CATEGORICAL ACTIONS AND CRYSTALS

JONATHAN BRUNDAN AND NICHOLAS DAVIDSON

Abstract. This is an expository article developing some aspects of the theory of categorical actions of Kac-Moody algebras in the spirit of works of Chuang–Rouquier, Khovanov–Lauda, Webster, and many others.

1. Introduction

This work is a contribution to the study of categorifications of Kac-Moody algebras and their integrable modules. The subject has its roots in Lusztig's construction of canonical bases of quantum groups using geometry of quiver varieties [Lu1] (which happened around 1990). It is intimately connected to the rich combinatorial theory of crystal bases initiated at the same time by Kashiwara [K1]. In the decade after that, several other examples were studied related to the representation theory of the symmetric group and associated Hecke algebras [LLT, A, G] (building in particular on ideas of Bernstein and Zelevinsky [BZ]), rational representations of the general linear group [BK], and the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ associated to the general linear Lie (super)algebra [BFK, B1]. The first serious attempt to put these examples into a unified axiomatic framework was undertaken by Chuang and Rouquier [CR]. They built a powerful structure theory for studying categorical actions of $\mathfrak{sl}_2$, which they applied notably to prove Broué's Abelian Defect Conjecture for the symmetric groups.

Another major breakthrough came in 2008, when Khovanov and Lauda [KL1, KL2, KL3] and Rouquier [R1] independently introduced some new algebras called *quiver Hecke algebras*, and used them to construct *Kac-Moody 2-categories* associated to arbitrary Kac-Moody algebras. The definitions of Kac-Moody 2-categories given by Khovanov and Lauda and by Rouquier look quite different, so that for a while subsequent works split into two different schools according to which definition they were following. In fact, Rouquier's and Khovanov and Lauda's definitions are equivalent, as was established by the first author [B3].

In this (mostly expository) article, we will revisit some of Rouquier's foundational definitions in the light of [B3]. We do this using the diagrammatic formalism of Khovanov and Lauda wherever possible. From the outset, we have systematically incorporated the better choice of normalization for the second adjunction of the Kac-Moody 2-category suggested by [BHLW]. For a survey with greater emphasis on the connections to geometry, we refer the reader to Kamnitzer's text [Kam].

Another of our goals is to extend several of the existing results so that they may be applied in some more general situations. To explain the novelty, we need some definitions. Let $\Bbbk$ be an algebraically closed field and $\mathcal{V}ec$ be the category of (small) vector spaces. A *finite-dimensional category* is a small $\Bbbk$-linear category $\mathcal{A}$ all of whose morphism spaces are finite-dimensional. Let Mod-$\mathcal{A}$ denote the functor category $\mathcal{H}om(\mathcal{A}^{\mathrm{op}}, \mathcal{V}ec)$ of right modules over $\mathcal{A}$. We say that $\mathcal{A}$ is *Artinian* if all of the finitely generated objects and the finitely cogenerated objects in Mod-$\mathcal{A}$ have finite length (see also Remark 2.1). A *locally Schurian category* is an Abelian category that is equivalent to Mod-$\mathcal{A}$ for some

2010 *Mathematics Subject Classification*: 17B10, 18D10.

Research supported in part by NSF grant DMS-1161094.

finite-dimensional category $\mathcal{A}$. If in addition $\mathcal{A}$ is Artinian, then the full subcategory of Mod-$\mathcal{A}$ consisting of all objects of finite length is a *Schurian category* in the sense of [BLW, §2.1].

In the Abelian setting, the general structural results about 2-representations of Kac-Moody 2-categories obtained in [CR, R1, R2] typically only apply to categories in which all objects have finite length and whose irreducible objects satisfy Schur's Lemma. If one wants there to be enough projectives and injectives too, this means that one is working in a Schurian category in the sense just defined. The main new contribution of this paper is to extend some of these structural results to locally Schurian categories.

The motivation for doing this from a Lie theoretic perspective is as follows. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra with Chevalley generators $\{e_i, f_i \mid i \in I\}$, weight lattice P, etc... Recall that a $\mathfrak{g}$-module V is *integrable* if it decomposes into weight spaces as $V = \bigoplus_{\lambda \in P} V_\lambda$, and each e_i and f_i acts locally nilpotently. In order to categorify an integrable module with finite-dimensional weight spaces, it is reasonable to hope that one can use a finite-dimensional category whose blocks are finite-dimensional algebras, in which case all subsequent constructions can be performed in the Schurian category consisting of finite-dimensional modules over these algebras. Examples include the *minimal categorification* $\mathcal{L}_{\min}(\kappa)$ of the integrable highest weight module $L(\kappa)$ of (dominant) highest weight κ defined already by Khovanov, Lauda and Rouquier via cyclotomic quiver Hecke algebras, and the minimal categorifications $\mathcal{L}_{\min}(\kappa_1, \dots, \kappa_n)$ of tensor products $L(\kappa_1) \otimes \cdots \otimes L(\kappa_n)$ of integrable highest weight modules introduced by Webser in [W2].

In [W1], Webster also investigated categorifications of more general tensor products involving both integrable lowest weight and highest weight modules; see also [BW] for the construction of canonical bases in such mixed tensor products. Away from finite type, these modules have infinite-dimensional weight spaces. The candidates for their minimal categorifications suggested by Webster are finite-dimensional categories which are not Artinian in general, so that the locally Schurian setting becomes essential. In type A, there are some closely related examples arising from the cyclotomic oriented Brauer categories of [BCNR], which in level one are Deligne's categories $\underline{\mathrm{Re}}\mathrm{p}(GL_t)$ (e.g. see [EHS]). These also fit into the framework of this article.

Here is a guide to the organization of the remainder of the article.

In Section 2, we set up the basic algebraic foundations of locally Schurian categories. Everything here is either well known (e.g. see [M]), or it is an obvious extension of classical results. However our language is new.

Section 3 is an exposition of the definition of Kac-Moody 2-category, based mainly on [B3]. We also discuss briefly the graded version of the Kac-Moody 2-category. This is important as it makes the connection to quantum groups, although we will not emphasize it elsewhere in the article.

Section 4 begins with a review of Rouquier's theory of 2-representations of Kac-Moody 2-categories. We recall his definition of the universal categorification $\mathcal{L}(\kappa)$ of $L(\kappa)$ from [R2, §4.3.3]. The minimal categorification $\mathcal{L}_{\min}(\kappa)$ is a certain finite-dimensional specialization of $\mathcal{L}(\kappa)$; it can be realized equivalently in terms of cyclotomic quiver Hecke algebras. We also introduce a 2-representation $\mathcal{L}(\kappa'|\kappa)$, which is expected to play the role of universal categorification for the tensor product $L(\kappa'|\kappa) := L'(\kappa') \otimes L(\kappa)$ of the integrable lowest weight module $L'(\kappa')$ of (anti-dominant) lowest weight κ' with the integrable highest weight module $L(\kappa)$ (see Construction 4.13). The minimal categorification $\mathcal{L}_{\min}(\kappa'|\kappa)$ from [W1, Proposition 5.6] is a certain finite-dimensional specialization of $\mathcal{L}(\kappa'|\kappa)$. After that, we focus on nilpotent categorical actions on locally Schurian categories. Any such structure has an *associated crystal* in the sense of Kashiwara; for example, the crystal associated to $\mathcal{L}_{\min}(\kappa)$ is the highest weight crystal $\mathbf{B}(\kappa)$. This has already found many striking applications in classical representation theory; e.g. see [FK] (the oldest) and [DVV] (the most recent at the time of writing).

Acknowledgements. We thank Ben Webster for sharing his ideas in [W3], and for suggesting the reduction argument used in the proof of Theorem 4.27. Also we thank Aaron Lauda for giving us the opportunity to write this survey, and the referee for many helpful suggestions.

Notation. Throughout, we work over an algebraically closed field $\Bbbk$. This means that all (2-)categories and (2-)functors will be assumed to be $\Bbbk$-linear by default.

2. Locally Schurian categories

In this section, we introduce our language of locally Schurian categories.

2.1. **Locally unital algebras.** A *locally unital algebra* is an associative (not necessarily unital) algebra A equipped with a small family $(1_x)_{x\in X}$ of mutually orthogonal idempotents such that

$$A = \bigoplus_{x,y\in X} 1_y A 1_x.$$

A *locally unital homomorphism* (resp. *isomorphism*) between two locally unital algebras is an algebra homomorphism (resp. isomorphism) which takes distinguished idempotents to distinguished idempotents. Also, we say that A is a *contraction* of B if there is an algebra isomorphism $A \xrightarrow{\sim} B$ sending each distinguished idempotent in A to a sum of distinguished idempotents in B.

We say that A is *locally Noetherian* (resp. *locally Artinian*) if all of the left ideals $A1_x$ and all of the right ideals 1_yA satisfy the Ascending Chain Conditon (resp. the Descending Chain Condition). One can also define analogs of *left* (resp. *right*) Noetherian or Artinian for locally unital algebras, requiring just that all the left ideals $A1_x$ (resp. the right ideals 1_yA) satisfy the appropriate chain condition. Unlike in the unital setting, locally left/right Artinian does not imply locally left/right Noetherian (but see Lemma 2.8 below). The following example of a locally unital algebra that is locally left Artinian but not locally left Noetherian is taken from the end of [M, §3]: consider the locally unital algebra of upper triangular matrices over $\Bbbk$ with rows and columns indexed by the totally ordered set $\mathbb{N}\cup\{\infty\}$, all but finitely many of whose entries are zero.

All *modules* over a locally unital algebra will be assumed to be locally unital without further mention; for a right module V this means that $V = \bigoplus_{x\in X} V1_x$ as a direct sum of subspaces. If V is any A-module satisfying ACC, it is clearly finitely generated. Conversely, assuming that A is locally Noetherian (resp. locally Artinian), finitely generated modules satisfy ACC (resp. DCC). We deduce in the locally Noetherian case that submodules of finitely generated modules are finitely generated.

Let Mod-A be the category of all right A-modules. We'll also need the following full subcategories of Mod-A:

- lfdMod-A consisting of all *locally finite-dimensional modules*, i.e. right modules V with $\dim V1_x < \infty$ for all $x \in X$;
- fgMod-A consisting of all finitely generated modules;
- pMod-A consisting of all finitely generated projective modules.

Replacing "right" with "left" everywhere here, we obtain analogous categories A-Mod, A-lfdMod, A-fgMod and A-pMod of left modules. There are contravariant equivalences

$$\circledast : \text{lfdMod-}A \to A\text{-lfdMod}, \qquad \# : \text{pMod-}A \to A\text{-pMod}$$

defined as follows: the *dual* $V^\circledast$ of a locally finite-dimensional right module V is the left module $\bigoplus_{x\in X} \mathrm{Hom}_\Bbbk(V1_x, \Bbbk)$; the *dual* $P^\#$ of a finitely generated projective right module P is the left module $\mathrm{Hom}_A(P, A)$. If V and P are left modules instead, their duals ${}^{\circledast}V$ and ${}^{\#}P$ are the right modules defined analogously.

Remark 2.1. The data of a locally unital algebra A is the same as the data of a small category $\mathcal{A}$ with object set X and morphisms $\mathrm{Hom}_{\mathcal{A}}(x, y) := 1_y A 1_x$. In this incarnation, locally unital algebra homomorphisms correspond to functors. A right A-module becomes a functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{V}ec$, and then a module homomorphism is a natural transformation of functors. We say $\mathcal{A}$ is *Noetherian* (resp. *Artinian*) if A is locally Noetherian (resp. locally Artinian) in the sense already defined. All of the other notions introduced in this subsection can be recast in this more categorical language too, as was done in [M]. For example, the projective module $1_x A$ corresponds to the functor $\mathrm{Hom}_{\mathcal{A}}(-, x) : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}ec$. Then the Yoneda Lemma asserts that there is a fully faithful functor $\mathcal{A} \to \mathrm{pMod}\text{-}A$ sending $x \in \mathrm{ob}\,\mathcal{A}$ to $1_x A$, and $a \in \mathrm{Hom}_{\mathcal{A}}(x, y)$ to the homomorphism $1_x A \to 1_y A$ defined by left multiplication by $a \in 1_y A 1_x$. This extends canonically to an equivalence of categories

$$\dot{\mathcal{A}} \to \mathrm{pMod}\text{-}A, \tag{2.1}$$

where $\dot{\mathcal{A}}$ denotes the *additive Karoubi envelope* of $\mathcal{A}$, that is, the idempotent completion of the additive envelope of $\mathcal{A}$.

For locally unital algebras A and B with distinguished idempotents $(1_x)_{x \in X}$ and $(1_y)_{y \in Y}$, respectively, an (A, B)-bimodule $M = \bigoplus_{x \in X, y \in Y} 1_x M 1_y$ determines an adjoint pair (T_M, H_M) of functors

$$T_M := - \otimes_A M : \mathrm{Mod}\text{-}A \to \mathrm{Mod}\text{-}B,$$
$$H_M := \bigoplus_{x \in X} \mathrm{Hom}_B(1_x M, -) : \mathrm{Mod}\text{-}B \to \mathrm{Mod}\text{-}A.$$

Here are a couple of useful facts about tensoring with bimodules. First, there is a natural isomorphism

$$V \otimes_A \mathrm{Hom}_B(Q, M) \xrightarrow{\sim} \mathrm{Hom}_B(Q, V \otimes_A M), \qquad v \otimes f \mapsto (q \mapsto v \otimes f(q)) \tag{2.2}$$

for all right A-modules V and finitely generated projective right B-modules Q; cf. [AF, 20.10]. Also, given another locally unital algebra C and a right exact functor $E : \mathrm{Mod}\text{-}B \to \mathrm{Mod}\text{-}C$ commuting with direct sums, EM is an (A, C)-bimodule, and there is a natural isomorphism of right C-modules

$$V \otimes_A EM \xrightarrow{\sim} E(V \otimes_A M), \qquad v \otimes n \mapsto E(f_v)(n) \tag{2.3}$$

for all right A-modules V, where $f_v : M \to V \otimes_A M$ is the right C-module homomorphism $m \mapsto v \otimes m$. The proof of this involves a reduction to the case $V = A$ using the Five Lemma.

Definition 2.2. We say that an (A, B)-bimodule M is *left rigid* if it has a left dual in the 2-category of bimodules; see [EGNO, §2.10] for our conventions here. It means that there exists a (B, A)-bimodule $M^{\#}$, an (A, A)-bimodule homomorphism $\mathrm{coev} : A \to M \otimes_B M^{\#}$, and a (B, B)-bimodule homomorphism $\mathrm{ev} : M^{\#} \otimes_A M \to B$, such that the compositions

$$M \xrightarrow{\mathrm{can}} A \otimes_A M \xrightarrow{\mathrm{coev} \otimes 1} M \otimes_B M^{\#} \otimes_A M \xrightarrow{1 \otimes \mathrm{ev}} M \otimes_B B \xrightarrow{\mathrm{can}} M,$$
$$M^{\#} \xrightarrow{\mathrm{can}} M^{\#} \otimes_A A \xrightarrow{1 \otimes \mathrm{coev}} M^{\#} \otimes_A M \otimes_B M^{\#} \xrightarrow{\mathrm{ev} \otimes 1} B \otimes_B M^{\#} \xrightarrow{\mathrm{can}} M^{\#}$$

are the identities. In other words, $T_{M^{\#}}$ is right adjoint to T_M. We say that M is *right rigid* if it has a right dual, i.e. there exists a (B, A)-bimodule ${}^{\#}M$ such that $T_{{}^{\#}M}$ is left adjoint to T_M. Finally, we say M is *rigid* if it is both left and right rigid, and *sweet*[1] if in addition $M^\# \cong {}^\# M$ as (B, A)-bimodules.

The following is essentially [EGNO, Ex. 2.10.16].

Lemma 2.3. *Let A and B be locally unital algebras with distinguished idempotents $(1_x)_{x\in X}$ and $(1_y)_{y\in Y}$, respectively. Let M be an (A, B)-bimodule.*

(1) The bimodule M is left rigid if and only if $1_x M$ is finitely generated and projective as a right B-module for each $x \in X$. In that case, $M^\# \cong \bigoplus_{x\in X}(1_x M)^\#$.
(2) It is right rigid if and only if $M1_y$ is finitely generated and projective as a left A-module for each $y \in Y$. In that case, ${}^\# M \cong \bigoplus_{y\in Y} {}^\#(M1_y)$.

Proof. (1) Suppose that M possesses a left dual $M^\#$. Then T_M has a right exact right adjoint $T_{M^\#}$, so T_M sends projectives to projectives. Hence, $1_x M \cong T_M(1_x A)$ is projective for each $x \in X$. Let $\operatorname{coev}(1_x) = \sum_{i=1}^{n_x} v_{x,i} \otimes f_{x,i}$ for $v_{x,i} \in 1_x M$ and $f_{x,i} \in (M^\#)1_x$. Then any $v \in 1_x M$ is equal to $(1 \otimes \operatorname{ev})(v_{x,i} \otimes f_{x,i} \otimes v) \in \sum_{i=1}^{n_i} v_{x,i}A$. This shows that $1_x M$ is finitely generated.

Conversely, suppose that each $1_x M$ is finitely generated and projective as a right module. Then (2.2) implies that $H_M \cong T_{M^\#}$ where $M^\# := \bigoplus_{x\in X}(1_x M)^\#$. Hence, we have constructed a bimodule $M^\#$ such that $T_{M^\#}$ is right adjoint to T_M, proving that M is left rigid.

(2) Similar (working with left modules instead of right ones). □

By a *projective generating family* for an Abelian category $\mathcal{C}$, we mean a small family $(P(x))_{x\in X}$ of compact[2] projective objects such that for each $V \in \operatorname{ob}\mathcal{C}$ there is some $x \in X$ with $\operatorname{Hom}_{\mathcal{C}}(P(x), V) \neq 0$. Just like in [F, Exercise 5.F], one can show that an Abelian category $\mathcal{C}$ is equivalent to Mod-A for some locally unital algebra A if and only if $\mathcal{C}$ possesses arbitrary direct sums and has a projective generating family; see also [M, Theorem 3.1]. We just need this in the following special case, which is the locally unital analog of the classical Morita Theorem:

Theorem 2.4. *Let B be a locally unital algebra. Suppose that $(P(x))_{x\in X}$ is a projective generating family for* Mod-B. *Let*

$$A := \bigoplus_{x,y\in X} \operatorname{Hom}_B(P(x), P(y)),$$

viewed as a locally unital algebra with distinguished idempotents $(1_x := 1_{P(x)})_{x\in X}$. Let $P := \bigoplus_{x\in X} P(x)$, which is an (A, B)-bimodule.

(1) The functors $T_P = - \otimes_A P$ and $H_P = \bigoplus_{x\in X} \operatorname{Hom}_B(1_x P, -)$ are quasi-inverse equivalences between the categories Mod-A *and* Mod-B.
(2) We have that $H_P \cong T_Q$ where $Q := P^\#$.

Thus, we have constructed a sweet (A, B)-bimodule P and a sweet (B, A)-bimodule Q such that $P \otimes_B Q \cong A$ and $Q \otimes_A P \cong B$ as bimodules.

Proof. The fact that $H_P \circ T_P \cong 1_{\text{Mod-}A}$ follows from (2.2). Then one deduces that $T_P \circ H_P \cong 1_{\text{Mod-}B}$ too by a standard argument; cf. [AF, 22.2]. Finally Lemma 2.3(1) implies that $H_P \cong T_Q$. □

Corollary 2.5. *For locally unital algebras A and B, the following are equivalent:*

[1] The language "sweet bimodule" appears in [K, §2.6], but (in view of Lemma 2.3) the ones defined there are just what we call rigid bimodules, since it is not assumed that ${}^\# M \cong M^\#$. Note though that the important examples constructed in [K] do satisfy this extra hypothesis, so they are sweet in our sense too.

[2] For categories of the form Mod-A for some locally unital algebra A, a projective is compact if and only if it is finitely generated.

(1) the categories Mod-A *and* Mod-B *are equivalent;*
(2) the categories pMod-A *and* pMod-B *are equivalent;*
(3) the categories A-pMod *and* B-pMod *are equivalent;*
(4) the categories A-Mod *and* B-Mod *are equivalent.*

Proof. (1) $\Rightarrow$ (2). The restriction of an equivalence Mod-$A \to$ Mod-B gives an equivalence pMod-$A \to$ pMod-B.

(2) $\Rightarrow$ (1). Let F : pMod-$A \to$ pMod-B be an equivalence of categories. Let $(1_x)_{x\in X}$ be the distinguished idempotents in A. Then $(P(x) := F(1_xA))_{x\in X}$ is a projective generating family for Mod-B such that $A \cong \bigoplus_{x,y\in X} \mathrm{Hom}_B(P(x), P(y))$. Now apply Theorem 2.4.

(3) $\Leftrightarrow$ (4). This is the same as (1) $\Leftrightarrow$ (2) with A and B replaced by the opposite algebras.

(2) $\Leftrightarrow$ (3). This follows as pMod-A (resp. pMod-B) is contravariantly equivalent to A-pMod (resp. B-pMod). □

Two locally unital algebras A and B are said to be *Morita equivalent* if the conditions of Corollary 2.5 are satisfied. For example, if A is a contraction of B, then the categories Mod-A and Mod-B are obviously isomorphic. Hence, A and B are Morita equivalent. For another simple example, let N be any (possibly infinite but small) set and $M_N(\Bbbk)$ be the algebra of $N \times N$ matrices with entries in $\Bbbk$, all but finitely many of which are zero. This is a locally unital algebra with distinguished idempotents given by the diagonal matrix units $\{e_{i,i} \,|\, i \in N\}$. Applying Theorem 2.4 with $B := \Bbbk, X := N$ and taking each $P(x)$ to be a copy of $\Bbbk$, we see that $M_N(\Bbbk)$ is Morita equivalent to the ground field $\Bbbk$.

Remark 2.6. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are the categories associated to locally unital algebras A and B as in Remark 2.1. We say that $\mathcal{A}$ and $\mathcal{B}$ are *Morita equivalent* if their additive Karoubi envelopes $\dot{\mathcal{A}}$ and $\dot{\mathcal{B}}$ are equivalent. In view of Corollary 2.5 and (2.1), this is equivalent to the algebras A and B being Morita equivalent as above.

