q^{-1}Y, L(\alpha)) \cong \mathrm{Ext}^{d+1}_{H_\alpha}(L(\lambda), L(\alpha)) \cong \mathrm{Ext}^d_{H_\alpha}(X, L(\alpha) \neq 0.$$

But all composition factors of Y are of the form $L(\alpha)$ (up to shift), so this contradicts the maximality of d.

Thus we have shown that $\mathrm{Ext}^d_{H_\alpha}(L(\alpha), L(\alpha)) = 0$ for all $d \geq 2$. To complete the proof (in a slightly different way to McNamara) we apply Lemma 3.2 with $d = 2$ to get that $\mathrm{Ext}^1_{H'_\alpha}(L'(\alpha), L'(\alpha)) = 0$. Then the same lemma with $d = 1$ shows that $\mathrm{Ext}^1_{H_\alpha}(L(\alpha), L(\alpha)) \cong q^{-2}\mathbb{K}$. □

Corollary 3.5. *For $\alpha \in R^+$ and $d \geq 1$, we have that*

$$\mathrm{Ext}^d_{H'_\alpha}(L'(\alpha), L'(\alpha)) = 0.$$

Proof. Combine the theorem with Lemma 3.2. □

Corollary 3.6. *For $\alpha \in R^+$ and $d \geq 1$, we have that*

$$\mathrm{Ext}^d_{H_\alpha}(\Delta(\alpha), \Delta(\alpha)) = \mathrm{Ext}^d_{H_\alpha}(\Delta(\alpha), L(\alpha)) = 0.$$

Proof. As in the proof of Theorem 3.3, we can identify H_α with $\mathbb{K}[z] \otimes H'_\alpha$ for some $z \in Z(H_\alpha)_2$, then have that $\Delta(\alpha) = \mathbb{K}[z] \boxtimes L'(\alpha)$ and $L(\alpha) = \mathbb{K} \boxtimes L'(\alpha)$. Now use Corollary 3.5. □

3.2. **Divided powers.** Throughout the subsection, we fix $\alpha \in R^+$ of height n. We are going to compute the endomorphism algebra of $\Delta(\alpha)^{\circ m}$. Choose a non-zero homogeneous vector $\bar{v}_\alpha$ of minimal degree in $L(\alpha)$ and set $v_\alpha := 1 \otimes \bar{v}_\alpha \in \Delta(\alpha)$. This ensures that v_α is of minimal degree in $\Delta(\alpha)$ and that $\Delta(\alpha)$ is generated as an H_α-module by v_α. Also fix a choice of endomorphism x as in Theorem 3.3. We obtain commuting endomorphisms $x_1, \dots, x_m \in \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})_2$ with $x_i := \mathrm{id}^{\circ(i-1)} \circ x \circ \mathrm{id}^{\circ(m-i)}$. Similarly, the endomorphism τ from the following lemma yields $\tau_1, \dots, \tau_{m-1} \in \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})_{-2}$ with $\tau_i := \mathrm{id}^{\circ(i-1)} \circ \tau \circ \mathrm{id}^{\circ(m-i-1)}$. (It is a bit confusing here that we are using the same notation x_i and τ_i for these endomorphisms as we use for the elements of the KLR algebra, but hopefully it is clear from the context which we mean.)

Lemma 3.7. *Let $w \in S_{2n}$ be the permutation mapping $(1, \dots, n, n+1, \dots, 2n)$ to $(n+1, \dots, 2n, 1, \dots, n)$. There is a unique $H_{2\alpha}$-module homomorphism*

$$\tau : \Delta(\alpha) \circ \Delta(\alpha) \to \Delta(\alpha) \circ \Delta(\alpha)$$

of degree -2 such that $\tau(1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)) = \tau_w 1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)$. For arbitrary $v_1, v_2 \in \Delta(\alpha)$, we have $\tau(1_{\alpha,\alpha} \otimes (v_1 \otimes v_2)) = \tau_w 1_{\alpha,\alpha} \otimes (v_2 \otimes v_1) + 1_{\alpha,\alpha} \otimes \sigma(v_1 \otimes v_2)$ for a unique $\sigma \in \mathrm{End}_{\mathbb{K}}(\Delta(\alpha) \boxtimes \Delta(\alpha))_{-2}$.

Proof. We apply the Mackey theorem to $\mathrm{res}^{2\alpha}_{\alpha,\alpha} \Delta(\alpha) \circ \Delta(\alpha)$. By exactly the same argument as in the proof of Lemma 2.12, there are just two non-zero layers in the Mackey filtration, corresponding to the double coset representatives 1 and w. We deduce that there is a short exact sequence

$$0 \longrightarrow \Delta(\alpha) \boxtimes \Delta(\alpha) \xrightarrow{f} \mathrm{res}^{2\alpha}_{\alpha,\alpha} \Delta(\alpha) \circ \Delta(\alpha) \xrightarrow{g} q^{-2} \Delta(\alpha) \boxtimes \Delta(\alpha) \longrightarrow 0$$

such that $f(v_1 \otimes v_2) = 1_{\alpha,\alpha} \otimes (v_1 \otimes v_2)$ and $g(\tau_w 1_{\alpha,\alpha} \otimes (v_2 \otimes v_1)) = v_1 \otimes v_2$ for $v_1, v_2 \in \Delta(\alpha)$. By Corollary 3.6, we have that

$$\mathrm{Ext}^1_{H_{\alpha,\alpha}}(\Delta(\alpha) \boxtimes \Delta(\alpha), \Delta(\alpha) \boxtimes \Delta(\alpha)) = 0.$$

So the short exact sequence splits. Let $\bar{g} : q^{-2} \Delta(\alpha) \boxtimes \Delta(\alpha) \to \mathrm{res}^{2\alpha}_{\alpha,\alpha} \Delta(\alpha) \circ \Delta(\alpha)$ be the unique splitting. Since $\mathrm{im}\, f = 1_{\alpha,\alpha} \otimes (\Delta(\alpha) \boxtimes \Delta(\alpha))$, we must have that $\bar{g}(v_1 \otimes v_2) = \tau_w 1_{\alpha,\alpha} \otimes (v_2 \otimes v_1) + 1_{\alpha,\alpha} \otimes \sigma(v_1 \otimes v_2)$ for a unique $\sigma \in \mathrm{End}_{\mathbb{K}}(\Delta(\alpha) \boxtimes \Delta(\alpha))_{-2}$. Applying Frobenius reciprocity, $\bar{g}$ induces a map τ as in the final statement of the lemma. It sends the generator $1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)$ to $\tau_w 1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)$, as $\sigma(v_\alpha \otimes v_\alpha) = 0$ by degree considerations. □

Lemma 3.8. *The endomorphisms $\tau_1, \dots, \tau_{m-1} \in \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})$ square to zero and satisfy the usual type* A *braid relations.*

Proof. For the quadratic relation, it suffices to show that $\tau^2 = 0$ in the setup of Lemma 3.7. As a vector space, the Mackey theorem analysis from the proof of that lemma tells us that

$$1_{\alpha,\alpha}(\Delta(\alpha) \circ \Delta(\alpha)) = 1_{\alpha,\alpha} \otimes (\Delta(\alpha) \boxtimes \Delta(\alpha)) \oplus \tau_w 1_{\alpha,\alpha} \otimes (\Delta(\alpha) \boxtimes \Delta(\alpha)).$$

Thus the vector $\tau_w 1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)$ is of minimal degree in $1_{\alpha,\alpha}(\Delta(\alpha) \circ \Delta(\alpha))$, namely, $2\deg(v_\alpha) - 2$. The vector $\tau_w^2 1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)$ is of strictly smaller degree $2\deg(v_\alpha) - 4$, hence it must be zero. This shows that τ^2 sends a generator of $\Delta(\alpha) \circ \Delta(\alpha)$ to zero, hence $\tau^2 = 0$.

For the braid relations, the commuting ones are trivial from the definitions. For the length three braid relation, it suffices to show that $\tau_1 \circ \tau_2 \circ \tau_1 = \tau_2 \circ \tau_1 \circ \tau_2$ working in $\mathrm{End}_{H_{3\alpha}}(\Delta(\alpha) \circ \Delta(\alpha) \circ \Delta(\alpha))$. Let $w_1, w_2 \in S_{3n}$ be the permutations mapping $(1, \dots, n, n+1, \dots, 2n, 2n+1, \dots, 3n)$ to $(n+1, \dots, 2n, 1, \dots, n, 2n+1, \dots, 3n)$ and $(1, \dots, n, 2n+1, \dots, 3n, n+1, \dots, 2n)$, respectively, and set $w_0 := w_1 w_2 w_1 = w_2 w_1 w_2$. By the defining relations for $H_{3\alpha}$, it is clear that $(\tau_{w_2}\tau_{w_1}\tau_{w_2} - \tau_{w_2}\tau_{w_1}\tau_{w_2})1_{\alpha,\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha \otimes v_\alpha)$ lies in

$$S := \sum_{w < w_0} \tau_w 1_{\alpha,\alpha,\alpha} \otimes (\Delta(\alpha) \boxtimes \Delta(\alpha) \boxtimes \Delta(\alpha)).$$

By the Mackey theorem, we have that

$$S = \bigoplus_{w \in \{1, w_1, w_2, w_1 w_2, w_2 w_1\}} \tau_w 1_{\alpha,\alpha,\alpha} \otimes (\Delta(\alpha) \boxtimes \Delta(\alpha) \boxtimes \Delta(\alpha)).$$

But the vector $(\tau_{w_2}\tau_{w_1}\tau_{w_2} - \tau_{w_2}\tau_{w_1}\tau_{w_2})1_{\alpha,\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha \otimes v_\alpha)$ is of degree $3\deg(v_\alpha) - 6$, while all the vectors in S are of degree $\geq 3\deg(v_\alpha) - 4$. Hence this vector is zero, and we have shown that the endomorphisms $\tau_2 \circ \tau_1 \circ \tau_2$ and $\tau_1 \circ \tau_2 \circ \tau_1$ agree on the generator $1_{\alpha,\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha \otimes v_\alpha)$. Hence they are equal. $\square$

In view of Lemma 3.8, we get well-defined endomorphisms τ_w of $\Delta(\alpha)^{\circ m}$ for each $w \in S_m$, defined as usual from any reduced expression for w. (This creates further ambiguity with the elements of the KLR algebra with the same name, but this is only temporary.)

Lemma 3.9. *The endomorphisms* $\{\tau_w \circ x_m^{k_m} \circ \cdots \circ x_1^{k_1} \,|\, w \in S_m, k_1, \dots, k_m \geq 0\}$ *give a basis for* $\mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})$.

