AN UPDATE ON HEISENBERG AND KAC-MOODY CATEGORIFICATION

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Dedicated to Dan Nakano on the occasion of his sixtieth birthday

Abstract. Heisenberg categories act on many Abelian categories appearing in type A representation theory. There is also a general procedure to construct from a Heisenberg action another action of a Kac-Moody 2-category for some associated Cartan matrix. One of the adjunctions on the Kac-Moody side is matched up in an easy way with adjunctions on the Heisenberg side, but the second adjunction is much harder to describe. In this paper, we derive explicit formulae for this difficult adjunction, leading to some further simplifications to the existing theory.

1. Introduction

In 2004, Chuang and Rouquier introduced the notion of an $\mathfrak{sl}_2$-*categorification*; see [CR08]. A few years later, in [Rou08], the definition was extended from $\mathfrak{sl}_2$ to other symmetrizable Kac-Moody algebras. The resulting axiomatic framework of *Kac-Moody categorifications* was motivated in part by classical examples of Abelian categories arising in the representation theory of symmetric and general linear groups (and related Hecke algebras and quantum groups). In these examples, the combinatorics is controlled by an underlying Kac-Moody algebra, either $\mathfrak{sl}_\infty$ when the (quantum) characteristic of the ground field is 0, or $\widehat{\mathfrak{sl}}_p$ in characteristic $p > 0$. This point of view was taken already by Grojnowski [Gro99], who made an important step in the development of the general theory by explaining the coincidence between the crystal graph for the basic representation of $\widehat{\mathfrak{sl}}_p$ determined in [MM90] and the modular branching graph for representations of symmetric groups from [Kle96]. Kac-Moody categorifications have been used to construct many interesting Morita and derived equivalences, including the ones used in [CR08] to prove Broué's Abelian Defect Conjecture for symmetric groups.

In [LLT96], Lascoux, Leclerc and Thibon formulated a precise categorification conjecture relating the representation theory of cyclotomic Hecke algebras of type A at a complex pth root of unity to canonical bases of integrable highest weight representations of $\widehat{\mathfrak{sl}}_p$. This was quickly proved by Ariki [Ari96]. Subsequently, an analogous conjecture relating graded representation theory of the *quiver Hecke algebras* from [Rou08, KL09] to canonical bases of quantum groups of other symmetric Cartan types was formulated by Khovanov and Lauda, and proved by Varagnolo and Vasserot in [VV11]; see also [Rou12]. In fact, the Lascoux-Leclerc-Thibon conjecture can be seen as a special case of the Khovanov-Lauda conjecture due to the isomorphism between cyclotomic Hecke algebras and cyclotomic quiver Hecke algebras in type A established in [BK09].

The general definition of a Kac-Moody categorification rests on the 2-categories introduced in [Rou08, KL10] known as *Kac-Moody 2-categories* or *2-quantum groups*. Roughly speaking, these are obtained from quiver Hecke algebras by adjoining certain right duals subject to some non-trivial relations, the *inversion relations*, which are categorifications of the commutation relations between e_i and f_j in the underlying Kac-Moody algebra. However, it is surprisingly difficult to fit the

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classical examples from type A representation theory into the general formalism. One approach was developed by Rouquier in [Rou08, Th. 5.27], a result he called "control by K_0". This is often useful but still requires a lot of preparation.

In [BSW20a], we developed another approach to the construction of Kac-Moody categorifications based on the notion of a *Heisenberg categorification*. This rests on another family of strict monoidal categories, the (degenerate or quantum) *Heisenberg categories* $\mathbf{Heis}_\kappa$, which are obtained from (degenerate or quantum) affine Hecke algebras of type A by adjoining a certain right dual subject to another inversion relation depending on the value of $\kappa \in \mathbb{Z}$, called the *central charge*. This is a fruitful approach because it is much easier to check that the Abelian categories arising in many sorts of type A representation theory are Heisenberg categorifications. Then the main construction in [BSW20a] gives a systematic procedure to pass from a Heisenberg categorification to a Kac-Moody categorification. After that, all of the powerful structural results about Kac-Moody categorifications can be applied.

The bridge from Heisenberg to Kac-Moody has as its starting point the isomorphism between cyclotomic quotients of affine Hecke algebras and quiver Hecke algebras mentioned already. This is then upgraded to obtain from the given action of the Heisenberg category a much less obvious action of a Kac-Moody 2-category. On both sides, there are two families of adjunctions, one defined by rightward cups and caps, and the other defined by leftward cups and caps. The rightward cups and caps are matched up in an obvious way, but then one needs to check that the Kac-Moody inversion relations hold on the Heisenberg side. In [BSW20a], this was done via an indirect argument involving the Chinese Remainder Theorem. The argument does not produce any explicit formula relating the leftward cups and caps on the two sides.

In this article, we begin in Sections 2 to 4 with a survey of the definitions of degenerate Heisenberg and Kac-Moody categorifications and the main "Heisenberg to Kac-Moody" construction from [BSW20a]. In Sections 5 and 6, we work through two classic examples illustrating the general theory, discussing the representation theory of the symmetric groups, and representations of the general linear group scheme and its Frobenius kernels in positive characteristic. With all of this background in place, in Section 7, we formulate and prove the main new result of the article. This gives explicit formulae relating the second adjunction on the Kac-Moody side—the one defined by leftward cups and caps—to the second adjunction on the Heisenberg side. Finally, in Section 8, we treat the quantum case.

The interesting works of Bao and Wang [BW18] and Ehrig and Stroppel [ES18] point to the existence of an equally rich theory of categorifications arising from classical groups of types B/C/D. The analogues of the degenerate and quantum Heisenberg categories are the affine Brauer category introduced in [RS19] and its quantum analogue, the Kauffman category. The analogues of the Kac-Moody 2-categories are the quasi-split 2-iquantum groups introduced in [BWW25]; we expect the ones of diagonal Cartan type A and of quasi-split Satake type AIII (possibly affine) to play a role. However, the expected bridge from Brauer categorifications to iquantum group categorifications is still to be built. For 2-iquantum groups at this level of generality, there is no satisfactory analogue of the inversion relation, so a different strategy will be needed compared to [BSW20a]. This was one of the main motivations prompting us to think again in this article about the identification of leftward cups and caps in Heisenberg and Kac-Moody categorification.

The two sorts of categorification discussed so far can be thought of as the GL and the OSp branches of a wider story. There are two more classical families of supergroups over algebraically closed fields: the periplectic supergroups P and the isomeric supergroups Q. In the isomeric case, the first two authors are currently working on the development of a theory of *isomeric Heisenberg and Kac-Moody categorification*, building in particular on [KKT16]. Significant progress in the periplectic case has been made recently by Nehme and Stroppel in [NS25].

2. Kac-Moody categorifications

Suppose that we are given:

- A symmetrizable generalized Cartan matrix $A = (a_{i,j})_{i,j\in I}$. This means that $a_{i,i} = 2$ for all $i \in I$, $a_{i,j} \in -\mathbb{N}$ for $i \neq j$ in I, $a_{i,j} = 0 \Leftrightarrow a_{j,i} = 0$, and there are given positive integers d_i $(i \in I)$ such that $d_i a_{i,j} = d_j a_{j,i}$ for all $i, j \in I$. We do not insist that the set I is finite, but the number of non-zero entries in each row and column of A should be finite.
- A free Abelian group X, the *weight lattice*, containing elements α_i $(i \in I)$, the *simple roots*, and ϖ_i $(i \in I)$, the *fundamental weights*, and homomorphisms $h_i : X \to \mathbb{Z}$ $(i \in I)$ such that $h_i(\alpha_j) = a_{i,j}$ and $h_i(\varpi_j) = \delta_{i,j}$ for all $i, j \in I$. We note that the fundamental weights are necessarily linearly independent, but the simple roots may be dependent.

To this data, we associate a locally unital $\mathbb{C}$-algebra $\dot{\mathrm{U}}$, which is the modified form of the universal enveloping algebra of a Kac-Moody algebra of this Cartan type. It has mutually orthogonal distinguished idempotents 1_λ $(\lambda \in X)$, and it is generated by elements

$$e_i 1_\lambda = 1_{\lambda+\alpha_i} e_i, \qquad\qquad 1_\lambda f_i = f_i 1_{\lambda+\alpha_i} \tag{2.1}$$

for $i \in I, \lambda \in X$ subject to idempotented versions of the usual Serre relations. Let $\dot{\mathrm{U}}_{\mathbb{Z}}$ be the $\mathbb{Z}$-form for $\dot{\mathrm{U}}$ generated by the divided powers $e_i^{(n)} 1_\lambda := e_i^n 1_\lambda / n!$ and $1_\lambda f_i^{(n)} := 1_\lambda f_i^n / n!$.

The main object of interest in this paper is a certain 2-category $\mathfrak{U}$ which is a categorification of the q-analogue of $\dot{\mathrm{U}}_{\mathbb{Z}}$; see Remark 2.1. It was introduced originally by Rouquier [Rou08] and, independently, Khovanov and Lauda [KL10]; our exposition follows [BWW25, Sec. 2]. Fix an algebraically closed ground field $\Bbbk$ and parameters $Q_{i,j}(x, y) \in \Bbbk[x, y]$ for all $i, j \in I$ such that $Q_{i,i}(x, y) = 0$, $Q_{i,j}(x, y) = Q_{j,i}(y, x)$, and the following hold for $i \neq j$:

- The polynomial $Q_{i,j}(x, y)$ is homogeneous of degree $-2d_i a_{i,j}$ if x and y are assigned the degrees $2d_i$ and $2d_j$, respectively.
- The coefficient $Q_{i,j}(1, 0)$ is non-zero.

We also let

$$t_{i,j} := \begin{cases} Q_{i,j}(1,0) & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases} \tag{2.2}$$

The *Kac-Moody 2-category* $\mathfrak{U}$ with these parameters is the $\Bbbk$-linear 2-category with object set X, generating 1-morphisms

$$E_i \mathbb{1}_\lambda = \mathbb{1}_{\lambda+\alpha_i} E_i \colon \lambda \to \lambda + \alpha_i, \qquad\qquad \mathbb{1}_\lambda F_i = F_i \mathbb{1}_{\lambda+\alpha_i} \colon \lambda + \alpha_i \to \lambda \tag{2.3}$$

for $\lambda \in X$ and $i \in I$, with identity 2-endomorphisms represented by the oriented strings $\lambda{+}\alpha_i \underset{i}{\uparrow} \lambda$ and $\lambda \underset{i}{\downarrow} \lambda{+}\alpha_i$, and the four families of generating 2-morphisms displayed in Table 1. The generating 2-morphisms are subject to relations still to be explained. We write these down using the following conventions:

- We will usually only label one of the 2-cells in a string diagram with a weight—the others can then be worked out implicitly. When we omit *all* labels in 2-cells, it should be understood that we are discussing something that holds for all possible choices of labels.
- We will label strings just at one end. If we omit a label or orientation, it means that we are discussing something that holds for all possibilities.
- When a dot is labelled by a multiplicity, we mean to take its power under vertical composition. For a polynomial $f(x) = \sum_{r=0}^{n} c_r x^r$, we use the shorthand

$$\boxed{f(x)}\!-\!\circ\!\big| \;=\; \big|\!\circ\!-\!\boxed{f(x)} \;:=\; \sum_{r=0}^{n} c_r \;\big|\!\circ^{\,r}$$

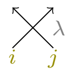
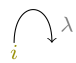
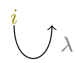
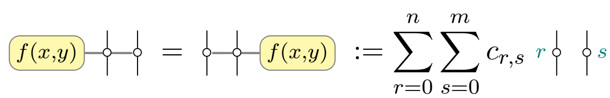
Generator	Degree
 $: E_i\mathbb{1}_\lambda \Rightarrow E_i\mathbb{1}_\lambda$	$2d_i$
 $: E_iE_j\mathbb{1}_\lambda \Rightarrow E_jE_i\mathbb{1}_\lambda$	$-d_ia_{i,j}$
 $: E_iF_i\mathbb{1}_\lambda \Rightarrow \mathbb{1}_\lambda$	$d_i(1-h_i(\lambda))$
 $: \mathbb{1}_\lambda \Rightarrow F_iE_i\mathbb{1}_\lambda$	$d_i(1+h_i(\lambda))$

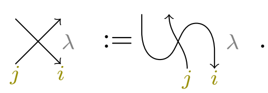
TABLE 1. Generating 2-morphisms of the Kac-Moody 2-category

to "pin" $f(x)$ to a dot on a string. Similarly, for $f(x,y) = \sum_{r=0}^n \sum_{s=0}^m c_{r,s}x^ry^s$, we use

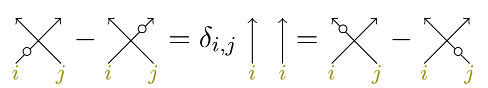


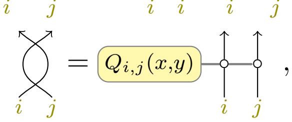
This notation extends to polynomials $f(x,y,z)$ in three variables pinned to three dots, with the convention that the variables in alphabetic order correspond to the dots ordered by the lexicographic order on their Cartesian coordinates. Thus, x corresponds to the leftmost dot, and the lowest one if there are several such dots in the same vertical line.