The final theorem in this subsection is concerned with adjoint functors. Again this is classical in the unital setting.

Theorem 2.7. *Let* B*,* $P = \bigoplus_{x\in X} P(x)$ *and* A *be as in Theorem 2.4, so that* H_P : Mod-$B \to$ Mod-A *is an equivalence of categories. Suppose we are given a functor* E : Mod-$B \to$ Mod-B*. Then* E *possesses a right adjoint if and only if it is right exact and commutes with direct sums. In that case, let*

$$M := \bigoplus_{x,y\in X} \mathrm{Hom}_B(P(x), EP(y))$$

viewed as an (A, A)*-bimodule in the natural way. Then the diagram*

$$\begin{array}{ccc} \text{Mod-}B & \xrightarrow{H_P} & \text{Mod-}A \\ {\scriptstyle E}\big\downarrow & & \big\downarrow{\scriptstyle T_M} \\ \text{Mod-}B & \xrightarrow[H_P]{} & \text{Mod-}A \end{array}$$

commutes up to a canonical isomorphism.

Proof. It is standard that functors possessing a right adjoint are right exact and commute with direct sums. Conversely, suppose that E is right exact and commutes with direct sums. Using (2.3) then (2.2), we get that

$$H_P \circ E \circ T_P \cong H_P \circ T_{EP} \cong T_{H_P(EP)},$$

which is isomorphic to T_M as $H_P(EP) \cong M$. Thus $H_P \circ E \circ T_P \cong T_M$. Composing on the right with the quasi-inverse H_P of T_P, we deduce that $H_P \circ E \cong T_M \circ H_P$. This proves the final part of the theorem. Hence E has a right adjoint as T_M has the right adjoint H_M. □

2.2. **Finite-dimensional categories.** Let A be a locally unital algebra with distinguished idempotents $(1_x)_{x\in X}$. We assume in this subsection that A is *locally finite-dimensional*, i.e. each subspace $1_y A 1_x$ is finite-dimensional. Equivalently, the associated category $\mathcal{A}$ from Remark 2.1 is a *finite-dimensional category*, i.e. it is a small $\Bbbk$-linear category all of whose morphism spaces are finite-dimensional. All of the right ideals $1_x A$ and the left ideals $A1_x$ are locally finite-dimensional. Hence, so are their duals $(A1_x)^{\circledast}$ and $(1_x A)^{\circledast}$. Consequently, all finitely generated modules are locally finite-dimensional, as are all finitely cogenerated modules.

Lemma 2.8. *For a locally finite-dimensional locally unital algebra A, locally Artinian implies locally Noetherian.*

Proof. As $A1_x$ satisfies DCC, its dual $(A1_x)^{\circledast}$ satisfies ACC. Hence, $(A1_x)^{\circledast}$ is finitely generated, so satisfies DCC. Hence, $A1_x$ satisfies ACC. Similarly each $1_x A$ has ACC. □

Here are various other basic facts about modules over a locally finite-dimensional locally unital algebra A. For the most part, these are proved by mimicking the usual proofs in the setting of finite-dimensional algebras, so we will be quite brief. Fix representatives $\{L(b) \mid b \in \mathbf{B}\}$ for the isomorphism classes of irreducible right A-modules[3].

(L1) If V is finitely generated (resp. locally finite-dimensional) and W is locally finite-dimensional (resp. finitely cogenerated) then $\operatorname{Hom}_A(V, W)$ is finite-dimensional.

(L2) *Schur's Lemma* holds: $\operatorname{End}_A(L(b)) \cong \Bbbk$ for each $b \in \mathbf{B}$.

(L3) Any finitely generated (resp. finitely cogenerated) module satisfies the *Krull-Schmidt Theorem*.

(L4) The category Mod-A is a Grothendieck category, i.e. it is Abelian, it possesses arbitrary direct sums, direct limits of short exact sequences are exact, and there is a generator (namely, the regular module A itself). Hence, by the general theory of Grothendieck categories, every A-module has an injective hull; moreover, a module V is finitely cogenerated if and only if its *socle* $\operatorname{soc}(V)$, i.e. the sum of the irreducible submodules of V, is an essential submodule of V of finite length.

(L5) For $b \in \mathbf{B}$, let $A_b := A/\operatorname{Ann}_A(L(b))$, which is a locally unital algebra with distinguished idempotents $(1_x)_{x\in X}$ that are the images of the ones in A. Also let $M_b := \bigoplus_{x,y\in X} \operatorname{Hom}_{\Bbbk}(L(b)1_x, L(b)1_y)$, viewed as a locally unital algebra with multiplication that is the opposite of composition. Note that M_b is a contraction of $M_N(\Bbbk)$ where $N := \{(x,i) | x \in X, 1 \leq i \leq \dim L(b)1_x\}$. Then the natural right action of A on $L(b)$ induces a locally unital isomorphism $A_b \xrightarrow{\sim} M_b$. Hence, A_b is a contraction of a locally unital matrix algebra. Next, let $J := \bigcap_{b\in\mathbf{B}} \operatorname{Ann}_A(L(b))$ be the *Jacobson radical* of A. The map

$$A/J \to \bigoplus_{b\in\mathbf{B}} A_b, \qquad a + J \mapsto (a + \operatorname{Ann}_A(L(b)))_{b\in\mathbf{B}}$$

is a well-defined algebra isomorphism. Hence, A/J is a contraction of a (possibly infinite) direct sum of locally unital matrix algebras. It follows that A/J is *semisimple*, i.e. every A/J-module is completely reducible. Moreover, J is the smallest two-sided ideal of A with this property.

(L6) For a right A-module V, its *radical* $\operatorname{rad}(V) := VJ$ is the intersection of all of its proper maximal submodules; its *head* $\operatorname{hd}(V) := V/\operatorname{rad}(V)$ is its largest completely reducible quotient. Applying $\circledast$ to the statements made in (L4), we deduce that every finitely generated A-module has a projective cover; moreover,

[3]We use the notation $\mathbf{B}$ here as ultimately this set will carry a crystal structure; cf. Definition 4.30.

V is finitely generated if and only if $\mathrm{rad}(V)$ is a superfluous submodule and $\mathrm{hd}(V)$ is of finite length[4].

(L7) Let $P(b)$ be a projective cover of $L(b)$. For any right A-module V, the *composition multiplicity* $[V : L(b)]$ is defined as usual to be the supremum of the multiplicities $\#\{i = 1, \dots, n \mid V_i/V_{i-1} \cong L(b)\}$ taken over all filtrations $0 = V_0 < \cdots < V_n = V$ and all $n \in \mathbb{N}$. By Schur's Lemma, we have that $[V : L(b)] = \dim \mathrm{Hom}_A(P(b), V) \in \mathbb{N} \cup \{\infty\}$. Noting that $\mathrm{Hom}_A(1_x A, L(b)) \cong L(b)1_x$, the projective module $1_x A$ decomposes as

$$1_x A \cong \bigoplus_{b \in \mathbf{B}} P(b)^{\bigoplus \dim L(b)1_x}.$$

All but finitely many summands on the right hand side are zero, so there are only finitely many $b \in \mathbf{B}$ such that $L(b)1_x \neq 0$. Hence, for any V, we have that

$$\dim V1_x = \sum_{b \in \mathbf{B}} [V : L(b)] \dim L(b)1_x.$$

In particular, we get from this that V is locally finite-dimensional if and only if $[V : L(b)] < \infty$ for all $b \in \mathbf{B}$.

(L8) Given $b \in \mathbf{B}$, we choose $x \in X$ so that $L(b)1_x \neq 0$. The decomposition of $1_x A$ derived in (L7) implies that there exists a primitive idempotent $1_b \in 1_x A 1_x$ such that $P(b) \cong 1_b A$. Then $I(b) := (A1_b)^\circledast$ is an injective hull of $L(b)$. For any V, we have that $[V : L(b)] = \dim \mathrm{Hom}_A(V, I(b))$.

(L9) Suppose we are given a family $(A_i)_{i \in I}$ of locally finite-dimensional locally unital algebras, with the distinguished idempotents in A_i indexed by X_i. Then $A := \bigoplus_{i \in I} A_i$ is a locally finite-dimensonal locally unital algebra with distinguished idempotents indexed by $X := \bigsqcup_{i \in I} X_i$. Moreover there is an equivalence of categories $\prod_{i \in I} \text{Mod-}A_i \to \text{Mod-}A$ which sends an object $(V_i)_{i \in I}$ of $\prod_{i \in I} \text{Mod-}A_i$ to the A-module $\bigoplus_{i \in I} V_i$.

(L10) Suppose $\mathbf{B} = \bigsqcup_{i \in I} \mathbf{B}_i$ is a partition such that that $\mathrm{Hom}_A(P(b), P(c)) = 0$ for all $b \in \mathbf{B}_i, c \in \mathbf{B}_j$ and $i \neq j$. For $x \in X$, we can write 1_x uniquely as a sum of mutually orthogonal idempotents $1_x = \sum_{i \in I} 1_{(x,i)}$ so that $1_{(x,i)} A \cong \bigoplus_{b \in \mathbf{B}_i} P(b)^{\oplus \dim L(b)1_x}$. Let $A_i := \bigoplus_{x,y \in X} 1_{(y,i)} A 1_{(x,i)}$, which is itself a locally unital algebra with idempotents $(1_{(x,i)})_{x \in X}$ and irreducibles represented by $\{L(b) \mid b \in \mathbf{B}_i\}$. Then we have that $A = \bigoplus_{i \in I} A_i$. Hence, A is a contraction of $\bigoplus_{i \in I} A_i$. If none of the $\mathbf{B}_i$ can be partitioned any further in this way, we call this the *block decomposition* of A, and refer to indecomposable subalgebras A_i as *blocks*.

2.3. **Locally Schurian categories.** Later in the article, we will be interested in categorical actions on categories of the following form:

Definition 2.9. We say that a category $\mathcal{C}$ is *locally Schurian* if it is equivalent to Mod-A for some locally finite-dimensional locally unital algebra A.

Given a locally Schurian category $\mathcal{C}$, Theorem 2.4 gives a recipe for constructing a locally finite-dimensional locally unital algebra A such that $\mathcal{C}$ is equivalent to Mod-A: choose a projective generating family $(P(x))_{x \in X}$ for $\mathcal{C}$; set

$$A := \bigoplus_{x,y \in X} \mathrm{Hom}_{\mathcal{C}}(P(x), P(y)) \tag{2.4}$$

[4]One can give a direct proof of this using the fact that J is *locally nilpotent* in the sense that eJe is a nilpotent ideal of the finite-dimensional algebra eAe for any idempotent $e \in A$.

viewed as a locally unital algebra with distinguished idempotents $(1_x := 1_{P(x)})_{x \in X}$; then the functor

$$H := \bigoplus_{x \in X} \mathrm{Hom}_{\mathcal{C}}(P(x), -) : \mathcal{C} \to \text{Mod-}A \tag{2.5}$$

is an equivalence of categories. Often it is convenient to proceed by choosing representatives $\{L(b) \mid b \in \mathbf{B}\}$ for the isomorphism classes of irreducible object in $\mathcal{C}$ and letting $P(b)$ (resp. $I(b)$) be a projective cover (resp. an injective hull) of $L(b)$. Then we call $(P(b))_{b \in \mathbf{B}}$ a *minimal* projective generating family for $\mathcal{C}$, and $\mathcal{C}$ is equivalent to Mod-B where

$$B := \bigoplus_{b,c \in \mathbf{B}} \mathrm{Hom}_{\mathcal{C}}(P(b), P(c)). \tag{2.6}$$

This a *basic* locally unital algebra: the irreducible B-modules are all one dimensional.

An object V in $\mathcal{C}$ is *locally finite-dimensional* if and only if all its composition multiplicities are finite. We let $\mathrm{p}\mathcal{C} \subseteq \mathrm{fg}\mathcal{C} \subseteq \mathrm{lfd}\mathcal{C}$ be the full subcategories of $\mathcal{C}$ consisting of finitely generated projective, finitely generated, and locally finite-dimensional objects; for A as in (2.4), these are equivalent to the subcategories pMod-A, fgMod-A and lfdMod-A of Mod-A.

We say that $\mathcal{C}$ is *Noetherian* if all finitely generated (resp. cogenerated) objects satisfy ACC (resp. DCC); equivalently, the algebra A from (2.4) is locally Noetherian. We say that $\mathcal{C}$ is *Artinian* if all finitely generated (resp. cogenerated) objects satisfy DCC (resp. ACC); equivalently, the algebra A is locally Artinian. We say that $\mathcal{C}$ is *finite* if there are only finitely many isomorphism classes of irreducible object; equivalently, the basic algebra B from (2.6) is finite-dimensional. By Lemma 2.8, Artinian implies Noetherian.

If $\mathcal{C}$ is Artinian then $\mathrm{fg}\mathcal{C}$ is a *Schurian category* in the sense of [BLW, §2.1]: it is Abelian, all objects are of finite length, there are enough projectives and injectives, and the endomorphism algebras of the irreducible objects are one dimensional. Moreover, for $V \in \mathrm{ob}\,\mathcal{C}$, the following are equivalent:

- V is finitely generated;
- V is finitely cogenerated;
- V has finite length.

2.4. **Sweet endofunctors.** We will consider categorical actions on locally Schurian categories involving functors of the following form:

Definition 2.10. Let $\mathcal{C}$ be a locally Schurian category. We say that an endofunctor E of $\mathcal{C}$ is *sweet* if there is an endofunctor F which is biadjoint to E.

Recalling the definition of sweet bimodule from Definition 2.2, the following theorem gives an algebraic characterization of sweet endofunctors.

Theorem 2.11. *Let E be an endofunctor of a locally Schurian category $\mathcal{C}$. Fix an equivalence $H : \mathcal{C} \to \text{Mod-}A$ as in (2.5). Then E is sweet if and only if there is a sweet bimodule M such that $H \circ E \cong T_M \circ H$. In that case, we have that*

$$M \cong \bigoplus_{x,y \in X} \mathrm{Hom}_{\mathcal{C}}(P(x), EP(y)). \tag{2.7}$$

Moreover, E is exact, continuous and cocontinuous, and it preserves the sets of locally finite-dimensional, finitely generated, finitely cogenerated, projective and injective objects.

Proof. If M is a sweet bimodule such that $H \circ E \cong T_M \circ H$, then E is sweet since T_M is a sweet endofunctor of Mod-A and H is an equivalence. Conversely, suppose that E possesses a biadjoint F. Theorem 2.7 shows that $H \circ E \cong T_M \circ H$ for M as in (2.7); similarly $H \circ F \cong T_N \circ H$ for some bimodule N. Then T_M and T_N are biadjoint. Hence,

N is both right and left dual to M, i.e. M is a sweet bimodule. It follows at once that E and F both send finitely generated objects to finitely generated objects, as T_M and T_N clearly do.

Since E has a biadjoint, it is exact, continuous and cocontinous. Also F is exact, so E preserves projectives and injectives. To see that E preserves locally finite-dimensional objects, we observe for locally finite-dimensional V that

$$\dim \operatorname{Hom}_{\mathcal{C}}(P(b), EV) = \dim \operatorname{Hom}_{\mathcal{C}}(FP(b), V) < \infty$$

for all $b \in \mathbf{B}$. Similarly F preserves locally finite-dimensional objects. Finally, since E preserves finitely generated objects, we have that

$$\dim \operatorname{Hom}_{\mathcal{C}}(EP(b), L(c)) = \dim \operatorname{Hom}_{\mathcal{C}}(P(b), FL(c)) = [FL(c) : L(b)]$$

is zero for all but finitely many c. Hence,

$$\dim \operatorname{Hom}_{\mathcal{C}}(L(c), EI(b)) = \dim \operatorname{Hom}_{\mathcal{C}}(FL(c), I(b)) = [FL(c) : L(b)]$$

is zero for all but finitely many c. This implies that $EI(b)$ is a finite direct sum of $I(c)$'s. Hence, E preserves finitely cogenerated objects, and similarly for F. □

Lemma 2.12. *Suppose that F and G are sweet endofunctors of a locally Schurian category $\mathcal{C}$. Let $\eta : F \Rightarrow G$ be a natural transformation. If $\eta_L : FL \to GL$ is an isomorphism for each irreducible object $L \in \operatorname{ob}\mathcal{C}$, then η is an isomorphism.*

Proof. We may assume that $\mathcal{C} = \text{Mod-}B$ for a basic locally unital algebra B. This means that the irreducible B-modules are parametrized by the same set $\mathbf{B}$ as indexes its distinguished idempotents, and $[V : L(b)] = \dim \operatorname{Hom}_B(P(b), V) = \dim V1_b$ for each $b \in \mathbf{B}$. The main step is to prove that $\eta_V : FV \to GV$ is an isomorphism for each locally finite-dimensional B-module V. Assuming this, the lemma may be deduced as follows: given any B-module V, consider a two-step projective resolution $Q \to P \to V \to 0$; since P and Q are direct sums of finitely generated projectives, and F and G commute with arbitrary direct sums, the locally finite-dimensional result shows that η_P and η_Q are isomorphisms. Hence, η_V is an isomorphism too by the Five Lemma.

So now suppose that V is locally finite-dimensional. It suffices to show for each fixed $a \in \mathbf{B}$ that the restriction of η_V defines a linear isomorphism between $(FV)1_a$ and $(GV)1_a$. Let

$$X := \{b \in \mathbf{B} \mid (FL(b))1_a \neq 0\} = \{b \in \mathbf{B} \mid \operatorname{Hom}_B(P(a), FL(b)) \neq 0\}.$$

Fixing a left adjoint E to F, we have that $X = \{b \in \mathbf{B} \mid \operatorname{Hom}_B(EP(a), L(b)) \neq 0\}$. Since $EP(a)$ is a finitely generated projective, we deduce from this that X is finite; moreover, $EP(a)$ a direct sum of indecomposable projectives of the form $P(b)$ for $b \in X$. Now we proceed by induction on $n := \sum_{b \in X} \dim V1_b \in \mathbb{N}$. In case $n = 0$, we have that $\operatorname{Hom}_B(P(a), FV) \cong \operatorname{Hom}_B(EP(a), V) = 0$. Hence, $(FV)1_a = 0$; similarly, $(GV)1_a = 0$. So the desired conclusion that $(FV)1_a \cong (GV)1_a$ is trivial. For the induction step, we take a vector $0 \neq v \in V1_b$ for some $b \in X$. Let $W := vB$ and $W' := \operatorname{rad}(W)$, so that we have a filtration $0 \leq W' < W \leq V$ with $W/W' \cong L(b)$. By induction, $\eta_{W'}$ and $\eta_{V/W}$ restrict to isomorphisms $(FW')1_a \xrightarrow{\sim} (GW')1_a$ and $(FV/FW)1_a \xrightarrow{\sim} (GV/GW)1_a$. Also $\eta_{W/W'}$ is an isomorphism as W/W' is irreducible. Hence, η_V defines an isomorphism $(FV)1_a \xrightarrow{\sim} (GV)1_a$ as required. □

2.5. **Serre quotients.** Finally in this section, we review briefly the standard notions of *Serre subcategory* and *Serre quotient category* in the setting of locally Schurian categories. Let $\mathcal{C}$ be a locally Schurian category with irreducible objects represented by $\{L(b) | b \in \mathbf{B}\}$ as above. Let $\mathbf{B}'$ be any subset of $\mathbf{B}$ and $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ consisting of all the objects whose irreducible subquotients are isomorphic to $L(b)$ for $b \in \mathbf{B}'$. It is a Serre subcategory of $\mathcal{C}$, i.e. it is closed under taking subobjects, quotients and extensions. Moreover it is itself a locally Schurian category with irreducible objects represented by $\{L(b) \mid b \in \mathbf{B}'\}$. To see this, define B according to (2.6) so that $\mathcal{C}$ is equivalent to Mod-B. Then $\mathcal{C}'$ is equivalent to Mod-B' where B' is the quotient of B by the two-sided ideal generated by the idempotents $\{1_b \mid b \in \mathbf{B} \setminus \mathbf{B}'\}$. The exact inclusion functor $\iota : \mathcal{C}' \to \mathcal{C}$ corresponds to the natural inflation functor from Mod-B' to Mod-B, and it has a left adjoint $\iota^!$ (resp. a right adjoint ι^*) which sends an object to its largest quotient (resp. subobject) belonging to $\mathcal{C}'$. We have that $\iota^! \circ \iota \cong 1_{\mathcal{C}'} \cong \iota^* \circ \iota$.

The *Serre quotient* $\mathcal{C}/\mathcal{C}'$ is an Abelian category equipped with an exact *quotient functor* $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{C}'$ satisfying the following universal property: if $F : \mathcal{C} \to \mathcal{D}$ is any exact functor to an Abelian category $\mathcal{D}$ then there is a unique functor $\bar{F} : \mathcal{C}/\mathcal{C}' \to D$ such that $F = \bar{F} \circ \pi$. In fact $\mathcal{C}/\mathcal{C}'$ is another locally Schurian category with irreducibles represented by $\{\pi L(b) | b \in \mathbf{B} \backslash \mathbf{B}'\}$. Again this is easy to see in terms of the algebra B: the category $\mathcal{C}/\mathcal{C}'$ is equivalent to modules over the algebra $eBe := \bigoplus_{b,c \in \mathbf{B} \backslash \mathbf{B}'} 1_b B 1_c$. The quotient functor π corresponds to the obvious truncation functor $e :$ Mod-$B \to$ Mod-eBe sending a B-module V to $Ve := \bigoplus_{b \in \mathbf{B} \backslash \mathbf{B}'} V 1_b$. Consequently, π has a left adjoint $\pi^! : \mathcal{C}/\mathcal{C}' \to \mathcal{C}$ and a right adjoint $\pi^* : \mathcal{C}/\mathcal{C}' \to \mathcal{C}$ corresponding to the functors $- \otimes_{eBe} eB$ and $\bigoplus_{b \in \mathbf{B} \backslash \mathbf{B}'} \mathrm{Hom}_{eBe}(B1_b, -)$, respectively. We have that $\pi \circ \pi^! \cong 1_{\mathcal{C}/\mathcal{C}'} \cong \pi \circ \pi^*$.