Proof. These endomorphisms are linearly independent because they produce linearly independent vectors when applied to $1_{\alpha,\dots,\alpha} \otimes (v_\alpha \otimes \cdots \otimes v_\alpha)$. It remains to show that

$$\mathrm{Dim}\,\mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m}) \leq \sum_{w \in S_m} \frac{q^{-2\ell(w)}}{(1-q^2)^m},$$

where we write $\leq$ to indicate that the coefficient of each power of q on the left hand side is less than or equal to the corresponding coefficient in the formal Laurent series on the right. As $\Delta(\alpha)^{\boxtimes m}$ has irreducible head isomorphic to $L(\alpha)^{\boxtimes m}$ and $[\Delta(\alpha)^{\circ m}] = [L(\alpha)^{\circ m}]/(1-q^2)^m$, we have by Frobenius reciprocity and Lemma 2.12 that

$$\begin{aligned}
\mathrm{Dim}\,\mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m}) &= \mathrm{Dim}\,\mathrm{Hom}_{H_{\alpha,\dots,\alpha}}(\Delta(\alpha)^{\boxtimes m}, \mathrm{res}^{m\alpha}_{\alpha,\dots,\alpha}\Delta(\alpha)^{\circ m}) \\
&\leq [\mathrm{res}^{m\alpha}_{\alpha,\dots,\alpha}\Delta(\alpha)^{\circ m} : L(\alpha)^{\boxtimes m}] \\
&= [\mathrm{res}^{m\alpha}_{\alpha,\dots,\alpha}L(\alpha)^{\circ m} : L(\alpha)^{\boxtimes m}]/(1-q^2)^m \\
&= q^{-\frac{1}{2}m(m-1)}[m]!/(1-q^2)^m.
\end{aligned}$$

By the formula for the Poincaré polynomial of S_m this is $\sum_{w \in S_m} \frac{q^{-2\ell(w)}}{(1-q^2)^m}$. $\square$

Lemma 3.10. *There is a* unique *choice for the endomorphism x from Theorem 3.3 such that the following relations hold:* $\tau_i \circ x_j = x_j \circ \tau_i$ *for* $j \neq i, i+1$, $\tau_i \circ x_{i+1} = x_i \circ \tau_i + 1$ *and* $x_{i+1} \circ \tau_i = \tau_i \circ x_i + 1$.

Proof. The commuting relations are automatic. For the remaining relations, it suffices to show working in $\mathrm{End}_{H_{2\alpha}}(\Delta(\alpha)\circ\Delta(\alpha))$ that the (unique up to scalars) endomorphism x from Theorem 3.3 can be chosen so that $\tau\circ x_2 = x_1\circ\tau + 1$ and $x_2\circ\tau = \tau\circ x_1 + 1$.

Pick $z \in Z(H_\alpha)_2$ as in the proof of Theorem 3.3 so that $x(v) = zv$ for each $v \in \Delta(\alpha)$. Let $z_1 := z \otimes 1_\alpha$ and $z_2 := 1_\alpha \otimes z$, both of which are elements of $H_\alpha \otimes H_\alpha \subseteq H_{2\alpha}$. As τ is a homomorphism, we have that

$$(z_1 + z_2)\tau(1_{\alpha,\alpha} \otimes (v_1 \otimes v_2)) = \tau(1_{\alpha,\alpha} \otimes (z_1 + z_2)(v_1 \otimes v_2))$$

for any $v_1, v_2 \in \Delta(\alpha)$. From the explicit form of z, there exists $Z \in Z(H_{2\alpha})_2$ such that $Z1_{\alpha,\alpha} = z_1 + z_2$. So for w as in Lemma 3.7 we have that

$$\begin{aligned}(z_1 + z_2)\tau_w 1_{\alpha,\alpha} \otimes (v_2 \otimes v_1) &= Z\tau_w 1_{\alpha,\alpha} \otimes (v_2 \otimes v_1))\\ &= \tau_w Z1_{\alpha,\alpha} \otimes (v_2 \otimes v_1) = \tau_w 1_{\alpha,\alpha} \otimes (z_1 + z_2)(v_2 \otimes v_1)\end{aligned}$$

for any $v_1, v_2 \in \Delta(\alpha)$. Using also the last part of Lemma 3.7, we deduce:

$$\begin{aligned}&\tau_w 1_{\alpha,\alpha} \otimes (z_1 + z_2)(v_2 \otimes v_1) + 1_{\alpha,\alpha} \otimes (z_1 + z_2)\sigma(v_1 \otimes v_2) =\\ &\qquad \tau_w 1_{\alpha,\alpha} \otimes (z_1 + z_2)(v_2 \otimes v_1) + 1_{\alpha,\alpha} \otimes \sigma((z_1 + z_2)(v_1 \otimes v_2)).\end{aligned}$$

Hence $(z_1 + z_2)\sigma(v_1 \otimes v_2) = \sigma((z_1 + z_2)(v_1 \otimes v_2))$ for all $v_1, v_2 \in \Delta(\alpha)$.

Now consider the endomorphisms

$$\theta_+ := \tau\circ x_2 - x_1\circ\tau, \qquad \theta_- := \tau\circ x_1 - x_2\circ\tau.$$

They map $1_{\alpha,\alpha} \otimes (v_1 \otimes v_2)$ to $1_{\alpha,\alpha} \otimes [\sigma(z_2(v_1 \otimes v_2)) - z_1\sigma(v_1 \otimes v_2)]$ and to $1_{\alpha,\alpha} \otimes [\sigma(z_1(v_1 \otimes v_2)) - z_2\sigma(v_1 \otimes v_2)]$, respectively. Hence using the observation from the previous paragraph we have that $\theta_+ + \theta_- = 0$. Moreover θ_+ is of degree zero and maps $1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)$ into $1_{\alpha,\alpha} \otimes (\Delta(\alpha) \boxtimes \Delta(\alpha))$, so we deduce from Lemma 3.9 that $\theta_+ = c$, hence $\theta_- = -c$, for some scalar $c \in \mathbb{K}$. It remains to show that $c \neq 0$, for then we can replace x by x/c and get that $\theta_+ = 1, \theta_- = -1$ as required.

Suppose for a contradiction that $c = 0$. Then $\tau\circ x_1 = x_2\circ\tau$ and $\tau\circ x_2 = x_1\circ\tau$. This means that τ leaves invariant the submodule $S := \mathrm{im}\, x_1 + \mathrm{im}\, x_2$ of $\Delta(\alpha)\circ\Delta(\alpha)$, hence it induces a well-defined endomorphism $\bar\tau$ of the quotient $\Delta(\alpha)\circ\Delta(\alpha)/S$ with $\bar\tau^2 = 0$. But $\Delta(\alpha)\circ\Delta(\alpha)/S \cong L(\alpha)\circ L(\alpha)$, and under this isomorphism $\bar\tau$ corresponds to an endomorphism sending $1_{\alpha,\alpha} \otimes (\bar v_\alpha \otimes \bar v_\alpha)$ to $\tau_w 1_{\alpha,\alpha} \otimes (\bar v_\alpha \otimes \bar v_\alpha)$. This shows that $\mathrm{End}_{H_{2\alpha}}(L(\alpha)\circ L(\alpha))$ is more than one-dimensional, contradicting the irreducibility of this module from Theorem 2.9. □

Henceforth, we assume that the endomorphism x has been normalized according to Lemma 3.10. Recalling the definition of the nil Hecke algebra NH_m from §2.2, Lemmas 3.8–3.10 show that there is a unique algebra *isomorphism*

$$NH_m \xrightarrow{\sim} \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})^{\mathrm{op}}, \qquad x_i \mapsto x_i, \tau_j \mapsto \tau_j.$$

The op here means that we view $\Delta(\alpha)^{\circ m}$ as a *right* NH_m-module, i.e. it is an $(H_{m\alpha}, NH_m)$-bimodule (a convention which finally eliminates the confusion between the elements x_i, τ_j of $H_{m\alpha}$ and the elements of NH_m with the same name: they act on different sides). Finally define the *divided power module*

$$\Delta(\alpha^m) := q^{\frac{1}{2}m(m-1)}\Delta(\alpha)^{\circ m}e_m \tag{3.2}$$

where $e_m \in NH_m$ is the idempotent (2.4).

Lemma 3.11. *We have that $\Delta(\alpha)^{\circ m} \cong [m]!\Delta(\alpha^m)$ as an $H_{m\alpha}$-module. Moreover $\Delta(\alpha^m)$ has irreducible head $L(\alpha^m)$, and in the Grothendieck group we have that $[\Delta(\alpha^m)] = [L(\alpha^m)]/(1-q^2)(1-q^4)\cdots(1-q^{2m})$.*

Proof. So far, we have identified the endomorphism algebra $\mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})^{\mathrm{op}}$ with NH_m. Since $NH_m \cong [m]!P$ where P is the indecomposable projective module $q^{\frac{1}{2}m(m-1)}NH_m e_m$, we deduce that

$$\Delta(\alpha)^{\circ m} = \Delta(\alpha)^{\circ m} \otimes_{NH_m} NH_m \cong [m]!\Delta(\alpha)^{\circ m} \otimes_{NH_m} P \cong [m]!\Delta(\alpha^m).$$

The fact that $[\Delta(\alpha^m)] = [L(\alpha^m)]/(1-q^2)(1-q^4)\cdots(1-q^{2m})$ follows from this using $[\Delta(\alpha)] = [L(\alpha)]/(1-q^2)$ and $L(\alpha^m) = q^{\frac{1}{2}m(m-1)}L(\alpha)^{\circ m}$. Finally to show that the head of $\Delta(\alpha^m)$ is $L(\alpha^m)$, it suffices to show that

$$\mathrm{Dim}\,\mathrm{Hom}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m}, L(\alpha^m)) = [m]!,$$

which follows from Theorem 3.3, Lemma 2.12 and Frobenius reciprocity. $\square$

Thus we have constructed a module $\Delta(\alpha^m)$ which is equal in the Grothendieck group to the divided power $r_\alpha^m/[m]!$. More generally, for a Kostant partition $\lambda \in \mathrm{KP}$, gather together its equal parts to write it as $(\gamma_1^{m_1}, \dots, \gamma_s^{m_s})$ with $\gamma_1 \succ \cdots \succ \gamma_s$, then define the *standard module*

$$\Delta(\lambda) := \Delta(\gamma_1^{m_1}) \circ \cdots \circ \Delta(\gamma_s^{m_s}). \tag{3.3}$$

Theorem 3.12. *For $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}$, we have that*

$$\Delta(\lambda_1) \circ \cdots \circ \Delta(\lambda_l) \cong [\lambda]!\Delta(\lambda).$$

Moreover $V_0 := \Delta(\lambda)$ admits a filtration $V_0 \supset V_1 \supset V_2 \supset \cdots$ with $V_0/V_1 \cong \bar{\Delta}(\lambda)$ and all other sections of the form $q^m\bar{\Delta}(\lambda)$ for $m > 0$. Finally $\Delta(\lambda)$ has irreducible head isomorphic to $L(\lambda)$, and in the Grothendieck group we have that

$$[\Delta(\lambda)] = [\bar{\Delta}(\lambda)] / \prod_{\substack{\beta \in R^+ \\ 1 \le r \le m_\beta(\lambda)}} (1-q^{2r}).$$

Proof. The isomorphism $\Delta(\lambda_1) \circ \cdots \circ \Delta(\lambda_l) \cong [\lambda]!\Delta(\lambda)$ and the Grothendieck group identity both follow from Lemma 3.11. The existence of the filtration follows from Lemma 3.11 and exactness of induction. Finally, to show that $\Delta(\lambda)$ has irreducible head, the filtration together with Theorem 2.10 implies that the only module that could possibly appear with non-zero multiplicity in the head of $\Delta(\lambda)$ is $L(\lambda)$. Now calculate using Frobenius reciprocity, Lemma 2.13 and Lemma 3.11:

$$\begin{aligned}\mathrm{Dim}\,\mathrm{Hom}_{H_\alpha}(\Delta(\lambda), L(\lambda)) &= \mathrm{Dim}\,\mathrm{Hom}_{H_\alpha}(\Delta(\lambda_1) \circ \cdots \circ \Delta(\lambda_l), L(\lambda))/[\lambda]! \\ &= \mathrm{Dim}\,\mathrm{Hom}_{H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_l}}(\Delta(\lambda_1) \boxtimes \cdots \boxtimes \Delta(\lambda_l), L(\lambda_1) \boxtimes \cdots \boxtimes L(\lambda_l)) = 1.\end{aligned}$$

$\square$

Recalling (2.7), the theorem implies in particular that $[\Delta(\lambda)] = r_\lambda$.