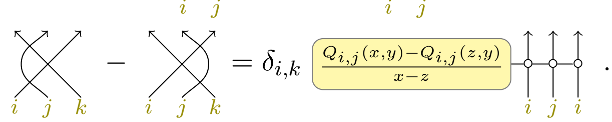
- We use the following shorthand to denote the composite 2-morphism obtained by "rotating" the upward crossing:

 (2.4)

Now for the relations. There are three families. First, we have the *quiver Hecke algebra* relations:

 (2.5)

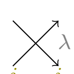
 (2.6)

 (2.7)

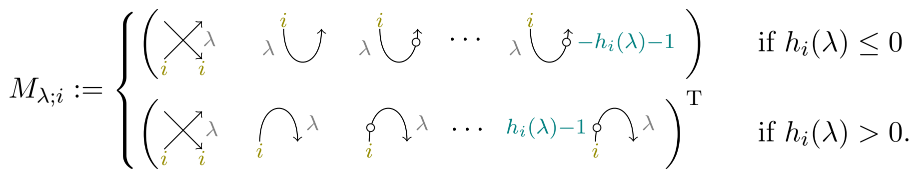
Next, the *right adjunction relations* implying that $\mathbb{1}_\lambda F_i$ is right dual to $E_i\mathbb{1}_\lambda$:

 (2.8)

Finally, we have the *inversion relations* which assert that

 $: E_jF_i\mathbb{1}_\lambda \Rightarrow F_iE_j\mathbb{1}_\lambda$ (2.9)

is an isomorphism for all $\lambda \in X$ and $i \neq j$, as are the following matrices for all λ and i:

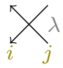
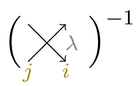
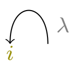
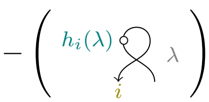
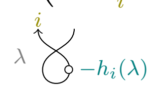
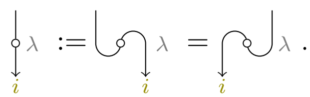
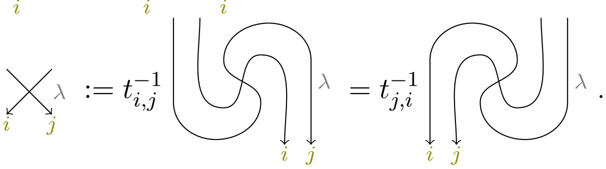


(2.10)

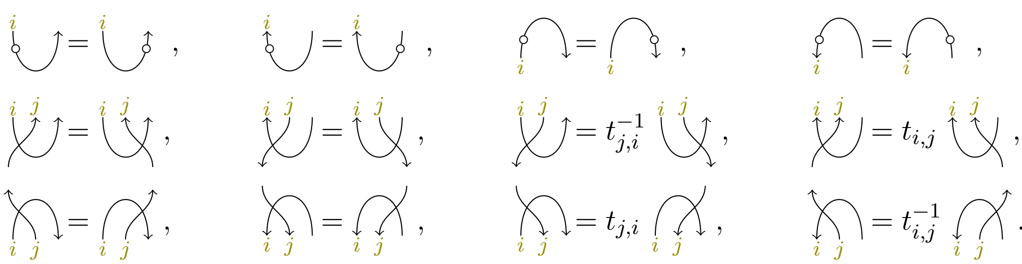
There are also leftward cups and caps such that

(2.11)

They define a *second adjunction* making $\mathbb{1}_\lambda F_i$ into a left dual of $E_i \mathbb{1}_\lambda$. They are defined as follows:

- Let  be  if $i \neq j$, or the first entry of the matrix $-M_{\lambda;i}^{-1}$ if $i = j$.
- Let  be the last entry of $M_{\lambda;i}^{-1}$ if $h_i(\lambda) < 0$ or  if $h_i(\lambda) \geq 0$.
- Let  be the last entry of $M_{\lambda;i}^{-1}$ if $h_i(\lambda) > 0$ or  if $h_i(\lambda) \leq 0$.
- Let 
- Let 

The equalities in the definitions of the downward dot and crossing just given are by no means obvious; they are justified in [Bru16]. Equivalently, we have that



(2.12)

(2.13)

(2.14)

Like in the general theory of symmetric functions, when working with $\mathfrak{U}$, it is often helpful to work in terms of generating functions, which in general will be formal Laurent series in auxiliary variables $u^{-1}, v^{-1}, \dots$. For such a generating function $f(u)$, we use the notation $[f(u)]_{u:n}$ to denote its u^n-coefficient, $[f(u)]_{u;\geq 0}$ for its polynomial part, and so on. For a polynomial $f(x)$, we have that

$$\left[\frac{f(u)}{u-x}\right]_{u:-1} = f(x), \qquad\qquad \left[\frac{f(u)}{u-x}\right]_{u:<0} = \frac{f(x)}{u-x}. \tag{2.15}$$

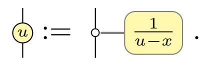
A trivial but perhaps countertuitive consequence is that

$$\prod_{i=1}^{n} \left[\frac{f_i(u)}{u-x}\right]_{u:-1} = \left[\frac{\prod_{i=1}^{n} f_i(u)}{u-x}\right]_{u:-1} \tag{2.16}$$

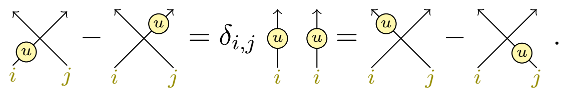
for $f_1(x), \dots, f_n(x) \in \Bbbk[x]$. We view the series

$$\frac{1}{u-x} = \sum_{r \geq 0} x^r u^{-r-1}. \tag{2.17}$$

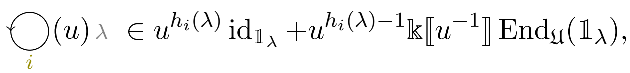
as a generating function for multiple dots on a string, introducing the additional shorthand

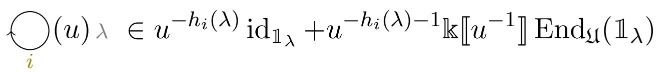

(2.18)

The following is a consequence of (2.5):


(2.19)

There are also the extremely useful *bubble generating functions*

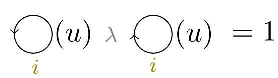

(2.20)


(2.21)

which are the unique formal Laurent series with leading coefficients as in (2.20) and (2.21) such that


(2.22)

and the *infinite Grassmannian relation*

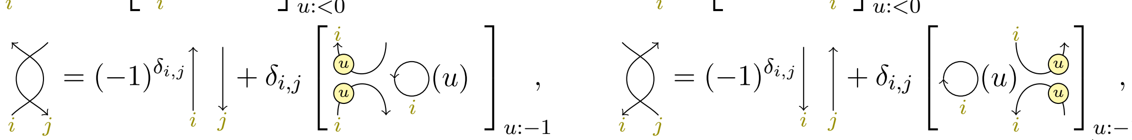

(2.23)

holds. If $h_i(\lambda) \geq 0$ then the clockwise bubble generating function is completely determined by (2.21) and (2.22), and the counterclockwise bubble generating function is the two-sided inverse of the clockwise one. On the other hand if $h_i(\lambda) \leq 0$ then the counterclockwise bubble generating function is completely determined by (2.20) and (2.22), and the clockwise one is its inverse. It is a non-trivial consequence of the defining relations of $\mathfrak{U}$ that such series exist; see [Bru16]. Here are some further relations involving the bubble generating functions, which explains their importance:


(2.24)

where $R_{i,j}(x,y) := \begin{cases} t_{i,j}^{-1} Q_{i,j}(x,y) & \text{if } i \neq j \\ (x-y)^{-2} & \text{if } i = j, \end{cases}$


(2.25)


(2.26)

Again, these follow from relations derived in [Bru16] (also [KL10]), although it is some work to translate into the generating function form (see [BSW20a]).

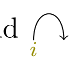
Remark 2.1. There is a $\mathbb{Z}$-grading on $\mathfrak{U}$ defined by declaring that the degrees of the generating 2-morphisms are as listed in Table 1. Taking it into account, one can introduce a free $\mathbb{Z}[q,q^{-1}]$-algebra $K_0(\mathfrak{U}_q)$, the split Grothendieck ring of the graded Karoubi envelope of $\mathfrak{U}$. It is isomorphic to Lusztig's modified integral form $\dot{\mathrm{U}}_{\mathbb{Z}[q,q^{-1}]}$ of the quantized enveloping algebra of the same type as $\dot{\mathrm{U}}$. This result is due to Khovanov and Lauda [KL10], although it depends on the Non-degeneracy Conjecture formulated therein which was proved later; see [Web24, Sec. 3] or [BWW25, Sec. 5.2]. The graded version of $\mathfrak{U}$ will not play a role in the remainder of this article.

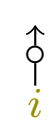
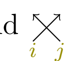
Definition 2.2. A *Kac-Moody categorification* with the type and parameters fixed above is a locally finite[1] $\Bbbk$-linear Abelian category $\mathbf{R}$ with a decomposition

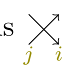
$$\mathbf{R} = \bigoplus_{\lambda \in X} \mathbf{R}_\lambda \tag{2.27}$$

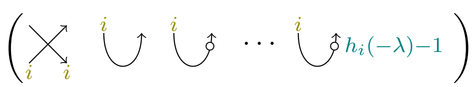
as the internal direct sum[2] of Serre subcategories $\mathbf{R}_\lambda$ ($\lambda \in X$), the *weight subcategories* of $\mathbf{R}$, plus adjoint pairs (E_i, F_i) of $\Bbbk$-linear endofunctors for each $i \in I$, such that:

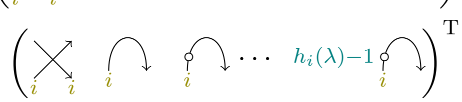
(KM0) The functor E_i takes objects of $\mathbf{R}_\lambda$ to $\mathbf{R}_{\lambda+\alpha_i}$; equivalently, F_i takes objects of $\mathbf{R}_{\lambda+\alpha_i}$ to $\mathbf{R}_\lambda$.

(KM1) The adjoint pair (E_i, F_i) has a prescribed adjunction with unit and counit of adjunction denoted $: \mathrm{id}_{\mathbf{R}} \Rightarrow F_i \circ E_i$ and $: E_i \circ F_i \Rightarrow \mathrm{id}_{\mathbf{R}}$.

(KM2) There are given natural transformations  $: E_i \Rightarrow E_i$ and  $: E_i \circ E_j \Rightarrow E_j \circ E_i$ satisfying the QHA relations (2.5) to (2.7).

(KM3) Defining  as in (2.4), the natural transformations  $: E_j \circ F_i \Rightarrow F_i \circ E_j$ are isomorphisms for all $i \neq j$, as are the matrices

 $: E_i \circ F_i|_{\mathbf{R}_\lambda} \oplus \mathrm{id}_{\mathbf{R}_\lambda}^{\oplus h_i(-\lambda)} \Rightarrow F_i \circ E_i|_{\mathbf{R}_\lambda}$ if $h_i(\lambda) \leq 0$, or

 $: E_i \circ F_i|_{\mathbf{R}_\lambda} \Rightarrow F_i \circ E_i|_{\mathbf{R}_\lambda} \oplus \mathrm{id}_{\mathbf{R}_\lambda}^{\oplus h_i(\lambda)}$ if $h_i(\lambda) > 0$,

for all $i \in I$ and $\lambda \in X$.

This data is equivalent to that of a strict $\Bbbk$-linear 2-functor from $\mathfrak{U}$ to the $\Bbbk$-linear 2-category of locally finite $\Bbbk$-linear Abelian categories.

Definition 2.2 may seem quite complicated. However, there are many examples. Perhaps the most celebrated one comes from finite-dimensional representations of the symmetric groups S_n ($n \geq 0$) with E_i and F_i being the so-called i-induction and i-restriction functors. This fits naturally into the general framework of Heisenberg categorification introduced in the next section, and will be discussed further in Section 5. Other examples arising from rational representations of the general linear group are developed in Section 6.

The work of Chuang and Rouquier [CR08], and Rouquier's subsequent work [Rou08] which formulated the definition in the manner just stated, establishes some impressive and useful structural results about Kac-Moody categorifications. Since it provides the essential motivation for the material in the rest of the article, we conclude the section by listing some of these properties. A more

[1] This means that objects have finite length and morphism spaces are finite-dimensional as vector spaces over $\Bbbk$.

[2] If $\mathbf{C}$ is any additive category and $\mathbf{C}_i$ ($i \in I$) is a family of full subcategories, $\mathbf{C}$ is their *internal direct sum*, denoted $\mathbf{C} = \bigoplus_{i\in I} \mathbf{C}_i$, if the subcategories $\mathbf{C}_i$ are mutually orthogonal and, for each $X \in \mathrm{ob}\,\mathbf{C}$, there are objects $X_i \in \mathrm{ob}\,\mathbf{C}_i$ ($i \in I$), all but finitely many of which are zero, such that X is a biproduct of X_i ($i \in I$). Denoting the biproduct inclusions and projections by $\iota_{i,X} : X_i \hookrightarrow X$ and $\pi_{i,X} : X \twoheadrightarrow X_i$, there is an associated projection functor $\mathrm{Pr}_i : \mathbf{C} \to \mathbf{C}_i$ mapping object X to X_i and morphism $f : X \to Y$ to $\pi_{i,Y} \circ f \circ \iota_{i,X}$. It is convenient to assume that $\iota_{i,X_i} = \pi_{i,X_i} = \mathrm{id}_{X_i}$ so that $\mathrm{Pr}_i \circ \mathrm{Inc}_i = \mathrm{id}_{\mathbf{C}_i}$, where $\mathrm{Inc}_i : \mathbf{C}_i \to \mathbf{C}$ is the inclusion functor.

detailed account can be found in the survey article [BD17], to which we defer for the appropriate citations to the literature. Let $\mathbf{R}$ be a Kac-Moody categorification as above.

(1) For $n \geq 1$, the axiom (KM2) implies that there is a homomorphism $\mathrm{NH}_n \to \mathrm{End}_{\Bbbk}(E_i^n)$, where NH_n is the nil-Hecke algebra over $\Bbbk$. Letting $e_n \in \mathrm{NH}_n$ be some preferred choice of a homogeneous primitive idempotent, its image under this homomorphism defines a summand $E_i^{(n)}$ of the functor E_i^n such that $E_i^n \cong (E_i^{(n)})^{\oplus n!}$. Similarly, there is a summand $F_i^{(n)}$ of F_i^n such that $F_i^n \cong (F_i^{(n)})^{\oplus n!}$, which may be chosen so that the given adjunction (E_i, F_i) induces an adjunction $(E_i^{(n)}, F_i^{(n)})$.

(2) As well as being right adjoint to $E_i^{(n)}$, the functor $F_i^{(n)}$ is also left adjoint to $E_i^{(n)}$ with unit and counit induced by the second adjunction described above. Hence, $E_i^{(n)}$ and $F_i^{(n)}$ are exact and they preserve injective and projective objects.

(3) Let $G_0(\mathbf{R})$ be the Grothendieck group of the Abelian category $\mathbf{R}$. It is naturally a module over the algebra $\dot{\mathrm{U}}_{\mathbb{Z}}$ with weight decomposition $G_0(\mathbf{R}) = \bigoplus_{\lambda \in X} G_0(\mathbf{R}_\lambda)$. The actions of $e_i^{(n)} 1_\lambda$ and $f_i^{(n)} 1_\lambda$ are induced by the exact functors $E_i^{(n)}|_{\mathbf{R}_\lambda}$ and $F_i^{(n)}|_{\mathbf{R}_\lambda}$. In fact, $G_0(\mathbf{R})$ is an *integrable* $\dot{\mathrm{U}}_{\mathbb{Z}}$-module: for each $V \in \mathrm{ob}\,\mathbf{R}$ there is $n \in \mathbb{N}$ such that $E_i^{(n)} V = F_i^{(n)} V = 0$. Moreover, for each $V \in \mathrm{ob}\,\mathbf{R}$, we have that $E_i V = F_i V = 0$ for all but finitely many $i \in I$.

For the remaining results in this summary, we assume for simplicity that $\mathbf{R}$ is a *nilpotent* categorification[3], that is, the endomorphism of $E_i V$ defined by $\overset{\bullet}{\uparrow}_i$ is nilpotent for all $V \in \mathrm{ob}\,\mathbf{R}$ and $i \in I$; equivalently, the endomorphism of $F_i V$ defined by $\overset{\bullet}{\downarrow}_i$ is nilpotent for all V and i.

(4) Let $\mathcal{B} = \bigsqcup_{\lambda \in X} \mathcal{B}_\lambda$ be a set such that $\mathcal{B}_\lambda$ indexes a full set $L(b)\,(b \in \mathcal{B}_\lambda)$ of pairwise inequivalent irreducible objects of $\mathbf{R}_\lambda$. For $b \in \mathcal{B}$, let $\varepsilon_i(b)$ and $\phi_i(b)$ be the nilpotency degrees of the endomorphisms $x : E_i L(b) \to E_i L(b)$ and $y : F_i L(b) \to F_i L(b)$ defined by the natural transformations $\overset{\bullet}{\uparrow}_i$ and $\overset{\bullet}{\downarrow}_i$, respectively. Then the endomorphism algebras $\mathrm{End}_{\mathbf{R}}(E_i L(b))$ and $\mathrm{End}_{\mathbf{R}}(F_i L(b))$ are isomorphic to $\Bbbk[x]/\left(x^{\varepsilon_i(b)}\right)$ and $\Bbbk[y]/\left(y^{\phi_i(b)}\right)$, respectively. In particular, $E_i L(b)$ is non-zero if and only if $\varepsilon_i(b) \neq 0$, and $F_i L(b)$ is non-zero if and only if $\phi_i(b) \neq 0$.