Lemma 2.13. *In the above setup, assume that we are given $V, W \in \mathrm{ob}\,\mathcal{C}$ such that V is finitely generated, W is finitely cogenerated, and all constituents of $\mathrm{hd}(V)$ and $\mathrm{soc}(W)$ are of the form $L(b)$ for $b \in \mathbf{B} \setminus \mathbf{B}'$. Then the functor π induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(V, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}/\mathcal{C}'}(\pi V, \pi W).$$

Proof. The counit of adjunction defines a morphism $f : \pi^! \pi V \to V$. By the assumptions on V, f is an epimorphism. Moreover f becomes an isomorphism on applying π, hence $\ker f$ belongs to $\mathcal{C}'$. Using also the assumptions on W, we deduce that $\mathrm{Hom}_{\mathcal{C}/\mathcal{C}'}(\pi V, \pi W) \cong \mathrm{Hom}_{\mathcal{C}}(\pi^! \pi V, W) \cong \mathrm{Hom}_{\mathcal{C}}(V, W)$. □

3. Kac-Moody 2-categories

In this section, we review Rouquier's definition of Kac-Moody 2-category from [R1]. Then, following [B3], we explain the relationship between this and the 2-category introduced by Khovanov and Lauda in [KL3], and discuss the graded version. Note our exposition uses a slightly different normalization for the second adjunction compared to [B3] based on the idea of [BHLW].

3.1. **Kac-Moody data.** Let I be a finite index set[5] and $A = (-d_{ij})_{i,j \in I}$ be a generalized Cartan matrix, so $d_{ii} = -2$, $d_{ij} \geq 0$ for $i \neq j$, and $d_{ij} = 0 \Leftrightarrow d_{ji} = 0$. We assume that A is *symmetrizable*, so that there exist positive integers $(d_i)_{i \in I}$ such that $d_i d_{ij} = d_j d_{ji}$ for all $i, j \in I$. Pick a finite-dimensional complex vector space $\mathfrak{h}$ and linearly independent subsets $\{\alpha_i \mid i \in I\}$ and $\{h_i \mid i \in I\}$ of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, such that $\langle h_i, \alpha_j \rangle = -d_{ij}$ for all $i, j \in I$. Let

$$P := \{\lambda \in \mathfrak{h}^* \mid \langle h_i, \lambda \rangle \in \mathbb{Z} \text{ for all } i \in I\}, \qquad Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i,$$

$$P^+ := \{\lambda \in \mathfrak{h}^* \mid \langle h_i, \lambda \rangle \in \mathbb{N} \text{ for all } i \in I\}, \qquad Q^+ := \bigoplus_{i \in I} \mathbb{N}\alpha_i.$$

We refer to P and Q as the *weight lattice* and the *root lattice*, respectively. We view P as an interval-finite poset via the usual *dominance ordering*: $\lambda \leq \mu \Leftrightarrow \mu - \lambda \in Q^+$.

---

[5]With minor adjustments to the basic definitions, everything here can be extended to infinite I too; type A_∞ is particularly important in applications.

Let $\mathfrak{g}$ be the associated *Kac-Moody algebra* with Cartan subalgebra $\mathfrak{h}$. Thus, $\mathfrak{g}$ is the Lie algebra generated by $\mathfrak{h}$ and elements $e_i, f_i\,(i \in I)$ subject to the usual Serre relations: for $h, h' \in \mathfrak{h}$ and $i, j \in I$ we have that

$$[h, h'] = 0, \qquad [e_i, f_j] = \delta_{i,j} h_i, \tag{3.1}$$
$$[h, e_i] = \langle h, \alpha_i \rangle e_i, \qquad [h, f_i] = -\langle h, \alpha_i \rangle f_i, \tag{3.2}$$
$$(\mathrm{ad}\ e_i)^{d_{ij}+1}(e_j) = 0, \qquad (\mathrm{ad}\ f_i)^{d_{ij}+1}(f_j) = 0. \tag{3.3}$$

Let $U(\mathfrak{g})$ be its *universal enveloping algebra*. Actually it is often more convenient to work with the idempotented version $\dot{U}(\mathfrak{g})$ of $U(\mathfrak{g})$, which is a certain locally unital algebra with distinguished idempotents $(1_\lambda)_{\lambda \in P}$. It is defined by analogy with [Lu2, §23.1] (which treats the quantum case). As well as being an algebra, $\dot{U}(\mathfrak{g})$ is a $(U(\mathfrak{g}), U(\mathfrak{g}))$-bimodule with $h 1_\lambda = \langle h, \lambda \rangle 1_\lambda = 1_\lambda h$, $e_i 1_\lambda = 1_{\lambda+\alpha_i} e_i$ and $f_i 1_\lambda = 1_{\lambda-\alpha_i} f_i$. The elements $\{1_\lambda, e_i 1_\lambda, f_i 1_\lambda \mid i \in I, \lambda \in P\}$ generate $\dot{U}(\mathfrak{g})$ subject to the relations derived from (3.1)–(3.3). Weight modules for $\mathfrak{g}$, i.e. $\mathfrak{g}$-modules V such that $V = \bigoplus_{\lambda \in P} V_\lambda$, are just the same as (locally unital) $\dot{U}(\mathfrak{g})$-modules.

By an *upper* (resp. *lower*) *integrable module*, we mean a weight module with finite-dimensional weight spaces all of whose weights lie in a finite union of sets of the form $\lambda - Q^+$ (resp. $\lambda + Q^+$). By the classical theory from [Kac, Chapters 9–10], a $\mathfrak{g}$-module is upper integrable if and only if it is isomorphic to a finite direct sum of the irreducible modules[6]

$$L(\kappa) := \dot{U}(\mathfrak{g}) 1_\kappa / \langle e_i 1_\kappa, f_i^{1+\langle h_i, \kappa \rangle} 1_\kappa \mid i \in I \rangle \tag{3.4}$$

for $\kappa \in P^+$; these are the *integrable highest weight modules*. Similarly, a $\mathfrak{g}$-module is lower integrable if and only if it is a finite direct sum of the (irreducible) *integrable lowest weight modules*

$$L'(\kappa') := \dot{U}(\mathfrak{g}) 1_{\kappa'} / \langle e_i^{1-\langle h_i, \kappa' \rangle} 1_{\kappa'}, f_i 1_{\kappa'} \mid i \in I \rangle. \tag{3.5}$$

for $\kappa' \in -P^+$. Generalizing (3.4)–(3.5), we let

$$L(\kappa'|\kappa) := \dot{U}(\mathfrak{g}) 1_{\kappa+\kappa'} / \langle e_i^{1-\langle h_i, \kappa' \rangle} 1_{\kappa+\kappa'}, f_i^{1+\langle h_i, \kappa \rangle} 1_{\kappa+\kappa'} \mid i \in I \rangle \tag{3.6}$$

for $\kappa \in P^+$ and $\kappa' \in -P^+$. These modules are not so well studied, but they play an important role in Lusztig's construction of canonical bases for $\dot{U}(\mathfrak{g})$ from [Lu2, Part IV]. They are integrable modules but they may have infinite-dimensional weight spaces outside of finite type, so that they are neither upper nor lower integrable. The next lemma is the classical counterpart of [Lu2, Proposition 23.3.6].

Lemma 3.1. *There is a $\mathfrak{g}$-module isomorphism $L(\kappa'|\kappa) \xrightarrow{\sim} L(\kappa') \otimes L(\kappa)$ such that $u\bar{1}_{\kappa+\kappa'} \mapsto u(\bar{1}_{\kappa'} \otimes \bar{1}_\kappa)$ for $u \in \dot{U}(\mathfrak{g})$.*

The following lemma about annihilators will be useful later on.

Lemma 3.2. *We have that*

$$\bigcap_{\substack{\kappa \in P^+ \\ \kappa' \in -P^+}} \mathrm{Ann}_{\dot{U}(\mathfrak{g})}(L(\kappa'|\kappa)) = \{0\}.$$

Moreover if $\mathfrak{g}$ is of finite type then

$$\bigcap_{\lambda \in P^+} \mathrm{Ann}_{\dot{U}(\mathfrak{g})}(L(\lambda)) = \{0\}.$$

[6]All of these assertions depend on the assumption that A is symmetrizable.

Proof. The proof of the first statement reduces using Lemma 3.1 to checking that the maps

$$U^-(\mathfrak{g}) \to \bigoplus_{\kappa \in P^+} L(\kappa), \qquad u \mapsto (u\bar{1}_\kappa)_{\kappa \in P^+},$$
$$U^+(\mathfrak{g}) \to \bigoplus_{\kappa' \in -P^+} L'(\kappa'), \qquad u \mapsto (u\bar{1}_{\kappa'})_{\kappa' \in -P^+}$$

are injective, where $U^\pm(\mathfrak{g})$ are the positive and negative parts of $U(\mathfrak{g})$ generated by the e_i and f_i, respectively. These are well-known facts; e.g. they can be deduced in a non-classical way from [Lu2, Proposition 19.3.7]. The second statement (which is even better known) follows from the first since each $L(\kappa'|\kappa)$ is a finite direct sum of $L(\lambda)$'s in view of Lemma 3.1 and complete reducibility. □

3.2. **Strict 2-categories.** Let $\mathcal{C}at$ be the category of (small) $\Bbbk$-linear categories and $\Bbbk$-linear functors. It is a monoidal category with tensor functor $\boxtimes : \mathcal{C}at \times \mathcal{C}at \to \mathcal{C}at$ defined on objects (= categories) by letting $\mathcal{C} \boxtimes \mathcal{C}'$ be the category with objects that are pairs $(\lambda, \lambda') \in \operatorname{ob} \mathcal{C} \times \operatorname{ob} \mathcal{C}'$, morphisms

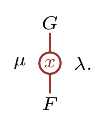
$$\operatorname{Hom}_{\mathcal{C}\boxtimes\mathcal{C}'}((\lambda, \lambda'), (\mu, \mu')) := \operatorname{Hom}_{\mathcal{C}}(\lambda, \mu) \otimes_\Bbbk \operatorname{Hom}_{\mathcal{C}}(\lambda', \mu'),$$

and composition law defined by $(g \otimes g') \circ (f \otimes f') := (g \circ f) \otimes (g' \circ f')$. The definition of $\boxtimes$ on morphisms (= functors) is obvious: $F \boxtimes F'$ is the functor that sends $(\lambda, \lambda') \mapsto (F\lambda, F'\lambda')$ and $g \otimes g' \mapsto Fg \otimes F'g'$.

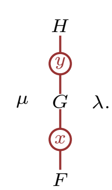
Definition 3.3. A *strict 2-category* is a category enriched in $\mathcal{C}at$. Thus, for objects λ, μ in a strict 2-category $\mathfrak{C}$, there is given a category $\mathcal{H}om_{\mathfrak{C}}(\lambda, \mu)$ of morphisms from λ to μ, whose objects F, G are the *1-morphisms* of $\mathfrak{C}$, and whose morphisms $x : F \Rightarrow G$ are the *2-morphisms* of $\mathfrak{C}$.

For example, $\mathcal{C}at$ can be viewed as a strict 2-category $\mathfrak{C}\mathfrak{at}$ with 2-morphisms being natural transformations.

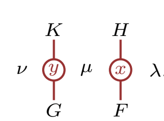
Given a strict 2-category $\mathfrak{C}$, we use the shorthand $\operatorname{Hom}_{\mathfrak{C}}(F, G)$ for the vector space $\operatorname{Hom}_{\mathcal{H}om_{\mathfrak{C}}(\lambda,\mu)}(F, G)$ of all 2-morphisms $x : F \Rightarrow G$. Let us also briefly recall the "string calculus" for 2-morphisms in $\mathfrak{C}$; e.g. see [L2, §2]. We represent a 2-morphism $x \in \operatorname{Hom}_{\mathfrak{C}}(F, G)$ by the picture



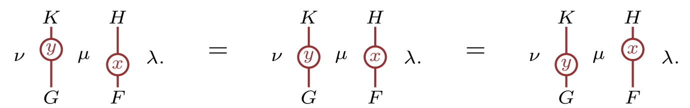
The vertical composition $y \circ x$ of x with another 2-morphism $y \in \operatorname{Hom}_{\mathfrak{C}}(G, H)$ is obtained by vertically stacking pictures:



Given 2-morphisms $x : F \Rightarrow H, y : G \Rightarrow K$ between 1-morphisms $F, H : \lambda \to \mu, G, K : \mu \to \nu$, we denote their horizontal composition by $yx : GF \Rightarrow KH$, and represent it by horizontally stacking pictures:



When confusion seems unlikely, we will use the same notation for a 1-morphism F as for its identity 2-morphism. With this convention, we have that $yH \circ Gx = yx = Kx \circ yF$, or in pictures:



This is the *interchange law*; it means that diagrams for 2-morphisms are invariant under rectilinear isotopy.

We note that any strict 2-category $\mathfrak{C}$ has an *additive envelope* constructed by taking the additive envelope of each of the morphism categories in $\mathfrak{C}$. The *additive Karoubi envelope* $\dot{\mathfrak{C}}$ is the strict 2-category obtained by taking idempotent completions after that. Finally, we define the *Grothendieck ring*

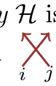
$$K_0(\dot{\mathfrak{C}}) := \bigoplus_{\lambda,\mu \in \mathrm{ob}\,\mathfrak{C}} K_0(\mathcal{H}om_{\dot{\mathfrak{C}}}(\lambda,\mu)), \tag{3.7}$$

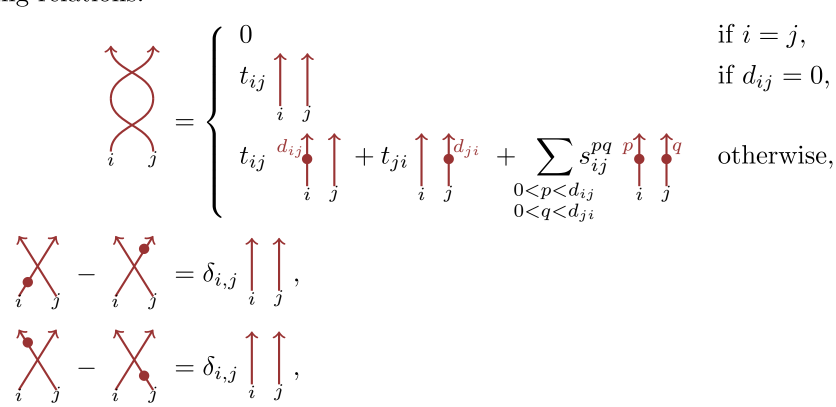
where the latter K_0 denotes the usual split Grothendieck group of an additive category. Horizontal composition induces a multiplication making $K_0(\dot{\mathfrak{C}})$ into a locally unital ring with distinguished idempotents $(1_\lambda)_{\lambda \in \mathrm{ob}\,\mathfrak{C}}$.

3.3. **Quiver Hecke categories and the nil Hecke algebra.** The data of a strict monoidal category $\mathcal{C}$ is equivalent to that of a strict 2-category $\mathfrak{C}$ with one object; the objects and morphisms in $\mathcal{C}$ correspond to the 1-morphisms and 2-morphisms in $\mathfrak{C}$. For strict monoidal categories, we will use the same diagrammatic formalism as explained in the previous subsection; the only difference is that there no need to label the regions of the diagrams by objects since there is only one.

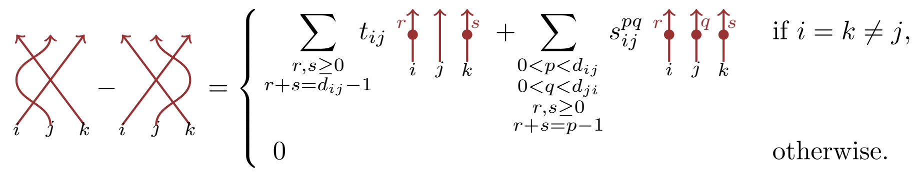
In the next definition, we introduce the *quiver Hecke category*, which is "half" of the Kac-Moody 2-category $\mathfrak{U}(\mathfrak{g})$ to be defined in the next subsection. Everything from this point on depends on the Kac-Moody data from §3.1 plus some additional parameters: we fix

- units $t_{ij} \in \Bbbk^\times$ such that $t_{ii} = 1$ and $d_{ij} = 0 \Rightarrow t_{ij} = t_{ji}$;
- scalars[7] $s_{ij}^{pq} \in \Bbbk$ for $0 < p < d_{ij}$, $0 < q < d_{ji}$ such that $s_{ij}^{pq} = s_{ji}^{qp}$.

Definition 3.4. The (positive) *quiver Hecke category* $\mathcal{H}$ is the strict monoidal category generated by objects I and morphisms  $: i \to i$ and  $: i \otimes j \to j \otimes i$ subject to the following relations:



[7]In [B3] and elsewhere, scalars s_{ij}^{pq} are incorporated into the relations also for $p = 0$ or $q = 0$; we don't allow so much freedom here because it makes it impossible to prove that dots are nilpotent in the cyclotomic quotients discussed below (see Lemma 4.16).



(We depict the nth power of  under vertical composition by labelling the dot with n.) There is also the (negative) *quiver Hecke category* $\mathcal{H}'$ generated by objects I and morphisms  : $i \to i$ and  : $i \otimes j \to j \otimes i$ subject to the relations obtained by reversing the directions of all the arrows in the above, then switching the order of the terms on the left hand sides of the second, third and fourth relations. In fact, $\mathcal{H}'$ is isomorphic to $\mathcal{H}$, but the different normalization of generators is sometimes more convenient.

For objects $\boldsymbol{i} = i_n \otimes \cdots \otimes i_1 \in I^{\otimes n}$ and $\boldsymbol{j} = j_m \otimes \cdots \otimes j_1 \in I^{\otimes m}$, there are no morphisms $\boldsymbol{i} \to \boldsymbol{j}$ in $\mathcal{H}$ unless $m = n$. The endomorphism algebra

$$H_n := \bigoplus_{\boldsymbol{i},\boldsymbol{j} \in I^{\otimes n}} \mathrm{Hom}_{\mathcal{H}}(\boldsymbol{i},\boldsymbol{j}) \tag{3.8}$$

is the (positive) *quiver Hecke algebra* which was introduced independently in [R1] and [KL1]. There is also the negative version $H'_n := \bigoplus_{\boldsymbol{i},\boldsymbol{j} \in I^{\otimes n}} \mathrm{Hom}_{\mathcal{H}'}(\boldsymbol{i},\boldsymbol{j})$, which is isomorphic to H_n with a different normalization of generators.

In the special case that I is a singleton, the quiver Hecke algebra H_n is the *nil Hecke algebra* NH_n, which plays a crucial role in the general theory. Numbering strands by $1, \dots, n$ from right to left, let us write X_i for the element of NH_n corresponding to a dot on the ith strand, and T_i for the element corresponding to the crossing of the ith and $(i+1)$th strands. Let S_n be the symmetric group with its usual simple reflections $s_1, \dots, s_{n-1}$, length function ℓ and longest element w_n. It acts naturally on the polynomial algebra $\mathrm{Pol}_n := \Bbbk[X_1, \dots, X_n]$; we write Sym_n for the subalgebra of invariants. The following are well known; e.g. see [KL1, §2], [R2, §2] or [B2, §2].

(N1) There is a faithful representation of NH_n on Pol_n defined by $X_i \cdot f := X_i f$ and $T_i \cdot f := \frac{s_i(f)-f}{X_i - X_{i+1}}$ (the ith *Demazure operator*).

(N2) For any $w \in S_n$, let T_w be the corresponding element of NH_n defined via a reduced expression for w. Then NH_n is a free left Pol_n-module on basis $\{T_w \mid w \in S_n\}$. In particular, $\mathrm{Pol}_n \hookrightarrow NH_n$.

(N3) The algebra Pol_n is a free Sym_n-module on basis $\{b_w \mid w \in S_n\}$ where $b_w := (-1)^{\ell(w)} T_w \cdot X_1^{n-1} X_2^{n-2} \cdots X_{n-1}$. Moreover $b_{w_n} = 1$.

(N4) There is an algebra isomorphism $NH_n \xrightarrow{\sim} \mathrm{End}_{\mathrm{Sym}_n}(\mathrm{Pol}_n)$ induced by the action of NH_n on Pol_n. Hence, $NH_n \cong M_{n!}(\mathrm{Sym}_n)$ and $Z(NH_n) = \mathrm{Sym}_n$.

(N5) The element $\pi_n := (-1)^{\ell(w_n)} X_1^{n-1} X_2^{n-2} \cdots X_{n-1} T_{w_n}$ acts on the basis for Pol_n from (N3) by $\pi_n \cdot b_w = \delta_{w,1} b_w$. Hence, it is a primitive idempotent in NH_n, and $NH_n \cong (NH_n \pi_n)^{\oplus n!}$ as a left NH_n-module.

In the remainder of the subsection, we wish to give a first indication of the power of the quiver Hecke relations. Let $\mathcal{C}$ be some category which is additive and idempotent-complete. Suppose that we are given a categorical action of the quiver Hecke category $\mathcal{H}$ on $\mathcal{C}$, i.e. there is a strict monoidal functor

$$\Phi : \mathcal{H} \to \mathcal{E}nd(\mathcal{C}) \tag{3.9}$$

where $\mathcal{E}nd(\mathcal{C})$ denotes the strict monoidal category of all endofunctors of $\mathcal{C}$. Let $E_i := \Phi(i)$ and $x_i := \Phi($  $) \in \mathrm{End}(E_i)$. Since E_i is $\Bbbk$-linear (as always), it is additive, hence it induces an endomorphism $e_i := [E_i]$ of the split Grothendieck group $K_0(\mathcal{C})$. More generally, for $i \in I$ and $n \geq 1$, we can obviously identify the nil Hecke algebra NH_n with $\mathrm{End}_{\mathcal{H}}(i^{\otimes n})$; then the image under Φ of the idempotent π_n from (N5) gives us an idempotent $\pi_{i,n} \in \mathrm{End}(E_i^n)$. Let $E_i^{(n)} := \pi_{i,n} E_i^n$, i.e. it is the endofunctor of $\mathcal{C}$ that sends an object V to the image of $(\pi_{i,n})_V \in \mathrm{End}_{\mathcal{C}}(E_i^n V)$, and a morphism $f : V \to W$ to the restriction of $E_i^n f$. The following lemma shows that this categorifies the divided power $e_i^{(n)} := e_i^n / n!$.