3.3. **Standard homological properties.** The following theorem recovers (and extends to positive characteristic) the homological properties proved originally in [Ka, Theorem 4.12]. Recall $\bar\nabla(\lambda) = \bar\Delta(\lambda)^{\circledast}$.

Theorem 3.13. *For $\alpha \in Q^+$, $\lambda, \mu \in \mathrm{KP}(\alpha)$ and $d > 0$, the following hold:*

(1) $\mathrm{Ext}^d_{H_\alpha}(\Delta(\lambda), \bar\Delta(\mu)) = 0$ *if* $\lambda \not\preceq \mu$.
(2) $\mathrm{Ext}^d_{H_\alpha}(\Delta(\lambda), \Delta(\mu)) = 0$ *if* $\lambda \not\preceq \mu$.
(3) $\mathrm{Hom}_{H_\alpha}(\Delta(\lambda), \bar\nabla(\lambda)) \cong \mathbb{K}$ *and* $\mathrm{Hom}_{H_\alpha}(\Delta(\lambda), \bar\nabla(\mu)) = 0$ *if* $\lambda \neq \mu$.
(4) $\mathrm{Ext}^d_{H_\alpha}(\Delta(\lambda), \bar\nabla(\mu)) = 0$.

Proof. (1) Suppose that $\lambda = (\lambda_1, \dots, \lambda_l)$. By Theorem 3.12 we have that

$$\mathrm{Dim}\,\mathrm{Ext}^d_{H_\alpha}(\Delta(\lambda), \bar\Delta(\mu)) = \mathrm{Dim}\,\mathrm{Ext}^d_{H_\alpha}(\Delta(\lambda_1) \circ \cdots \circ \Delta(\lambda_l), \bar\Delta(\mu))/[\lambda]^!.$$

By generalized Frobenius reciprocity and Lemma 2.13, this is zero unless $\lambda \preceq \mu$. If $\lambda = \mu$ it equals

$$\sum_{d_1+\cdots+d_l=d} \left(\prod_{k=1}^{l} \mathrm{Dim}\,\mathrm{Ext}^{d_k}_{H_{\lambda_k}}(\Delta(\lambda_k), L(\lambda_k)) \right),$$

which is zero by Corollary 3.6.

(2) Fix a choice of a filtration as in Theorem 3.12 and set $V := \Delta(\lambda)$ and $V_r := \Delta(\lambda)_r$ for short. Observe that $V = \varprojlim V/V_r$ (inverse limit in the category of graded H_α-modules). By [W, Theorem 3.5.8], there is a short exact sequence

$$0 \to {\varprojlim}^1 \mathrm{Ext}^{d-1}_{H_\alpha}(V, V/V_r) \to \mathrm{Ext}^d_{H_\alpha}(V, V) \to \varprojlim \mathrm{Ext}^d_{H_\alpha}(V, V/V_r) \to 0$$

for all $d > 0$. We must show the first and last spaces are zero. For the latter, a routine long exact sequence argument using (1) and induction on r gives that $\mathrm{Ext}^d_{H_\alpha}(V, V/V_r) = 0$ for all $r \geq 1$. This also implies the vanishing of ${\varprojlim}^1 \mathrm{Ext}^{d-1}_{H_\alpha}(V, V/V_r)$ except perhaps in the case $d = 1$. To treat the $d = 1$ case, we apply $\mathrm{Hom}_{H_\alpha}(V, -)$ to the short exact sequence

$$0 \to V_r/V_{r+1} \to V/V_{r+1} \to V/V_r \to 0,$$

noting that $\mathrm{Ext}^1_{H_\alpha}(V, V_r/V_{r+1}) = 0$ by (1), to deduce that the natural map $\mathrm{Hom}_{H_\alpha}(V, V/V_{r+1}) \to \mathrm{Hom}_{H_\alpha}(V, V/V_r)$ is onto for each $r \geq 1$. This shows that the tower $(\mathrm{Hom}_{H_\alpha}(V, V/V_r))$ satisfies the Mittag-Leffler condition, and we are done by [W, Proposition 3.5.7].

(3) By dualizing Theorem 2.10, $\bar\nabla(\mu)$ has irreducible socle isomorphic to $L(\mu)$ and all its other composition are of the form $q^m L(\nu)$ for $\nu \prec \mu$ and $m \in \mathbb{Z}$. Now use Theorem 3.12.

(4) Since restriction commutes with duality, the same argument as for (1) shows that $\mathrm{Ext}^d_{H_\alpha}(\Delta(\lambda), \bar\nabla(\mu)) = 0$ if $\lambda \not\preceq \mu$. Instead, if $\lambda \prec \mu$, it suffices to show equivalently that $\mathrm{Ext}^d_{H_\alpha}(\bar\Delta(\mu), \nabla(\lambda)) = 0$ where $\nabla(\lambda) := \Delta(\lambda)^{\circledast}$. This follows from Lemma 2.13 and generalized Frobenius reciprocity once again. $\square$

We say that an H_α-module V has a Δ-*flag* if there is a (finite!) filtration $V = V_0 \supset V_1 \supset \cdots \supset V_n = 0$ such that $V_i/V_{i-1} \cong q^{m_i}\Delta(\lambda_i)$ for each $i = 1, \dots, n$

and some $m_i \in \mathbb{Z}$, $\lambda_i \in \mathrm{KP}(\alpha)$. Note then by Theorem 3.13(3)–(4) that

$$(V : \Delta(\lambda)) := \sum_{\substack{1\leq i\leq n\\ \lambda_i=\lambda}} q^{m_i} = \overline{\mathrm{Dim}\,\mathrm{Hom}_{H_\alpha}(V, \bar{\nabla}(\lambda))}$$

for each $\lambda \in \mathrm{KP}(\alpha)$, so that this multiplicity is well-defined independent of the particular choice of Δ-flag. The following theorem (and its proof) is analogous to a well-known result (and proof) in the context of quasi-hereditary algebras; see e.g. [Do, Proposition A2.2(iii)].

Theorem 3.14. *For $\alpha \in Q^+$, suppose that V is a finitely generated H_α-module with $\mathrm{Ext}^1_{H_\alpha}(V, \bar{\nabla}(\mu)) = 0$ for all $\mu \in \mathrm{KP}(\alpha)$. Then V has a Δ-flag.*

Proof. Let $\ell(V) \in \mathbb{N}$ denote the sum of the dimensions of all $\mathrm{Hom}_{H_\alpha}(V, \bar{\nabla}(\lambda))$ for all $\lambda \in \mathrm{KP}(\alpha)$; this makes sense because V is finitely generated. We prove the theorem by induction on $\ell(V)$, the result being trivial if $\ell(V) = 0$. Suppose that $\ell(V) > 0$. Let λ be minimal such that $\mathrm{Hom}_{H_\alpha}(V, L(\lambda)) \neq 0$. Then let $m \in \mathbb{Z}$ be minimal such that $\mathrm{hom}_{H_\alpha}(q^m V, L(\lambda)) \neq 0$.

We show in this paragraph that $\mathrm{Ext}^1_{H_\alpha}(V, L(\mu)) = 0$ for all $\mu \preceq \lambda$. There is a short exact sequence $0 \to L(\mu) \to \bar{\nabla}(\mu) \to Q \to 0$ where all composition factors of Q are of the form $q^n L(\nu)$ for $n \in \mathbb{Z}$ and $\nu \prec \mu \preceq \lambda$. By the minimality of λ, $\mathrm{Hom}_{H_\alpha}(V, Q) = 0$. Hence applying $\mathrm{Hom}_{H_\alpha}(V, -)$ to this short exact sequence, we obtain an exact sequence $0 \to \mathrm{Ext}^1_{H_\alpha}(V, L(\mu)) \to \mathrm{Ext}^1_{H_\alpha}(V, \bar{\nabla}(\mu)) = 0$. This shows that $\mathrm{Ext}^1_{H_\alpha}(V, L(\mu)) = 0$ as required.

Next we show that there is a homogeneous surjection $q^m V \twoheadrightarrow \Delta(\lambda)$ by showing that the natural map $\mathrm{hom}_{H_\alpha}(q^m V, \Delta(\lambda)) \to \mathrm{hom}_{H_\alpha}(q^m V, L(\lambda))$ is surjective. The long exact sequence obtained on applying $\mathrm{hom}_{H_\alpha}(q^m V, -)$ to the short exact sequence $0 \to \mathrm{rad}\,\Delta(\lambda) \to \Delta(\lambda) \to L(\lambda) \to 0$ gives us an exact sequence

$$\mathrm{hom}_{H_\alpha}(q^m V, \Delta(\lambda)) \to \mathrm{hom}_{H_\alpha}(q^m V, L(\lambda)) \to \mathrm{ext}^1_{H_\alpha}(q^m V, \mathrm{rad}\,\Delta(\lambda)).$$

Thus we are reduced to showing that $\mathrm{ext}^1_{H_\alpha}(q^m V, \mathrm{rad}\,\Delta(\lambda)) = 0$. From Theorem 3.12, we get a filtration $K := \mathrm{rad}\,\Delta(\lambda) \supset K_1 \supset \cdots$ such that $K \cong \varprojlim K/K_r$, each K/K_r is finite dimensional, and all composition factors of K/K_r are of the form $q^n L(\lambda)$ for $n > 0$ or $q^n L(\mu)$ for $n \in \mathbb{Z}$ and $\mu \prec \lambda$. The minimality of λ and m implies that $\mathrm{hom}_{H_\alpha}(q^m V, K/K_r) = 0$ for all $r \geq 1$. Hence the tower $(\mathrm{hom}_{H_\alpha}(q^m V, K/K_r))$ trivially satisfies the Mittag-Leffler condition, and we deduce invoking [W, Theorem 3.5.8] that

$$\mathrm{ext}^1_{H_\alpha}(q^m V, \mathrm{rad}\,\Delta(\lambda)) \cong \varprojlim \mathrm{ext}^1_{H_\alpha}(q^m V, K/K_r).$$

Each $\mathrm{ext}^1_{H_\alpha}(q^m V, K/K_r)$ is zero by the previous paragraph. We are done.