(5) For $b \in \mathcal{B}_\lambda$, Schur's Lemma implies that the bubble generating functions act on $L(b)$ by series in $\Bbbk(\!(u^{-1})\!)$. In fact, $\circlearrowright_i(u)$ acts on $L(b)$ by $u^{\phi_i(b) - \varepsilon_i(b)}$ and $\circlearrowleft_i(u)$ acts by $u^{\varepsilon_i(b) - \phi_i(b)}$.

Hence:
$$h_i(\lambda) = \phi_i(b) - \varepsilon_i(b). \tag{2.28}$$

(6) For $b \in \mathcal{B}$ with $\varepsilon_i(b) > 0$, $E_i L(b)$ is an indecomposable object with irreducible socle and head isomorphic to $L(\tilde{e}_i(b))$ for some $\tilde{e}_i(b) \in \mathcal{B}$; moreover, $\varepsilon_i(\tilde{e}_i(b)) = \varepsilon_i(b) - 1$ and $\phi_i(\tilde{e}_i(b)) = \phi_i(b) + 1$. Similarly, if $\phi_i(b) > 0$ then $F_i L(b)$ is indecomposable with irreducible socle and head isomorphic to $L(\tilde{f}_i(b))$ for some $\tilde{f}_i(b) \in \mathcal{B}$; moreover, $\phi_i(\tilde{f}_i(b)) = \phi_i(b) - 1$ and $\varepsilon_i(\tilde{f}_i(b)) = \varepsilon_i(b) + 1$. This defines mutually inverse bijections
$$\{b \in \mathcal{B} \mid \varepsilon_i(b) > 0\} \underset{\tilde{f}_i}{\overset{\tilde{e}_i}{\rightleftarrows}} \{b \in \mathcal{B} \mid \phi_i(b) > 0\}.$$

Also defining $\mathrm{wt} : \mathcal{B} \to X$ so that $\mathrm{wt}(b) = \lambda \Leftrightarrow b \in \mathcal{B}_\lambda$, the datum $(\mathcal{B}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \phi_i, \mathrm{wt})$ is a *normal crystal* in the sense[4] of [Kas95, Sec. 7.6].

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[3]There are similar results without this nilpotency assumption, but one needs to "unfurl" the Cartan matrix to obtain a larger Kac-Moody algebra. See [Web24, Sec. 4] which discusses the general case.

[4]Note though that we prefer to set things up so that domains of $\tilde{e}_i$ and $\tilde{f}_i$ are subsets of $\mathcal{B}$, leaving them undefined on the elements $b \in \mathcal{B}$ with $\varepsilon_i(b) = 0$ or $\phi_i(b) = 0$, respectively.

(7) For $b \in \mathcal{B}_\lambda$ and $0 \leq m \leq n := \varepsilon_i(b)$, the object $E_i^{(m)} L(b)$ is indecomposable with simple socle and head isomorphic to $L\big(\tilde{e}_i^m(b)\big)$. The composition multiplicity $[E_i^{(m)} L(b) : L\big(\tilde{e}_i^m(b)\big)]$ equals $\binom{n}{m}$, and all other composition factors are of the form $L(b')$ for $b' \in \mathcal{B}_{\lambda+m\alpha_i}$ with $\varepsilon_i(b') < n - m$. Also the endomorphism algebra $\mathrm{End}_{\mathbf{R}}(E_i^{(m)} L(b))$ is explicitly determined; it is isomorphic to the cohomology algebra $H^*(\mathrm{Gr}_{m,n}, \Bbbk)$ of the Grassmannian. There are analogous statements about $F_i^{(m)} L(b)$ for $0 \leq m \leq n := \phi_i(b)$.

(8) The isomorphism classes $[L(b)]$ $(b \in \mathcal{B})$ give a *perfect basis* for the integrable $\dot{\mathrm{U}}_{\mathbb{Z}}$-module $G_0(\mathbf{R})$ in the sense of [BK07, Def. 5.49].

Let W be the Weyl group associated to the Cartan matrix A. It is generated by simple reflections s_i $(i \in I)$, and acts on X so that $s_i(\lambda) = \lambda - h_i(\lambda)\alpha_i$. Since $G_0(\mathbf{R})$ is an integrable $\dot{\mathrm{U}}_{\mathbb{Z}}$-module, the set of weights $\lambda \in X$ such that $\mathcal{B}_\lambda \neq \varnothing$ is a union of W-orbits.

(9) Suppose that $\lambda \in X$ and $i \in I$ such that $\varepsilon_i(b) = 0$ for all $b \in \mathcal{B}_\lambda$. By (2.28), we have that $n := h_i(\lambda) = \phi_i(b)$ for any $b \in \mathcal{B}_\lambda$. Then the functor $F_i^{(n)} : \mathbf{R}_\lambda \to \mathbf{R}_{s_i(\lambda)}$ is an equivalence of categories with quasi-inverse defined by $E_i^{(n)}$.

(10) Take $\lambda \in X$ and $i \in I$ such that $m := h_i(\lambda) \geq 0$. There is a complex of functors
$$\cdots \longrightarrow E_i^{(2)} F_i^{(m+2)} \longrightarrow E_i F_i^{(m+1)} \longrightarrow F_i^{(m)} \longrightarrow 0$$
which induces a triangulated functor $D^b(\mathbf{R}_\lambda) \to D^b(\mathbf{R}_{s_i(\lambda)})$ between bounded derived categories. In [CR08], Chuang and Rouquier proved that this is an equivalence of categories; in fact, there is a perverse equivalence between $\mathbf{R}_\lambda$ and $\mathbf{R}_{s_i(\lambda)}$ in the sense of [CR09]. Hence, weight subcategories of $\mathbf{R}$ for weights lying in the same W-orbit are derived equivalent.

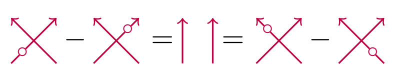
Remark 2.3. We have formulated the definition of Kac-Moody categorification in the context of locally finite $\Bbbk$-linear Abelian categories. There is a parallel theory in which $\mathbf{R}$ is merely a $\Bbbk$-linear Karoubian category with finite-dimensional morphism spaces. There are analogous results to the above in this finite-dimensional Karoubian setting. One needs to replace $G_0(\mathbf{R})$ with the split Grothendieck group $K_0(\mathbf{R})$, which again has the induced structure of an integrable $\dot{\mathrm{U}}_{\mathbb{Z}}$-module. The isomorphism classes of indecomposable objects of $\mathbf{R}$ define a *dual perfect basis* for $K_0(\mathbf{R})$ in the sense of [KKKS15]. This is discussed further in [BD17].

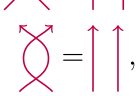
3. Heisenberg categorifications

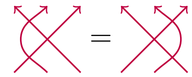
Recall that $\Bbbk$ is an algebraically closed field. Fix a *central charge* $\kappa \in \mathbb{Z}$. The *degenerate Heisenberg category* $\mathbf{Heis}_\kappa$ is a strict $\Bbbk$-linear monoidal category defined by generators and relations. It was introduced originally by Khovanov [Kho14] for $\kappa = \pm 1$, then the definition was extended to all non-zero central charges in [MS18], while for $\kappa = 0$ it is the affine oriented Brauer category from [BCNR17]. We define it here following the approach of [Bru18, Theorem 1.2], which is similar in spirit to the definition of $\mathfrak{U}$ given in the previous section. It has two generating objects E and F, whose identity endomorphisms are represented by the oriented strings $\uparrow$ and $\downarrow$, and the following four generating morphisms:

 $: E \to E,$  $: E \otimes E \to E \otimes E,$  $: E \otimes F \to \mathbb{1},$  $: \mathbb{1} \to F \otimes E.$

These are subject to certain relations. First, we have the *degenerate affine Hecke algebra* relations:

 (3.1)

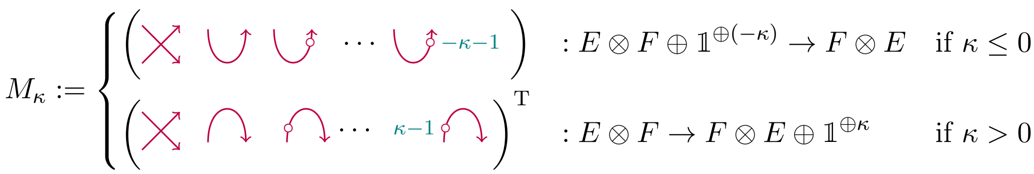
 (3.2)

 (3.3)

Next, the *right adjunction relations* implying that F is right dual to E:

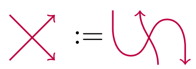
 (3.4)

Finally, we have the *inversion relation* which asserts that the morphism



(3.5)

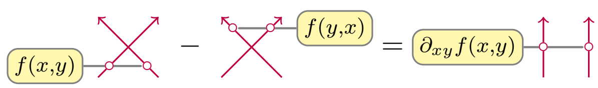
in the additive envelope of $\mathbf{Heis}_\kappa$ is invertible, where the rightward crossing is defined by

 (3.6)

Remark 3.1. Recall that the *Demazure operator* $\partial_{xy} : \Bbbk[x,y] \to \Bbbk[x,y]$ is the linear map with

$$\partial_{xy} f(x,y) := \frac{f(x,y) - f(y,x)}{x-y}. \tag{3.7}$$

The dot slide relation (3.1) implies that



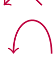
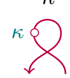
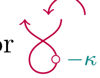
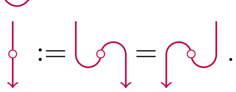
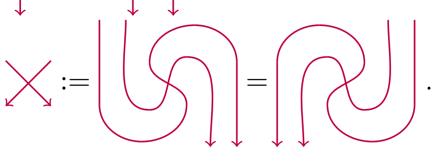
(3.8)

for any $f(x,y) \in \Bbbk[x,y]$.

Again like in the previous section, there is also a *second adjunction* making F into a left dual of E, that is, there are leftward cups and caps such that

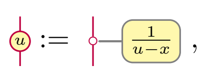
 (3.9)

These, and some other useful shorthands, are defined as follows:

- Let  be the first entry of the inverse matrix M_κ^{-1}.
- Let  be the last entry of M_κ^{-1} if $\kappa < 0$ or  if $\kappa \geq 0$.
- Let  be the last entry of $-M_\kappa^{-1}$ if $\kappa > 0$ or  if $\kappa \leq 0$.
- Let  .
- Let  .

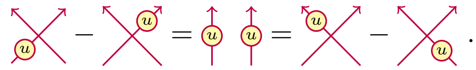
The equalities in the definitions of the downward dot and crossing are justified in [Bru18]. It follows that string diagrams for morphisms in $\mathbf{Heis}_\kappa$ are invariant under planar isotopy. Hence, $\mathbf{Heis}_\kappa$ is strictly pivotal.

We use the pin notation and generating functions as explained in the previous section. In particular, we have the dot generating function



(3.10)

which satisfies



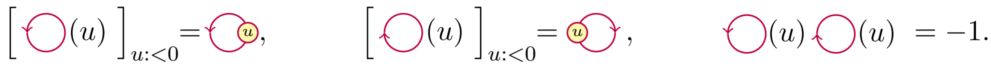
(3.11)

In the Heisenberg setting, the *bubble generating functions*[5] are the unique formal Laurent series

$$(u) \in u^{\kappa}\,\mathrm{id}_{\mathbb{1}} + u^{\kappa-1}\Bbbk[\![u^{-1}]\!]\,\mathrm{End}_{\mathbf{Heis}_\kappa}(\mathbb{1}), \tag{3.12}$$

$$(u) \in -u^{-\kappa}\,\mathrm{id}_{\mathbb{1}} + u^{-\kappa-1}\Bbbk[\![u^{-1}]\!]\,\mathrm{End}_{\mathbf{Heis}_\kappa}(\mathbb{1}) \tag{3.13}$$

such that



(3.14)

Existence of these series is justified in [Bru18]. The Heisenberg analogues of (2.24) to (2.26) are

(3.15)

where $R(x,y) := 1-(x-y)^{-2} = \frac{(x-y-1)(x-y+1)}{(x-y)^2}$,

(3.16)

(3.17)

These identities are copied from [BSW20a, Sec. 3.1].

Remark 3.2. Let $K_0(\mathbf{Heis}_k)$ be the Grothendieck ring of the additive Karoubi envelope of $\mathbf{Heis}_\kappa$. When the characteristic of the ground field is zero, $K_0(\mathbf{Heis}_\kappa)$ is isomorphic to the *Heisenberg ring* H_κ, that is, the ring generated by elements $\{h_n^+, e_n^- \mid n \geq 0\}$ subject to the relations

$$h_0^+ = e_0^- = 1, \qquad h_m^+h_n^+ = h_n^+h_m^+, \qquad e_m^-e_n^- = e_n^-e_m^-, \qquad h_m^+e_n^- = \sum_{r=0}^{\min(m,n)} \binom{\kappa}{r} e_{n-r}^- h_{m-r}^+.$$

This ring is a $\mathbb{Z}$-form for the universal enveloping algebra of the infinite-dimensional Heisenberg Lie algebra specialized at central charge κ. The existence of an isomorphism $K_0(\mathbf{Heis}_\kappa) \cong \mathrm{H}_\kappa$ was conjectured by Khovanov in [Kho14] (for $\kappa = \pm 1$), and proved in [BSW23]. Under the isomorphism, the classes $[E], [F] \in K_0(\mathbf{Heis}_\kappa)$ correspond to $h_1^+, e_1^- \in \mathrm{H}_\kappa$. More generally, h_n^+, e_n^- are the isomorphism classes of the summands of E^n and F^n defined by idempotents arising from the trivial and sign representations of the symmetric group S_n.

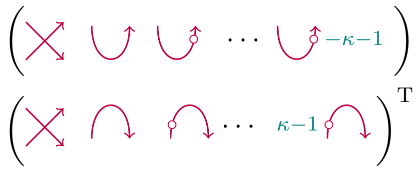
Definition 3.3. A *degenerate Heisenberg categorification* of central charge κ is a locally finite $\Bbbk$-linear Abelian category $\mathbf{R}$ plus an adjoint pair (E,F) of $\Bbbk$-linear endofunctors such that:

(H1) The adjoint pair (E,F) has a prescribed adjunction with unit and counit of adjunction denoted  $: \mathrm{id}_{\mathbf{R}} \Rightarrow F \circ E$ and  $: E \circ F \Rightarrow \mathrm{id}_{\mathbf{R}}$.

(H2) There are given natural transformations  $: E \Rightarrow E$ and  $: E \circ E \Rightarrow E \circ E$ satisfying the dAHA relations (3.1) to (3.3).

[5]Note that the clockwise bubble generating function here is -1 times the one in [BSW20a]—with hindsight we think this is a more consistent convention.

(H3) Defining the rightward crossing according to (3.6), the matrix



$$: E\circ F \oplus \mathrm{id}_{\mathbf{R}}^{\oplus(-\kappa)} \Rightarrow F\circ E \qquad \text{if } \kappa \leq 0, \text{ or}$$

$$: E\circ F \Rightarrow F\circ E \oplus \mathbb{1}^{\oplus\kappa} \qquad \text{if } \kappa > 0,$$

is invertible.