Lemma 3.5. *We have that* $E_i^n \cong \left(E_i^{(n)}\right)^{\bigoplus n!}$.

Proof. By (N5), the identity element of NH_n is a sum of $n!$ primitive idempotents, each of which is conjugate to π_n. □

Lemma 3.6. *Suppose for some $V \in \mathrm{ob}\,\mathcal{C}$ that there is a monic polynomial $f(t)$ of degree n such that $f((x_i)_V) = 0$. Then $E_i^{(n+1)} V = 0$. In particular, if $\mathcal{C}$ is finite-dimensional, then all e_i act locally nilpotently on $K_0(\mathcal{C})$.*

Proof. The second statement follows from the first: if $\mathcal{C}$ is finite-dimensional then, for any $V \in \mathrm{ob}\,\mathcal{C}$, we have that $(x_i)_V$ is an element of the finite-dimensional algebra $\mathrm{End}_{\mathcal{C}}(E_i V)$. Hence, it certainly satisfies some polynomial relation. To prove the first statement, we first note the following identity in NH_{n+1}:

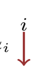
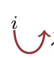
$$(-1)^{\ell(w_{n+1})} \pi_{n+1} f(X_1) X_2^{n-1} \cdots X_n T_{w_{n+1}} = \pi_{n+1}.$$

This holds because $\pi_{n+1}^2 = \pi_{n+1}$ and moreover $T_{w_{n+1}} X_1^m X_2^{n-1} \cdots X_n T_{w_{n+1}} = 0$ for $m < n$ by degree considerations. Now as above we identify NH_{n+1} with $\mathrm{End}_{\mathcal{H}}(i^{\otimes(n+1)})$, apply Φ to our identity, then evaluate the resulting natural transformations at V. By assumption, $\Phi(f(X_1))_V = E_i^n f((x_i)_V) = 0$. Hence, the left hand side vanishes, and we deduce that $\Phi(\pi_{n+1})_V = 0$. This is the identity endomorphism of $E_i^{(n+1)} V$, so the latter object is isomorphic to zero. □

Perhaps most striking of all, we have the following, which is an immediate consequence of the even stronger categorical Serre relations from [R1, Proposition 4.2]:

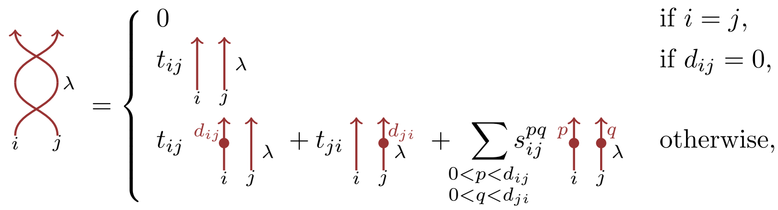
Lemma 3.7. *The endomorphisms e_i of $K_0(\mathcal{C})$ satisfy the Serre relations from (3.3).*

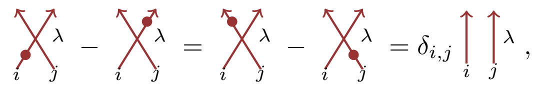
3.4. **Kac-Moody 2-categories.** We are ready to formulate Rouquier's definition of the Kac-Moody 2-category $\mathfrak{U}(\mathfrak{g})$; cf. [R1, §4.1.3].

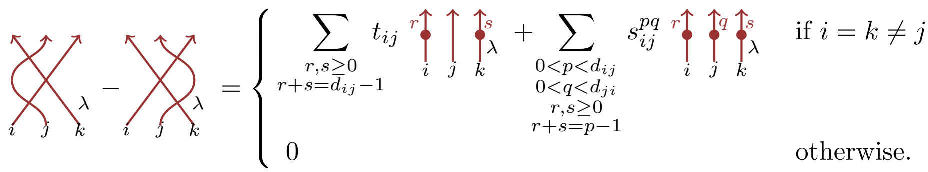
Definition 3.8. The *Kac-Moody 2-category* $\mathfrak{U}(\mathfrak{g})$ is the strict 2-category[8] with objects P; generating 1-morphisms $E_i 1_\lambda : \lambda \to \lambda + \alpha_i$ and $F_i 1_\lambda : \lambda \to \lambda - \alpha_i$ for each $i \in I$ and $\lambda \in P$, whose identity 2-morphisms will be represented diagrammatically by $\lambda{+}\alpha_i$  λ and $\lambda{-}\alpha_i$  λ, respectively; and generating 2-morphisms  $\lambda : E_i 1_\lambda \to E_i 1_\lambda$,  $\lambda : E_i E_j 1_\lambda \to E_j E_i 1_\lambda$,  $\lambda : 1_\lambda \to F_i E_i 1_\lambda$ and  $\lambda : E_i F_i 1_\lambda \to 1_\lambda$. The generating 2-morphisms are subject to the following relations. First, we have the *positive quiver*

[8] Some authors require it is additive from the outset but we don't assume this.

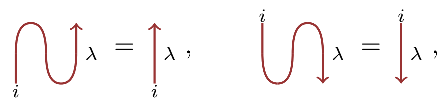
Hecke relations (cf. Definition 3.4):

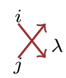


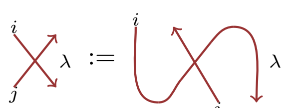




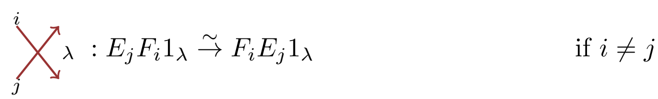
Next we have the *right adjunction relations*

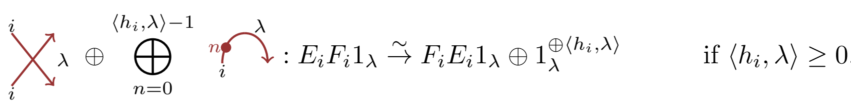


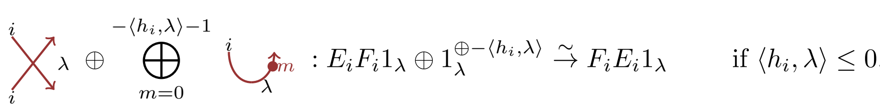
which imply that $F_i 1_{\lambda+\alpha_i}$ is a right dual of $E_i 1_\lambda$. Finally there are some *inversion relations*. To formulate these, define a new 2-morphism  $\lambda : E_j F_i 1_\lambda \to F_i E_j 1_\lambda$ by setting



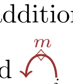
Then the inversion relations assert that the following are isomorphisms:

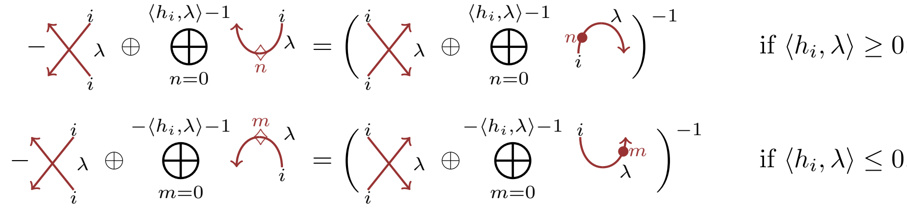






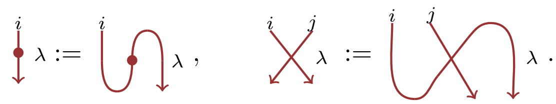
the last two being 2-morphisms in the additive envelope of $\mathfrak{U}(\mathfrak{g})$.

Remark 3.9. More formally, the inversion relations mean that there are some additional generating 2-morphisms  $\lambda : F_i E_j 1_\lambda \to E_j F_i 1_\lambda$,  $\lambda : 1_\lambda \to E_i F_i 1_\lambda$ and  $\lambda : F_i E_i 1_\lambda \to 1_\lambda$ for $0 \leq n < \langle h_i, \lambda\rangle$ and $0 \leq m < -\langle h_i, \lambda\rangle$ such that the following hold:

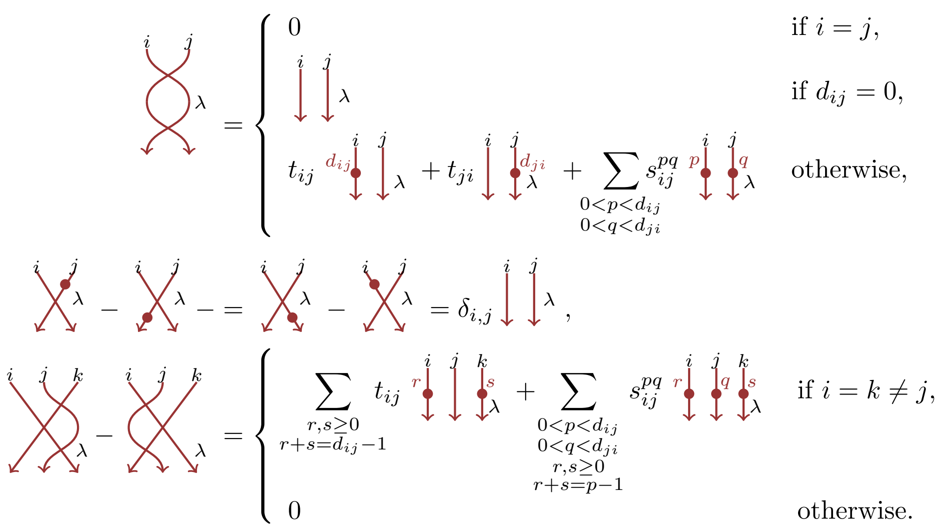


As usual with objects defined by generators and relations, one then needs to play the game of deriving consequences from the defining relations. Here we record some which were established in [B3]; we also cite below the more recent exposition in [BE2] since that uses exactly the same normalization as here.

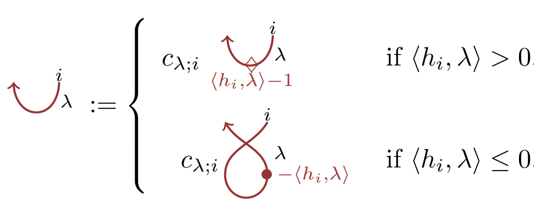
(K1) *Negative quiver Hecke relations.* Define 2-morphisms  $\lambda : F_i 1_\lambda \to F_i 1_\lambda$ and  $\lambda : F_j F_i 1_\lambda \to F_i F_j 1_\lambda$ by setting

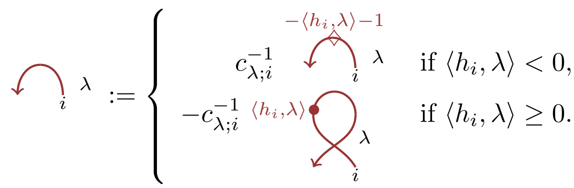


On rotating the positive quiver Hecke relations clockwise through 180°, one sees that these satisfy the negative quiver Hecke relations:

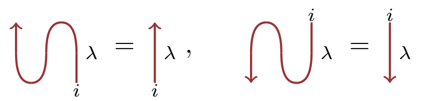


(K2) *Second adjunction.* We next introduce 2-morphisms  $\lambda : 1_\lambda \to E_i F_i 1_\lambda$ and  $\lambda : F_i E_i 1_\lambda \to 1_\lambda$. The definition of these was suggested already by Rouquier in [R1, §4.1.4]. Following the idea of [BHLW] we will normalize them in a different way which depends on an additional choice of units $c_{\lambda;i} \in \Bbbk^\times$ for each $i \in I$ and $\lambda \in P$ such that $c_{\lambda+\alpha_j;i} = t_{ij} c_{\lambda;i}$. Fixing such a choice from now on, we set



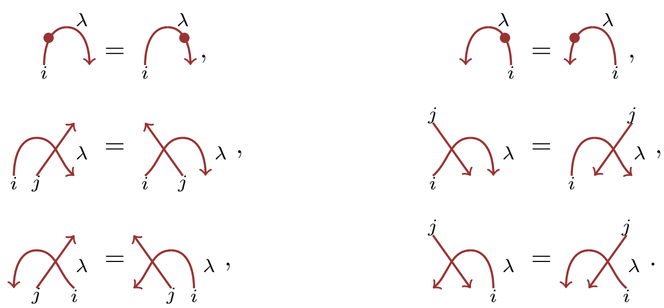


Then by [B3, Theorem 4.3] (or [BE2, Proposition 6.2] with the present normalization) we have the *left adjunction relations*:

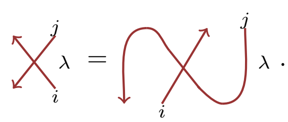


This means that $F_i 1_{\lambda+\alpha_i}$ is also a left dual of $E_i 1_\lambda$.

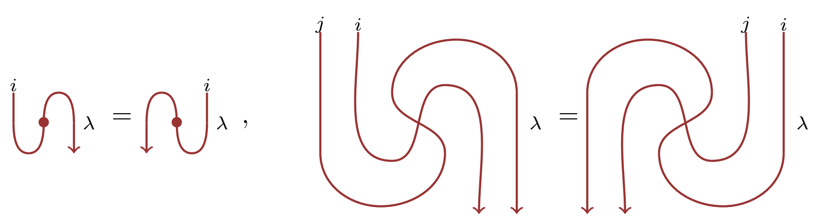
(K3) *Pitchfork relations and cyclicity.* The following relations follow from the definitions and [B3, Theorem 5.3] (or [BE2, Proposition 4.1] and [BE2, Proposition 7.2]):



It follows that



Moreover, the 2-morphisms  λ and  λ are *cyclic* i.e. their right mates are equal to their left mates:



This is the main advantage of the normalization of the second adjunction from [BHLW] as chosen in (K2).

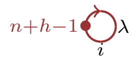
(K4) *Infinite Grassmannian relations.* Let Sym be the algebra of symmetric functions over $\Bbbk$. Recall Sym is generated both by the elementary symmetric functions e_n $(n \geq 1)$ and by the complete symmetric functions h_n $(n \geq 1)$. Adopting the convention that $\mathrm{e}_0 = \mathrm{h}_0 = 1$ and $\mathrm{e}_n = \mathrm{h}_n = 0$ for $n < 0$, these two families of generators are related by the equation

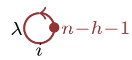
$$\sum_{r+s=n} (-1)^r \mathrm{e}_r \mathrm{h}_s = 0 \text{ for all } n > 0. \tag{3.10}$$

Take $i \in I$, $\lambda \in P$ and set $h := \langle h_i, \lambda\rangle$. Then the infinite Grassmannian relations assert that there is a well-defined homomorphism

$$\beta_{\lambda;i} : \mathrm{Sym} \to \mathrm{End}_{\mathfrak{U}(\mathfrak{g})}(1_\lambda)$$

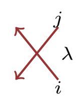
such that

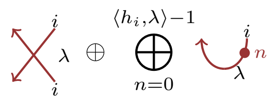
$$\beta_{\lambda;i}(\mathrm{h}_n) = c_{\lambda;i}^{-1}$$  if $n > -h$, (3.11)

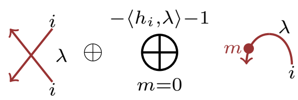
$$\beta_{\lambda;i}(\mathrm{e}_n) = (-1)^n c_{\lambda;i}$$  if $n > h$. (3.12)

This was proved originally by Lauda[9] in [L1, Proposition 8.2]; see [BE2, Proposition 5.1] where it is established using our normalization. It motivated Lauda's introduction also of certain *negatively dotted bubbles*, which are 2-morphisms in $\mathrm{End}_{\mathfrak{U}(\mathfrak{g})}(1_\lambda)$ defined so that (3.11)–(3.12) hold for all $n \in \mathbb{Z}$.

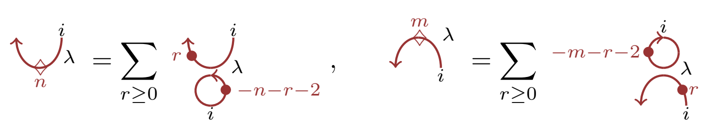
(K5) *Dual inversion relations.* The following 2-morphisms are invertible:

 $: F_jE_i1_\lambda \xrightarrow{\sim} E_iF_j1_\lambda$ if $i \neq j$,

 $: F_iE_i1_\lambda \oplus 1_\lambda^{\oplus\langle h_i,\lambda\rangle} \xrightarrow{\sim} E_iF_i1_\lambda$ if $\langle h_i, \lambda\rangle \geq 0$,

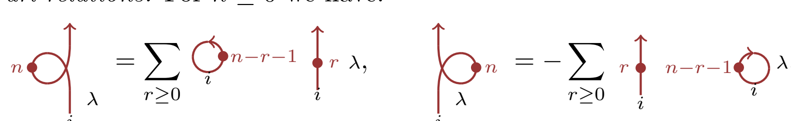
 $: F_iE_i1_\lambda \xrightarrow{\sim} E_iF_i1_\lambda \oplus 1_\lambda^{\oplus -\langle h_i,\lambda\rangle}$ if $\langle h_i, \lambda\rangle \leq 0$.

This may be deduced from the definitions above using also the following relations proved in [B3, Corollary 3.3] (or [BE2, Corollary 5.2]):



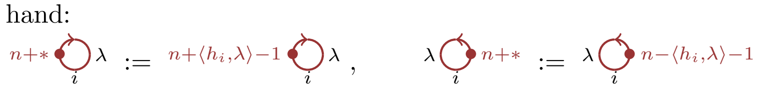
Note here we are using the negatively dotted bubbles. Another consequence of the last relations displayed, plus the description of the leftward crossing given in (K3), is that all 2-morphisms in $\mathfrak{U}(\mathfrak{g})$ are generated (under both vertical and horizontal composition) by upward dots, upward crossings, and leftward and rightward cups and caps.

(K6) *Curl relations.* For $n \geq 0$ we have:



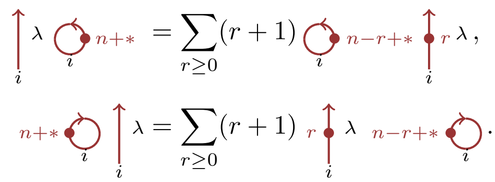
These are proved e.g. in [BHLW, Lemma 3.2] or [BE2, Corollary 5.4].

(K7) *Bubble slides.* For the next relations, we adopt the following convenient shorthand:

 (3.13)

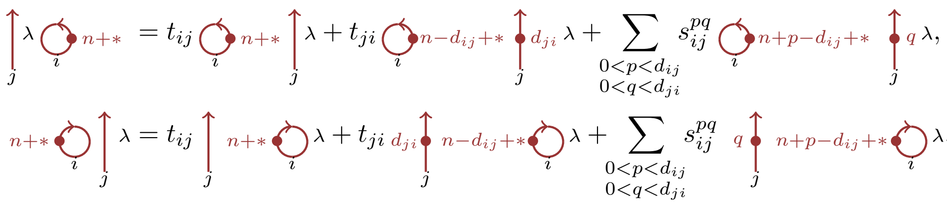
[9]In fact, Lauda showed for $\mathfrak{g} = \mathfrak{sl}_2$ that this homomorphism is an isomorphism. In general, the product of the homomorphisms $\beta_{\lambda;i}$ over all $i \in I$ should given an isomorphism $\bigotimes_{i\in I} \mathrm{Sym} \xrightarrow{\sim} \mathrm{End}_{\mathfrak{U}(\mathfrak{g})}(1_\lambda)$, but the proof of this assertion depends on the Nondegeneracy Condition discussed later in the subsection.

Then following hold for all $n \in \mathbb{Z}$. First:



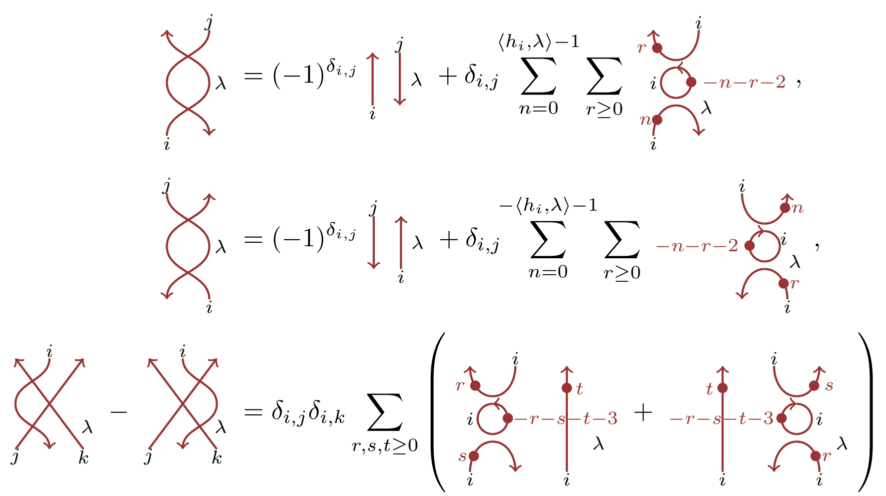
Then for $i \neq j$ with $d_{ij} = 0$ we have:

Finally for $i \neq j$ with $d_{ij} > 0$ we have:



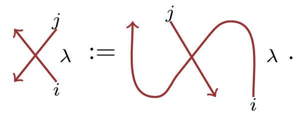
In the simply-laced case, these were proved in [KL3, Propositions 3.3–3.5]. In general, they are recorded in various places in the literature; e.g. see [W2, Proposition 2.8] or [BHLW, §3.2]. When $d_{ij} > 1$, the proof of the final two bubble slides above is not as straightforward as those references may suggest as it requires also an application of the deformed braid relation; we refer to [BE2, Proposition 7.3] for the detailed argument.

(K8) *Alternating crossings.* Finally we record the following relations from [BE2, Corollary 5.3] and [BE2, Proposition 7.6]; the first two are immediate from the definitions and the "diamond relations" recorded in (K5); the last one was derived already in [KL3].