We have now proved that there is a short exact sequence

$$0 \to U \to V \to q^{-m}\Delta(\lambda) \to 0$$

for some submodule U of V. Applying $\mathrm{Hom}_{H_\alpha}(-, \bar{\nabla}(\mu))$ we get from the long exact sequence and Theorem 3.13 that $\ell(U) < \ell(V)$ and $\mathrm{Ext}^1_{H_\alpha}(U, \bar{\nabla}(\mu)) = 0$ for all $\mu \in \mathrm{KP}(\alpha)$. Thus by induction U has a Δ-flag, hence so does V. $\square$

As a corollary we obtain "BGG reciprocity." For $\mathbb{K}$ of characteristic zero, this was noted already in [Ka, Remark 4.17] (for convex orderings that are adapted to the orientation of the quiver).

Corollary 3.15. *For any $\alpha \in Q^+$ and $\lambda \in \mathrm{KP}(\alpha)$, the projective module $P(\lambda)$ has a Δ-flag with $(P(\lambda) : \Delta(\mu)) = [\bar{\Delta}(\mu) : L(\lambda)]$ (the latter notation denotes graded Jordan-Hölder multiplicity).*

Proof. The theorem immediately implies that $P(\lambda)$ has a Δ-flag. Moreover

$$(P(\lambda) : \Delta(\mu)) = \overline{\operatorname{Dim} \operatorname{Hom}_{H_\alpha}(P(\lambda), \bar{\nabla}(\mu))} = \overline{[\bar{\nabla}(\mu) : L(\lambda)]} = [\bar{\Delta}(\mu) : L(\lambda)]$$

as $L(\lambda)^{\circledast} \cong L(\lambda)$. $\square$

Corollary 3.16. *For any $\alpha \in Q^+$, we have that*

$$\operatorname{Dim} H_\alpha = \sum_{\lambda \in \mathrm{KP}(\alpha)} (\operatorname{Dim} \Delta(\lambda))(\operatorname{Dim} \bar{\Delta}(\lambda)) = \sum_{\lambda \in \mathrm{KP}(\alpha)} (\operatorname{Dim} \bar{\Delta}(\lambda))^2 \Big/ \prod_{\substack{\beta \in R^+ \\ 1 \leq r \leq m_\beta(\lambda)}} (1 - q^{2r}).$$

Proof. Again H_α has a Δ-flag by the theorem, so its dimension is given by

$$\begin{aligned} \operatorname{Dim} H_\alpha &= \sum_{\lambda \in \mathrm{KP}(\alpha)} (\operatorname{Dim} \Delta(\lambda))(\overline{\operatorname{Dim} \operatorname{Hom}_{H_\alpha}(H_\alpha, \bar{\nabla}(\lambda))}) \\ &= \sum_{\lambda \in \mathrm{KP}(\alpha)} (\operatorname{Dim} \Delta(\lambda))(\overline{\operatorname{Dim} \bar{\nabla}(\lambda)}) = \sum_{\lambda \in \mathrm{KP}(\alpha)} (\operatorname{Dim} \Delta(\lambda))(\operatorname{Dim} \bar{\Delta}(\lambda)). \end{aligned}$$

To deduce the second equality use the last part of Theorem 3.12. $\square$

Corollary 3.17. *For any $\lambda \in \mathrm{KP}$, we have that*

$$\Delta(\lambda) \cong P(\lambda) \Big/ \sum_{\mu \not\preceq \lambda} \; \sum_{f: P(\mu) \to P(\lambda)} \operatorname{im} f,$$

$$\bar{\Delta}(\lambda) \cong P(\lambda) \Big/ \sum_{\mu \not\prec \lambda} \; \sum_{f: P(\mu) \to \operatorname{rad} P(\lambda)} \operatorname{im} f,$$

summing over all (not necessarily homogeneous) homomorphisms f.

Proof. In view of Theorem 3.13(2), the Δ-flag of $P(\lambda)$ can be arranged in order refining $\preceq$, with $\Delta(\lambda)$ at the top. Now apply Theorems 3.12 and 2.10. $\square$

Remark 3.18. Assuming char $\mathbb{K} = 0$, Corollary 3.17 implies that our modules $\Delta(\lambda)$ and $\bar{\Delta}(\lambda)$ coincide with the modules $\widetilde{E}_b$ and E_b from [Ka, Corollary 4.18] (for $b \in B(\infty)$ chosen so that $L(\lambda) \cong L_b$).

4. Standard modules II

We continue to work with a fixed convex order $\prec$ on R^+. The results in this section *do not* reply on the assumption that H_α has finite global dimension. Instead, we suppose that the following two hypotheses hold:

(MP) For each $\alpha \in R^+$ of height at least two, we are given $\mathrm{mp}(\alpha) \in \mathrm{MP}(\alpha)$ satisfying the length two property.

(EXT) For each $\alpha \in R^+$ we have that $\operatorname{Ext}^1_{H_\alpha}(L(\alpha), L(\alpha)) \cong q^{-2}\mathbb{K}$.

The main new result of the section uses (MP) and (EXT) to give an alternative inductive description of the root modules $\Delta(\alpha)\,(\alpha \in R^+)$ via certain short exact sequences. The finiteness of global dimension of the KLR algebras is an easy consequence. Then in the next section we justify the assumptions (MP) and (EXT) by checking them directly for a special choice of convex ordering. Put together, this will show that the assumption (GD) from the previous subsection always holds. Of course, as soon as that has been established, we will know in fact that the hypothesis (EXT) is true for all convex orderings by Theorem 3.4 above; since we believe in Conjecture 2.16 we also expect that the hypothesis (MP) is always true for all orderings and all minimal pairs.

4.1. **A short exact sequence.** The arguments in §3.2 don't use the full force of the assumption (GD); they only require the vanishing of $\mathrm{Ext}^1_{H_\alpha}(\Delta(\alpha), \Delta(\alpha))$ for each $\alpha \in R^+$. This vanishing is a consequence of our alternative assumption (EXT).

Lemma 4.1. *For any $\alpha \in R^+$, we have that*

$$\mathrm{Ext}^1_{H'_\alpha}(L'(\alpha), L'(\alpha)) = \mathrm{Ext}^1_{H_\alpha}(\Delta(\alpha), \Delta(\alpha)) = 0.$$

Proof. Repeat the arguments used to prove Corollaries 3.5 and 3.6 but using (EXT) in place of Theorem 3.4. $\square$

Hence we can appeal to all the results established in §3.2. In particular we define standard modules $\Delta(\lambda)$ for all $\lambda \in \mathrm{KP}$ as in Theorem 3.12. For the rest of the subsection, we fix a choice of $\alpha \in R^+$ of height $n \geq 2$, and denote the fixed minimal pair $\mathrm{mp}(\alpha)$ provided by assumption (MP) by (β, γ). Thus (β, γ) satisfies the length two property. Let $m := \mathrm{ht}(\gamma)$.

Lemma 4.2. *Let $w \in S_n$ be $(1, \dots, n) \mapsto (n-m+1, \dots, n, 1, \dots, n-m)$, so that $\tau_w 1_{\gamma,\beta} = 1_{\beta,\gamma}\tau_w$. There is a unique degree 1 homomorphism*

$$\varphi : \Delta(\beta) \circ \Delta(\gamma) \to \Delta(\gamma) \circ \Delta(\beta)$$

such that $\varphi(1_{\beta,\gamma} \otimes (v_1 \otimes v_2)) = \tau_w 1_{\gamma,\beta} \otimes (v_2 \otimes v_1)$ for all $v_1 \in \Delta(\beta), v_2 \in \Delta(\gamma)$.

Proof. It suffices by Frobenius reciprocity to show that there is an isomorphism

$$q\Delta(\beta) \boxtimes \Delta(\gamma) \xrightarrow{\sim} \mathrm{res}^\alpha_{\beta,\gamma} \Delta(\gamma) \circ \Delta(\beta), \quad v_1 \otimes v_2 \mapsto \tau_w 1_{\gamma,\beta} \otimes (v_2 \otimes v_1).$$

To see this we apply Theorem 2.1. Suppose we are given $\beta_1, \beta_2, \gamma_1, \gamma_2 \in Q^+$ such that $\gamma = \gamma_1 + \gamma_2 = \gamma_2 + \beta_2$, $\beta = \beta_1 + \beta_2 = \gamma_1 + \beta_1$, and both of the restrictions $\mathrm{res}^\gamma_{\gamma_1,\gamma_2}\Delta(\gamma)$ and $\mathrm{res}^\beta_{\beta_1,\beta_2}\Delta(\beta)$ are non-zero. By Lemma 2.11, γ_1 is a sum of positive roots $\preceq \gamma \prec \beta$ and β_1 is a sum of positive roots $\preceq \beta$. Since $\gamma_1 + \beta_1 = \beta$ we deduce from Lemma 2.5 that $\beta_1 = \beta, \beta_2 = 0, \gamma_1 = 0$ and $\gamma_2 = \gamma$. Thus the only non-zero layer in the Mackey filtration is the top layer, which is isomorphic to $q\Delta(\beta) \boxtimes \Delta(\gamma)$. $\square$

The endomorphism x of $\Delta(\beta)$ from Theorem 3.3, which we assume is normalized uniquely as in Lemma 3.10, induces injective endomorphisms $x'_1 := x \circ 1$ of $\Delta(\beta) \circ \Delta(\gamma)$ and $x_2 := 1 \circ x$ of $\Delta(\gamma) \circ \Delta(\beta)$. These endomorphisms are intertwined by the homomorphism φ from Lemma 4.2, i.e. $x_2 \circ \varphi = \varphi \circ x'_1$. Similarly the endomorphism x of $\Delta(\gamma)$ gives us $x_1 := x \circ 1 \in \mathrm{End}_{H_\alpha}(\Delta(\gamma) \circ \Delta(\beta))_2$ and

$x_2' := 1 \circ x \in \mathrm{End}_{H_\alpha}(\Delta(\beta) \circ \Delta(\gamma))_2$ such that $x_1 \circ \varphi = \varphi \circ x_2'$. Then x_1 and x_2 commute, as do x_1' and x_2'.

Lemma 4.3. *The homomorphism φ from Lemma 4.2 is injective. Moreover the module $\Delta(\gamma) \circ \Delta(\beta)$ has irreducible head isomorphic to $L(\alpha)$ and, picking a graded vector space complement A to its unique maximal submodule, we have as a vector space that*

$$\Delta(\gamma) \circ \Delta(\beta) = \mathrm{im}\,\varphi \oplus \bigoplus_{r \geq 0} (c_1 x_1 + c_2 x_2)^r (A)$$

for all but one point $[c_1, c_2] \in \mathbb{P}^1$.