Thus, $\mathbf{R}$ is a strict left $\mathbf{Heis}_\kappa$-module category.

There is an analogy between the properties (H1)–(H3) in Definition 3.3 and (KM1)–(KM3) from Definition 2.2. However, the definition of Heisenberg categorification is simpler than that of Kac-Moody categorification—there is no underlying root datum or weight decomposition, and the inversion relation (H3) depends only on the fixed central charge κ whereas (KM3) depends locally on the weight. There is also no analogue of (KM0), although we will introduce a variant (H0) of that later on when we discuss the examples in Sections 5 and 6.

4. Heisenberg to Kac-Moody

Now that we have explained Definitions 2.2 and 3.3, we can recall the main construction from [BSW20a, Sec. 4] giving the bridge from a degenerate Heisenberg categorification to a Kac-Moody categorification; see also Section 8 where we discuss the quantum variant. Let $\mathbf{R}$ be a degenerate Heisenberg categorification of central charge $\kappa \in \mathbb{Z}$. Let $L(b)$ $(b \in \mathcal{B})$ be a full set of pairwise inequivalent irreducible objects of $\mathbf{R}$. For $b \in \mathcal{B}$, we let $m_b(x)$ (resp., $n_b(x)$) be the (monic) minimal polynomial of the endomorphism of $EL(b)$ (resp., $FL(b)$) defined by (resp., by).

Let $Z(\mathbf{R})$ be the center of the category $\mathbf{R}$, that is, the endomorphism algebra of the identity functor $\mathrm{id}_{\mathbf{R}} : \mathbf{R} \to \mathbf{R}$. The bubble generating function (u) defines an element of $Z(\mathbf{R})(\!(u^{-1})\!)$. By Schur's Lemma, this evaluates to a formal Laurent series $\chi_b(u) \in \Bbbk(\!(u^{-1})\!)$ on the irreducible object $L(b)$ $(b \in \mathcal{B})$. The coefficients of $\chi_b(u)$ encode useful information about the central character of $L(b)$. The first key result, which is [BSW20a, Lem. 4.4], is that

$$\chi_b(u) = n_b(u)/m_b(u). \tag{4.1}$$

In particular, $\chi_b(u)$ is a rational function.

The *spectrum* of $\mathbf{R}$ is the subset $I \subseteq \Bbbk$ consisting of the roots of the minimal polynomials $m_b(x)$ for all $b \in \mathcal{B}$; equivalently, by adjunction, it is the set of roots of the minimal polynomials $n_b(x)$ for all $b \in \mathcal{B}$. According to [BSW20a, Lem. 4.6], the set I is closed under the operations $i \mapsto i \pm 1$. We define a symmetric Cartan matrix $A = (a_{i,j})_{i,j\in I}$ by setting

$$a_{i,j} := \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i = j \pm 1 \text{ and } \operatorname{char}\Bbbk \neq 2 \\ -2 & \text{if } i = j+1 = j-1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

The connected components of A are all of type A_∞ if $\operatorname{char}\Bbbk = 0$ or $A_{p-1}^{(1)}$ if $\operatorname{char}\Bbbk = p > 0$. We also fix a root datum of this Cartan type with root lattice

$$X := \bigoplus_{i\in I} \mathbb{Z}\varpi_i \tag{4.3}$$

spanned by the fundamental weights ϖ_i $(i \in I)$, and simple roots defined by $\alpha_j := \sum_{i\in I} a_{i,j}\varpi_i$. The unique homorphisms $h_i : X \to \mathbb{Z}$ with $h_i(\varpi_j) = \delta_{i,j}$ have the required property that $h_i(\alpha_j) = a_{i,j}$ for all $i, j \in I$.

Since $\Bbbk$ is algebraically closed, we can write

$$m_b(x) = \prod_{i \in I} (x - i)^{\varepsilon_i(b)}, \qquad n_b(x) = \prod_{i \in I} (x - i)^{\phi_i(b)} \tag{4.4}$$

for some functions $\varepsilon_i, \phi_i : \mathcal{B} \to \mathbb{N}$. We also define $\mathrm{wt} : \mathcal{B} \to X$ by setting

$$\mathrm{wt}(b) := \sum_{i \in I} (\phi_i(b) - \varepsilon_i(b)) \varpi_i \in X. \tag{4.5}$$

By (4.1), the coefficient $h_i(\mathrm{wt}(b))$ of ϖ_i in $\mathrm{wt}(b)$ is the multiplicity of i as a zero or pole of the rational function $\chi_b(u)$. Thus, $\mathrm{wt}(b)$ encodes the same central character information as the formal series $\chi_b(u)$. In view of (3.12), $\chi_b(u)$ has leading term u^κ, so we have that

$$\kappa = \sum_{i \in I} h_i(\mathrm{wt}(b)) \tag{4.6}$$

for all $b \in \mathcal{B}$.

For $\lambda \in X$, let $\mathcal{B}_\lambda := \{b \in \mathcal{B} \mid \mathrm{wt}(b) = \lambda\}$, and definte the *weight subcategory* $\mathbf{R}_\lambda$ be the Serre subcategory of $\mathbf{R}$ generated by the irreducible objects $L(b)$ for $b \in \mathcal{B}_\lambda$. By central character considerations, the category $\mathbf{R}$ decomposes as the internal direct sum

$$\mathbf{R} = \bigoplus_{\lambda \in X} \mathbf{R}_\lambda. \tag{4.7}$$

By (4.6), the weight subcategory $\mathbf{R}_\lambda$ is zero unless $\sum_{i \in I} h_i(\lambda) = \kappa$.

The functors E and F decompose as

$$E = \bigoplus_{i \in I} E_i, \qquad F = \bigoplus_{i \in I} F_i \tag{4.8}$$

where E_i is the subfunctor of E such that $E_i V$ is the generalized i-eigenspace of the endomorphism of EV defined by $\overset{\bullet}{\uparrow}$ for each $V \in \mathrm{ob}\, \mathbf{R}$, and F_i is the subfunctor of F defined similarly by taking the generalized i-eigenspace of $\overset{\bullet}{\downarrow}$. The definition of the spectrum I ensures that all of the functors E_i and F_i appearing in (4.8) are non-zero.

To complete the picture, we are going to use the usual string calculus for the 2-category of $\Bbbk$-linear categories to introduce some further natural transformations between these functors. In such diagrams, an unlabelled 2-cell should be interpreted as the category $\mathbf{R}$, and a 2-cell labelled by λ indicates the category $\mathbf{R}_\lambda$. The identity endomorphisms of $E : \mathbf{R} \to \mathbf{R}$ and $F : \mathbf{R} \to \mathbf{R}$ will be represented by the strings $\uparrow$ and $\downarrow$. The same strings with the rightmost 2-cell labelled λ denote the identity endomorphisms of the functors obtained by pre-composing with the inclusion $\mathrm{Inc}_\lambda : \mathbf{R}_\lambda \to \mathbf{R}$, while if the the leftmost 2-cell is labelled μ we mean the identity endomorphisms of the functors obtained by post-composing with the projection $\mathrm{Pr}_\mu : \mathbf{R} \to \mathbf{R}_\mu$. Similar conventions apply to $E_i : \mathbf{R} \to \mathbf{R}$ and $F_i : \mathbf{R} \to \mathbf{R}$, whose identity endomorphisms will be represented using the labelled strings $\underset{i}{\uparrow}$ and $\underset{i}{\downarrow}$, respectively. We use the string diagrams

$$\underset{i}{\uparrow} : E_i \Rightarrow E, \qquad \underset{i}{\downarrow} : F_i \Rightarrow F, \qquad \overset{i}{\uparrow} : E \Rightarrow E_i, \qquad \overset{i}{\downarrow} : F \Rightarrow F_i \tag{4.9}$$

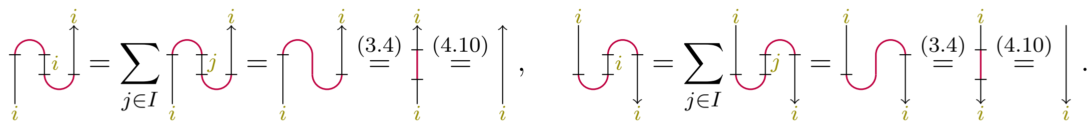
to denote the various inclusions and projections between E and F and their summands E_i and F_i. We then have that

$$\overset{j}{\underset{i}{\uparrow}} = \delta_{i,j} \underset{i}{\uparrow}, \qquad \overset{j}{\underset{i}{\downarrow}} = \delta_{i,j} \underset{i}{\downarrow}. \tag{4.10}$$

Composing the rightward cup and cap with projectors gives natural transformations


(4.11)

These are zero if $i \neq j$, as may be proved by a similar argument to the proof of [BSW20a, Lem. 4.1] using that dots slide across rightward cups and caps in $\mathbf{Heis}_\kappa$. It follows that


(4.12)

This shows that the *rightward Kac-Moody cups and caps* defined by


(4.13)

are the unit and counit of an adjunction making (E_i, F_i) into an adjoint pair for each $i \in I$. Using the vanishing properties of (4.11), it is also easy to check that

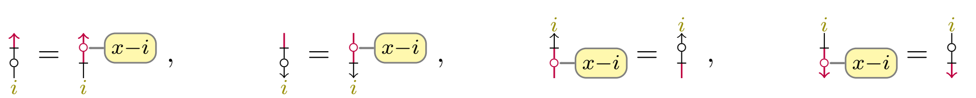

(4.14)

In a similar way, the leftward cup and cap in $\mathbf{Heis}_\kappa$ can be projected to obtain the unit and counit of an adjunction (F_i, E_i), but we will not introduce any special diagrammatic notation for these natural transformations—they are *not* simply equal to the leftward Kac-Moody cups and caps and appearing in Theorem 7.1 below.

For the projected dots, we incorporate an additional change-of-origin, defining the *Kac-Moody dots*

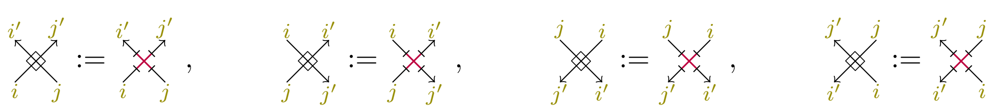

(4.15)

The point of this shift is that these natural transformations are *locally nilpotent*, that is, they evaluate to nilpotent endomorphisms on any object of $\mathbf{R}$. Consequently, we can extend the pin notation by allowing any formal power series in $\Bbbk[\![x]\!]$ (rather than merely a polynomial in $\Bbbk[x]$) to be pinned to the Kac-Moody dots or . From (4.10), it follows that


(4.16)

Using these and the fact that Heisenberg dots slide across Heisenberg cups and caps, it is easy to deduce that the Kac-Moody dots slide across the rightward Kac-Moody cups and caps.

The analysis of projected crossings is more interesting. We will denote them by


(4.17)

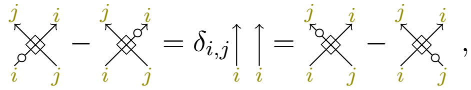
In [BSW20a, Lem. 4.1], it is shown that these natural transformations are zero unless $\{i, j\} = \{i', j'\}$. Also, by [BSW20a, Lem. 4.2], we have that


(4.18)

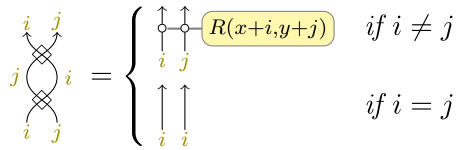
when $i \neq j$; the expression on the right hand side makes sense because $i-j+x-y \in \Bbbk[\![x, y]\!]^\times$ and x, y are locally nilpotent. In the following lemma, we record some further properties of the projected crossings. The proof makes a good exercise to get better acquainted with the diagrammatics.

(Perhaps this is a good time to remind again of our labelling convention for multipins: alphabetical order of variables corresponds to lexicographic order of Cartesian coordinates.)

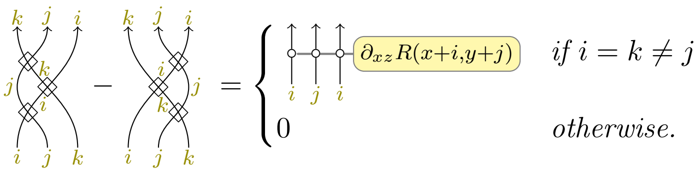
Lemma 4.1. *Recall that $R(x,y) = 1 - (x-y)^{-2}$, and ∂_{xy} denotes the Demazure operator from Remark 3.1. The following hold for $i, j, k \in I$:*



(4.19)

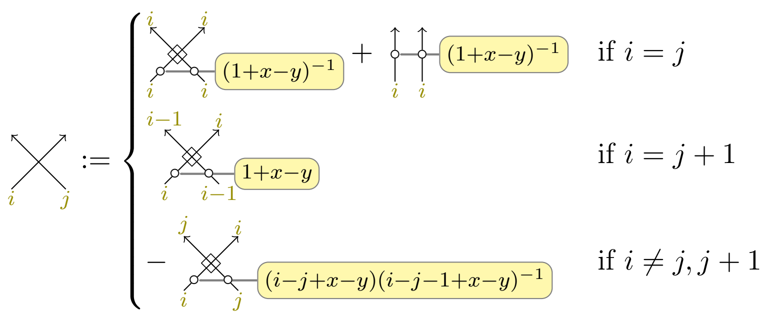


(4.20)



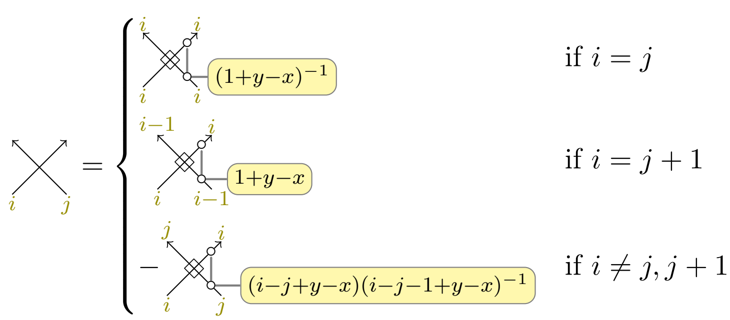
(4.21)

Now we introduce the *upward Kac-Moody crossings*



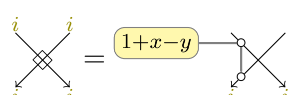
(4.22)

By (4.19) and Remark 3.1, this definition can be written equivalently as



(4.23)

From the $i = j$ case of this and (2.12) to (2.14), we also have that

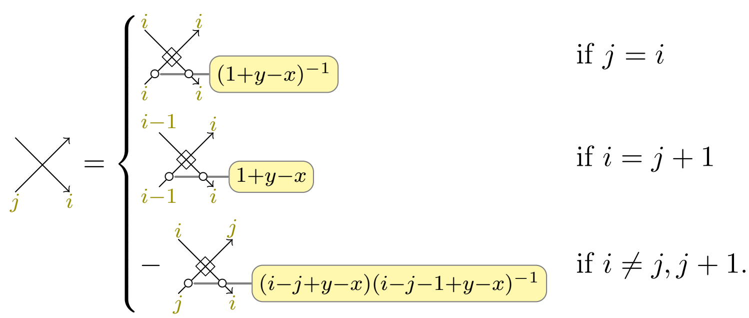


(4.24)