Remark 3.10. A useful consequence of these relations is that $\mathfrak{U}(\mathfrak{g})$ can be presented equivalently with generating 2-morphisms  λ,  λ,  λ and  λ subject to the negative quiver Hecke relations from (K1), the left adjunction relations from (K2), and

replacing the original inversion relations by the dual inversion relations from (K5) where



In pictures, this new presentation is the original presentation from Definition 3.8 rotated through 180°.

It still seems somewhat remarkable that all of the above relations (K1)–(K8) can be derived from Rouquier's minimalistic definition. Independently, Khovanov and Lauda [KL3] introduced a strict 2-category incorporating extra generators and relations essentially matching the ones above; see also [CL] which extended the definition in [KL3] to more general parameters. The following is a consequence of (K1)–(K8).

Theorem 3.11 ([B3, Main Theorem])**.** *Rouquier's Kac-Moody 2-category* $\mathfrak{U}(\mathfrak{g})$ *from Definition 3.8 is isomorphic to the 2-category introduced in* [KL3, CL].

Khovanov and Lauda exploited their extra generators and relations to construct some explicit sets of 2-morphisms[10] which they showed span the 2-morphism spaces in $\mathfrak{U}(\mathfrak{g})$; see [KL3, Proposition 3.11]. They then conjectured that these spanning sets actually give bases for the 2-morphism spaces in $\mathfrak{U}(\mathfrak{g})$. This is the *Nondegeneracy Condition* from [KL3, §3.2.3]. For $\mathfrak{g} = \mathfrak{sl}_n$, it is known that the Nondegeneracy Condition holds, thanks to [KL3, §6.4].

To explain the significance of the Nondegeneracy Condition, let $\dot{\mathfrak{U}}(\mathfrak{g})$ be the additive Karoubi envelope of $\mathfrak{U}(\mathfrak{g})$ and $K_0(\dot{\mathfrak{U}}(\mathfrak{g})) = \bigoplus_{\lambda,\mu \in P} 1_\mu K_0(\dot{\mathfrak{U}}(\mathfrak{g}))1_\lambda$ be its Grothendieck ring as in (3.7). Also let $\dot{U}(\mathfrak{g})_\mathbb{Z}$ be the Kostant $\mathbb{Z}$-form for $\dot{U}(\mathfrak{g})$, i.e. the subring generated by the idempotents 1_λ and the divided powers $e_i^{(r)} 1_\lambda, f_i^{(r)} 1_\lambda$ for $\lambda \in P, i \in I$ and $r \geq 1$. Then the arguments used to prove [KL3, Theorem 1.1] (which are based ultimately on Lemmas 3.5 and 3.7) show that there is a unique surjective locally unital homomorphism

$$\gamma : \dot{U}(\mathfrak{g})_\mathbb{Z} \twoheadrightarrow K_0(\dot{\mathfrak{U}}(\mathfrak{g})) \tag{3.14}$$

which sends $e_i 1_\lambda$ and $f_i 1_\lambda$ to $[E_i 1_\lambda]$ and $[F_i 1_\lambda]$, respectively. Moreover, *assuming the parameters satisfy the homogeneity property*

$$s_{ij}^{pq} \neq 0 \Rightarrow p d_{ji} + q d_{ij} = d_{ij} d_{ji} \tag{3.15}$$

and *providing that the Nondegeneracy Condition holds*, [KL3, Theorem 1.2] implies that γ is an isomorphism. This makes precise the sense in which $\mathfrak{U}(\mathfrak{g})$ should categorify the universal enveloping algebra of $\mathfrak{g}$.

Remark 3.12. In finite type, it is known that γ is an isomorphism (regardless of whether (3.15) holds); see Corollary 4.21 below.

Remark 3.13. As we were finalizing this article, Webster released a preliminary version of [W3]. In this work, he appears to have found a general proof of the Nondegeneracy Condition valid for all types and all choices of parameters satisfying (3.15).

3.5. **Gradings.** In this subsection, we discuss the graded version $\mathfrak{U}_q(\mathfrak{g})$ of $\mathfrak{U}(\mathfrak{g})$ and its connection to quantum groups. Our language is based on [BE1, §6], and is slightly different to that of [KL3, R1].

Let $\underline{\mathcal{GV}ec}$ be the symmetric monoidal category of (small) $\mathbb{Z}$-graded vector spaces and degree-preserving linear maps. The grading shift functor gives an automorphism

$$Q : \underline{\mathcal{GV}ec} \to \underline{\mathcal{GV}ec},$$

[10] They worked with a restricted choice of parameters compared to here, but it is clear how to extend their constructions to the general case using (K1)–(K8).

our convention for this being that $(QV)_n = V_{n-1}$. By a *graded category*, we mean a category enriched in $\underline{\mathcal{GV}ec}$. Thus, the morphism spaces in a graded category are equipped with a $\mathbb{Z}$-grading in a way that is compatible with composition. If $\mathcal{C}$ is any graded category, the *underlying category* $\underline{\mathcal{C}}$ is the category with the same objects as $\mathcal{C}$, but only the homogeneous morphisms of degree zero. For example, $\underline{\mathcal{GV}ec}$ is the underlying category of the graded category $\mathcal{GV}ec$ whose objects are (small) $\mathbb{Z}$-graded vector spaces and whose morphisms are sums of homogeneous linear maps of various degrees.

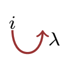
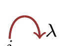
Let $\mathcal{GC}at$ be the category of all (small) graded categories. It is monoidal with product $\boxtimes$ defined just like in §3.2. A *strict graded 2-category* is a category enriched in $\mathcal{GC}at$; cf. Definition 3.3. If $\mathfrak{C}$ is any graded 2-category, the *underlying 2-category* $\underline{\mathfrak{C}}$ is the 2-category with the same objects and 1-morphisms as $\mathfrak{C}$, but only the homogeneous 2-morphisms of degree zero. Also let $\dot{\underline{\mathfrak{C}}}$ be the additive Karoubi envelope of $\underline{\mathfrak{C}}$, and $K_0(\dot{\underline{\mathfrak{C}}})$ be its Grothendieck ring defined as in (3.7).

For any graded 2-category $\mathfrak{C}$, there is a universal construction of another graded 2-category $\mathfrak{C}_q$, which we call the *Q-envelope* of $\mathfrak{C}$. It has the same object set as $\mathfrak{C}$. Given objects λ, μ, the 1-morphisms $\lambda \to \mu$ in $\mathfrak{C}_q$ are defined formally to be symbols $Q^n F$ for all 1-morphisms $F : \lambda \to \mu$ in $\mathfrak{C}$ and all $n \in \mathbb{Z}$. Then the 2-morphisms in $\mathfrak{C}_q$ are defined from

$$\operatorname{Hom}_{\mathfrak{C}_q}(Q^n F, Q^m G) := Q^{m-n} \operatorname{Hom}_{\mathfrak{C}}(F, G),$$

where the Q^{m-n} on the right hand side is the grading shift in $\underline{\mathcal{GV}ec}$. Horizontal composition of 1-morphisms in $\mathfrak{C}_q$ is induced by the horizontal composition in $\mathfrak{C}$ so that $(Q^n F)(Q^m G) := Q^{n+m} FG$. Similarly, the horizontal and vertical compositions of 2-morphisms in $\mathfrak{C}_q$ are induced in an obvious way by the horizontal and vertical compositions in $\mathfrak{C}$. Note also for each 1-morphism F of $\mathfrak{C}$ and $n \in \mathbb{Z}$ that 1_F defines a canonical 2-isomorphism $Q^n F \xrightarrow{\sim} F$ in $\mathfrak{C}_q$ that is homogeneous of degree $-n$.

The point of the construction of Q-envelope is that $\mathfrak{C}_q$ (hence, the underlying 2-category $\underline{\mathfrak{C}}_q$) is equipped with distinguished 1-morphisms $q_\lambda := Q1_\lambda : \lambda \to \lambda$ for each object λ, such that $q_\mu F = F q_\lambda$ for each 1-morphism $F : \lambda \to \mu$. In particular, the Grothendieck ring $K_0(\dot{\underline{\mathfrak{C}}}_q)$ is actually a $\mathbb{Z}[q, q^{-1}]$-algebra, with q acting on $1_\mu K_0(\dot{\underline{\mathfrak{C}}}_q) 1_\lambda$ by left multiplication by $[q_\mu]$ (= right multiplication by $[q_\lambda]$).

Definition 3.14. Assume that the parameters fixed in §3.3 satisfy (3.15). Then the relations defining the Kac-Moody 2-category $\mathfrak{U}(\mathfrak{g})$ are all homogeneous, so we can make $\mathfrak{U}(\mathfrak{g})$ into a graded 2-category by declaring that the generating 2-morphisms  λ,  λ,  λ and  λ are of degrees $2d_i$, $d_i d_{ij}$, $d_i(1 + \langle h_i, \lambda \rangle)$ and $d_i(1 - \langle h_i, \lambda \rangle)$, respectively. Then we define the *graded Kac-Moody 2-category* $\mathfrak{U}_q(\mathfrak{g})$ to be the Q-envelope of this graded 2-category.

Using Theorem 3.11, it is easy to see that the underlying 2-category $\underline{\mathfrak{U}}_q(\mathfrak{g})$ is isomorphic to the graded version of the Kac-Moody 2-category as defined in [R1, KL3]. Then [KL3, Theorem 1.1] shows that there is a surjective $\mathbb{Z}[q, q^{-1}]$-algebra homomorphism

$$\gamma_q : \dot{U}_q(\mathfrak{g})_{\mathbb{Z}} \twoheadrightarrow K_0(\dot{\underline{\mathfrak{U}}}_q(\mathfrak{g})), \tag{3.16}$$

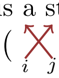
where $\dot{U}_q(\mathfrak{g})_{\mathbb{Z}}$ denotes the Kostant-Lusztig $\mathbb{Z}$-form of the idempotented version of the quantized enveloping algebra $U_q(\mathfrak{g})$. Moreover, *providing that the Nondegeneracy Condition holds*, [KL3, Theorem 1.2] shows that γ_q is an isomorphism. The injectivity of γ claimed earlier in the ungraded setting follows from this on specializing q to 1.

4. Categorical actions and crystals

In this final section, we focus on 2-representations of Kac-Moody 2-categories. We will review the existing results mostly following Rouquier [R2]. After that, we focus on the locally Schurian case, explaining various results in that setting which generalize aspects of [CR, R1].

4.1. **2-representations.** We keep the choices of Kac-Moody data and parameters as in the previous section. The following is [R1, Definition 5.1.1].

Definition 4.1. A *2-representation* of $\mathfrak{U}(\mathfrak{g})$ is the following data:

- (M1) a category $\mathcal{R}$ with a given decomposition $\mathcal{R} = \coprod_{\lambda \in P} \mathcal{R}_\lambda$ into *weight subcategories*;
- (M2) endofunctors E_i and F_i of $\mathcal{R}$ for each $i \in I$, such that $E_i|_{\mathcal{R}_\lambda} : \mathcal{R}_\lambda \to \mathcal{R}_{\lambda+\alpha_i}$ and $F_i|_{\mathcal{R}_\lambda} : \mathcal{R}_\lambda \to \mathcal{R}_{\lambda-\alpha_i}$;
- (M3) natural transformations $x_i : E_i \to E_i$ and $\tau_{ij} : E_i E_j \to E_j E_i$ satisfying the positive quiver Hecke relations from Definition 3.4, i.e. so that there is a strict monoidal functor $\Phi : \mathcal{H} \to \mathcal{E}nd(\mathcal{R})$ with $\Phi(i) = E_i$, $\Phi($  $) = x_i$ and $\Phi($  $) = \tau_{ij}$ for all $i, j \in I$;
- (M4) the unit $\eta_i : 1_\mathcal{R} \to F_i E_i$ and counit $\varepsilon_i : E_i F_i \to 1_\mathcal{R}$ of an adjunction making F_i into a right adjoint of E_i.

Then we require that the following axiom holds:

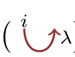
- (M5) all of the natural transformations σ_{ij} $(i \neq j)$ and $\rho_{i,\lambda}$ are invertible, where

$$\sigma_{ij} := F_i E_j \varepsilon_i \circ F_i \tau_{ij} F_i \circ \eta_i E_j F_i : E_j F_i \to F_i E_j,$$

$$\rho_{i,\lambda} := \begin{cases} \sigma_{ii} \oplus \bigoplus_{n=0}^{h-1} \varepsilon_i \circ (x_i F_i)^{\circ n} : E_i F_i|_{\mathcal{R}_\lambda} \to F_i E_i|_{\mathcal{R}_\lambda} \oplus 1_{\mathcal{R}_\lambda}^{\oplus h} & \text{if } h \geq 0, \\ \sigma_{ii} \oplus \bigoplus_{m=0}^{-h-1} (F_i x_i)^{\circ m} \circ \eta_i : E_i F_i|_{\mathcal{R}_\lambda} \oplus 1_{\mathcal{R}_\lambda}^{\oplus -h} \to F_i E_i|_{\mathcal{R}_\lambda} & \text{if } h \leq 0, \end{cases}$$

for $h := \langle h_i, \lambda \rangle$.

We say that a 2-representation $\mathcal{R}$ is *small*, *finite-dimensional*, *additive*, *Abelian* etc... if all of the categories $\mathcal{R}_\lambda (\lambda \in P)$ are small, finite-dimensional, additive, Abelian etc... In the additive case, the functors E_i and F_i extend to $\bigoplus_{\lambda \in P} \mathcal{R}_\lambda$, and it is more convenient to denote this by $\mathcal{R}$ in place of $\coprod_{\lambda \in P} \mathcal{R}_\lambda$. If $\mathcal{R}$ is not additive, one can always replace it by its additive envelope, or indeed its additive Karoubi envelope $\dot{\mathcal{R}}$; the endofunctors E_i and F_i extend canonically to make these into 2-representations too.

The point of Definition 4.1 is that a small 2-representation $\mathcal{R}$ is exactly the same data as a strict 2-functor $\mathbb{R} : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{Cat}$. The dictionary for going between the two notions is given by $\mathcal{R}_\lambda = \mathbb{R}(\lambda)$, $E_i|_{\mathcal{R}_\lambda} = \mathbb{R}(E_i 1_\lambda)$, $F_i|_{\mathcal{R}_\lambda} = \mathbb{R}(F_i 1_\lambda)$, $x_i|_{E_i|_{\mathcal{R}_\lambda}} = \mathbb{R}($  $\lambda)$, $\tau_{ij}|_{E_iE_j|_{\mathcal{R}_\lambda}} = \mathbb{R}($  $\lambda)$, $\eta_i|_{1_{\mathcal{R}_\lambda}} = \mathbb{R}($  $\lambda)$ and $\varepsilon_i|_{F_iE_i|_{\mathcal{R}_\lambda}} = \mathbb{R}($  $\lambda)$.

The following strengthens [R1, Theorem 5.16].

Lemma 4.2. *Suppose that $\mathcal{R}$ is a 2-representation in the sense of Definition 4.1. Then there is a canonical choice for the unit $\eta_i' : 1_\mathcal{R} \to E_i F_i$ and counit $\varepsilon_i' : F_i E_i \to 1_\mathcal{R}$ of an adjunction making F_i into a left adjoint of E_i.*

Proof. We may assume that $\mathcal{R}$ is small, so that there is a corresponding strict 2-functor $\mathbb{R} : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{Cat}$. Then let $\eta'_i|_{1_{\mathcal{R}_\lambda}} := \mathbb{R}\big(\overset{i}{\curvearrowleft\!\cup}\,\lambda\big)$ and $\varepsilon'_i|_{F_iE_i|_{\mathcal{R}_\lambda}} := \mathbb{R}\big(\underset{i}{\curvearrowleft\!\cap}\;\lambda\big)$, where notation is as in (K2) from §3.4. □

The other generators and relations from (K1)–(K8) can be transported to any 2-representation $\mathcal{R}$ in a similar way. For example, from the images of the downward dots and crossings from (K1), one obtains canonical natural transformations $x'_i : F_i \to F_i$ and $\tau'_{ij} : F_iF_j \to F_jF_i$ which satisfy the negative quiver Hecke relations.

Remark 4.3. Definition 4.1 can be formulated equivalently by replacing (M3)–(M5) by

(M3′) natural transformations $x'_i : F_i \to F_i$ and $\tau'_{ij} : F_iF_j \to F_jF_i$ satisfying the negative quiver Hecke relations, i.e. so that there is a strict monoidal functor $\Psi : \mathcal{H}' \to \mathcal{E}nd(\mathcal{R})$ with $\Psi(i) = F_i$, $\Psi\big(\overset{i}{\downarrow\!\bullet}\big) = x'_i$ and $\Phi\big(\overset{j\;\,i}{\times}\big) = \tau'_{ij}$ for all $i, j \in I$;

(M4′) the unit $\eta'_i : 1_{\mathcal{R}} \to E_iF_i$ and counit $\varepsilon'_i : F_iE_i \to 1_{\mathcal{R}}$ of an adjunction making F_i into a left adjoint of E_i;

(M5′) all of the natural transformations σ'_{ij} $(i \neq j)$ and $\rho'_{i,\lambda}$ are invertible, where

$$\sigma'_{ij} := E_iF_j\varepsilon'_i \circ E_i\tau'_{ij}E_i \circ \eta'_iF_jE_i : F_jE_i \to E_iF_j,$$

$$\rho'_{i,\lambda} := \begin{cases} \sigma'_{ii} \oplus \bigoplus_{n=0}^{h-1} (E_ix'_i)^{\circ n} \circ \eta'_i : F_iE_i|_{\mathcal{R}_\lambda} \oplus 1_{\mathcal{R}_\lambda}^{\oplus h} \to E_iF_i|_{\mathcal{R}_\lambda} & \text{if } h \geq 0, \\ \sigma'_{ii} \oplus \bigoplus_{m=0}^{-h-1} \varepsilon'_i \circ (x'_iE_i)^{\circ m} : F_iE_i|_{\mathcal{R}_\lambda} \to E_iF_i|_{\mathcal{R}_\lambda} \oplus 1_{\mathcal{R}_\lambda}^{\oplus -h} & \text{if } h \leq 0, \end{cases}$$

for $h := \langle h_i, \lambda \rangle$.

This follows because, in view of the alternative presentation of $\mathfrak{U}(\mathfrak{g})$ from Remark 3.10, the new formulation is also the data of a strict 2-functor.

If $\mathcal{R}$ is a 2-representation, the endofunctors E_i and F_i induce endomorphisms $[E_i]$ and $[F_i]$ of the split Grothendieck group

$$K_0(\dot{\mathcal{R}}) = \bigoplus_{\lambda \in P} K_0(\dot{\mathcal{R}}_\lambda) \tag{4.1}$$

of its additive Karoubi envelope.

Lemma 4.4. *Given a 2-representation $\mathcal{R}$, there is a unique way to make $K_0(\dot{\mathcal{R}})$ into a module over the Kostant $\mathbb{Z}$-form $U(\mathfrak{g})_{\mathbb{Z}}$ of the universal enveloping algebra of $\mathfrak{g}$ so that the Chevalley generators e_i, f_i act as $[E_i], [F_i]$, respectively, and (4.1) is its decomposition into weight spaces.*

Proof. We may assume that $\mathcal{R}$ (hence, $\dot{\mathcal{R}}$) is small, so that there is a corresponding strict 2-functor $\dot{\mathbb{R}} : \dot{\mathfrak{U}}(\mathfrak{g}) \to \mathfrak{Cat}$ with $\dot{\mathbb{R}}(\lambda) = \dot{\mathcal{R}}_\lambda$, etc... The definition of strict 2-functor then ensures that $K_0(\dot{\mathcal{R}})$ is a module over $K_0(\dot{\mathfrak{U}}(\mathfrak{g}))$: for $F \in \operatorname{ob} \mathcal{H}om_{\dot{\mathfrak{U}}(\mathfrak{g})}(\lambda, \mu)$ defining $[F] \in K_0(\dot{\mathfrak{U}}(\mathfrak{g}))$, and $P \in \operatorname{ob} \dot{\mathcal{R}}_\lambda$ defining $[P] \in K_0(\dot{\mathcal{R}}_\lambda)$, we set $[F][P] := [\dot{\mathbb{R}}(F)(P)] \in K_0(\dot{\mathcal{R}}_\mu)$. It remains to lift the action of $K_0(\dot{\mathfrak{U}}(\mathfrak{g}))$ to $\dot{U}(\mathfrak{g})_{\mathbb{Z}}$ using the homomorphism γ from (3.14); this does not depend on the injectivity of γ. □

Lemma 4.4 shows for any 2-representation $\mathcal{R}$ that $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\dot{\mathcal{R}})$ is a $\mathfrak{g}$-module in a canonical way. Typically, it is an *integrable* $\mathfrak{g}$-module in view of the next lemma.

Lemma 4.5. *If $\mathcal{R}$ is a finite-dimensional 2-representation, then $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\dot{\mathcal{R}})$ is an integrable $\mathfrak{g}$-module.*

Proof. The local nilpotency of e_i follows from Lemma 3.6. Similar arguments with the negative nil Hecke category show that f_i is locally nilpotent too. $\square$

Definition 4.6. Let $\mathcal{R}$ and $\mathcal{S}$ be 2-representations. A *strongly equivariant functor* $G : \mathcal{R} \to \mathcal{S}$ is a functor such that $G|_{\mathcal{R}_\lambda} : \mathcal{R}_\lambda \to \mathcal{S}_\lambda$ for each $\lambda \in P$, plus natural isomorphisms $\zeta_i = \zeta_{G,i} : E_i G \Rightarrow G E_i$, such that the following hold for each $i \in I$:

(E1) the natural transformation $F_i G \varepsilon_i \circ F_i \zeta_i F_i \circ \eta_i G F_i : G F_i \Rightarrow F_i G$ is invertible, with inverse denoted $\zeta_i' : F_i G \Rightarrow G F_i$;

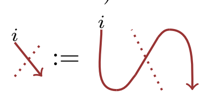
(E2) we have that $G x_i \circ \zeta_i = \zeta_i \circ x_i G$;

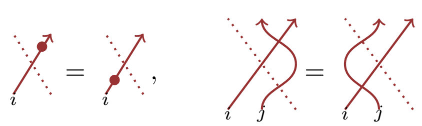
(E3) we have that $G\tau_{ij} \circ \zeta_i E_j \circ E_i \zeta_j = \zeta_j E_i \circ E_j \zeta_i \circ \tau_{ij} G$.