Proof. Let $V := \Delta(\gamma) \circ \Delta(\beta)$ and $L := L(\gamma) \circ L(\beta)$. As in the proof of Theorem 3.3, we can identify $\Delta(\gamma)$ with $k[z_1] \boxtimes L'(\gamma)$ and $\Delta(\beta)$ with $k[z_2] \boxtimes L'(\beta)$ for $z_1 \in Z(H_\gamma)_2, z_2 \in Z(H_\beta)_2$, so that x_1 and x_2 are induced from multiplication by z_1 and z_2, respectively. It follows that there is a natural surjection $\pi : V \twoheadrightarrow L$ with kernel $x_1(V) + x_2(V)$. Moreover, letting C be a graded vector space complement to $x_1(V) + x_2(V)$ in V, we have that $V = \bigoplus_{r,s \geq 0} x_1^r x_2^s (C)$ as a vector space, with $x_1^r x_2^s (C) \cong q^{2(r+s)} C$ for each r, s. Hence, setting $V_t := \sum_{r+s=t} x_1^r x_2^s (V)$, we obtain a filtration $V = V_0 \supset V_1 \supset \cdots$ with $V_t / V_{t+1} \cong q^{2t} L^{\oplus(t+1)}$. Recall further from (2.13) that L is uniserial of length two with $\mathrm{soc}\, L \cong qL(\lambda)$ and $L/\mathrm{soc}\, L \cong L(\alpha)$. The filtration just constructed therefore tells us that the head of V can only involve the module $L(\alpha)$ with some multiplicity. The fact that the multiplicity is one follows by a Frobenius reciprocity calculation using Lemma 2.15(4). Let $R_t := \sum_{r+s=t} x_1^r x_2^s (\mathrm{rad}\, V)$. This gives a more refined filtration

$$V = V_0 \supset R_0 \supset V_1 \supset R_1 \supset \cdots$$

with $V_t / R_t \cong q^{2t} L(\alpha)^{\oplus(t+1)}$ and $R_t / V_{t+1} \cong q^{2t+1} L(\lambda)^{\oplus(t+1)}$.

In a similar way, we analyse the module $V' := q\Delta(\beta) \circ \Delta(\gamma)$. Let $L' := qL(\beta) \circ L(\gamma)$, which is uniserial of length two with $\mathrm{soc}\, L' \cong q^2 L(\alpha)$ and $L'/\mathrm{soc}\, L' \cong qL(\lambda)$ by (2.12). Setting $V_t' := \sum_{r+s=t} (x_1')^r (x_2')^s (V')$, we get a filtration $V' = V_0' \supset V_1' \supset \cdots$ with $V_t' / V_{t+1}' \cong q^{2t} (L')^{\oplus(t+1)}$ for each $t \geq 1$. Then using Lemma 2.15(2) we deduce that V' has irreducible head isomorphic to $qL(\lambda)$. Letting $R_t' := \sum_{r+s=t} (x_1')^r (x_2')^s (\mathrm{rad}\, V')$, we get the refined filtration

$$V' = V_0' \supset R_0' \supset V_1' \supset R_1' \supset \cdots$$

such that $V_t' / R_t' \cong q^{2t+1} L(\lambda)^{\oplus(t+1)}$ and $R_t' / V_{t+1}' \cong q^{2t+2} L(\alpha)^{\oplus(t+1)}$.

Note that φ maps V_1' into V_1. The induced map $V'/V_1' \to V/V_1$ is identified with the map $L' \to L, 1_{\beta,\gamma} \otimes (v_1 \otimes v_2) \mapsto \tau_w 1_{\beta,\gamma} \otimes (v_2 \otimes v_1)$ arising by Frobenius reciprocity as in the proof of Lemma 4.2. Hence this induced map is non-zero with kernel equal to the socle $q^2 L(\alpha)$ of V'/V_1' and image equal to the socle $qL(\lambda)$ of V/V_1. This shows that $\varphi(V') \subseteq R_0$, hence $\varphi(V_1') \subseteq R_1$, and $\varphi(R_0') \subseteq V_1$, but $\varphi(V') \not\subseteq V_1$.

Thus φ induces a non-zero homomorphism $\bar{\varphi} : V'/V_1' \to R_0/R_1$. We show in this paragraph that $\bar{\varphi}$ is injective. If not, its kernel must be equal to the socle $q^2 L(\alpha)$ of V'/V_1', hence its image is isomorphic to $qL(\lambda)$. It follows that $R_0/R_1 \cong qL(\lambda) \oplus q^2 L(\alpha) \oplus q^2 L(\alpha)$. But then V/R_1 is a module with irreducible head $L(\alpha)$ and radical $qL(\lambda) \oplus q^2 L(\alpha)^{\oplus 2}$. The existence of such a module contradicts the one-dimensionality of $\mathrm{Ext}^1_{H_\alpha}(L(\alpha), L(\alpha))$ from assumption (EXT).

Now to complete the proof, let A and A' be graded vector space complements to R_0 in V and R'_0 in V', respectively. Then $B := \varphi(A')$ is a complement to V_1 in R_0. In view of the injectivity established in the previous paragraph, there is a unique point $[c_1, c_2] \in \mathbb{P}^1$ such that $\varphi(R'_0) = (c_1 x_1 + c_2 x_2)(A) + R_1$. Let B' be a complement to V'_1 in R'_0 such that φ restricts to an isomorphism between B' and $(c_1 x_1 + c_2 x_2)(A)$. Finally we choose the complement C in the opening paragraph to be $A \oplus B$ and set $C' := A' \oplus B'$, which is a complement to V'_1 in V'. We've shown that φ restricts to isomorphisms $A' \xrightarrow{\sim} B$ and $B' \xrightarrow{\sim} (c_1 x_1 + c_2 x_2)(A)$. As $V = \bigoplus_{r,s \geq 0} (x_1^r x_2^s(A) \oplus x_1^r x_2^s(B))$ and $V' = \bigoplus_{r,s \geq 0} ((x'_1)^r (x'_2)^s (A') \oplus (x'_1)^r (x'_2)^s (B'))$, we deduce that φ gives an isomorphism between V' and $\bigoplus_{r,s \geq 0} ((x_1)^r (x_2)^s (B) \oplus (x_1)^r (x_2)^s ((c_1 x_1 + c_2 x_2)(A))$. So φ is injective. Moreover $\bigoplus_{r \geq 0} (c'_1 x_1 + c'_2 x_2)^r (A)$ is a complement to $\operatorname{im} \varphi$ in V for any point $[c'_1, c'_2] \in \mathbb{P}^1 \setminus \{[c_1, c_2]\}$. $\square$

Theorem 4.4. *For a minimal pair $(\beta, \gamma) \in \mathrm{MP}(\alpha)$ with the length two property chosen as above, there is a short exact sequence*

$$0 \longrightarrow q\Delta(\beta) \circ \Delta(\gamma) \xrightarrow{\varphi} \Delta(\gamma) \circ \Delta(\beta) \xrightarrow{\psi} \Delta(\alpha) \longrightarrow 0.$$

Moreover there are unique scalars $k_i \in \mathbb{K}$ for $i \in \{1, 2\}$, at least one of which is non-zero, such that $\psi \circ x_i = k_i x \circ \psi$, where $x \in \mathrm{End}_{H_\alpha}(\Delta(\alpha))_2$ is the endomorphism from Lemma 3.10 and x_i is as defined just before Lemma 4.3.

Proof. In Lemma 4.2, we have already constructed the first map φ in the short exact sequence; it is injective by Lemma 4.3. Let $V := \Delta(\gamma) \circ \Delta(\beta)$ and $Q := \operatorname{Coker} \varphi$. By Lemma 4.3, for at least one $i \in \{1, 2\}$, the degree two endomorphism x_i of V induces a *non-zero* endomorphism x of Q. To prove the theorem, it remains to show that $Q \cong \Delta(\alpha)$; then, in view of the uniqueness from Theorem 3.3, x can be rescaled so it corresponds exactly to the endomorphism from Lemma 3.10 under the isomorphism $Q \cong \Delta(\alpha)$.

We also know from Lemma 4.3 that Q has a unique maximal submodule Q_1 with $Q/Q_1 \cong L(\alpha)$, and that $Q = \bigoplus_{r \geq 0} x^r(A)$ for any complement A to Q_1 in Q. Thus Q has a filtration $Q = Q_0 \supset Q_1 \supset \cdots$ defined from $Q_r := \sum_{s \geq r} x^s(Q)$ such that $Q_r / Q_{r+1} \cong q^{2r} L(\alpha)$ for each $r \geq 0$. Letting Q' (resp. Q'_r) denote the restriction of Q (resp. Q_r) to H'_α and $L' := L'(\alpha)$, we get a short exact sequence

$$0 \longrightarrow Q'_1 \longrightarrow Q' \longrightarrow L' \longrightarrow 0.$$

We claim that $\mathrm{ext}^1_{H'_\alpha}(L', Q'_1) = 0$, so that this short exact sequence splits. To prove the claim, we have that $Q'_1 \cong \varprojlim Q'_1/Q'_r$. By [W, Theorem 3.5.8], there is a short exact sequence

$$0 \to \varprojlim{}^1 \hom_{H'_\alpha}(L', Q'_1/Q'_r) \to \mathrm{ext}^1_{H'_\alpha}(L', Q'_1) \to \varprojlim \mathrm{ext}^1_{H'_\alpha}(L', Q'_1/Q'_r) \to 0.$$

We must show that the first and last terms vanish. Each Q'_1/Q'_r is finite dimensional with all composition factors isomorphic to L' (up to degree shift). In view of Lemma 4.1, we have that $\mathrm{Ext}^1_{H'_\alpha}(L', L') = 0$, so each Q'_1/Q'_r is completely reducible and $\mathrm{ext}^1_{H'_\alpha}(L', Q'_1/Q'_r) = 0$ for each r. Thus $\varprojlim \mathrm{ext}^1_{H'_\alpha}(L', Q'_1/Q'_r) =$

0. Moreover the surjection $Q'_1/Q'_{r+1} \twoheadrightarrow Q'_1/Q'_r$ splits, so the canonical map $\hom_{H'_\alpha}(L', Q'_1/Q'_{r+1}) \to \hom_{H'_\alpha}(L', Q'_1/Q'_r)$ is surjective. This shows the tower $(\hom_{H'_\alpha}(L', Q'_1/Q'_r))$ satisfies Mittag-Leffler, so $\varprojlim^1 \hom_{H'_\alpha}(L', Q'_1/Q'_r) = 0$ by [W, Proposition 3.5.7]. We have proved the claim.