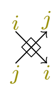
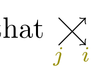
Theorem 4.2 ([BSW20a, Th. 4.11])**.** *Let **R** be a degenerate Heisenberg categorification of central charge κ. The definitions just explained make **R** into a Kac-Moody categorification with the* *parameters defined from*

$$Q_{i,j}(x,y) := \begin{cases} 0 & \textit{if } i = j \\ y - x & \textit{if } i = j+1 \neq j-1 \\ x - y & \textit{if } i = j-1 \neq j+1 \\ (y-x)(x-y) & \textit{if } i = j+1 = j-1 \\ 1 & \textit{if } i \neq j, j \pm 1. \end{cases}$$

Sketch proof. We have in our hands all of the required data, and it just remains to verify the conditions of (KM0)–(KM3) from Definition 2.2. The axiom (KM0) follows by [BSW20a, Lem. 4.7]; the proof is an application of the bubble slide relation (3.15). We already constructed the adjunctions required for (KM1). The axiom (KM2) follows from the isomorphism established in [BK09]. Alternatively, one can simply verify the QHA relations from scratch—with Lemma 4.1 in view, and using string diagrams rather than following the algebraic approach of [BK09], it becomes a routine calculation. More conceptual approaches to the calculation are available in the literature, but they do not save any time compared to this most naive direct assault! This leaves the hardest axiom (KM3), the inversion relation. Using (3.6) and (4.23), the rightward Kac-Moody crossing defined via (2.4) is



(4.25)

If $i \neq j$, it is easy to show that the rightward crossing  is invertible, with two-sided inverse given by the leftward crossing ; see [BSW20a, Lem. 4.8]. From this and (4.25), it follows easily that  is invertible when $i \neq j$, checking the inversion relation in these cases. The inversion relation when $i = j$ follows instead from [BSW20a, Lem. 4.9, Lem. 4.10], which are more difficult to prove. □

Remark 4.3. We have explained Theorem 4.2 in its simplest vanilla flavor. If more is known about the category $\mathbf{R}$, the construction can be modified so as to incorporate extra information into the choice of the weight lattice X. We will not attempt to formulate such modifications formally, preferring to explain the idea in the context of the examples in characteristic $p > 0$ described the next two sections. In these, we will use an additional *degree decomposition* $\mathbf{R} = \bigoplus_{n \in \mathbb{Z}} \mathbf{R}_n$ to upgrade X from the weight lattice of the Kac-Moody algebra $\widehat{\mathfrak{sl}}_p'$ to that of $\widehat{\mathfrak{sl}}_p$.

Remark 4.4. Another point we would like to make is that we have formulated the definition of Heisenberg categorification in the context of locally finite $\Bbbk$-linear Abelian categories. There is a parallel theory in which $\mathbf{R}$ is a $\Bbbk$-linear *Schurian category*, that is, it is equivalent to the category of locally finite-dimensional modules over a locally finite-dimensional locally unital $\Bbbk$-algebra. The analogue of Theorem 4.2 in the Schurian setting makes the full subcategory of $\mathbf{R}$ consisting of finitely generated projective objects into a finite-dimensional Karoubian Kac-Moody categorification as in Remark 2.3. This is discussed further in [BSW20a].

5. Example: representations of symmetric groups

In this section, we explain the first example of a Heisenberg categorification, which was also the original motivation for Khovanov's definition of the Heisenberg category in [Kho14]. As usual, let $\Bbbk$ be an algebraically closed field of characteristic $p \geq 0$. Let $\Bbbk S := \bigoplus_{n\geq 0} \Bbbk S_n$ be the direct sum of the group algebras of all symmetric groups. It is a locally unital algebra with distinguished idempotents 1_n $(n \in \mathbb{N})$ that are the identity elements of the groups S_n. We are interested in the locally finite $\Bbbk$-linear Abelian category $\Bbbk S\textbf{-mod}_{\mathrm{fd}}$ of finite-dimensional left $\Bbbk S$-modules $V = \bigoplus_{n\geq 0} 1_n V$. Identifying $\Bbbk S_n\textbf{-mod}_{\mathrm{fd}}$ with the full subcategory of $\Bbbk S\textbf{-mod}_{\mathrm{fd}}$ consisting of the modules V such that $1_n V = V$, we have the internal direct sum decomposition

$$\Bbbk S\textbf{-mod}_{\mathrm{fd}} = \bigoplus_{n\geq 0} \Bbbk S_n\textbf{-mod}_{\mathrm{fd}} \tag{5.1}$$

We make $\Bbbk S\textbf{-mod}_{\mathrm{fd}}$ into a degenerate Heisenberg categorification of central charge $\kappa := -1$ in the following way:

- Let $E : \Bbbk S\textbf{-mod}_{\mathrm{fd}} \to \Bbbk S\textbf{-mod}_{\mathrm{fd}}$ be the $\Bbbk$-linear functor defined on $\Bbbk S_n\textbf{-mod}_{\mathrm{fd}}$ by the induction functor $\mathrm{ind}_{S_n}^{S_{n+1}} := \Bbbk S_{n+1} \otimes_{\Bbbk S_n} -$ (the embedding of S_n into S_{n+1} is the usual one fixing the last letter $n+1$). This functor has the natural right adjoint F defined on $\Bbbk S_{n+1}\textbf{-mod}_{\mathrm{fd}}$ by the restriction functor $\mathrm{res}_{S_n}^{S_{n+1}}$, and taking objects of $\Bbbk S_0\textbf{-mod}_{\mathrm{fd}}$ to $\{0\}$.
- The endomorphism $: E \Rightarrow E$ is defined on objects of $\Bbbk S_n\textbf{-mod}_{\mathrm{fd}}$ by the endomorphism of $\mathrm{ind}_{S_n}^{S_{n+1}}$ arising from the endomorphism of the $(\Bbbk S_{n+1}, \Bbbk S_n)$-bimodule $\Bbbk S_{n+1}$ given by right multiplication by the Jucys-Murphy element $x_{n+1} := \sum_{i=1}^{n} (i\ \ n{+}1)$.
- The endomorphism $: E \circ E \Rightarrow E \circ E$ is defined on objects of $\Bbbk S_n\textbf{-mod}_{\mathrm{fd}}$ by the endomorphism of $\mathrm{ind}_{S_{n+1}}^{S_{n+2}} \circ \mathrm{ind}_{S_n}^{S_{n+1}} \cong \mathrm{ind}_{S_n}^{S_{n+2}}$ arising from the endomorphism of the $(\Bbbk S_{n+2}, \Bbbk S_n)$-bimodule $\Bbbk S_{n+2}$ given by right multiplication by the transposition $(n{+}1\ \ n{+}2)$.

We still need to check (H2) and (H3) from Definition 3.3. For (H2), the relations (3.1) to (3.3) are straightforward to verify. For example, (3.1) holds because $(n{+}1\ n{+}2)x_{n+2} - x_{n+1}(n{+}1\ n{+}2) = 1_{n+2}$ in $\Bbbk S_{n+2}$. For the inversion relation (H3), we need to show that

$$\left(\quad \right) : E \circ F \oplus \mathrm{id}_{\Bbbk S\textbf{-mod}_{\mathrm{fd}}} \Rightarrow F \circ E$$

is an isomorphism. On calculating the rightward crossing explicitly, this reduces to showing that the $(\Bbbk S_n, \Bbbk S_n)$-bimodule homomorphism

$$\theta : \Bbbk S_n \otimes_{\Bbbk S_{n-1}} \Bbbk S_n \oplus \Bbbk S_n \to \Bbbk S_{n+1}, \qquad \begin{pmatrix} g_1 \otimes g_2 \\ g_3 \end{pmatrix} \mapsto g_1 (n\ \ n{+}1) g_2 + g_3$$

is an isomorphism for each $n \geq 0$; in the case $n = 0$, the first summand should be interpreted as zero. The vectors $\begin{pmatrix} (i\ i{+}1\ \cdots\ n) \otimes 1_n \\ 0 \end{pmatrix}$ for $1 \leq i \leq n$ together with $\begin{pmatrix} 0 \\ 1_n \end{pmatrix}$ give a basis for $\Bbbk S_n \otimes_{\Bbbk S_{n-1}} \Bbbk S_n \oplus \Bbbk S_n$ as a free right $\Bbbk S_n$-module. The homomorphism θ maps them to $(i\ \ i{+}1\ \ \cdots\ \ n{+}1) \in \Bbbk S_{n+1}$ for $1 \leq i \leq n{+}1$, which give a basis for $\Bbbk S_{n+1}$ as a free right $\Bbbk S_n$-module. So θ is indeed an isomorphism. We have just proved that

$$\mathrm{ind}_{S_{n-1}}^{S_n} \circ \mathrm{res}_{S_{n-1}}^{S_n} \oplus \mathrm{id}_{\Bbbk S_n\textbf{-mod}_{\mathrm{fd}}} \cong \mathrm{res}_{S_n}^{S_{n+1}} \circ \mathrm{ind}_{S_n}^{S_{n+1}},$$

which is a special case of the Mackey theorem.

Now we apply the general construction from Section 4 to equip $\Bbbk S\textbf{-mod}_{\mathrm{fd}}$ with the structure of a Kac-Moody categorication. Hence, we can freely apply the general results (1) to (10) about Kac-Moody categorifications from the end of Section 2. Let $L(b)$ $(b \in \mathcal{B}_n)$ be a full set of pairwise inequivalent irreducible left $\Bbbk S_n$-modules. Thus, the set $\mathcal{B} := \mathcal{B}_0 \sqcup \mathcal{B}_1 \sqcup \cdots$ indexes a full set $L(b)$ $(b \in \mathcal{B})$ of irreducibles in $\Bbbk S\textbf{-mod}_{\mathrm{fd}}$. Of course, there is a well-known parametrization by certain partitions, but we do not want to assume knowledge of any such explicit construction yet. However, we will use the notation $\varnothing$ to denote the unique element of $\mathcal{B}_0$, so $L(\varnothing) \cong \Bbbk S_0$.

The next step is to determine the spectrum I. By a *generator* for a Heisenberg categorification we mean an object P such that the set of objects obtained by applying all finite sequences of the functors E and F to P is a generating family in the usual sense of Abelian categories.

Lemma 5.1. *Let P be a generator for a Heisenberg categorification $\mathbf{R}$. The spectrum I of $\mathbf{R}$ is the closure of the set of roots of the minimal polynomials of the endomorphisms* $\overset{\bullet}{\uparrow} : EP \to EP$ *and* $\overset{\bullet}{\downarrow} : FP \to FP$ *under the operation $i \mapsto i \pm 1$.*

Proof. Let I be the spectrum of $\mathbf{R}$ and J be the closure of the set of roots of the minimal polynomials in the statement of the lemma. We have already observed that the spectrum is closed under the operation $i \mapsto i \pm 1$, hence, $I \supseteq J$. To complete the proof, we assume for a contradiction that $I \neq J$. Because P is a generator, we can find some $i \in I - J$ and a finite composition G of the functors E_j, F_j $(j \in J)$ such that either $E_i GP$ or $F_i GP$ is non-zero. We have that $a_{i,j} = 0$ for all $j \in J$ so, by the defining relations for $\mathfrak{U}$, E_i and F_i both commute with G (up to isomorphism). We deduce that either $E_i P$ or $F_i P$ is non-zero, hence, i is a root of one of the minimal polynomials used to define J. This is a contradiction. $\square$

To apply this to the example in hand, note that the irreducible module $L(\varnothing)$ is a generator because $\Bbbk S_n \cong E^n(\Bbbk S_0)$ for $n \geq 0$. Also, the minimal polynomials $m_\varnothing(x)$ and $n_\varnothing(x)$ are easily checked to be x and 1, respectively. Applying Lemma 5.1, we deduce that the spectrum I of $\mathbf{R}$ is the closure of $\{0\}$, so it is $\mathbb{Z}$ if $p = 0$ or the prime subfield $\mathbb{Z}/p\mathbb{Z}$ of $\Bbbk$ if $p > 0$. Taking X be the weight lattice exactly as in (4.3), the locally unital $\mathbb{C}$-algebra $\dot{\mathrm{U}}$ from Section 2 is the modified form of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_\infty$ if $p = 0$, or of the Lie algebra $\widehat{\mathfrak{sl}}'_p$ if $p > 0$.

Everything is now set up the way we want in the $p = 0$ case, but when $p > 0$ it is helpful to modify the general construction slightly so that $\dot{\mathrm{U}}$ becomes the modified form of the universal enveloping algebra of $\widehat{\mathfrak{sl}}_p$, which has the additional scaling element d compared to $\widehat{\mathfrak{sl}}'_p$. This is done by using some extra structure of $\mathbf{R} = \Bbbk S\text{-}\mathbf{mod}_{\mathrm{fd}}$: letting $\mathbf{R}_n := \Bbbk S_n\text{-}\mathbf{mod}_{\mathrm{fd}}$ if $n \geq 0$, or the full subcategory consisting of all of the zero objects if $n < 0$, we have the decomposition

$$\mathbf{R} = \bigoplus_{n \in \mathbb{N}} \mathbf{R}_n \tag{5.2}$$

as an internal direct sum of Serre subcategories. Moreover, the following additional axiom, which is analogous to (KM0), is obviously satisfied:

(H0) E maps objects of $\mathbf{R}_n$ to $\mathbf{R}_{n+1}$; equivalently, F maps objects of $\mathbf{R}_{n+1}$ to $\mathbf{R}_n$.

We incorporate this extra degree information into the Kac-Moody categorification by extending the weight lattice to the free Abelian group

$$X := \tfrac{1}{2p}\mathbb{Z}\delta \oplus \bigoplus_{i \in I} \mathbb{Z}\varpi_i \tag{5.3}$$

for some additional weight δ, the *null root*. We redefine the simple coroots $h_i : X \to \mathbb{Z}$ so that $h_i(\delta) = 0$ and $h_i(\varpi_j) = \delta_{i,j}$ for all $i, j \in I$, and we redefine the simple roots by setting

$$\alpha_j := \delta_{j,0}\delta + \sum_{i \in I} a_{i,j}\varpi_i. \tag{5.4}$$

Notice now that the simple roots are linearly independent, whereas before they summed to 0. The scaling element d is a homomorphism $d : X \to \frac{1}{2p}\mathbb{Z}$ with $d(\delta) = 1$, $d(\varpi_i) = 0$ and $d(\alpha_i) = \delta_{i,0}$. Let

$$\deg := pd + \sum_{i=0}^{p-1} \tfrac{i(p-i)}{2} h_i : X \to \tfrac{1}{2}\mathbb{Z}, \tag{5.5}$$

so $\deg(\delta) = p$, $\deg(\varpi_i) = \frac{i(p-i)}{2}$ and, most importantly, $\deg(\alpha_i) = 1$. Then we modify the definition of the function $\mathrm{wt} : \mathcal{B} \to X$ from (4.5) by redefining the weight $\mathrm{wt}(b) \in X$ of $b \in \mathcal{B}$ so that $\deg(\mathrm{wt}(b))$ is the degree of $L(b)$, that is, n if $b \in \mathcal{B}_n$, and $h_i(\mathrm{wt}(b)) = \phi_i(b) - \varepsilon_i(b)$ for each $i \in I$ as before. Finally, we use this new definition of wt to obtain the weight decomposition (4.7) with weight subcategories parametrized now by the δ-extended weight lattice. Since E_i takes $\Bbbk S_n$-modules to $\Bbbk S_{n+1}$-module and $\deg(\alpha_i) = 1$, the functors E_i and F_i still satisfy (KM0). The remaining checks in the proof of Theorem 4.2 go through as before, making $\mathbf{R}$ into a Kac-Moody categorification with this extended weight lattice.