There is an obvious way to make the composition GH of two strongly equivariant functors into a strongly equivariant functor in its own right: set $\zeta_{GH,i} := \zeta_{G,i} H \circ G \zeta_{H,i}$. Also the identity functor 1 is strongly equivariant with $\zeta_{1,i} := 1_{E_i}$. Let $\mathcal{R}ep(\mathfrak{U}(\mathfrak{g}))$ be the resulting category of (small) 2-representations and strongly equivariant functors.

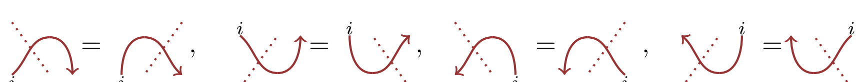
Definition 4.7. A *strongly equivariant equivalence* is an equivalence of categories $G : \mathcal{R} \to \mathcal{S}$ such that $G|_{\mathcal{R}_\lambda} : \mathcal{R}_\lambda \to \mathcal{S}_\lambda$ for each λ, plus isomorphisms $\zeta_i : E_i G \Rightarrow G E_i$ satisfying (E2)–(E3) for each $i \in I$.

Remark 4.8. In the situation of Definition 4.7, the axiom (E1) holds automatically, i.e. strongly equivariant equivalences are strongly equivariant functors. To see this, fix a right adjoint H to G. The given isomorphism $\zeta_i : E_i G \Rightarrow G E_i$ induces a canonical isomorphism $F_i H \Rightarrow H F_i$ between the right adjoints of $G E_i$ and $E_i G$. Horizontally composing on the left and right with G and using the isomorphisms $1 \Rightarrow HG$ and $GH \Rightarrow 1$ defined by the adjunction, we get an isomorphism $G F_i \Rightarrow F_i G$. This is precisely the natural transformation written down in (E1).

It is helpful to interpret Definition 4.6 in terms of the string calculus for 2-morphisms in $\mathfrak{Cat}$. We represent ζ_i by  (so the dotted line is the identity morphism 1_G, to the right of which is the category $\mathcal{R}$ and to the left is $\mathcal{S}$). Then it is natural to denote its inverse by . The 2-morphism in (E1) is . We denote its inverse ζ_i' by . The axioms (E2) and (E3) are the identities

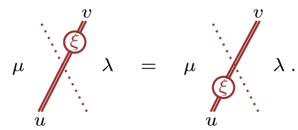
 (4.2)

where the dots and solid crossings represent x_i and τ_{ij}, respectively. Representing $\eta_i, \eta_i', \varepsilon_i$ and ε_i' by oriented cups and caps as usual, one can check further that

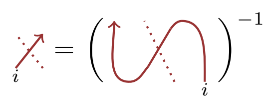
 (4.3)

The first two of these are almost immediate; the second two follow using the inversion relations and the definitions of the leftward cups and caps in $\mathfrak{U}(\mathfrak{g})$.

Remark 4.9. If $\mathcal{R}$, $\mathcal{S}$ are small 2-representations with associated 2-functors $\mathbb{R}$, $\mathbb{S}$, the data of a strongly equivariant functor $G : \mathcal{R} \to \mathcal{S}$ is the same as the data of a morphism of 2-functors $G : \mathbb{R} \to \mathbb{S}$ as in [R1, Definition 2.3] (with $G(\lambda) = G|_{\mathcal{R}_\lambda}$). Indeed, given any 1-morphism $u : \lambda \to \mu$ in $\mathfrak{U}(\mathfrak{g})$, i.e. an appropriate composition of several E_i and F_j applied to 1_λ, the corresponding horizontal composition of ζ_i and ζ_j' defines a natural isomorphism $\zeta(u) : \mathbb{S}(u)G(\lambda) \Rightarrow G(\mu)\mathbb{R}(u)$ satisfying the axioms of a morphism of 2-functors. The non-trivial part about this assertion is the naturality of $\zeta(u)$, i.e. the statement that $G(\mu)\mathbb{R}(\xi) \circ \zeta(u) = \zeta(v) \circ \mathbb{S}(\xi)G(\lambda)$ for all 2-morphisms $\xi : u \Rightarrow v$. In pictures:



The proof of this reduces to checking it in case ξ is an upward dot or crossing or any cup or cap, since these generate all 2-morphisms in $\mathfrak{U}(\mathfrak{g})$ thanks to (K5). These cases are covered by (4.2)–(4.3).

Remark 4.10. Using (4.3), we see in particular that  , i.e. the natural transformation ζ_i' determines ζ_i. Indeed, using also Remark 4.9, one can reformulate Definition 4.6 equivalently in terms of isomorphisms $\zeta_i' : F_iG \Rightarrow GF_i$. The axioms (E1)–(E3) become:

- (E1′) the natural transformation $E_iG\varepsilon_i' \circ E_i\zeta_i'E_i \circ \eta_i'GE_i : GE_i \Rightarrow E_iG$ is invertible, with inverse denoted $\zeta_i : E_iG \Rightarrow GE_i$;
- (E2′) we have that $Gx_i' \circ \zeta_i' = \zeta_i' \circ x_i'G$;
- (E3′) we have that $G\tau_{ij}' \circ \zeta_i'F_j \circ F_i\zeta_j' = \zeta_j'F_i \circ F_j\zeta_i' \circ \tau_{ij}'G$.

This version is compatible with the definition of 2-representation from Remark 4.3.

Definition 4.11. A *strongly equivariant natural transformation* between strongly equivariant functors $G, H : \mathcal{R} \to \mathcal{S}$ is a natural transformation $\pi : G \Rightarrow H$ such that $\pi E_i \circ \zeta_{G,i} = \zeta_{H,i} \circ E_i\pi : E_iG \Rightarrow HE_i$. Let $\mathfrak{Rep}(\mathfrak{U}(\mathfrak{g}))$ be the strict 2-category of (small) 2-representations, strongly equivariant functors and strongly equivariant natural transformations. We denote the morphism categories in this 2-category by $\mathcal{H}om_{\mathfrak{U}(\mathfrak{g})}(\mathcal{R}, \mathcal{S})$.

Remark 4.12. In the setup of Remark 4.9, a strongly equivariant natural transformation is the same as a modification between morphisms of 2-functors in the sense of [R1, Definition 2.4].

4.2. **Generalized cyclotomic quotients.** In this subsection, we define some important examples of 2-representations. We need a couple more basic notions to prepare for this.

Fix a weight $\kappa \in P$. Then there is a 2-representation[11] $\mathcal{R}(\kappa)$ of $\mathfrak{U}(\mathfrak{g})$ defined as follows: $\mathcal{R}(\kappa)_\lambda := \mathcal{H}om_{\mathfrak{U}(\mathfrak{g})}(\kappa, \lambda)$; E_i (resp. F_i) is the functor defined by horizontally composing 1-morphisms on the left by E_i (resp. F_i) and 2-morphisms on the left by  λ (resp.  λ); x_i, τ_{ij}, η_i and ε_i are the natural transformations defined by horizontally composing on the left by , ,  and , respectively.

An *invariant ideal*[12] $\mathcal{I}$ of a 2-representation $\mathcal{R}$ is a family of subspaces $\mathcal{I}(b,c) \leq \operatorname{Hom}_{\mathcal{R}}(b,c)$ for each $b, c \in \operatorname{ob} \mathcal{R}$ such that

- $f \in \operatorname{Hom}_{\mathcal{R}}(a,b)$ and $g \in \mathcal{I}(b,c) \Rightarrow g \circ f \in \mathcal{I}(a,c)$;
- $h \in \operatorname{Hom}_{\mathcal{R}}(c,d)$ and $g \in \mathcal{I}(b,c) \Rightarrow h \circ g \in \mathcal{I}(b,d)$;
- $g \in \mathcal{I}(b,c)$ and $i \in I \Rightarrow E_i(g) \in \mathcal{I}(E_i(b), E_i(c))$ and $F_i(g) \in \mathcal{I}(F_i(b), F_i(c))$.

[11] In [R2, §4.3.3], Rouquier denotes this by $\mathcal{M}(\kappa)$, but that seems confusing notation since it is a categorification of the left ideal $\dot{U}(\mathfrak{g})1_\kappa$ rather than the Verma module $M(\kappa)$.

[12] In Rouquier's language, an invariant ideal is the data of a full sub-2-representation.

Given an invariant ideal $\mathcal{I}$, the quotient category $\mathcal{R}/\mathcal{I}$ is the category with the same objects as $\mathcal{R}$ and morphisms $\operatorname{Hom}_{\mathcal{R}/\mathcal{I}}(b,c) := \operatorname{Hom}_{\mathcal{R}}(b,c)/\mathcal{I}(b,c)$. It has a naturally induced structure of 2-representation in its own right.

Now we are ready for the main construction.

Construction 4.13. Fix weights $\kappa \in P^+$, $\kappa' \in -P^+$. Let $k_i := \langle h_i, \kappa\rangle$, $k_i' := -\langle h_i, \kappa'\rangle$, and take a family of indeterminates $\{z_{i,r}, z'_{j,s} \mid i,j \in I, 1 \le r \le k_i, 1 \le s \le k_j'\}$. Let $\Bbbk[z] := \Bbbk[z_{i,r}, z'_{j,s} \mid i,j \in I, 1 \le r \le k_i, 1 \le k_j']$ be the corresponding polynomial algebra. Adopting the convention that $z_{i,0} = z'_{i,0} := 1$, we define new variables $\delta_{i,s}, \delta'_{i,s} \in \Bbbk[z]$ for $s \ge 0$ from the generating functions

$$\sum_{s\ge 0} \delta_{i,s} t^s := c_{\kappa+\kappa';i} \frac{\sum_{r=0}^{k_i'} z'_{i,r} t^r}{\sum_{r=0}^{k_i} z_{i,r} t^r}, \qquad \sum_{s\ge 0} \delta'_{i,s} t^s := c_{\kappa+\kappa';i}^{-1} \frac{\sum_{r=0}^{k_i} z_{i,r} t^r}{\sum_{r=0}^{k_i'} z'_{i,r} t^r}. \tag{4.4}$$

Here, we are working in $\Bbbk[z][[t]]$ where t is a formal parameter. Let $\mathcal{R} := \mathcal{R}(\kappa+\kappa') \otimes_{\Bbbk} \Bbbk[z]$ be the $\Bbbk[z]$-linear 2-representation obtained from $\mathcal{R}(\kappa+\kappa')$ by extending scalars in the obvious way. Let $\mathcal{I}$ be the $\Bbbk[z]$-linear invariant ideal of $\mathcal{R}$ generated by the morphisms

$$\sum_{r=0}^{k_i} \left(k_i - r \,\bullet\!\!\downarrow_i \;\kappa+\kappa' \right) z_{i,r}, \tag{4.5}$$

$$\kappa+\kappa' \;\circlearrowleft_i^{\bullet\, s+*} \; - 1_{\kappa+\kappa'} \delta_{i,s} \tag{4.6}$$

for all $i \in I$ and $s = 1, \dots, k_i'$, using the shorthand (3.13). Taking the quotient, we obtain the $\Bbbk[z]$-linear 2-representation

$$\mathcal{L}(\kappa'|\kappa) := \mathcal{R}/\mathcal{I}. \tag{4.7}$$

Finally, we viewing $\Bbbk$ as a $\Bbbk[z]$-algebra so that each $z_{i,r}, z'_{i,r}$ act as zero, we have the *minimal specialization*

$$\mathcal{L}_{\min}(\kappa'|\kappa) := \mathcal{L}(\kappa'|\kappa) \otimes_{\Bbbk[z]} \Bbbk. \tag{4.8}$$

Lemma 4.14. *The ideal $\mathcal{I}$ in Construction 4.13 is generated also by the morphisms*

$$\sum_{r=0}^{k_i'} \left(k_i' - r \,\bullet\!\!\uparrow_i \;\kappa+\kappa' \right) z'_{i,r}, \tag{4.9}$$

$$\kappa+\kappa' \;\circlearrowright_i^{\bullet\, s+*} \; - 1_{\kappa+\kappa'} \delta'_{i,s}, \tag{4.10}$$

for all $i \in I$ and $s = 1, \dots, k_i$. Moreover, it contains (4.6) and (4.10) for all $s \ge 0$.

Proof. We first show that the images of the elements (4.6) are zero in $\mathcal{L}(\kappa'|\kappa)$ for all $s \ge 0$. For the induction step, we may assume that $s > k_i'$. Note by the definition from (K7) that

$$\kappa+\kappa' \;\circlearrowleft_i^{\bullet\, s+*} \; = \; \kappa+\kappa' \;\circlearrowleft_i^{\bullet\, s+k_i-k_i'-1},$$

so for $s > k_i'$ there are $\ge k_i$ dots on the right hand side here. Therefore using the relation (4.5), we get that

$$\sum_{r=0}^{k_i} \kappa+\kappa' \;\circlearrowleft_i^{\bullet\, s-r+*} \; z_{i,r} = 0,$$

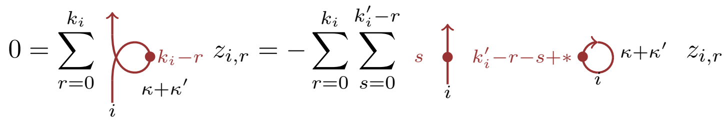
working in the quotient $\mathcal{L}(\kappa'|\kappa)$. Applying the inductive hypothesis, we deduce that

$$\kappa+\kappa' \;\circlearrowleft_i^{\bullet\, s+*} \; + \sum_{r=1}^{k_i} 1_{\kappa+\kappa'} \delta_{i,s-r} z_{i,r} = 0.$$

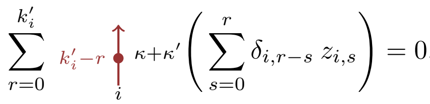
It remains to observe that $\sum_{r=0}^{k_i} \delta_{i,s-r} z_{i,r} = 0$ already in $\Bbbk[z]$ when $s > k_i'$. This follows because $\left(\sum_{s\geq 0} \delta_{i,s} t^s\right)\left(\sum_{r=0}^{k_i} z_{i,r} t^r\right)$ is a polynomial of degree k_i' by the definition (4.4).

Now let $\mathrm{e}(t) := \sum_{r\geq 0} \mathrm{e}_r t^r$ and $\mathrm{h}(t) := \sum_{r\geq 0} \mathrm{h}_r t^r$, so that $\mathrm{e}(-t)\mathrm{h}(t) = 1$ by (3.10); remember also the definitions (3.11)–(3.12). Also set $\delta_i(t) := \sum_{s\geq 0} \delta_{i,s} t^s$ and $\delta_i'(t) := \sum_{s\geq 0} \delta'_{i,s} t^s$, so that $\delta_i'(t)\delta_i(t) = 1$ by (4.4). In the previous paragraph, we have shown that the image of $\beta_{\kappa+\kappa';i}(\mathrm{h}(t))$ is $c^{-1}_{\kappa+\kappa';i} 1_{\kappa+\kappa'} \delta_i(t)$. Hence, the image of $\beta_{\kappa+\kappa';i}(\mathrm{e}(-t))$ is $c_{\kappa+\kappa';i} 1_{\kappa+\kappa'} \delta_i'(t)$. This shows that (4.10) belongs to $\mathcal{I}$ for all $s \geq 0$.

In this paragraph we show that (4.9) belongs to $\mathcal{I}$ too. Working in $\mathcal{L}(\kappa'|\kappa)$ once again, we have by (4.5) and (K6) that



Changing the summation using also (4.6), we have shown that



It remains to apply (4.4) to simplify this to (4.9).

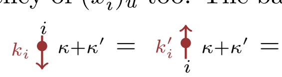
Conversely, one checks by similar arguments that the $\Bbbk[z]$-linear invariant ideal $\mathcal{I}'$ generated by the elements (4.9)–(4.10) contains (4.5)–(4.6). □

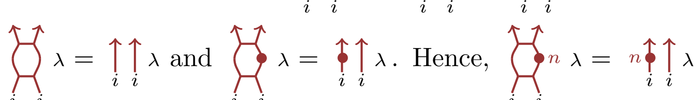
Lemma 4.15. *There is a unique $\mathfrak{g}$-module homomorphism*

$$L(\kappa'|\kappa) \to \mathbb{C} \otimes_{\mathbb{Z}} K_0(\dot{\mathcal{L}}_{\min}(\kappa'|\kappa)), \qquad \bar{1}_{\kappa+\kappa'} \mapsto [1_{\kappa+\kappa'}].$$

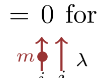
Proof. We need to show that the homomorphism $\dot{U}(\mathfrak{g})1_{\kappa+\kappa'} \to \mathbb{C} \otimes_{\mathbb{Z}} K_0(\dot{\mathcal{L}}_{\min}(\kappa'|\kappa))$ sending $1_{\kappa+\kappa'} \mapsto [1_{\kappa+\kappa'}]$ factors through the quotient $L(\kappa'|\kappa)$ from (3.6). This follows because $E_i^{(1+k_i')} 1_{\kappa+\kappa'} = 0 = F_i^{(1+k_i)} 1_{\kappa+\kappa'}$ in $\dot{\mathcal{L}}_{\min}(\kappa'|\kappa)$. The first equality here follows from Lemma 3.6 and the defining relation (4.9); the second one follows similarly using (4.5) and a rotated version of Lemma 3.6. □

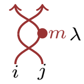
Lemma 4.16. *The 2-representation $\mathcal{L}_{\min}(\kappa'|\kappa)$ is* nilpotent *in the sense that $(x_i)_u \in \mathrm{End}_{\mathcal{L}_{\min}(\kappa'|\kappa)}(E_i u)$ and $(x_i')_u \in \mathrm{End}_{\mathcal{L}_{\min}(\kappa'|\kappa)}(F_i u)$ are nilpotent for all $i \in I$ and $u \in \mathrm{ob}\, \mathcal{L}_{\min}(\kappa'|\kappa)$.*

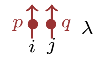
Proof. We show by induction on r that $(x_i)_u$ is nilpotent for any object u that is a monomial obtained by applying r of the generating E's and F's to $1_{\kappa+\kappa'}$; similar arguments give the nilpotency of $(x_i')_u$ too. The base case $r = 0$ follows from (4.5) and (4.9), since they show that  in $\mathcal{L}_{\min}(\kappa'|\kappa)$. For the induction step, we consider $(x_i)_u$ for a monomial u of length $(r+1)$. There are three cases:

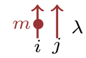
Case one: $u = E_i 1_\lambda v$. Introduce the *intertwiner* . Using the relations, one checks that 

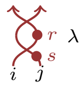
As $(x_i^n)_v \in \mathrm{End}_{\mathcal{L}_{\min}(\kappa'|\kappa)}(E_i v)$ is zero for some n by induction, we deduce that $(x_i^n)_u \in \mathrm{End}_{\mathcal{L}_{\min}(\kappa'|\kappa)}(E_i u)$ is zero too.

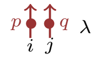
Case two: $u = E_j 1_\lambda v$ for $j \neq i$. By induction, we have that $(x_i^n)_v = (x_j^n)_v = 0$ for some n. Let $m := (d_{ij}+1)n$. Then we can use the defining relations to rewrite 

as a linear combination of 

and terms of the form 

with $p \geq nd_{ij}$ and $q \geq 1$. Repeating the calculation $(n-1)$ more times, we get that 

is a linear combination of 

and terms of the form 

with $r \geq n, s \geq 0, p \geq d_{ij}$ and $q \geq n$. It remains to act on v. All the terms vanish and we have shown that $(x_i^m)_u = 0$.

Case three: $u = F_j 1_\lambda v$. Use the first alternating crossing relation from (K8) to move dots to the right in a similar way. ☐

Corollary 4.17. $\mathcal{L}_{\min}(\kappa'|\kappa)$ *is a finite-dimensional category.*

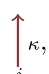
Proof. We take $u, v \in \operatorname{ob} \mathcal{L}_{\min}(\kappa'|\kappa)_\lambda$, i.e. 1-morphisms $\kappa + \kappa' \to \lambda$ in $\mathfrak{U}(\mathfrak{g})$ for some $\lambda \in P$, and consider the explicit spanning set for $\operatorname{Hom}_{\mathfrak{U}(\mathfrak{g})}(u, v)$ constructed as in [KL3, Proposition 3.11]; we arrange this so that all the dotted bubbles appear at the right hand side of the diagrams. We need to show that the image of this set spans a finite-dimensional vector space when we pass to the quotient $\mathcal{L}_{\min}(\kappa'|\kappa)$. This is clear because any diagram with a bubble vanishes in the quotient by the relations (4.6) and (4.10), and any diagram with too many dots on any given strand vanishes by the nilpotency established in Lemma 4.16. ☐

Remark 4.18. The category $\mathcal{L}(\kappa'|\kappa)$ is not finite-dimensional. However, each of its morphism spaces is *finitely generated* as a $\mathbb{k}[z]$-module. This follows by a similar but more delicate inductive argument compared to the proof of Corollary 4.17.

Taking $\kappa' = 0$ in Construction 4.13, we obtain the $\mathbb{k}[z]$-linear 2-representation

$$\mathcal{L}(\kappa) := \mathcal{L}(0|\kappa). \tag{4.11}$$

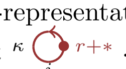
Note in this situation that $\mathbb{k}[z] = \mathbb{k}[z_{i,r} \mid i \in I, r = 1, \dots, k_i]$ and $z_{i,r} = c_{\kappa;i}\delta'_{i,r}$ for $r = 1, \dots, k_i$. Since each $k'_i = 0$, Lemma 4.14 shows that $\mathcal{L}(\kappa)$ is the quotient of $\mathcal{R} := \mathcal{R}(\kappa) \otimes_{\mathbb{k}} \mathbb{k}[z]$ by the $\mathbb{k}[z]$-linear invariant ideal generated by the morphisms



(4.12)

$1_\kappa z_{i,r} - c_{\kappa;i}$ 

, (4.13)

for $i \in I$ and $r = 1, \dots, k_i$. In view of (4.13), there is no need to extend scalars to $\mathbb{k}[z]$ after all: we could equivalently define $\mathcal{L}(\kappa)$ to be the quotient of $\mathcal{R}(\kappa)$ by the invariant ideal generated by the morphisms (4.12) for all $i \in I$, viewing it as a $\mathbb{k}[z]$-linear 2-representation so that each $z_{i,r}$ acts by horizontally composing on the right with $c_{\kappa;i}$ 

.