So we can choose the complement A in the previous paragraph so that it is an H'_α-submodule of Q isomorphic to $L'(\alpha)$. Then Frobenius reciprocity gives us a non-zero H_α-module homomorphism $\Delta(\alpha) \to Q$ with image containing A. It is surjective as A generates Q as an H_α-module. It is an isomorphism by comparing graded dimensions. □

4.2. **Global dimension.** We are ready to (re)prove that H_α has finite global dimension; cf. [Ka, Corollary 2.9] and [M, Theorem 4.6]. We stress that our argument here applies Theorem 4.4, hence depends on the assumption (MP).

Theorem 4.5. *Assume that $\alpha \in Q^+$ is of height n and take $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}(\alpha)$. Then the projective dimensions of $\Delta(\lambda)$ and $L(\lambda)$ satisfy*

$$\operatorname{pd} \Delta(\lambda) \le n - l, \qquad \operatorname{pd} L(\lambda) \le n.$$

In the extreme case $n = l$, we have that $\Delta(\lambda) \cong P(\lambda)$ and $\operatorname{pd} L(\lambda) = n$.

Proof. For the first bound, we need to show that $\operatorname{ext}^d_{H_\alpha}(\Delta(\lambda), V) = 0$ for any H_α-module V and $d > n - l$. Using Theorem 3.12 and generalized Frobenis reciprocity, this reduces to checking in the case that α is a positive root that $\operatorname{ext}^d_{H_\alpha}(\Delta(\alpha), V) = 0$ for all $d > n - 1$. To see this, apply $\hom_{H_\alpha}(-, V)$ to the short exact sequence from Theorem 4.4 and use generalized Frobenius reciprocity and induction on n.

For the second bound, we first show that $\operatorname{ext}^d_{H_\alpha}(\bar\nabla(\lambda), V) = 0$ for any V, any $\alpha \in Q^+, \lambda \in \mathrm{KP}(\alpha)$ and $d > n$. By Lemma 2.3 and up to a degree shift, $\bar\nabla(\lambda)$ is induced from $L(\lambda_l) \boxtimes \cdots \boxtimes L(\lambda_1)$. So we can use generalized Frobenius reciprocity to reduce to showing for a positive root α that $\operatorname{ext}^d_{H_\alpha}(L(\alpha), V) = 0$ for any V and $d > n$. This follows by applying $\hom_{H_\alpha}(-, V)$ to the short exact sequence from Theorem 3.3 and using the result from the previous paragraph.

Now take $\alpha \in Q^+$ of height n and $\lambda \in \mathrm{KP}(\alpha)$, and assume by induction that we have proved that $\operatorname{ext}^d_{H_\alpha}(L(\mu), V) = 0$ for all V, $\mu \prec \lambda$ and $d > n$. Define Q from the short exact sequence $0 \to L(\lambda) \to \bar\nabla(\lambda) \to Q \to 0$. By the induction hypothesis and Theorem 2.10, $\operatorname{ext}^d_{H_\alpha}(Q, V) = 0$ for all $d > n$. We finally deduce that $\operatorname{ext}^d_{H_\alpha}(L(\lambda), V) = 0$ for all $d > n$ on applying $\hom_{H_\alpha}(-, V)$ to the short exact sequence and using the previous paragraph.

Finally suppose that $n = l$. We've shown $\Delta(\lambda)$ is projective, hence it is isomorphic to $P(\lambda)$ as it has irreducible head $L(\lambda)$. Also $\operatorname{Ext}^n_{H_\alpha}(L(\lambda), L(\lambda)) \neq 0$ by generalized Frobenius reciprocity and Lemma 5.8, so $\operatorname{pd} L(\lambda) = n$. □

Corollary 4.6. *For $\alpha \in Q^+$ of height n, the algebra H_α has global dimension n, i.e. $\sup \operatorname{pd} V = n$ where the supremum is taken over all H_α-modules V.*

Proof. Exactly as explained in the proof of [M, Theorem 4.6], this is a consequence of the facts about $\operatorname{pd} L(\lambda)$ established in Theorem 4.5. □

4.3. **Projective resolutions.** We continue to assume (MP). Implicit in the proof of Theorem 4.5 is a projective resolution $P_*(\alpha)$ of the root module $\Delta(\alpha)$, i.e. a complex

$$\cdots \to P_2(\alpha) \stackrel{\partial_2}{\to} P_1(\alpha) \stackrel{\partial_1}{\to} P_0(\alpha) \stackrel{\partial_0}{\to} 0$$

of projective modules with $\mathrm{H}_0(P_*(\alpha)) \cong \Delta(\alpha)$ and $\mathrm{H}_d(P_*(\alpha)) = 0$ for $d \neq 0$. This is constructed as follows.

For $i \in I$, we have that $\Delta(\alpha_i) = H_{\alpha_i}$, which is projective already. So we just have to set $P_0(\alpha_i) := \Delta(\alpha_i)$ and $P_d(\alpha_i) := 0$ for $d \neq 0$ to obtain a projective resolution $P_*(\alpha_i)$ of $\Delta(\alpha_i)$.

Now suppose that $\alpha \in R^+$ is of height at least two and denote the minimal pair $\mathrm{mp}(\alpha)$ given by (MP) by (β, γ). We may assume by induction that the projective resolutions $P_*(\beta)$ and $P_*(\gamma)$ are already defined. Taking the total complex of their tensor product using [W, Acyclic Assembly Lemma 2.7.3], we obtain a projective resolution $P_*(\beta, \gamma)$ of $\Delta(\beta) \circ \Delta(\gamma)$ with

$$P_d(\beta, \gamma) := \bigoplus_{d_1+d_2=d} P_{d_1}(\beta) \circ P_{d_2}(\gamma),$$

$$\partial_d := \Big(\mathrm{id} \circ \partial_{d_2} - (-1)^{d_2} \partial_{d_1} \circ \mathrm{id}\Big)_{d_1+d_2=d} : P_d(\beta, \gamma) \to P_{d-1}(\beta, \gamma).$$

Similarly we obtain a projective resolution $P_*(\gamma, \beta)$ of $\Delta(\gamma) \circ \Delta(\beta)$ with

$$P_d(\gamma, \beta) := \bigoplus_{d_1+d_2=d} P_{d_1}(\gamma) \circ P_{d_2}(\beta),$$

$$\partial_d := \Big(\partial_{d_1} \circ \mathrm{id} + (-1)^{d_1} \mathrm{id} \circ \partial_{d_2}\Big)_{d_1+d_2=d} : P_d(\gamma, \beta) \to P_{d-1}(\gamma, \beta).$$

(We've chosen signs carefully here so that Theorem 4.7 works out nicely.) Applying [W, Comparision Theorem 2.2.6], the map φ from Lemma 4.2 lifts to a chain map $\varphi_* : qP_*(\beta, \gamma) \to P_*(\gamma, \beta)$. Then we take the mapping cone of φ_* to obtain a complex $P_*(\alpha)$ with

$$P_d(\alpha) := P_d(\gamma, \beta) \oplus qP_{d-1}(\beta, \gamma),$$

$$\partial_d := (\partial_d, \partial_{d-1} + (-1)^{d-1}\varphi_{d-1}) : P_d(\alpha) \to P_{d-1}(\alpha).$$

In view of Theorem 4.4 and [W, Acyclic Assembly Lemma 2.7.3] once again, $P_*(\alpha)$ is a projective resolution of $\Delta(\alpha)$.

Let us describe $P_d(\alpha)$ more explicitly. First for $i \in I$ and the empty tuple σ, set $\boldsymbol{i}_{\alpha_i,\sigma} := i$. Now suppose that α is of height $n \geq 2$ and $\mathrm{mp}(\alpha) = (\beta, \gamma)$ for γ of height m. For $\sigma = (\sigma_1, \dots, \sigma_{n-1}) \in \{0,1\}^{n-1}$, let $|\sigma| := \sigma_1 + \cdots + \sigma_{n-1}$, $\sigma_{<m} := (\sigma_1, \dots, \sigma_{m-1})$ and $\sigma_{>m} := (\sigma_{m+1}, \dots, \sigma_{n-1})$. Define $\boldsymbol{i}_{\alpha,\sigma} \in \langle I \rangle_\alpha$ recursively from

$$\boldsymbol{i}_{\alpha,\sigma} := \begin{cases} \boldsymbol{i}_{\gamma,\sigma_{<m}} \boldsymbol{i}_{\beta,\sigma_{>m}} & \text{if } \sigma_m = 0, \\ \boldsymbol{i}_{\beta,\sigma_{>m}} \boldsymbol{i}_{\gamma,\sigma_{<m}} & \text{if } \sigma_m = 1. \end{cases}$$

Then we have that

$$P_d(\alpha) = \bigoplus_{\substack{\sigma \in \{0,1\}^{n-1} \\ |\sigma| = d}} q^d H_\alpha 1_{\boldsymbol{i}_{\alpha,\sigma}}. \tag{4.1}$$

In particular, $P_0(\alpha) = H_\alpha 1_{\boldsymbol{i}_{\alpha,\mathbf{0}}}$ where $\mathbf{0} = (0, \dots, 0)$. One can show using Lemma 2.15(4) and induction that $\mathrm{Dim}\, 1_{\boldsymbol{i}_{\alpha,\mathbf{0}}} L(\alpha) = 1$, so that there is a unique

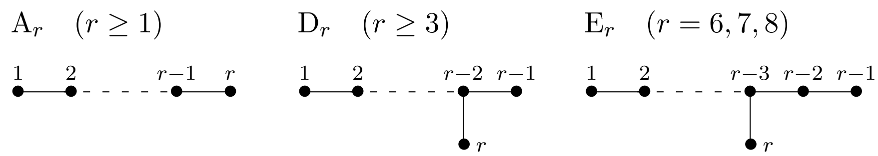


TABLE 1. The standard ordering.

(up to scalar) homogeneous homomorphism $P_0(\alpha) \twoheadrightarrow \Delta(\alpha)$. For the differentials $\partial_d : P_d(\alpha) : P_{d-1}(\alpha)$, there are elements $\tau_{\sigma,\rho} \in 1_{\boldsymbol{i}_{\alpha,\sigma}} H_\alpha 1_{\boldsymbol{i}_{\alpha,\rho}}$ for each $\sigma, \rho \in \{0,1\}^{n-1}$ with $|\sigma| = d, |\rho| = d-1$ such that, on viewing elements of (4.1) as row vectors, the differential ∂_d is defined by right multiplication by the matrix $(\tau_{\sigma,\rho})_{|\sigma|=d,|\rho|=d-1}$. Moreover $\tau_{\sigma,\rho} = 0$ unless the tuples σ and ρ differ in just one entry. Unfortunately we have not been able to find a satisfactory description of such elements $\tau_{\sigma,\rho}$, except in the following special case.