We would next like to understand the $\dot{\mathrm{U}}$-module $\mathbb{C} \otimes_{\mathbb{Z}} G_0(\Bbbk S\text{-}\mathbf{mod}_{\mathrm{fd}})$. The general theory gives that it is an integrable $\dot{\mathrm{U}}$-module with a perfect basis given by the isomorphism classes $[L(b)] (b \in \mathcal{B})$. Since $m_\varnothing(x) = x$ and $n_\varnothing(x) = 1$, we have that

$$\varepsilon_i(\varnothing) = \delta_{i,0}, \qquad \phi_i(\varnothing) = 0 \tag{5.6}$$

for $i \in I$. Applying (4.5), we deduce that $\mathrm{wt}(\varnothing) = -\varpi_0$. Let $V(-\varpi_0)$ be the irreducible $\dot{\mathrm{U}}$-module of lowest weight $-\omega_0$, that is, the quotient of $\dot{\mathrm{U}}1_{-\omega_0}$ by the left ideal generated by $e_i^{(1+\delta_{i,0})}1_{-\omega_0}$ and $f_i 1_{-\omega_0}$ for all $i \in I$. By (5.6), there is a injective $\dot{\mathrm{U}}$-module homomorphism $V(-\varpi_0) \to \mathbb{C} \otimes_{\mathbb{Q}} G_0(\Bbbk S\text{-}\mathbf{mod}_{\mathrm{fd}})$ mapping the generator of $V(-\varpi_0)$ to $[L(\varnothing)]$. In fact, this homomorphism is an isomorphism, so that

$$\mathbb{C} \otimes_{\mathbb{Z}} G_0(\Bbbk S\text{-}\mathbf{mod}_{\mathrm{fd}}) \cong V(-\varpi_0). \tag{5.7}$$

There are several ways to prove this, depending on how much one is willing to assume about integrable representations of $\dot{\mathrm{U}}$. The simplest approach is to compare the dimensions of the sum of the weight spaces of these two modules for all weights $\lambda \in X$ with $\deg(\lambda) = n$, that is, the weights λ of the form $-\varpi_0 + \alpha_{i_1} + \cdots + \alpha_{i_n}$ for $i_1, \ldots, i_n \in I$. For the module $\mathbb{C} \otimes_{\mathbb{Z}} G_0(\Bbbk S\text{-}\mathbf{mod}_{\mathrm{fd}})$, this is the dimension of the subspace $\mathbb{C} \otimes_{\mathbb{Z}} G_0(\Bbbk S_n\text{-}\mathbf{mod}_{\mathrm{fd}})$, which by some basic representation theory of finite groups is equal to the number of conjugacy classes of S_n if $p = 0$, or the number of p-regular conjugacy classes of S_n if $p > 0$. This is the number of partitions of n if $p = 0$ or the number of p-regular partitions of n if $p > 0$. It remains to observe that the same number computes the dimension of the sum of the degree n weight spaces of $V(-\varpi_0)$, hence, our homomorphism is an isomorphism. This argument uses some knowledge of the basic representation $V(-\varpi_0)$, which we discuss a little further below.

We focus in this paragraph on the $p = 0$ case. Then $V(-\varpi_0)$ is a minuscule representation of $\mathfrak{sl}_\infty$ with the following completely explicit construction: it is the $\mathbb{C}$-vector space with basis v_λ indexed by the set of partitions, the vector $v_\varnothing$ is of weight $-\varpi_0$, and the Chevalley generators e_i (resp., f_i) act by mapping v_λ to v_μ for μ obtained adding (resp., removing) a node of content i to (resp., from) the Young diagram[6] of λ if such a node exists, or to 0 otherwise. The same combinatorics gives a realization of the crystal $\mathcal{B}(-\varpi_0)$ of $V(-\varpi_0)$ as Young's partition lattice. Since all perfect bases for an integrable lowest or highest weight module have canonically isomorphic underlying crystals by [BK07, Th. 5.37], the crystal $\mathcal{B}(-\varpi_0)$ is also the associated crystal of the Kac-Moody categorification $\Bbbk S\text{-}\mathbf{mod}_{\mathrm{fd}}$. It follows that the labelling set $\mathcal{B}$ is identified with the set of partitions, and we have in our hands explicit branching rules for the i-induction and i-restriction functors E_i and F_i. The fact that the labelling of irreducible $\Bbbk S_n$-modules thus obtained coincides with the standard labelling follows by comparing these branching rules with the classically known ones. The Okounkov-Vershik approach from [OV96] is particularly well suited for this comparison.

When $p > 0$, the crystal $\mathcal{B}(-\varpi_0)$ of the basic representation $V(-\varpi_0)$ was described explicitly in [MM90]. Its vertex set is naturally identified with the set of p-regular partitions; using this, the counting argument used to prove (5.7) is clear. Now, the result from [BK07, Th. 5.37]

[6] We use the English convention for Young diagrams, in which the content of a node is its column number minus its row number.

about uniqueness of crystals implies that the crystal associated to the Kac-Moody categorification $\Bbbk S$-$\mathbf{mod}_{\mathrm{fd}}$ is identified with $\mathcal{B}(-\varpi_0)$. With these arguments, which used some basic facts about integrable representations of $\widehat{\mathfrak{sl}}_p$ but almost no knowledge of the modular representation theory of symmetric groups, we have constructed an explicit parametrization of the isomorphism classes of irreducible representations of the symmetric groups over the field $\Bbbk$ of characteristic $p > 0$ by the set of p-regular partitions, and modular branching rules for the i-induction and i-restriction functors.

Comparing with the modular branching rules proved in [Kle96], it follows that the labelling of irreducibles that we have constructed coincides with the usual labelling from [Jam78]. The results in [Kle96] are themselves not so easy to prove (see also Remark 6.6), but we have only used them in this final step to identify the labelling arising from the general theory of categorification with the classical labelling. Many other fundamental results about modular representations of symmetric groups follow immediately from the general theory. The classical Nakayama Conjecture can be interpreted as the assertion that the non-zero weight subcategories are exactly the blocks of $\mathbf{R}$. Then property (9) from Section 2 recovers the Morita equivalences between blocks of symmetric groups constructed originally by Scopes [Sco91], and (10) gives the derived equivalences between blocks from [CR08].

6. Example: rational representations of GL_n and its Frobenius kernels

Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$. Let G be the general linear group scheme GL_n over $\Bbbk$, fixing also the standard choices of maximal torus T (diagonal matrices) and Borel subgroup B (upper triangular matrices). We begin with some reminders about the symmetric tensor category $\mathbf{Rep}(G)$ of finite-dimensional representations of G. Let V be the natural G-module with standard basis $v_1, \dots, v_n$. The character group $X(T)$ has basis $\delta_1, \dots, \delta_n$ defined by letting $\delta_i : T \to \Bbbk^\times, \mathrm{diag}(t_1, \dots, t_n)\big) \mapsto t_i$ be the weight of v_i. We let

$$\rho := -\delta_2 - 2\delta_3 - \cdots - (n-1)\delta_n. \tag{6.1}$$

For $\lambda \in X(T)$ and $1 \leq i \leq n$, we use the notation λ_i to denote $(\lambda + \rho, \delta_i) \in \mathbb{Z}$, where $(\cdot, \cdot)$ is the usual symmetric bilinear form on $X(T)$ defined so that $\delta_1, \dots, \delta_n$ are orthonormal. The irreducible G-modules are parametrized by highest weight theory by the set

$$X^+(T) = \{\lambda \in X(T) \mid \lambda_1 > \cdots > \lambda_n\} \tag{6.2}$$

of *dominant weights*. We denote the irreducible G-module of highest weight $\lambda \in X^+(T)$ by $L(\lambda)$.

The *algebra of distributions* $\mathrm{Dist}(G)$ is identified with $\Bbbk \otimes_{\mathbb{Z}} U_{\mathbb{Z}}$, where $U_{\mathbb{Z}}$ is the Kostant $\mathbb{Z}$-form for the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ generated by the divided power monomials $e_{i,j}^{(m)} := e_{i,j}^m/m!$ for $m \geq 1$ in the matrix units and $1 \leq i, j \leq n$ with $i \neq j$, plus the binomials $\binom{e_{i,i}}{m}$ for $m \geq 1$ and $1 \leq i \leq n$. It is useful because the canonical functor from $\mathbf{Rep}(G)$ to $\mathrm{Dist}(G)$-$\mathbf{mod}_{\mathrm{fd}}$ is an isomorphism of categories. Note also that the Lie algebra $\mathfrak{g}$ of G is identified with the Lie subalgebra of $\mathrm{Dist}(G)$ with basis $e_{i,j}$ $(1 \leq i, j \leq n)$. Let

$$\Omega := \sum_{i,j=1}^n e_{i,j} \otimes e_{j,i} \in \mathfrak{g} \otimes \mathfrak{g} \tag{6.3}$$

be the Casimir tensor. Acting by Ω defines a G-equivariant endomorphism of any tensor product $M_1 \otimes M_2$ of G-modules.

Now we can make $\mathbf{Rep}(G)$ into a degenerate Heisenberg categorification of central charge $\kappa = 0$. Here is the required data:

- The endofunctors E and F are $V \otimes -$ and $V^* \otimes -$, respectively. The adjunction (E, F) is the canonical one.

- The natural transformation : $E \Rightarrow E$ is defined on $M \in \operatorname{ob}\mathbf{Rep}(G)$ by the endomorphism $V \otimes M \to V \otimes M$ arising from the action of the Casimir tensor Ω.
- The natural transformation : $E \otimes E \Rightarrow E \otimes E$ is defined on $M \in \operatorname{ob}\mathbf{Rep}(G)$ by the endomorphism $V \otimes V \otimes M \to V \otimes V \otimes M, u \otimes v \otimes m \mapsto v \otimes u \otimes m$.

The axioms of Heisenberg categorification are easy to check. The isomorphism $V \otimes V^* \otimes M \xrightarrow{\sim} V^* \otimes V \otimes M$ defined by the rightward crossing is simply the tensor flip $v \otimes f \otimes m \mapsto f \otimes v \otimes m$.

Now we apply the construction from Section 4, making $\mathbf{Rep}(G)$ into a Kac-Moody categorification. As with any example, the next steps are to determine the spectrum I, the integrable representation that is the Grothendieck group of $\mathbf{Rep}(G)$, and the associated crystal. This requires a bit more knowledge about $\mathbf{Rep}(G)$. For each $\lambda \in X^+(T)$, the *Weyl module* $\Delta(\lambda)$ is the universal highest weight module of highest weight λ in $\mathbf{Rep}(G)$. In fact, $\mathbf{Rep}(G)$ is a *highest weight category*, and the Weyl modules are its standard modules.

Lemma 6.1. *For $\mathbf{Rep}(G)$, the spectrum I is $\mathbb{Z}/p\mathbb{Z}$. For $i \in I$ and $\lambda \in X^+(T)$, $E_i\Delta(\lambda)$ (resp., $F_i\Delta(\lambda)$) has a multiplicity-free filtration with sections that are isomorphic to the Weyl modules $\Delta(\lambda+\delta_j)$ (resp., $\Delta(\lambda-\delta_j)$) for $1 \leq j \leq n$ such that $\lambda+\delta_j \in X^+(T)$ and $\lambda_j \equiv i \pmod p$ (resp., $\lambda-\delta_j \in X^+(T)$ and $\lambda_j - 1 \equiv -n-i \pmod p$).*

Proof. We prove this for E_i. The first step is to observe that $E\Delta(\lambda) = V \otimes \Delta(\lambda)$ has a multiplicity-free filtration with sections that are isomorphic to the Weyl modules $\Delta(\lambda+\delta_j)$ for all $1 \leq j \leq n$ such that $\lambda+\delta_j \in X^+(T)$. To see this, we let $M_0 := \{0\}$, then for $j = 1, \dots, n$ we recursively define M_j to be the submodule of $V \otimes \Delta(\lambda)$ generated by M_{j-1} and $v_j \otimes v_+$, where v_+ is a highest weight vector. The image of $v_j \otimes v_+$ in M_j/M_{j-1} is easily seen to be a (possibly zero) highest weight vector of weight $\lambda+\delta_j$. Hence, M_j/M_{j-1} is a quotient of $\Delta(\lambda+\delta_j)$ if $\lambda+\delta_j$ is a dominant weight, and it is zero if $\lambda+\delta_j$ is not a dominant weight. It remains to observe that the sections M_j/M_{j-1} are actually isomorphic to $\Delta(\lambda+\delta_j)$ (rather than being proper quotients) for all j with $\lambda+\delta_j \in X^+(T)$. This follows by a calculation using Weyl's dimension formula.

Now we take $i \in \Bbbk$ and consider $E_i\Delta(\lambda)$, i.e., the projection of $V \otimes \Delta(\lambda)$ onto the generalized i-eigenspace of the endomorphism Ω. Let z_r be the unique element of the Harish-Chandra center of $\operatorname{Dist}(G)$ which acts on $\Delta(\lambda)$ by $\mathrm{e}_r(\lambda)$, the rth elementary symmetric polynomial evaluated at the scalars $\lambda_1, \dots, \lambda_n$. By [Bru08, Lem. 5.1], z_2 acts on $V \otimes \Delta(\lambda)$ in the same way as $1 \otimes z_2 + 1 \otimes z_1 - \Omega$. Consequently, Ω leaves each of the submodules M_j invariant and, on an Ω-eigenvector of eigenvalue i in $M_j/M_{j-1} \cong \Delta(\lambda+\delta_j)$, we have that

$$\mathrm{e}_2(\lambda+\delta_j) = \mathrm{e}_2(\lambda) + \mathrm{e}_1(\lambda) - i.$$

This identity implies that $i = \lambda_j \bmod p$. We deduce that $E_i\Delta(\lambda)$ is zero unless $i \in \mathbb{Z}/p\mathbb{Z}$, hence, $I = \mathbb{Z}/p\mathbb{Z}$. Moreover, for $i \in \mathbb{Z}/p\mathbb{Z}$, we have shown that $E_i\Delta(\lambda)$ has the filtration claimed.