The discussion in the previous paragraph shows that $\mathcal{L}(\kappa)$ is Rouquier's *universal categorification* of $L(\kappa)$ from [R2, §4.3.3]. These 2-representations play a fundamental role in his general structure theory for upper integrable 2-representations. To start with, they formally satisfy the following universal property: for any 2-representation $\mathcal{V}$, let

$$\mathcal{V}_\kappa^{\mathrm{hw}} := \{V \in \operatorname{ob} \mathcal{V}_\kappa \mid E_i V = 0 \text{ for all } i \in I\}$$

which is a full subcategory of $\mathcal{V}_\kappa$; then there is an equivalence of categories

$$\operatorname{ev}_{1_\kappa} : \mathcal{H}om_{\mathfrak{U}(\mathfrak{g})}(\mathcal{L}(\kappa), \mathcal{V}) \to \mathcal{V}_\kappa^{\mathrm{hw}}, \qquad G \mapsto G1_\kappa, \quad \pi \mapsto \pi_{1_\kappa}. \tag{4.14}$$

This is a key ingredient in [R2, Theorem 4.22], which shows that any upper integrable, additive, idempotent-complete 2-representation has a finite filtration whose sections are specializations of the $\dot{\mathcal{L}}(\kappa)$'s ordered in some way refining the dominance order (most dominant at the bottom). This result is a categorical analog of the filtration of a based module constructed by Lusztig in [Lu2, Ch. 27].

Using a fundamental theorem of Kang and Kashiwara [KK, Theorem 5.2], Rouquier has also given an equivalent realization of $\dot{\mathcal{L}}(\kappa)$ as follows. Recalling Definition 3.4, introduce the (negative) *cyclotomic quiver Hecke category* $\mathcal{H}'(\kappa)$, namely, the quotient of the $\Bbbk[z]$-linear monoidal category $\mathcal{H}' \otimes_{\Bbbk} \Bbbk[z]$ by the $\Bbbk[z]$-linear left tensor ideal generated by the morphisms

$$\sum_{r=0}^{k_i} \left({\scriptstyle k_i - r} \underset{\downarrow}{\overset{i}{\bullet}} \right) z_{i,r} \qquad (i \in I). \tag{4.15}$$

The endomorphism algebras

$$\widehat{H}'_n(\kappa) := \bigoplus_{\boldsymbol{i},\boldsymbol{j} \in I^{\otimes n}} \operatorname{Hom}_{\mathcal{H}'(\kappa)}(\boldsymbol{i}, \boldsymbol{j}) \tag{4.16}$$

are Rouquier's *deformed cyclotomic quiver Hecke algebras*; in particular, $\widehat{H}'_0(\kappa)$, the endomorphism algebra of the unit object $\varnothing$ of $\mathcal{H}'(\kappa)$, is $\Bbbk[z]$. The theorem of Kang and Kashiwara shows that the additive Karoubi envelope $\dot{\mathcal{H}}'(\kappa)$ can be endowed with the structure of a $\Bbbk[z]$-linear 2-representation with E_i and F_i arising from certain restriction and induction functors[13]. Applying the universal property from (4.14), we get a canonical strongly equivariant functor $G : \mathcal{L}(\kappa) \to \dot{\mathcal{H}}'(\kappa)$ such that $\mathrm{ev}_{1_\kappa}(G) = \varnothing$.

Theorem 4.19 ([R2, Theorem 4.25])**.** *The functor G is $\Bbbk[z]$-linear and it induces a strongly equivariant equivalence $G : \dot{\mathcal{L}}(\kappa) \to \dot{\mathcal{H}}'(\kappa)$. Hence,* $\operatorname{End}_{\mathcal{L}(\kappa)}(1_\kappa) \cong \Bbbk[z]$.

It follows immediately that the *minimal categorification*

$$\mathcal{L}_{\min}(\kappa) := \mathcal{L}_{\min}(0|\kappa) \tag{4.17}$$

is Morita equivalent to the quotient $\mathcal{H}'_{\min}(\kappa)$ of $\mathcal{H}'$ by the left tensor ideal generated by ${\scriptstyle k_i} \underset{\downarrow}{\overset{i}{\bullet}}$ $(i \in I)$. The endomorphism algebras

$$H'_n(\kappa) := \bigoplus_{\boldsymbol{i},\boldsymbol{j} \in I^{\otimes n}} \operatorname{Hom}_{\mathcal{H}'_{\min}(\kappa)}(\boldsymbol{i}, \boldsymbol{j}) \tag{4.18}$$

are the *cyclotomic quiver Hecke algebras* introduced by Khovanov and Lauda in [KL1]. They are finite-dimensional algebras, so the blocks of $\mathcal{L}_{\min}(\kappa)$ are finite-dimensional algebras too; in particular, $\mathcal{L}_{\min}(\kappa)$ is Artinian. Moreover, $\operatorname{End}_{\mathcal{L}_{\min}(\kappa)}(1_\kappa) \cong H'_0(\kappa) \cong \Bbbk$. This shows that 1_κ is non-zero in $\mathcal{L}_{\min}(\kappa)$, which is the crucial point needed in order to deduce the following theorem, which was established already in [KK, Theorem 6.2]. Note also that Webster has given a different proof of all of these results in [W2, §3].

Theorem 4.20 ([KK, W2])**.** *For any $\kappa \in P^+$, the homomorphism*

$$L(\kappa) \to \mathbb{C} \otimes_{\mathbb{Z}} K_0(\dot{\mathcal{L}}_{\min}(\kappa))$$

from Lemma 4.15 is an isomorphism.

Proof. We just saw that the homomorphism is non-zero. Hence, it is injective. It is surjective too by a standard argument recalled in Corollary 4.34 below. □

Corollary 4.21. *If $\mathfrak{g}$ is of finite type, then the homomorphism $\gamma : \dot{U}(\mathfrak{g})_{\mathbb{Z}} \to K_0(\dot{\mathfrak{U}}(\mathfrak{g}))$ from (3.14) is an isomorphism; similarly, so is γ_q from (3.16).*

---

[13] The equivalent formulation of the definition of 2-representation from Remark 4.3 is convenient here since induction is obviously left adjoint to restriction; Lemma 4.2 implies that it is right adjoint too, but this is far from clear from the outset.

Proof. It remains to show that γ is injective. Take $u \in \dot{U}(\mathfrak{g})_{\mathbb{Z}}$ with $\gamma(u) = 0$. By its definition, the isomorphism $L(\kappa) \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{Z}} K_0(\dot{\mathcal{L}}_{\min}(\kappa))$ from Theorem 4.20 intertwines the action of u on the left hand space with the action of $\gamma(u)$ on the right hand space. Hence, u annihilates $L(\kappa)$. This is true for each $\kappa \in P^+$, so we get that $u = 0$ by the second statement of Lemma 3.2. □

Remark 4.22. The results just explained obviously have lowest weight analogs too. Taking $\kappa = 0$ in Construction 4.13, the universal and minimal categorifications of $L'(\kappa')$ are $\mathcal{L}'(\kappa') := \mathcal{L}(\kappa'|0)$ and $\mathcal{L}'_{\min}(\kappa') := \mathcal{L}_{\min}(\kappa'|0)$, respectively. They are Morita equivalent to analogously defined cyclotomic quotients $\mathcal{H}(\kappa')$ and $\mathcal{H}_{\min}(\kappa')$ of the quiver Hecke category $\mathcal{H}$. Moreover $\operatorname{End}_{\mathcal{L}'(\kappa')}(1_{\kappa'}) \cong \Bbbk[z]$, the finite-dimensional category $\mathcal{L}'_{\min}(\kappa')$ is Artinian, and $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\dot{\mathcal{L}}'_{\min}(\kappa')) \cong L(\kappa')$.

Remark 4.23. The general definition of the 2-representations $\mathcal{L}(\kappa'|\kappa)$ for $\kappa, \kappa' \neq 0$ given in Construction 4.13 is new, but their minimal specializations $\mathcal{L}_{\min}(\kappa'|\kappa)$ appear already[14] in [W1, Proposition 5.6]. In [W1, Theorem 5.13], Webster asserts that the homomorphism in Lemma 4.15 is an isomorphism for all κ, κ', so that $\mathcal{L}_{\min}(\kappa'|\kappa)$ is a categorification of $L(\kappa'|\kappa) \cong L'(\kappa') \otimes L(\kappa)$ in general. Like in the proof of Corollary 4.21 (using the first statement of Lemma 3.2 instead of the second), such a result implies that $\dot{U}(\mathfrak{g})_{\mathbb{Z}} \cong K_0(\dot{\mathfrak{U}}(\mathfrak{g}))$ for arbitrary $\mathfrak{g}$. However, Webster's proof of this is intertwined with his new approach in [W3] to verifying the Nondegeneracy Condition; cf. Remark 3.13. Actually, [W3] is based on some even more general deformations, which should be closely related to our $\mathcal{L}(\kappa'|\kappa)$ when there is just one lowest and one highest weight tensor factor.

Remark 4.24. The finite-dimensional category $\mathcal{L}_{\min}(\kappa'|\kappa)$ is not Artinian in general (outside of finite type). We conjecture that it is always Noetherian.

4.3. **Categorical actions.** Henceforth, $\mathcal{C}$ denotes a locally Schurian category in the sense of Definition 2.9. We fix a set of representatives $\{L(b) \,|\, b \in \mathbf{B}\}$ for the isomorphism classes of irreducible objects, and let $P(b)$ be a projective cover of $L(b)$. Let $K_0(\mathrm{p}\mathcal{C})$ denote the split Grothendieck group of the additive category $\mathrm{p}\mathcal{C}$ (= finitely generated projectives in $\mathcal{C}$). The classes $\{[P(b)] \,|\, b \in \mathbf{B}\}$ give a distinguished basis for $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{p}\mathcal{C})$.

Definition 4.25. A *categorical action* of $\mathfrak{g}$ on $\mathcal{C}$ is the following additional data:

(A1) a partition $\mathbf{B} = \bigsqcup_{\lambda \in P} \mathbf{B}_\lambda$ inducing a decomposition $\mathcal{C} = \prod_{\lambda \in P} \mathcal{C}_\lambda$ as in (L10) from §2.2;

(A2) sweet endofunctors E_i of $\mathcal{C}$ for each $i \in I$ such that $E_i|_{\mathcal{C}_\lambda} : \mathcal{C}_\lambda \to \mathcal{C}_{\lambda+\alpha_i}$ (recall Definition 2.10);

(A3) a strict monoidal functor $\Phi : \mathcal{H} \to \mathcal{E}nd(\mathcal{C})$ with $\Phi(i) = E_i$ for each i, where $\mathcal{H}$ is the quiver Hecke category from Definition 3.4.

Let $x_i := \Phi($ $)$ and $\tau_{ij} := \Phi($  $)$. We also fix the choice of a right adjoint F_i to E_i for each $i \in I$, and set $e_i := [E_i]$ and $f_i := [F_i]$, which are endomorphisms of $K_0(\mathrm{p}\mathcal{C})$. Then we impose the following axiom:

(A4) for each $i, j \in I$ and $\lambda \in P$, the commutator $[e_i, f_j]$ acts on $K_0(\mathrm{p}\mathcal{C}_\lambda)$ as multiplication by the scalar $\delta_{i,j}\langle h_i, \lambda\rangle$.

We say that the categorical action is *nilpotent*[15] if $(x_i)_V$ is a nilpotent element of the finite-dimensional algebra $\operatorname{End}_{\mathcal{C}}(E_i V)$ for all $i \in I$ and $V \in \operatorname{ob} \mathrm{fg}\mathcal{C}$.

The following gives a general recipe producing a categorical action on the locally Schurian category Mod-$\mathcal{A}$ for any finite-dimensional 2-representation $\mathcal{A}$. In particular,

[14]Note that Webster's diagrams are the mirror images of ours in a vertical axis.

[15]Just as good would be to assume there is $c \in \Bbbk$ that $(x_i - c1)_V$ is nilpotent for all i and V.

applying it to the finite-dimensional 2-representation $\mathcal{L}_{\min}(\kappa'|\kappa)$ from Construction 4.13, this shows that Mod-$\mathcal{L}_{\min}(\kappa'|\kappa)$ admits a categorical action; this example is also nilpotent thanks to Lemma 4.16.

Construction 4.26. Let $\mathcal{A} = \coprod_{\lambda \in P} \mathcal{A}_\lambda$ be a finite-dimensional 2-representation. Let $A = \bigoplus_{\lambda\in P} A_\lambda$ be the associated locally unital algebra as in Remark 2.1; the distinguished idempotents in A are indexed by $X = \bigsqcup_{\lambda\in P} X_\lambda$ where $X_\lambda := \mathrm{ob}\,\mathcal{A}_\lambda$. We define a categorical action on $\mathcal{C} :=$ Mod-A as follows.

- Fix representatives $\{L(b) \mid b \in \mathbf{B}_\lambda\}$ for the isomorphism classes of irreducible A_λ-modules, and let $\mathbf{B} := \bigsqcup_{\lambda\in P} \mathbf{B}_\lambda$. This partition induces the decomposition $\mathcal{C} = \prod_{\lambda\in P} \mathcal{C}_\lambda$ required for (A1); of course, we have that $\mathcal{C}_\lambda =$ Mod-A_λ.
- The functor E_i defines locally unital homomorphisms $e_i : A_\lambda \to A_{\lambda+\alpha_i}$ for each $\lambda \in P$; moreover, $e_i(1_u) = 1_{E_i u}$ for each $u \in X_\lambda$. Let $e_i^* A_{\lambda+\alpha_i}$ be the $(A_\lambda, A_{\lambda+\alpha_i})$-bimodule obtained from $A_{\lambda+\alpha_i}$ by restricting the natural left action through this homomorphism. Tensoring on the right with this bimodule defines a functor $\hat{E}_i :$ Mod-$A_\lambda \to$ Mod-$A_{\lambda+\alpha_i}$ for each $\lambda \in P$. This is the data required for (A2). The endofunctor $\hat{E}_i$ is sweet because the functor F_i extends similarly to a functor $\hat{F}_i :$ Mod-$A_{\lambda+\alpha_i} \to$ Mod-A_λ which is biadjoint to $\hat{E}_i$ thanks to Lemma 4.2.
- The natural transformation $x_i : E_i \to E_i$ on each $u \in X_\lambda$ produces a family of elements $x_{i;u} \in 1_{e_i u} A_{\lambda+\alpha_i} 1_{e_i u}$ $(u \in X_\lambda)$ such that $e_i(f)x_{i;u} = x_{i;v}e_i(f)$ for all $f \in 1_v A_\lambda 1_u$. Hence, there is a bimodule homomorphism $e_i^* A_{\lambda+\alpha_i} \to e_i^* A_{\lambda+\alpha_i}$ defined on $1_{e_i u} A_{\lambda+\alpha_i}$ by left multiplication by $x_{i;u}$, from which we get $\hat{x}_i : \hat{E}_i \to \hat{E}_i$. Similarly, $\tau_{ij} : E_i E_j \to E_j E_i$ translates to $\tau_{ij;u} \in 1_{e_j e_i u} A_{\lambda+\alpha_i+\alpha_j} 1_{e_i e_j u}$ such that $(e_j e_i(f))\tau_{ij;u} = \tau_{ji;v}(e_i e_j(f))$ for all $f \in 1_v A_\lambda 1_u$ and $u, v \in X_\lambda$. Left multiplication by these elements defines a bimodule homomorphism $(e_i e_j)^* A_{\lambda+\alpha_i+\alpha_j} \to (e_j e_i)^* A_{\lambda+\alpha_i+\alpha_j}$. The composite functor $\hat{E}_i\hat{E}_j$ is defined by tensoring with $(e_i e_j)^* A_{\lambda+\alpha_i+\alpha_j} \cong e_j^* A_{\lambda+\alpha_j} \otimes_{A_{\lambda+\alpha_j}} e_i^* A_{\lambda+\alpha_i+\alpha_j}$, so this is what we need to get $\hat{\tau}_{ij} : \hat{E}_i\hat{E}_j \to \hat{E}_j\hat{E}_i$. Thus we have the data for (A3).
- Finally the axiom (A4) follows from Lemma 4.4 and (2.1).

In fact, assuming nilpotency, the notion of a finite-dimensional 2-representation is equivalent to the notion of categorical action on a locally Schurian category. This depends on the following theorem, which is a variation on [R1, Theorem 5.27]. It is a remarkable example of relations on the Grothendieck group (specifically, axiom (A4)) implying relations between 2-morphisms (specifically, axiom (M5)); Rouquier refers to this as "control by K_0." It is very useful since (M5) can be very difficult to check directly.

Theorem 4.27. *Suppose that we are given a nilpotent categorical action on some locally Schurian category $\mathcal{C}$. Then the natural transformations σ_{ij} $(i \neq j)$ and $\rho_{i,\lambda}$ from Definition 4.1 are invertible. Hence, $\mathcal{C}$ is a locally Schurian 2-representation.*

Proof. Note to start with that $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{p}\mathcal{C})$ is an integrable $\mathfrak{g}$-module, by the same argument as in the proof of Lemma 4.5.

In this paragraph, we explain how to see that $\rho_{i,\lambda}$ is invertible (for fixed i and λ). By Lemma 2.12, it suffices to show that $\rho_{i,\lambda}$ is invertible on $L(b)$ for $b \in \mathbf{B}_\lambda$. In the Artinian case, this follows immediately from [R1, Theorem 5.22]. The proof in general reduces to the Artinian case as follows. By (A4) and integrability, the set

$$\mathbf{B}'' := \{a \in \mathbf{B} \mid P(a) \text{ is a summand of some sequence of } E_i \text{ and } F_i \text{ applied to } P(b)\}$$

is finite. Of course, $b \in \mathbf{B}''$. Let $\mathbf{B}' := \mathbf{B} \setminus \mathbf{B}''$ and $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{C}'$ be the corresponding Serre quotient as in §2.5. The isomorphism classes of irreducible objects of $\mathcal{C}/\mathcal{C}'$ are represented by $\{\pi L(a) \mid a \in \mathbf{B}''\}$; in particular, $\mathcal{C}/\mathcal{C}'$ is finite. Moreover, using the left adjoint functor $\pi^!$, we may identify its complexified K_0 with the subspace of $\mathbb{C}\otimes_{\mathbb{Z}} K_0(\mathrm{p}\mathcal{C})$ spanned by $\{[P(a)] \mid a \in \mathbf{B}''\}$. The definition of $\mathbf{B}''$ ensures that this subspace is stable under the action of e_i and f_i. Consequently, the functors E_i and F_i preserve the subcategory $\mathcal{C}'$; for example, for E_i, this follows because $\mathrm{Hom}_{\mathcal{C}}(P(c), E_i L(d)) \cong \mathrm{Hom}_{\mathcal{C}}(F_i P(c), L(d)) = 0$ for all $c \in \mathbf{B}'', d \in \mathbf{B}'$. Hence, E_i and F_i induce endofunctors of $\mathcal{C}/\mathcal{C}'$, showing that $\mathcal{C}/\mathcal{C}'$ admits a categorical $\mathfrak{sl}_2$-action. By [R1, Theorem 5.22], $\rho_{i,\lambda}$ is invertible on $\pi L(b) \in \mathrm{ob}\,\mathcal{C}/\mathcal{C}'$. It just remains to invoke Lemma 2.13 to deduce that $\rho_{i,\lambda}$ is invertible on $L(b) \in \mathrm{ob}\,\mathcal{C}$ too. In order to check the hypotheses of Lemma 2.13 here, we should note that if X is any functor obtained by taking a finite composition and/or direct sum of the categorification functors E_i and F_i, then $XL(b)$ is finitely generated and cogenerated by Theorem 2.11. Moreover all constituents of $\mathrm{soc}(XL(b))$ and $\mathrm{hd}(XL(b))$ are $\cong L(c)$ for $c \in \mathbf{B}''$; for example, to see this for the head, the right adjoint Y to X preserves $\mathcal{C}'$ as before, so for $c \in \mathbf{B}'$ we get that $\mathrm{Hom}_{\mathcal{C}}(XL(b), L(c)) \cong \mathrm{Hom}_{\mathcal{C}}(L(b), YL(c)) = 0$.

It remains to show that $\sigma_{ij} (i \neq j)$ is invertible. For this, we appeal to the proof of [R1, Theorem 5.25] to get that σ_{ij} is invertible on $E_i^r K$ for all $r \geq 0$ and any irreducible K with $F_i K = 0$; this is a very general result which requires no finiteness assumptions about $\mathcal{C}$ other than integrability. To deduce the invertibility of σ_{ij} on arbitrary objects, we claim that every irreducible object $L \in \mathrm{ob}\,\mathcal{C}$ can be realized as a quotient (resp. subobject) of some such object $E_i^r K$. Using the claim and naturality, the invertibility of σ_{ij} on $E_i^r K$ implies the surjectivity (resp. injectivity) of σ_{ij} on L too. Then we apply Lemma 2.12 to get that σ_{ij} is invertible on arbitary objects. Finally, we must prove the claim. By integrability, there is a unique $r \geq 0$ such that $F_i^r L \neq 0$ but $F_i^{r+1} L = 0$. Then we let K be any irreducible constituent of the socle (resp. head) of $F_i^r L$, so that $K \hookrightarrow F_i^r L$ (resp. $F_i^r L \twoheadrightarrow K$); this relies on the fact that $F_i^r L$ is finitely cogenerated (resp. generated) according to Theorem 2.11. Applying adjointness, we get that there is a non-zero homomorphism $E_i^r K \twoheadrightarrow L$ (resp. $L \hookrightarrow E_i^r K$), as required. □

Remark 4.28. The proof of Theorem 4.27 relies ultimately on [CR, Theorem 5.27], in which nilpotency is certainly assumed. However, we expect that this result can be generalized, so that the nilpotency assumption in the statement of Theorem 4.27 (and in the remainder of this subsection) should actually be unnecessary.