Theorem 4.7. *Suppose that $\alpha \in R^+$ is* multiplicity-free*, i.e. $\varpi_i \cdot \alpha \leq 1$ for all $i \in I$. Then the elements $\tau_{\sigma,\rho}$ inducing the differential $\partial_d : P_d(\alpha) \to P_{d-1}(\alpha)$ as above may be chosen so that*

$$\tau_{\sigma,\rho} := (-1)^{\sigma_1 + \cdots + \sigma_{r-1}} \tau_w$$

if σ and ρ differ just in the rth entry, where $w \in S_n$ is the unique *permutation with $1_{\boldsymbol{i}_{\alpha,\sigma}} \tau_w = \tau_w 1_{\boldsymbol{i}_{\alpha,\rho}}$.*

Proof. This goes by induction on height. The key point for the induction step is that the chain map $\varphi_* : qP_*(\beta,\gamma) \to P_*(\gamma,\beta)$ in the above construction can be chosen so that $\varphi_d : qP_{d_1}(\beta) \circ P_{d_2}(\gamma) \to P_{d_2}(\gamma) \circ P_{d_1}(\beta)$ is defined by right multiplication by $(-1)^{d_1}\tau_w$, where w is the permutation from Lemma 4.2. The proof that this is indeed a chain map relies on the fact that the braid relations hold exactly in H_α under the assumption that α is multiplicity-free. $\square$

5. LYNDON ORDERINGS

In this section, we verify the hypotheses (MP) and (EXT) from the previous section for a particular choice of convex ordering.

5.1. **Lyndon orderings and minimal pairs.** Suppose we are given a total ordering $<$ on the index set I. This extends lexicographically to a total ordering $<$ on $\langle I \rangle$. Then by [L, §4.3], there is a well-defined injection $R^+ \hookrightarrow \langle I \rangle, \alpha \mapsto \boldsymbol{i}_\alpha$, where $\boldsymbol{i}_\alpha$ is defined recursively so that $\boldsymbol{i}_{\alpha_i} := i$ for each $i \in I$ and

$$\boldsymbol{i}_\alpha = \max\{\boldsymbol{i}_\gamma \boldsymbol{i}_\beta \mid \beta, \gamma \in R^+, \beta + \gamma = \alpha, \boldsymbol{i}_\beta > \boldsymbol{i}_\gamma\} \tag{5.1}$$

for $\alpha \in R^+ \setminus \{\alpha_i \mid i \in I\}$. We define the *Lyndon ordering* $\prec$ on R^+ associated to the total ordering $<$ of I by declaring that $\alpha \prec \beta$ if and only if $\boldsymbol{i}_\alpha < \boldsymbol{i}_\beta$. According to a result from the (sadly unavailable) preprint [R], Lyndon orderings are convex (but not all convex orderings arise in this way).

Example 5.1. Suppose we are in type A$_r$ and $\prec$ is the Lyndon ordering arising from the standard ordering of I indicated in Table 1. If α is the highest root $\alpha_1 + \cdots + \alpha_r$ and (β, γ) is a minimal pair for α, then we must have that

$\beta = \alpha_{s+1} + \cdots + \alpha_r$ and $\gamma = \alpha_1 + \cdots + \alpha_s$ for $1 \leq s \leq r-1$. The corresponding words $\boldsymbol{i}_\alpha, \boldsymbol{i}_\beta$ and $\boldsymbol{i}_\gamma$ are the *increasing segments* $1 \cdots r$, $(s+1)\cdots r$ and $1 \cdots s$, respectively. Observe in particular that $\boldsymbol{i}_\alpha = \boldsymbol{i}_\gamma \boldsymbol{i}_\beta$.

Example 5.2. Suppose we are in type D_r and $\prec$ is the Lyndon ordering arising from the standard ordering of I indicated in Table 1. Suppose α is a positive root not contained in a lower rank subsystem. Then $\alpha = \alpha_1 + \cdots + \alpha_{s-1} + 2\alpha_s + \cdots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$ for some $2 \leq s \leq r-1$, and $\boldsymbol{i}_\alpha = 1 \cdots (r-2)\underline{r \cdots s}$ (the underline denotes a *decreasing segment*). If (β, γ) is a minimal pair for α, then we either have that $\boldsymbol{i}_\beta = s$ and $\boldsymbol{i}_\gamma = 1 \cdots (r-2)\underline{r \cdots (s+1)}$, or $\boldsymbol{i}_\beta = (t+1)\cdots(r-2)\underline{r \cdots s}$ and $\boldsymbol{i}_\gamma = 1 \cdots t$ for some $1 \leq t \leq s-2$. Again it happens always that $\boldsymbol{i}_\alpha = \boldsymbol{i}_\gamma \boldsymbol{i}_\beta$.

Example 5.3. Using a computer, we have worked out all the minimal pairs (β, γ) for all $\alpha \in R^+$ in type E_8 (hence also in the subsystems of types E_6 and E_7), assuming $\prec$ is the Lyndon ordering arising from the standard ordering from Table 1. Again in all cases it turns out that $\boldsymbol{i}_\alpha = \boldsymbol{i}_\gamma \boldsymbol{i}_\beta$. These "minimal factorizations" are listed below; to avoid duplication we display only ones that have not already been exhibited in some lower rank type A or D subsystem.

1|2345867, 12|345867, 123|45867, 1234|5867, 123458|67, 1234586|7; 1|23458675, 12|3458675, 123|458675, 12345867|5; 1|234586754, 12|34586754, 123458675|4; 1|2345867543, 1234586754|3; 12345867543|2; 1|234586756, 12|34586756, 123|4586756, 12345867|56, 123458675|6; 1|2345867564, 12|345867564, 1234586756|4; 1|23458675643, 12345867564|3; 123458675643|2; 1|23458675645, 12|3458675645, 1234586756|45, 12345867564|5; 1|234586756453, 123458675645|3; 1234586756453|2; 1|2345867564534, 123458675645|34, 1234586756453|4; 12345867564534|2; 12345867564534|23, 123458675645342|3; 1234586756453423|1234586756458; 1|234586756458, 12|34586756458, 1234586756|458, 12345867564|58, 123458675645|8; 1|2345867564583, 1234586756458|3; 12345867564583|2; 1|23458675645834, 1234586756458|34, 12345867564583|4; 123458675645834|2; 123458675645834|23, 1234586756458342|3; 1|234586756458345, 1234586756458|345, 12345867564583|45, 123458675645834|5; 1234586756458345|2; 1234586756458345|23, 12345867564583452|3; 1234586756458345|234, 12345867564583452|34, 123458675645834523|4; 1|2345867564583456, 1234586756458|3456, 12345867564583|456, 123458675645834|56, 1234586756458345|6; 12345867564583456|2; 12345867564583456|23, 123458675645834562|3; 12345867564583456|234, 123458675645834562|34, 1234586756458345623|4; 12345867564583456|2345, 123458675645834562|345, 1234586756458345623|45, 12345867564583456234|5; 12345867564583456|23458, 123458675645834562|3458, 1234586756458345623|458, 12345867564583456234|58, 123458675645834562345|8; 1|23458675645834567, 1234586756458|34567, 12345867564583|4567, 123458675645834|567, 1234586756458345|67, 12345867564583456|7; 123458675645834567|2; 123458675645834567|23, 1234586756458345672|3; 123458675645834567|234, 1234586756458345672|34, 12345867564583456723|4; 123458675645834567|2345, 1234586756458345672|345, 12345867564583456723|45, 123458675645834567234|5; 123458675645834567|23456, 1234586756458345672|3456, 12345867564583456723|456, 123458675645834567234|56, 1234586756458345672345|6; 123458675645834567|23458, 1234586756458345672|3458, 12345867564583456723|458, 123458675645834567234|58, 1234586756458345672345|8; 123458675645834567|234586, 1234586756458345672|34586, 12345867564583456723|4586, 123458675645834567234|586, 12345867564583456723458|6; 123458675645834567|2345865, 1234586756458345672|345865, 12345867564583456723|45865, 123458675645834567234586|5; 123458675645834567|23458654, 1234586756458345672|3458654, 1234586756458345672345865|4; 123458675645834567|234586543, 12345867564583456723458654|3; 123458675645834567234586543|2; 2|345867, 23|45867, 234|5867, 23458|67, 234586|7; 2|3458675, 23|458675, 2345867|5; 2|34586754, 23458675|4; 234586754|3; 2|34586756, 23|4586756, 2345867|56, 23458675|6; 2|345867564, 234586756|4; 2345867564|3; 2|3458675645, 234586756|45, 2345867564|5; 23458675645|3; 23458675645|34, 234586756453|4; 2|34586756458, 234586756|458, 2345867564|58, 23458675645|8; 234586756458|3; 234586756458|34, 2345867564583|4; 234586756458|345, 2345867564583|45, 23458675645834|5; 234586756458|3456, 2345867564583|456, 23458675645834|56, 234586756458345|6; 234586756458|34567, 2345867564583|4567, 23458675645834|567, 234586756458345|67, 2345867564583456|7; 3|45867, 34|5867, 3458|67, 34586|7; 3|458675, 345867|5; 3458675|4; 3|4586756, 345867|56, 3458675|6; 34586756|4; 34586756|45, 345867564|5; 34586756|458, 345867564|58 , 3458675645|8; 4|5867, 458|67, 4586|7; 45867|5; 45867|56, 458675|6; 58|67, 586|7.

Remark 5.4. For an arbitrary Lyndon ordering and $(\beta,\gamma) \in \mathrm{MP}(\alpha)$, it need not be the case that $\boldsymbol{i}_\alpha = \boldsymbol{i}_\gamma \boldsymbol{i}_\beta$. However the two-part Kostant partition (β,γ) of α in which γ is maximal does always produce a minimal pair with $\boldsymbol{i}_\alpha = \boldsymbol{i}_\gamma \boldsymbol{i}_\beta$. This distinguished choice of minimal pair corresponds to the "costandard factorization" from [L, §3.2].

5.2. **Some minimal pairs with the length two property.** Next we recall the construction of *homogeneous representations* from [KR1]. Let $\sim$ be the equivalence relation on $\langle I \rangle$ generated by interchanging an adjacent pair of letters which are not connected by an edge in the quiver. A word $\boldsymbol{i} \in \langle I \rangle$ of length n is said to be *homogeneous* if it is impossible to find $\boldsymbol{j} \sim \boldsymbol{i}$ such that either $j_r = j_{r+1}$ for some $1 \leq r \leq n-1$ or $j_s = j_{s+2}$ for some $1 \leq s \leq n-2$. If $\alpha \in R^+$ is such that $\boldsymbol{i}_\alpha$ is homogeneous, then the cuspidal module $L(\alpha)$ can be constructed explicitly as the graded vector space with basis $\{v_{\boldsymbol{i}} \mid \boldsymbol{i} \sim \boldsymbol{i}_\alpha\}$ concentrated in degree zero, such that each $v_{\boldsymbol{i}}$ is in the $\boldsymbol{i}$-word space, all x_j act as zero, and $\tau_k v_{\boldsymbol{i}} := v_{(k\,k+1)(\boldsymbol{i})}$ if $i_k \not- i_{k+1}$, $\tau_k v_{\boldsymbol{i}} := 0$ otherwise.