The proof for F_i is very similar, using instead that z_2 acts on $V^* \otimes \Delta(\lambda)$ in the same way as $1 \otimes z_2 + 1 \otimes (1-n-z_1) - \Omega$. □

Henceforth, $I := \mathbb{Z}/p\mathbb{Z} \subset \Bbbk$. Taking the vanilla flavor of Kac-Moody categorification with weight lattice $X = \bigoplus_{i \in I} \varpi_i$, the algebra $\dot{\mathrm{U}}$ is the modified form of the universal enveloping algebra of $\widehat{\mathfrak{sl}}_p'$. It is a tolerable nuisance that there are two weight lattices in play now, $X(T)$ for G and X for $\dot{\mathrm{U}}_{\mathbb{Z}}$, so that λ could either mean an element of $X(T)$ or an element of the weight lattice X depending on the context. We use $\mathrm{M}_{\mathbb{Z}}$ to denote the level zero representation of $\dot{\mathrm{U}}_{\mathbb{Z}}$ which is the restricted dual of the natural representation of $\widehat{\mathfrak{sl}}_p'$. It is a free $\mathbb{Z}$-module with basis m_j ($j \in \mathbb{Z}$), on which the Chevalley generators act by

$$e_i m_j = \begin{cases} m_{j+1} & \text{if } j \equiv i \pmod p \\ 0 & \text{otherwise,} \end{cases} \qquad f_i m_j = \begin{cases} m_{j-1} & \text{if } j \equiv i+1 \pmod p \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

The weight of m_j is $-\epsilon_j$ where $\epsilon_j := \varpi_{j \bmod p} - \varpi_{(j-1) \bmod p}$. As $\mathbf{Rep}(G)$ is a highest weight category, the isomorphism classes of the Weyl modules give a basis for $G_0(\mathbf{Rep}(G))$. From Lemma 6.1, it follows that there is an isomorphism of $\dot{\mathrm{U}}_{\mathbb{Z}}$-modules

$$G_0(\mathbf{Rep}(G)) \xrightarrow{\sim} \bigwedge^n \mathrm{M}_{\mathbb{Z}}, \qquad [\Delta(\lambda)] \mapsto m_{\lambda_1} \wedge \cdots \wedge m_{\lambda_n}. \tag{6.5}$$

To complete this first part of the story, we describe the associated crystal $(X^+(T), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \phi_i, \mathrm{wt})$. Recall that Kashiwara introduced an associative (but not commutative) tensor product operation on normal crystals; see [Kas95]. Given normal crystals $\mathcal{B}_1$ and $\mathcal{B}_2$, their tensor product $\mathcal{B}_1 \otimes \mathcal{B}_2$ is the normal crystal with underlying set $\{b_1 \otimes b_2 \,|\, b_1 \in \mathcal{B}_1, b_2 \in \mathcal{B}_2\}$ with $\mathrm{wt}(b_1 \otimes b_2) := \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$. The crystal operators $\tilde{e}_i, \tilde{f}_i$ are defined by

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varepsilon_i(b_2) > \phi_i(b_1) \\ \tilde{e}_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_2) \leq \phi_i(b_1), \end{cases} \tag{6.6}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if } \phi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i(b_2) & \text{if } \phi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \tag{6.7}$$

assuming this makes sense (the crystal operators are undefined otherwise). Associated to the based module $\mathrm{M}_{\mathbb{Z}}$, there is a normal crystal with vertex set $\mathbb{Z}$, crystal operators defined by $\tilde{e}_i(j) = j+1 \Leftrightarrow j \equiv i \pmod p$, $\tilde{f}_i(j) = j-1 \Leftrightarrow j \equiv i+1 \pmod p$, and $\mathrm{wt} : \mathbb{Z} \to X$ defined by $\mathrm{wt}(j) := -\epsilon_j$. Taking the nth tensor power of this gives a crystal whose underlying set is $\mathbb{Z}^n$. We transport this to $X(T)$ using the bijection $X(T) \xrightarrow{\sim} \mathbb{Z}^n, \lambda \mapsto (\lambda_1, \ldots, \lambda_n)$.

Remark 6.2. Here is a more user-friendly description of the crystal operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\lambda \in X(T)$. Let $(\sigma_1, \ldots, \sigma_n) \in \{0, +, -\}^n$ be the *reduced i-signature* of λ. This is obtained by starting from the sequence with $\sigma_j := +$ if $\lambda_j = i$, $\sigma_j := -$ if $\lambda_j - 1 = i$, and $\sigma_j := 0$ otherwise, then repeatedly replacing subsequences of the form $-, 0, \ldots, 0, +$ with $0, 0, \ldots, 0, 0$ so that at the end there is no $-$ appearing to the left of a $+$. Then $\varepsilon_i(\lambda)$ is the number of entries equal to $+$ and, if this is non-zero, $\tilde{e}_i(\lambda) = \lambda + \delta_j$ where j is the index of the rightmost $+$. Similarly, $\phi_i(\lambda)$ is the number of entries equal to $-$ and, if this is non-zero, $\tilde{f}_i(\lambda) = \lambda - \delta_j$ where j is the index of the leftmost $-$.

Theorem 6.3. *The crystal associated to the Heisenberg categorification $\mathbf{Rep}(G)$ is the connected component of the $\widehat{\mathfrak{sl}}'_p$-crystal $(X(T), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \phi_i, \mathrm{wt})$ just described with vertex set $X^+(T) \subset X(T)$.*

Proof. This follows from [Kle96] using the explicit description in Remark 6.2. □

Everything so far is well known. To say something not quite so standard, we look instead at Frobenius kernels. The basic reference is [Jan03, Ch. II.3]. The *Frobenius morphism* $F : G \to G$ is the morphism of group schemes defined by raising matrix entries to the pth power. Let

$$G_r := \ker F^r$$

be the rth *Frobenius kernel*, which is a closed normal subgroup scheme. Also set $B_r := B \cap G_r$ and $T_r := T \cap G_r$. The category $\mathbf{Rep}(G_r)$ of finite-dimensional rational representations of G_r can be viewed equivalently as the category $\mathrm{Dist}(G_r)$-$\mathbf{mod}_{\mathrm{fd}}$, where $\mathrm{Dist}(G_r)$ is the subalgebra of $\mathrm{Dist}(G)$ generated by the divided powers $e_{i,j}^{(m)}$ $(i \neq j)$ and the binomials $\binom{e_{i,i}}{m}$ for all m with $1 \leq m \leq p^r - 1$. It is a finite-dimensional algebra of dimension p^{rn^2}.

Let $X(T_r) := X(T)/p^r X(T)$. This Abelian group naturally labels the one-dimensional representations of T_r. For $\bar{\lambda} \in X(T_r)$ with pre-image $\lambda \in X(T)$, we use $\bar{\lambda}_i$ to denote the unique element of $\mathbb{Z}/p^r\mathbb{Z}$ such that $(\lambda + \rho, \delta_i) \equiv \bar{\lambda}_i \pmod{p^r}$. Let $\Bbbk_{\bar{\lambda}}$ be the B_r-module that is the inflation of the one-dimensional T_r-module corresponding to $\bar{\lambda} \in X(T_r)$. The *baby Verma module* $Z_r(\bar{\lambda})$ is the

G_r-module $\mathrm{Dist}(G_r) \otimes_{\mathrm{Dist}(B_r)} \Bbbk_{\bar\lambda}$. It has simple head $L_r(\bar\lambda)$, and the modules $L_r(\bar\lambda)$ ($\bar\lambda \in X(T_r)$) give representatives for the isomorphism classes of irreducible G_r-modules.

We make $\mathbf{Rep}(G_r)$ into a degenerate Heisenberg categorification of central charge 0 in exactly the same way as we did for $\mathbf{Rep}(G)$. Hence, it is a Kac-Moody categorification by the construction from Section 4. The counterpart of Lemma 6.1 is as follows.

Lemma 6.4. *For $\mathbf{Rep}(G_r)$, the spectrum I is $\mathbb{Z}/p\mathbb{Z}$. For $i \in I$ and $\bar\lambda \in X(T_r)$, $E_iZ_r(\bar\lambda)$ (resp., $F_iZ_r(\bar\lambda)$) has a multiplicity-free filtration with sections that are isomorphic to the baby Verma modules $Z_r(\bar\lambda+\delta_j)$ (resp., $Z_r(\bar\lambda-\delta_j)$) for $1 \leq j \leq n$ such that $\bar\lambda_j \equiv i \pmod p$ (resp., $\bar\lambda_j - 1 \equiv -n-i \pmod p$).*

Proof. This follows from the same argument as used to prove Lemma 6.1. It is even a bit easier since there is no dominance constraint, and $\dim Z_r(\bar\lambda) = p^{rn(n-1)/2}$ always so that the dimension calculation is quite trivial. □

Lemma 6.4 allows us to describe the Grothendieck group as a $\dot{\mathrm{U}}_{\mathbb{Z}}$-module: letting $\mathrm{M}_{\mathbb{Z}}^{(r)}$ be the level zero representation of $\dot{\mathrm{U}}_{\mathbb{Z}}$ which is free as a $\mathbb{Z}$-module with basis $m_j^{(r)}$ ($j \in \mathbb{Z}/p^r\mathbb{Z}$) and action of Chevalley generators defined analogously to (6.4), there is an isomorphism

$$K_0(\mathbf{Rep}(G_r)) \xrightarrow{\sim} \left(\mathrm{M}_{\mathbb{Z}}^{(r)}\right)^{\otimes n}, \qquad [Z_r(\bar\lambda)] \mapsto m_{\bar\lambda_1}^{(r)} \otimes \cdots \otimes m_{\bar\lambda_n}^{(r)}. \tag{6.8}$$

Associated to the module $\mathrm{M}_{\mathbb{Z}}^{(r)}$, there is a normal $\widehat{\mathfrak{sl}}_p'$-crystal with vertex set $\mathbb{Z}/p^r\mathbb{Z}$. The crystal operators and weight function are defined analogously to the crystal of $\mathrm{M}_{\mathbb{Z}}$ described above. We take the nth tensor power of this to obtain a crystal with underlying set $(\mathbb{Z}/p^r\mathbb{Z})^n$, then transport this to $X(T_r)$ using the bijection $X(T_r) \xrightarrow{\sim} (\mathbb{Z}/p^r\mathbb{Z})^n, \bar\lambda \mapsto (\bar\lambda_1, \dots, \bar\lambda_n)$. It is easy to see from this definition that the quotient map $X(T) \twoheadrightarrow X(T_r), \lambda \mapsto \bar\lambda$ is a morphism of $\widehat{\mathfrak{sl}}_p'$-crystals.

Theorem 6.5. *The crystal associated to the Heisenberg categorification $\mathbf{Rep}(G_r)$ is the crystal with underlying set $X(T_r)$ just described, that is, it is the nth tensor power of the $\widehat{\mathfrak{sl}}_p'$-crystal of $\mathrm{M}_{\mathbb{Z}}^{(r)}$.*

Proof. We will deduce this from Theorem 6.3 using the following basic facts. A weight $\lambda \in X^+(T)$ is p^r*-restricted* if $\lambda_i - \lambda_{i+1} \leq p^r$ for $i = 1, \dots, n-1$. By [Jan03, Prop. II.3.15] and the Steinberg tensor product theorem, for $\lambda \in X^+(T)$, the restriction $\mathrm{res}^G_{G_r} L(\lambda)$ is completely reducible, and it is irreducible if and only if λ is p^r-restricted, in which case it is isomorphic to $L(\bar\lambda)$ where $\bar\lambda$ is the canonical image of λ in $X(T_r)$.

Now take $\bar\lambda \in X(T_r)$ and let $\lambda \in X^+(T)$ be a p^r-restricted lift. We know that the G-module $E_iL(\lambda)$ is non-zero if and only if $\varepsilon_i(\lambda) > 0$, in which case it has irreducible head $\cong L(\tilde e_i(\lambda))$. Hence $E_iL_r(\bar\lambda) \cong E_i\left(\mathrm{res}^G_{G_r} L(\lambda)\right) \cong \mathrm{res}^G_{G_r}\left(E_iL(\lambda)\right)$ is non-zero if and only if $\varepsilon_i(\lambda) > 0$. Assuming it is non-zero, we also know that the G_r-module $E_iL_r(\bar\lambda)$ has irreducible head. This implies that $\tilde e_i(\lambda)$ must be p^r-restricted (this can also be seen directly using the combinatorics from Remark 6.2) and the irreducible head of $E_iL_r(\bar\lambda)$ is isomorphic to $L_r(\overline{\tilde e_i(\lambda)}) \cong \mathrm{res}^G_{G_r} L(\tilde e_i(\lambda))$. The lemma follows since the map $X(T) \twoheadrightarrow X(T_r)$ is a crystal morphism. □

The category $\mathbf{Rep}(G)$ has a natural degree decomposition defined by the action of the one-dimensional torus that is the center of G, e.g., the trivial module is in degree 0. We find it more convenient to shift degrees so as to put the trivial module instead into degree $-\frac{n(n-1)}{2}$. Applying E adds 1 to the degree of a homogeneous object, i.e., the axiom (H0) formulated after (5.2) holds. Consequently, we can modify the above construction to make $\mathbf{Rep}(G)$ into a Kac-Moody categorification for the extended weight lattice from (5.3). This is done in the same way as explained in the previous section, redefining simple roots and simple coroots as there, and modifying the weight function $\mathrm{wt} : X^+(T) \to X$ so that $\deg(\mathrm{wt}(\lambda))$ is the degree of $L(\lambda)$, and $h_i(\mathrm{wt}(\lambda)) = \phi_i(\lambda) - \varepsilon_i(\lambda)$ for each i as usual. We redefine the weights ϵ_j by setting

$$\epsilon_j := \varpi_{j \bmod p} - \varpi_{(j-1) \bmod p} + \left(\tfrac{p-1}{2p} - \left\lceil \tfrac{j}{p} \right\rceil\right)\delta \in X. \tag{6.9}$$

The term $-\left\lceil \frac{j}{p} \right\rceil \delta$ is needed here so that $\epsilon_j - \epsilon_{j+1} = \alpha_{j \bmod p}$ for every $j \in \mathbb{Z}$; cf. (5.4). The additional term $\frac{p-1}{2p}\delta$ in (6.9) is somewhat arbitrary; we included it so that the function deg from (5.5) satisfies the memorable formula $\deg(\epsilon_j) = -j$ for any $j \in \mathbb{Z}$. We then have for any $\lambda \in X(T)$ that

$$\mathrm{wt}(\lambda) = -\epsilon_{\lambda_1} - \cdots - \epsilon_{\lambda_n}. \tag{6.10}$$

Now, $\dot{\mathrm{U}}$ is the modified form for the enveloping algebra of $\widehat{\mathfrak{sl}}_p$ (rather than $\widehat{\mathfrak{sl}}_p'$). The action of $\dot{\mathrm{U}}_{\mathbb{Z}}$ on $\mathrm{M}_{\mathbb{Z}}$ from before extends by declaring that the weight of the basis vector m_j is the new $-\epsilon_j$. It is again the case that

$$K_0(\mathbf{Rep}(G)) \xrightarrow{\sim} \bigwedge\nolimits^n \mathrm{M}_{\mathbb{Z}}, \qquad [\Delta(\lambda)] \mapsto m_{\lambda_1} \wedge \cdots \wedge m_{\lambda_n}. \tag{6.11}$$

The associated crystal has the same crystal operators as in Theorem 6.3, but using (6.10) as its weight function so that it is now an $\widehat{\mathfrak{sl}}_p$-crystal.