Construction 4.29. Let $\mathcal{C}$ be a locally Schurian category admitting a nilpotent categorical action. Fix a set X_0 indexing finitely generated projective objects $(P_x)_{x\in X_0}$ such that each P_x belongs to some weight subcategory of $\mathcal{C}$. For $n \geq 1$, define X_n and $(P_x)_{x\in X_n}$ recursively by letting X_n consist of the symbols $e_i x, f_i x$ for all $x \in X_{n-1}$ and $i \in I$, and setting $P_{e_i x} := E_i P_x, P_{f_i x} := F_i P_x$. Let $X := \bigsqcup_{n\geq 0} X_n$. We assume further that $(P_x)_{x\in X}$ is a projective generating family for $\mathcal{C}$. Having made this choice, we can define a finite-dimensional 2-representation $\mathcal{A}$ as follows.

- Let $\mathcal{A}$ be the finite-dimensional category with object set X, $\mathrm{Hom}_{\mathcal{A}}(x, y) := \mathrm{Hom}_{\mathcal{C}}(P_x, P_y)$, and composition induced by composition in $\mathcal{C}$. Note that $\mathcal{A} = \coprod_{\lambda\in P} \mathcal{A}_\lambda$ where $\mathcal{A}_\lambda$ is the full subcategory generated by $X_\lambda := \{x \in X \mid P_x \in \mathrm{ob}\,\mathcal{C}_\lambda\}$.
- Let $\bar{E}_i, \bar{F}_i : \mathcal{A} \to \mathcal{A}$ be the endofunctors defined on objects by $\bar{E}_i x := e_i x, \bar{F}_i x := f_i x$. On morphisms, $\bar{E}_i$ and $\bar{F}_i$ are defined by applying the given categorification functors E_i and F_i in $\mathcal{C}$.
- Let $\bar{x}_i : \bar{E}_i \Rightarrow \bar{E}_i, \bar{\tau}_{ij} : \bar{E}_i\bar{E}_j \Rightarrow \bar{E}_j\bar{E}_i, \bar{\eta}_i : 1_{\mathcal{A}} \Rightarrow \bar{F}_i\bar{E}_i$ and $\bar{\varepsilon}_i : \bar{E}_i\bar{F}_i \Rightarrow 1_{\mathcal{A}}$ be the natural transformations obtained by restricting x_i, τ_{ij}, η_i and ε_i in the obvious way.

This produces all of the data required by Definition 4.1(M1)–(M4). The final axiom (M5) is satisfied thanks to Theorem 4.27.

We leave it as an exercise for the reader to show that Constructions 4.26 and 4.29 are quasi-inverses in the appropriate sense. In particular, if one starts with $\mathcal{C}$ equipped with a nilpotent categorical action, applies Construction 4.29 to obtain $\mathcal{A}$, then applies Construction 4.26 to define a categorical action on Mod-$\mathcal{A}$, then $\mathcal{C}$ and Mod-$\mathcal{A}$ are strongly equivariantly equivalent.

4.4. **Associated crystals.** Finally we recall a definition of Kashiwara; e.g. see [K2].

Definition 4.30. A *normal crystal* is a set $\mathbf{B}$ with a decomposition $\mathbf{B} = \bigsqcup_{\lambda \in P} \mathbf{B}_\lambda$, plus *crystal operators* $\tilde{e}_i, \tilde{f}_i : \mathbf{B} \to \mathbf{B} \sqcup \{0\}$ for each $i \in I$ satisfying the following axioms:

(C1) for every $\lambda \in P$, the crystal operator $\tilde{e}_i$ restricts to a map $\mathbf{B}_\lambda \to \mathbf{B}_{\lambda+\alpha_i} \sqcup \{0\}$;
(C2) for $b \in \mathbf{B}$, we have that $\tilde{e}_i(b) = b' \neq 0$ if and only if $\tilde{f}_i(b') = b \neq 0$;
(C3) for every $b \in \mathbf{B}$, there is an $r \in \mathbb{N}$ such that $\tilde{e}_i^r(b) = \tilde{f}_i^r(b) = 0$.

For each i, define functions $\varepsilon_i, \varphi_i : \mathbf{B} \to \mathbb{N}$ by

$$\varepsilon_i(b) = \max\{r \in \mathbb{N} \mid \tilde{e}_i^r(b) \neq 0\}, \qquad \varphi_i(b) = \max\{r \in \mathbb{N} \mid \tilde{f}_i^r(b) \neq 0\}.$$

Then we also require that

(C4) $\varphi_i(b) - \varepsilon_i(b) = \langle h_i, \lambda \rangle$ for each $b \in \mathbf{B}_\lambda$ and $i \in I$.

The following theorem is essentially due to Chuang and Rouquier [CR], but it has its origins in [GV, G]. It shows that every nilpotent categorical action has a canonical *associated crystal*.

Theorem 4.31. *Suppose that we are given a nilpotent categorical action on a locally Schurian category $\mathcal{C}$ with irreducible objects $\{L(b) \mid b \in \mathbf{B}\}$ as in Definition 4.25. There is a unique structure of normal crystal on $\mathbf{B} = \bigsqcup_{\lambda \in P} \mathbf{B}_\lambda$ such that*

(1) $\tilde{e}_i b \neq 0 \Leftrightarrow E_i L(b) \neq 0 \Rightarrow \operatorname{soc}(E_i L(b)) \cong \operatorname{hd}(E_i L(b)) \cong L(\tilde{e}_i b)$*;*
(2) $\tilde{f}_i b \neq 0 \Leftrightarrow F_i L(b) \neq 0 \Rightarrow \operatorname{soc}(F_i L(b)) \cong \operatorname{hd}(F_i L(b)) \cong L(\tilde{f}_i b)$*.*

Moreover, the following hold for any $b \in \mathbf{B}$, $i \in I$ and $0 \leq n \leq m := \varepsilon_i(b)$:

(3) $E_i^{(n)} L(b)$ *has irreducible socle and head both isomorphic to* $L(\tilde{e}_i^n b)$*;*
(4) $[E_i^{(n)} L(b) : L(\tilde{e}_i^n b)] = \binom{m}{n}$*, and all irreducible subquotients of* $E_i^{(n)} L(b)$ *other than* $L(\tilde{e}_i^n b)$ *are of the form* $L(c)$ *for* c *with* $\varepsilon_i(c) < m - n$*;*
(5) the natural action of $Z(NH_n)$ *on* $E_i^{(n)} L(b)$ *induces an isomorphism*

$$\operatorname{Sym}_n / \langle \mathrm{h}_{m-n+1}, \mathrm{h}_{m-n+2}, \ldots, \mathrm{h}_m \rangle \xrightarrow{\sim} \operatorname{End}_{\mathcal{C}}(E_i^{(n)} L(b)),$$

where h_r *denotes the* r*th elementary symmetric polynomial in* $X_1, \ldots, X_n$*.*

Analogous statements to (3)–(5) with E and ε replaced by F and φ also hold.

Proof. Note this only involves some fixed i, so we are reduced immediately to the case that $\mathfrak{g}$ is of rank one. In the Artinian case, it suffices to work in the Schurian category fg$\mathcal{C}$, and then everything that we need is a consequence of [CR, Proposition 5.20] and the construction of [R1, Theorem 5.22]. To give a little more detail, *loc. cit.* shows that $E_i L(b)$ is either zero, or it has irreducible head and socle which are isomorphic, and similarly for $F_i L(b)$. Hence we can use (1)–(2) to define $\tilde{e}_i, \tilde{f}_i : \mathbf{B} \to \mathbf{B} \sqcup \{0\}$. The axiom (C1) is clear, while (C2) follows by an adjunction argument. Temporarily redefining $\varepsilon_i(b)$ from $\varepsilon_i(b) := \max\{r \in \mathbb{N} \mid E_i^{(r)} L(b) \neq 0\}$, we get that properties (3) and (4) hold by [CR, Proposition 3.20] again. Using them, an easy induction on $\varepsilon_i(b)$ shows that (C3) holds and that $\varepsilon_i(b)$ agrees with the function from Definition 4.30. Similarly,

$$\phi_i(b) := \max\{n \in \mathbb{N} \mid F_i^{(n)} L(b) \neq 0\} = \max\{n \in \mathbb{N} \mid \tilde{f}_i^n b \neq 0\}.$$

Now we can establish the final axiom (C4). Suppose that $b \in \mathbf{B}_\lambda$ and set $c := \tilde{e}_i^m b$. We have that $\varepsilon_i(c) = 0$, hence $E_i L(c) = 0$. Thus, in the Grothendieck group, the class of $L(c)$

is an $\mathfrak{sl}_2$-highest weight vector. By $\mathfrak{sl}_2$-theory, we deduce that $\phi_i(c) = \langle h_i, \lambda + m\alpha_i \rangle$. Hence, $\phi_i(b) - \varepsilon_i(b) = \phi_i(c) - 2m = \langle h_i, \lambda \rangle$ as required. Finally, for (5), the proof of [CR, Proposition 3.20] shows that the natural action of NH_n on $E_i^n L(b)$ induces an isomorphism $NH_n / \langle X_1^m \rangle \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}}(E_i^n L(b))$. By an elementary relation chase (omitted), the two-sided ideal of NH_n generated by X_1^m is also generated by $\mathrm{h}_{m-n+1}, \dots, \mathrm{h}_m$. Recalling that NH_n is a matrix algebra over its center Sym_n, we deduce on truncating with the idempotent $\pi_{i,n}$ that $\mathrm{End}_{\mathcal{C}}(E_i^{(n)} L(b)) \cong \mathrm{Sym}_n / \langle \mathrm{h}_{m-n+1}, \dots, \mathrm{h}_m \rangle$.

To extend the result to the general locally Schurian case, we make a reduction similar to the one made in the second paragraph of the proof of Theorem 4.27. Fix $b \in \mathbf{B}$ and define $\mathbf{B}'', \mathbf{B}', \mathcal{C}'$ and the quotient functor $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{C}'$ exactly as there. As we explained already, E_i and F_i induce endofunctors of $\mathcal{C}/\mathcal{C}'$, hence giving a categorical $\mathfrak{sl}_2$-action on $\mathcal{C}/\mathcal{C}'$, which is finite. Moreover all $E_i^{(n)} L(b)$ and $F_i^{(n)} L(b)$ are finitely generated and cogenerated, and their socles and heads have constituents only of the form $L(c)$ for $c \in \mathbf{B}''$. So we can use Lemma 2.13 to transport the results from $\mathcal{C}/\mathcal{C}'$ established in the previous paragraph to $\mathcal{C}$, and the general result follows. Perhaps the only statement that requires additional comment is the second assertion of (4). For this, the other properties imply that $E_i^{(m-n)}$ annihilates all composition factors $L(c)$ of $E_i^{(n)} L(b)$ different from $L(\tilde{e}_i^n b)$, hence we get that $\varepsilon_i(c) < m - n$. □

Remark 4.32. By a classical result, the algebra $\mathrm{Sym}_n / \langle \mathrm{h}_{m-n+1}, \mathrm{h}_{m-n+2}, \dots, \mathrm{h}_m \rangle$ in Theorem 4.31(5) is isomorphic to the cohomology of the Grassmannian $\mathrm{Gr}_{n,m}$. This is explained in [CR, §3.3.2].

Remark 4.33. It is interesting to consider what happens in Theorem 4.31 if the nilpotency assumption is dropped. In general, one still obtains a crystal structure on $\mathbf{B}$, but for a certain unfurling $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ in the sense of [W3]. This is a consequence of the isomorphism theorem established in [W3, §3].

The following well-known corollary is a first application; again this argument appeared already in a special case in [G].

Corollary 4.34. *For $\kappa \in P^+$, the Grothedieck group $K_0(\dot{\mathcal{L}}_{\min}(\kappa))$ is the $\mathbb{Z}$-span of the vectors $f_{i_n}^{(r_n)} \cdots f_{i_1}^{(r_1)} [1_\kappa]$ for $i_1, \dots, i_n \in I$ and $r_1, \dots, r_n \geq 1$.*

Proof. We apply Theorem 4.31 with $\mathcal{C} := \text{Mod-}\mathcal{L}_{\min}(\kappa)$; cf. Construction 4.26. Let M be the span of all the vectors $f_{i_n}^{(r_n)} \cdots f_{i_1}^{(r_1)} [1_\kappa]$. Proceeding by downward induction on weight, consider some $\lambda < \kappa$. We need to show for each $b \in \mathbf{B}_\lambda$ that $[P(b)] \in M$. Pick i so that $m := \varepsilon_i(b) \neq 0$. Using Theorem 4.31 and an argument with adjunctions, one shows that $F_i^{(m)} P(\tilde{e}_i^m(b)) \cong P(b) \oplus (*)$ where $(*)$ is a direct sum of projectives of the form $P(b')$ for $b' \in \mathbf{B}_\lambda$ with $\varepsilon_i(b') > \varepsilon_i(b)$. By downward induction on $\varepsilon_i(b)$, we may assume that all of these $[P(b')]$ lie in M. Hence, we get that $[P(b)] \in M$ too. □

The proof of Corollary 4.34 implicitly uses the defining property of a *dual perfect basis* from [KKKS, Definition 4.2]. In fact, Theorem 4.31 easily implies for any nilpotent locally Schurian categorical action that $\{[P(b)] \mid b \in \mathbf{B}\}$ is a dual perfect basis for $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{p}\mathcal{C})$. In particular, we recover the following well-known result on appealing also to [KKKS, Theorem 6.1]; this was originally proved in [LV] by a different method.

Corollary 4.35. *For $\kappa \in P^+$, the crystal associated to the minimal categorification $\mathcal{L}_{\min}(\kappa)$ is a copy of Kashiwara's highest weight crystal $\mathbf{B}(\kappa)$.*

Remark 4.36. If $\mathcal{C}$ is Artinian, one can also show that $\{[L(b)] \mid b \in \mathbf{B}\}$ is a *perfect basis* for the complexified Grothendieck group of the Schurian category $\mathrm{fg}\mathcal{C}$ in the (older) sense of [BeK, Definition 5.49]. This was observed originally by Shan [S, Proposition 6.2]. Combined with [BeK, Theorem 5.37] (in place of [KKKS, Theorem 6.1]), Corollary 4.35

may also be deduced from this; cf. [K, Remark 10.3.6]. However, perfect bases are not a natural thing to consider in the locally Schurian setup: in general it is not even clear that E_i and F_i send irreducible objects to objects of finite length.

Remark 4.37. It is natural to expect that the crystal associated to $\mathcal{L}_{\min}(\kappa'|\kappa)$ is Kashiwara's tensor product $\mathbf{B}'(\kappa') \otimes \mathbf{B}(\kappa)$ of the lowest weight crystal $\mathbf{B}'(\kappa')$ with the highest weight crystal $\mathbf{B}(\kappa)$. We hope to prove this in subsequent work using some of Losev's techniques from [L].

References

[AF] F. Anderson and K. Fuller, *Rings and Categories of Modules*, Springer, 1992.

[A] S. Ariki, On the decomposition numbers of the Hecke algebra of $G(m,1,n)$, *J. Math. Kyoto Univ.* **36** (1996), 789–808.

[BW] H. Bao and W. Wang, Canonical bases in tensor products revisited, to appear in *Amer. J. Math.*.

[BHLW] A. Beliakova, K. Habiro, A. Lauda and B. Webster, Cyclicity for categorified quantum groups, *J. Algebra* **452** (2016), 118–132.

[BeK] A. Berenstein and D. Kazhdan, Geometric and unipotent crystals II: from unipotent bicrystals to crystal bases, *Contemp. Math.* **433** (2007), 13–88.

[BFK] J. Bernstein, I. Frenkel and M. Khovanov, A categorification of the Temperley-Lieb algebra and Schur quotients of $U(\mathfrak{sl}_2)$ via projective and Zuckerman functors, *Selecta Math.* **5** (1999), 199–241.

[BZ] J. Bernstein and A. Zelevinsky, Induced representations of reductive p-adic groups, I, *Ann. Sci. Ecole Norm. Sup.* **10** (1977), 441–472.

[B1] J. Brundan, Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{gl}(m|n)$, *J. Amer. Math. Soc.* **16** (2003), 185-231.

[B2] J. Brundan, Quiver Hecke algebras and categorification, in: "Advances in Representation Theory of Algebras," eds: D. Benson et al, *EMS Congress Reports*, 2013, pp.103–133.

[B3] J. Brundan, On the definition of Kac-Moody 2-category, *Math. Ann.* **364** (2016), 353–372.

[BCNR] J. Brundan, J. Comes, D. Nash and A. Reynolds, A basis theorem for the oriented Brauer category and its cyclotomic quotients, to appear in *Quantum Top.*.

[BE1] J. Brundan and A. Ellis, Monoidal supercategories; `arXiv:1603.05928`.

[BE2] J. Brundan and A. Ellis, Super Kac-Moody 2-categories; in preparation.

[BK] J. Brundan and A. Kleshchev, Translation functors for general linear and symmetric groups, *Proc. London Math. Soc.* **80** (2000), 75–106.

[BLW] J. Brundan, I. Losev and B. Webster, Tensor product categorifications and the super Kazhdan-Lusztig conjecture, to appear in *Int. Math. Res. Notices.*

[CL] S. Cautis and A. Lauda, Implicit structure in 2-representations of quantum groups, *Selecta Math.* **21** (2015), 201–244.

[CR] J. Chuang and R. Rouquier, Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification, *Ann. of Math.* **167** (2008), 245–298.

[DVV] O. Dudas, M. Varagnolo and E. Vasserot, Categorical actions on unipotent representations of finite classical groups; `arXiv:1603.00742`.

[EHS] I. Entova-Aizenbud, V. Hinich and V. Serganova, Deligne categories and the limit of categories $\mathrm{Rep}(GL(m|n))$; `arXiv:1511.07699`.

[EGNO] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik, *Tensor Categories*, Amer. Math. Soc., 2015.

[FK] B. Ford and A. Kleshchev, A proof of the Mullineux conjecture, *Math. Z.* **226** (1997), 267–308.

[F] P. Freyd, Abelian categories, *Reprints in Theory and Applications of Categories* **3** (2003), 23–164.

[G] I. Grojnowski, Affine $\mathfrak{sl}_p$ controls the representation theory of the symmetric group and related Hecke algebras; `arXiv:math.RT/9907129`.

[GV] I. Grojnowski and M. Vazirani, Strong multiplicity one theorem for affine Hecke algebras of type A, *Transform. Groups* **6** (2001), 143–155.

[Kac] V. Kac, *Infinite Dimensional Lie Algebras*, Cambridge University Press, 1995.

[KKKS] B. Kahn, S.-J. Kang, M. Kashiwara and U. Suh, Dual perfect bases and dual perfect graphs, *Mosc. Math. J.* **15** (2015), 319–335.

[Kam] J. Kamnitzer, Categorification of Lie algebras (d'aprés Rouquier, Khovanov-Lauda, ...), *Astérisque* **361** (2014), 397–419.

[KK] S.-J. Kang and M. Kashiwara, Categorification of highest weight modules via Khovanov-Lauda-Rouquier algebras, *Invent. Math.* **190** (2012), 699–742.

[K1] M. Kashiwara, Crystalizing the q-analogue of universal enveloping algebras, *Comm. Math. Phys.* **133** (1990), 249–260.

[K2] M. Kashiwara, On crystal bases, in: "Representations of Groups (Banff 1994)," *CMS Conf. Proc.* **16** (1995), 155–197.

[K] M. Khovanov, A functor-valued invariant of tangles, *Alg. Geom. Topology* **2** (2002), 665–741.

[KL1] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups I, *Represent. Theory* **13** (2009), 309–347.

[KL2] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups II, *Trans. Amer. Math. Soc.* **363** (2011), 2685–2700.

[KL3] M. Khovanov and A. Lauda, A categorification of quantum $\mathfrak{sl}(n)$, *Quantum Top.* **1** (2010), 1–92.

[K] A. Kleshchev, *Linear and Projective Representations of Symmetric Groups*, Cambridge University Press, Cambridge, 2005.

[LLT] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, *Commun. Math. Phys.* **181** (1996), 205-263.

[L1] A. Lauda, A categorification of quantum $\mathfrak{sl}(2)$, *Advances Math.* **225** (2010), 3327–3424.

[L2] A. Lauda, An introduction to diagrammatic algebra and categorified quantum $\mathfrak{sl}(2)$, *Bull. Inst. Math. Acad. Sin.* **7** (2012), 165–270.

[LV] A. Lauda and M. Vazirani, Crystals from categorified quantum groups, *Advances Math.* **228** (2011), 803–861.

[L] I. Losev, Highest weight $\mathfrak{sl}_2$-categorifications I: crystals, *Math. Z.* **274** (2013), 1231–1247.

[Lu1] G. Lusztig, Canonical bases arising from quantized enveloping algebras, *J. Amer. Math. Soc.* **3** (1990), 447–498.

[Lu2] G. Lusztig, *Introduction to Quantum Groups*, Birkhäuser, 1993.

[M] B. Mitchell, Rings with several objects, *Advances Math.* **8** (1972), 1–161.

[R1] R. Rouquier, 2-Kac-Moody algebras; `arXiv:0812.5023`.

[R2] R. Rouquier, Quiver Hecke algebras and 2-Lie algebras, *Algebra Colloq.* **19** (2012), 359–410.

[S] P. Shan, Crystals of Fock spaces and cyclotomic rational double affine Hecke algebras, *Ann. Sci. Éc. Norm. Sup.* **44** (2010), 147–182.

[W1] B. Webster, Canonical bases and higher representation theory, *Compositio Math.* **151** (2015), 121–166.

[W2] B. Webster, Knot invariants and higher representation theory, to appear in *Mem. Amer. Math. Soc.*.

[W3] B. Webster, Unfurling Khovanov-Lauda-Rouquier algebras; `arXiv:1603.06311`.

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail address: `brundan@uoregon.edu, davidson@uoregon.edu`