Theorem 5.5. *The Lyndon ordering $\prec$ arising from the standard ordering of I indicated in Table 1 satisfies the length two property.*

Proof. Let (β,γ) be a minimal pair for $\alpha \in R^+$. The key feature of the chosen ordering is that in all cases $\boldsymbol{i}_\beta$ and $\boldsymbol{i}_\gamma$ are homogeneous, so

$$\mathrm{Ch}\, L(\beta) = \sum_{\boldsymbol{j} \sim \boldsymbol{i}_\beta} \boldsymbol{j}, \qquad \mathrm{Ch}\, L(\gamma) = \sum_{\boldsymbol{k} \sim \boldsymbol{i}_\gamma} \boldsymbol{k}.$$

This follows from the explicit description of the words $\boldsymbol{i}_\beta$ and $\boldsymbol{i}_\gamma$ from Examples 5.1–5.3. By the highest word theory from [KR2] the $\boldsymbol{i}_\alpha$-word space of $L(\alpha)$ is non-zero; this can also be checked directly using the formula (5.2) recorded below. So to prove the theorem it suffices to show that the $\boldsymbol{i}_\alpha$-word space of $\bar{\Delta}(\alpha) = L(\beta) \circ L(\gamma)$ has graded dimension q. Equivalently, by Lemma 2.3, we show that the $\boldsymbol{i}_\alpha$-word space of $L(\gamma) \circ L(\beta)$ has graded dimension 1. Since in all cases we have that $\boldsymbol{i}_\alpha = \boldsymbol{i}_\gamma \boldsymbol{i}_\beta$, and recalling (2.5), this amounts to checking that the only pair $(\boldsymbol{k}, \boldsymbol{j})$ with $\boldsymbol{k} \sim \boldsymbol{i}_\gamma$ and $\boldsymbol{j} \sim \boldsymbol{i}_\beta$ such that $\boldsymbol{i}_\gamma \boldsymbol{i}_\beta$ appears in the shuffle product $\boldsymbol{k} \circ \boldsymbol{j}$ is the pair $(\boldsymbol{i}_\gamma, \boldsymbol{i}_\beta)$ itself, and for this pair the only shuffle producing $\boldsymbol{i}_\gamma \boldsymbol{i}_\beta$ is the trivial one. This follows by inspection from the information in Examples 5.1–5.3. $\square$

Remark 5.6. If the convex ordering $\prec$ is arbitrary but α is a *multiplicity-free* positive root, a similar (actually easier) argument to the proof of Theorem 5.5 shows that *every* minimal pair (β, γ) for α satisfies the length two property. In particular all convex orderings in type A have the length two property.

5.3. **An $\mathbf{Ext^1}$ computation.** Now we verify hypothesis (EXT) from the previous section for the standard ordering. The argument here is essentially the same as the first half of the proof of [M, Proposition 4.5]; the hope that the same strategy should also work in the simply-laced case was already expressed there.

Theorem 5.7. *Suppose that $\prec$ is the Lyndon ordering arising from the standard ordering of I from Table 1. For every $\alpha \in R^+$ and $\lambda = (\beta, \gamma) \in \mathrm{MP}(\alpha)$, there exists an H_α-module V with $\operatorname{soc} V \cong q^2 L(\alpha)$ and $V/\operatorname{soc} V \cong L(\gamma) \circ L(\beta)$.*

Proof. As noted in the proof of Theorem 5.5, both $\boldsymbol{i}_\beta$ and $\boldsymbol{i}_\gamma$ are homogeneous, so the modules $L(\beta)$ and $L(\gamma)$ can be constructed explicitly as above as homogeneous representations. In a similar way we construct a module $L^2(\beta)$ with $\operatorname{soc} L^2(\beta) \cong qL(\beta)$ and $L^2(\beta)/\operatorname{soc} L^2(\beta) \cong q^{-1}L(\beta)$ by declaring that it has homogeneous basis $\{v^\pm_{\boldsymbol{i}} \mid \boldsymbol{i} \sim \boldsymbol{i}_\beta\}$ with each $v^\pm_{\boldsymbol{i}}$ of degree ± 1 and belonging to the $\boldsymbol{i}$-word space, such that $x_j v^-_{\boldsymbol{i}} := v^+_{\boldsymbol{i}}$, $x_j v^+_{\boldsymbol{i}} := 0$, $\tau_k v^\pm_{\boldsymbol{i}} := v^\pm_{(k\,k+1)(\boldsymbol{i})}$ if $i_k \not- i_{k+1}$, and $\tau_k v^\pm_{\boldsymbol{i}} := 0$ otherwise.

By Theorem 5.5, the length two property holds, so there is the non-split short exact sequence (2.13). Combined with exactness of induction, it follows that $qL(\gamma) \circ L^2(\beta)$ has a unique submodule $S \cong q^3 L(\lambda)$. Set $V := qL(\gamma) \circ L^2(\beta)/S$ and $v^\pm := 1_{\gamma,\beta} \otimes (v_{\boldsymbol{i}_\gamma} \otimes v^\pm_{\boldsymbol{i}_\beta}) + S$. Then V has a unique submodule $T \cong q^2 L(\alpha)$ generated by v^+, and $V/T \cong L(\gamma) \circ L(\beta)$. It remains to show that $T = \operatorname{soc} V$. Suppose for a contradiction that the socle is larger. Then V must also have a submodule $U \cong qL(\lambda)$. In the next two paragraphs, we prove that there exists a word $\boldsymbol{i} \in \langle I \rangle_\alpha$ and elements $a, b \in H_\alpha$ such that $1_{\boldsymbol{i}} L(\alpha) = 0$, $bv^- \in 1_{\boldsymbol{i}} V$, and $v^+ = abv^-$. This is enough to complete the proof, for then we must have that $bv^- \in U$, hence $v^+ = abv^- \in U$ too, contradicting $U \cap T = 0$.

To construct a, b and $\boldsymbol{i}$, we first assume that α is *not* the highest root in type E_8. Suppose that γ is of height m and recall that $\boldsymbol{i}_\alpha = \boldsymbol{i}_\gamma \boldsymbol{i}_\beta$. Choose $1 \le p \le m$ to be maximal such that the pth letter x of $\boldsymbol{i}_\alpha$ is connected to its $(m+1)$th letter y by an edge in the quiver. By inspection of the information in Examples 5.1–5.3, this is always possible and moreover none of the letters in between x and y are equal to y. Let w be the cycle $(p\, p+1 \cdots m+1)$ and set $a := \tau_{w^{-1}}, b := \tau_w$. Finally let $\boldsymbol{i}$ be the word obtained from $\boldsymbol{i}_\alpha$ by deleting the $(m+1)$th letter y then reinserting it just before the pth letter x; then we have that $bv^- \in 1_{\boldsymbol{i}} V$. An easy application of the relations shows that $abv^- = v^+$ (up to a sign). We are left with showing that $1_{\boldsymbol{i}} L(\alpha) = 0$. But in all these cases $\boldsymbol{i}_\alpha$ is also homogeneous so this follows as $\boldsymbol{i} \not\sim \boldsymbol{i}_\alpha$ by construction.

Finally we treat the highest root for E_8. Here $\boldsymbol{i}_\beta = 1234586756458$ and $\boldsymbol{i}_\gamma = 1234586756453423$, but the word $\boldsymbol{i}_\alpha = \boldsymbol{i}_\gamma \boldsymbol{i}_\beta$ is no longer homogeneous. We set $\boldsymbol{i} := 12345867564534212345867564358$, $a := \tau_w$, $b := \tau_{w^{-1}}$ where w is the cycle $(16\ 17\ \cdots\ 27)$; again we have that $bv^- \in 1_{\boldsymbol{i}} V$. Another explicit relation check (best made now by drawing a picture) shows that $abv^- = v^+$ (up to a sign). It remains to show that $1_{\boldsymbol{i}} L(\alpha) = 0$. Applying Lemma 2.14, we have that

$$\operatorname{Ch} L(\alpha) = \sum_{\boldsymbol{k} \sim \boldsymbol{i}_\gamma, \boldsymbol{j} \sim \boldsymbol{i}_\beta} (\boldsymbol{k} \circ \boldsymbol{j} - q \boldsymbol{j} \circ \boldsymbol{k})/(1-q^2). \tag{5.2}$$

Now one more calculation shows that the $\boldsymbol{i}$-coefficient on the right hand side is indeed zero. $\square$

Corollary 5.8. *Assuming $\prec$ is the Lyndon ordering arising from the standard ordering of I, we have that $\operatorname{Ext}^1_{H_\alpha}(L(\alpha), L(\alpha)) \cong q^{-2}\mathbb{K}$ for each $\alpha \in R^+$.*

Proof. Proceed by induction on height, the result being trivial for simple roots. Assuming α is not simple, let $\lambda = (\beta, \gamma)$ be a minimal pair for α. It satisfies the length two property by Theorem 5.5, so we have the short exact sequence (2.13). Applying $\operatorname{Hom}_{H_\alpha}(-, L(\alpha))$ we get an exact sequence

$$0 \to \operatorname{Ext}^1_{H_\alpha}(L(\alpha), L(\alpha)) \to \operatorname{Ext}^1_{H_\alpha}(L(\gamma) \circ L(\beta), L(\alpha)) \xrightarrow{f} \operatorname{Ext}^1_{H_\alpha}(qL(\lambda), L(\alpha)).$$

By generalized Frobenius reciprocity and Lemma 2.15(4), we have that

$$\operatorname{Ext}^*_{H_\alpha}(L(\gamma) \circ L(\beta), L(\alpha)) \cong \operatorname{Ext}^*_{H_\gamma}(L(\gamma), L(\gamma)) \otimes \operatorname{Ext}^*_{H_\beta}(L(\beta), L(\beta)).$$

We deduce using induction that $\operatorname{Dim} \operatorname{Ext}^1_{H_\alpha}(L(\gamma) \circ L(\beta), L(\alpha)) = 2q^{-2}$. Hence $\operatorname{Dim} \operatorname{Ext}^1_{H_\alpha}(L(\alpha), L(\alpha)) \in \{0, q^{-2}, 2q^{-2}\}$. It is not zero thanks to Lemma 3.2. It remains to rule out the possibility $2q^{-2}$, which we do by showing that the homomorphism f in the above exact sequence is non-zero. For this, we observe that the module V from Theorem 5.7 gives an extension

$$0 \to q^2 L(\alpha) \to V \xrightarrow{g} L(\gamma) \circ L(\beta) \to 0$$

representing an element $E \in \operatorname{Ext}^1_{H_\alpha}(L(\gamma) \circ L(\beta), L(\alpha))$. Let $S := \operatorname{soc}(L(\gamma) \circ L(\beta)) \cong qL(\lambda)$. Then the class $f(E) \in \operatorname{Ext}^1_{H_\alpha}(qL(\lambda), L(\alpha))$ is represented by the short exact sequence $0 \to q^2 L(\alpha) \to g^{-1}(S) \xrightarrow{g} S \to 0$. But this is non-split as V, hence $g^{-1}(S)$, has irreducible socle. Thus $f(E) \neq 0$. $\square$

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