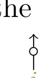
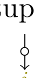
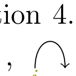
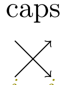
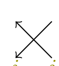
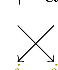
For Frobenius kernels, there is no degree decomposition, but this can be fixed by working instead with the *thickened Frobenius kernels* G_rT as in [Jan03, Sec. II.9]. Irreducible modules in $\mathbf{Rep}(G_rT)$ are parametrized by their highest weights: for $\lambda \in X(T)$, we write $\hat{L}_r(\lambda)$ for the irreducible G_rT-module of highest weight λ, which may be constructed explicitly as the unique irreducible quotient of $\hat{Z}_r(\lambda) := \mathrm{Dist}(G_rT) \otimes_{\mathrm{Dist}(B_rT)} \Bbbk_\lambda$. Then we make $\mathbf{Rep}(G_rT)$ into a degenerate Heisenberg categorification of central charge zero as above. It becomes a Kac-Moody categorification using the extended weight lattice X from (5.3) and the weight function (6.10). Letting $\dot{\mathrm{U}}_{\mathbb{Z}}$ and $\mathrm{M}_{\mathbb{Z}}$ be as in the previous paragraph, there is an isomorphism of $\dot{\mathrm{U}}_{\mathbb{Z}}$-modules

$$K_0(\mathbf{Rep}(G_rT)) \xrightarrow{\sim} \mathrm{M}_{\mathbb{Z}}^{\otimes n}, \qquad [\hat{Z}_r(\lambda)] \mapsto m_{\lambda_1} \otimes \cdots \otimes m_{\lambda_n}. \tag{6.12}$$

This follows from the obvious extended analogue of Lemma 6.4. Also the associated crystal is the crystal $(X(T), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \phi_i, \mathrm{wt})$ defined just before Theorem 6.3 but with wt as in (6.10), that is, it is the nth tensor power of the $\widehat{\mathfrak{sl}}_p$-crystal of $\mathrm{M}_{\mathbb{Z}}$. This may be deduced from Theorem 6.5 using that $\mathrm{res}^{G_rT}_{G_r} \hat{L}_r(\lambda) \cong L_r(\bar{\lambda})$ for $\lambda \in X(T)$ with image $\bar{\lambda} \in X(T_r)$. We observe this crystal is *the same* for all $r \geq 1$, although the categories $\mathbf{Rep}(G_rT)$ themselves are quite different as r varies.

Remark 6.6. In [Los13] (see also [LW15, Sec. 7]), Losev has developed a more conceptual approach for computing the associated crystal of a *tensor product categorification* in the sense of [LW15, Rem. 3.6] via Kashiwara's tensor product rule. This can be applied to $\mathbf{Rep}(G_rT)$, which is an n-fold tensor product categorification, thereby giving a more direct way to identify the associated crystal in this case. The earlier descriptions of associated crystals in Theorems 6.3 and 6.5, and also Kleshchev's result for symmetric groups discussed in the previous section, can be deduced as consequences. This is desirable since the ad hoc argument in [Kle96] is rather complicated.

7. Explicit formulae for the second adjunction

We go back now to the setup of Section 4. Prior to Theorem 4.2, we defined the Kac-Moody natural transformations  ,  ,  ,  and  . There are also natural transformations depicted by the leftward Kac-Moody caps and cups  and  , and the rightward, leftward and downward Kac-Moody crossings  ,  and  , which are defined by (2.4) and the properties itemized after (2.11). Then we can introduce the bubble generating functions $\circlearrowleft_i(u)\,\lambda$ and $\circlearrowright_i(u)\,\lambda$, which are the formal power series in $Z(\mathbf{R}_\lambda)(\!(u^{-1})\!)$ with leading terms $u^{h_i(\lambda)}$ and $u^{-h_i(\lambda)}$, respectively, satisfying (2.22) and (2.23).

A shortcoming of Theorem 4.2 is that it gives no clue as to exactly how to express the leftward Kac-Moody caps, cups and crossings in terms of the natural transformations arising from the initially given Heisenberg action on $\mathbf{R}$. We are going to fill this gap in this section. We find it easier to go in the other direction, so we focus initially on the problem of expressing the natural transformations defined by the leftward Heisenberg caps, cups and crossings in terms of the ones defined by the leftward Kac-Moody ones. Some preparation is needed before we can write this down.

The definition of the Kac-Moody bubble generating functions gives that

$$\circlearrowleft_i(u)\,\lambda = \overset{\leftarrow}{\boxed{u}}_i\,\lambda + \circlearrowleft_i\!\boxed{u}\,\lambda, \qquad \circlearrowright_i(u)\,\lambda = \overset{\rightarrow}{\boxed{u}}_i\,\lambda + \boxed{u}\!\circlearrowright_i\,\lambda, \tag{7.1}$$

with $\overset{\leftarrow}{\boxed{u}}_i\,\lambda$ (resp., $\overset{\rightarrow}{\boxed{u}}_i\,\lambda$) being a monic polynomial of degree $h_i(\lambda)$ (resp., $-h_i(\lambda)$) if this is a non-negative integer, or 0 otherwise. We refer to these as the *fake bubble polynomials*. Replacing u by $u-i$ gives the equivalent identities

$$\circlearrowleft_i(u-i)\,\lambda = \overset{\leftarrow}{\boxed{u-i}}_i\,\lambda + \circlearrowleft_i\!\!-\!\boxed{(u-x-i)^{-1}}\,\lambda, \quad \circlearrowright_i(u-i)\,\lambda = \overset{\rightarrow}{\boxed{u-i}}_i\,\lambda + \boxed{(u-x-i)^{-1}}\!-\!\!\circlearrowright_i\,\lambda. \tag{7.2}$$

These are decompositions of formal Laurent series in u^{-1} as the sum of its polynomial and non-polynomial parts. It is still the case that the *shifted fake bubble polynomial* $\overset{\leftarrow}{\boxed{u-i}}_i\,\lambda$ (resp., $\overset{\rightarrow}{\boxed{u-i}}_i\,\lambda$) is monic of degree $h_i(\lambda)$ (resp., $-h_i(\lambda)$) if this is a non-negative integer, or 0 otherwise.

Now comes the main new definition. For $i \neq j$ in I, we define the counterclockwise and clockwise *internal bubbles of color j* to be the natural transformations

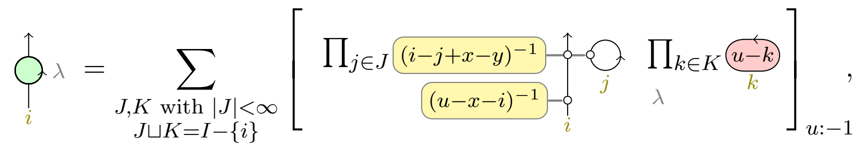
$$\uparrow_i\!\!\circledcirc_j\,\lambda := \boxed{(i-j+x-y)^{-1}}\!-\!\uparrow_i\!-\!\circlearrowleft_j\,\lambda + \left[\boxed{(u-x-i)^{-1}}\!-\!\uparrow_i\ \overset{\leftarrow}{\boxed{u-j}}_j\,\lambda\right]_{u:-1}, \tag{7.3}$$

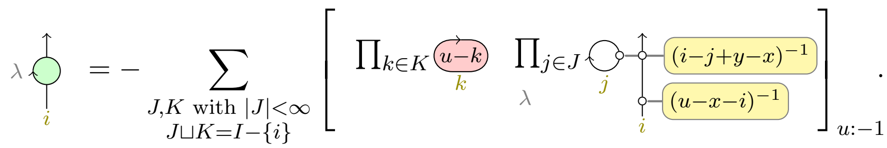
$$\lambda\,\circledcirc_j\!\!\uparrow_i := \lambda\,\circlearrowright_j\!-\!\uparrow_i\!-\!\boxed{(i-j+y-x)^{-1}} + \left[\lambda\,\overset{\rightarrow}{\boxed{u-j}}_j\ \uparrow_i\!-\!\boxed{(u-x-i)^{-1}}\right]_{u:-1}. \tag{7.4}$$

The first terms on the right hand sides of (7.3) and (7.4) make sense because $i-j \in \Bbbk^\times$ and x and y are locally nilpotent. If $h_j(\lambda) = 0$, as is the case for all but finitely many $j \in I$, then the shifted fake bubble polynomials appearing in the second terms on the right hand sides are identities, hence, these terms are simply equal to $\uparrow_i\,\lambda$ or $\lambda\,\uparrow_i$, respectively. Moreover, still with $h_j(\lambda) = 0$, all of the bubbles with a non-negative number of dots appearing in the expansions of the first terms on the right hand sides of (7.3) and (7.4) are locally nilpotent. This follows because all such bubbles act as zero on irreducible objects of $\mathbf{R}_\lambda$ according to property (5) from Section 2. Consequently, for any indecomposable object $V \in \operatorname{ob}\mathbf{R}_\lambda$ or $\operatorname{ob}\mathbf{R}_{\lambda-\alpha_i}$, respectively, and all but finitely many $i \neq j \in I$, the internal bubbles of color j from (7.3) and (7.4) define elements of $\operatorname{End}_{\mathbf{R}}(E_i V)$ of the form $1+z$ for z in the Jacobson radical of this finite-dimensional algebra. Thus, it makes sense to take the (possibly infinite) product over all $i \neq j \in I$ of these commuting natural transformations, defining the counterclockwise and clockwise *internal bubbles* to be the vertical compositions


(7.5)

Opening some parentheses and using (2.16) gives the following more explicit formulae:


(7.6)


(7.7)

In these expressions, the product over J applies to the double pin at the top (the pin labelled $(u-x-i)^{-1}$ is only taken once), and the product over K applies to the shifted fake bubble polynomial. The (possibly infinite) product over K makes sense because all but finitely many of the shifted fake bubble polynomials are 1. The (possibly infinite) sum at the front makes sense because, on any given indecomposable V, the expression inside is 0 for all but finitely many J.

We also introduce counterclockwise and clockwise internal bubbles on downward strings


(7.8)

whose definitions are obtained from the ones in the previous paragraph by rotating all diagrams through 180°. This ensures that all sorts of internal bubbles slide across both rightward and leftward caps and cups.

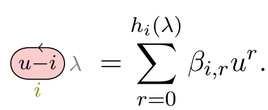
Theorem 7.1. *The leftward Heisenberg caps and cups are related to the leftward Kac-Moody caps and cups by*


(7.9)

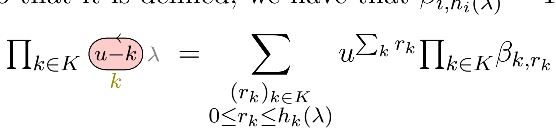
for $i, j \in I$ and $\lambda \in X$. Hence, the Heisenberg bubble generating functions are related to the Kac-Moody bubble generating functions by


(7.10)

Proof. We prove this just in the case that $\kappa \leq 0$; an analogous argument treats $\kappa > 0$. All of the equations to be proved are trivial when $\mathbf{R}_\lambda$ is zero, so we assume from now on that we are given some fixed $\lambda \in X$ such that $\mathbf{R}_\lambda$ is non-zero. We will use the shorthand $\beta_{i,r}$ to denote coefficients of the shifted fake bubble polynomials on the Kac-Moody side:

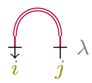
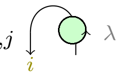
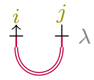
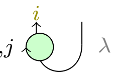

(7.11)

Assuming that $h_i(\lambda) \geq 0$ so that it is defined, we have that $\beta_{i,h_i(\lambda)} = 1$. We then have that


(7.12)

We will use this several times below to rewrite the fake bubble polynomials appearing in (7.6).

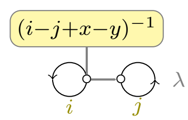
Now we define two new natural transformations denoted  and  on the Heisenberg side so that they satisfy (7.9), that is,

 $:= \delta_{i,j}$  ,    $= \delta_{i,j}$    (7.13)

for $i, j \in I$. Using this new version of the leftward Heisenberg cap, we obtain a new version of the counterclockwise Heisenberg bubble generating function, denoted $(u)\ \lambda$. It is defined like in (3.14) so that it is a formal Laurent series in $Z(\mathbf{R})[\![u^{-1}]\!]$ with leading term u^κ and $\left[(u)\ \lambda \right]_{u:<0} =$ u. Now we proceed to prove a series of claims, which together prove the theorem.

<u>Claim 1</u>. $(u)\ \lambda\ = \prod_{i\in I} \underset{i}{\bigcirc}(u-i)\ \lambda$.

<u>Proof</u>. To prove this, we will use the following fancy notation. Suppose we are given a finite subset $J \subseteq I$. Let A_J be the polynomial algebra $\Bbbk[x_j \mid j \in J]$. There is a linear map (usually *not* an algebra homomorphism!) $\theta_J : A_J \to Z(\mathbf{R}_\lambda)$ mapping the polynomial $f(x_j \mid j \in J)$ to the natural transformation obtained by pinning $f(x_j + j \mid j \in J)$ to the string diagram $\prod_{j\in J} \underset{j}{\bigcirc}\ \lambda$ with the variable x_j corresponding to the dot on the jth bubble. Let $\widehat{A}_J$ be the completion of A_J at the maximal ideal $(x_j - j \,|\, j \in J)$. Since the Kac-Moody dots are locally nilpotent, θ_J naturally extends to a linear map $\widehat{\theta}_J : \widehat{A}_J \to Z(\mathbf{R}_\lambda)$. For example, we have that

$$\widehat{\theta}_{\{i,j\}} \left(\frac{1}{x_i - x_j} \right) =$$    (7.14)

for $i \neq j$ in I. To prove the claim, as $\kappa \leq 0$, the left hand side of the equation to be proved is equal to $\delta_{\kappa,0} +$ $u\ \lambda$. As $\sum_{i\in I} h_i(\lambda) = \kappa$ by (4.6) and each $\underset{i}{\bigcirc}(u-i)\ \lambda$ has leading term $u^{h_i(\lambda)}$, the right hand side is also equal to $\delta_{\kappa,0}$ plus a term in $u^{-1}Z(\mathbf{R}_\lambda)[\![u^{-1}]\!]$. Thus, we are reduced to proving that

$$u\ \lambda\ = \left[\prod_{i\in I} \underset{i}{\bigcirc}(u-i)\ \lambda \right]_{u:<0} . \tag{7.15}$$

Using (7.2) and opening some parentheses, the right hand side of (7.15) is equal to

$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ J\sqcup K = I}} \left[\left(\prod_{j\in J} \underset{j}{\bigcirc} \text{—} (u-x-j)^{-1}\ \lambda \right) \left(\prod_{k\in K} \underset{k}{(u-k)}\ \lambda \right) \right]_{u:<0} .$$

Using the fancy notation and (7.12), this is equal to

$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ J\sqcup K = I}} \ \sum_{\substack{(r_k)_{k\in K} \\ 0\leq r_k \leq h_k(\lambda)}} \widehat{\theta}_J \left(\left[u^{\sum_k r_k} \prod_{j\in J} \frac{1}{u - x_j} \right]_{u:<0} \right) \prod_{k\in K} \beta_{k,r_k}. \tag{7.16}$$

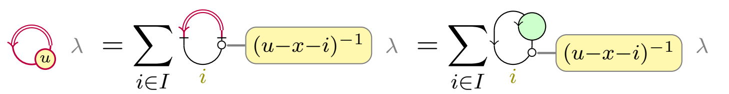
By partial fractions, we have in $\widehat{A}_J[\![u^{-1}]\!]$ that

$$\left[u^r \prod_{j\in J} \frac{1}{u-x_j} \right]_{u:<0} = \left[\sum_{i\in J} \frac{u^r}{u-x_i} \prod_{i\neq j\in J} \frac{1}{x_i - x_j} \right]_{u:<0} \overset{(2.15)}{=} \sum_{i\in J} \frac{x_i^r}{u-x_i} \prod_{i\neq j\in J} \frac{1}{x_i - x_j}.$$

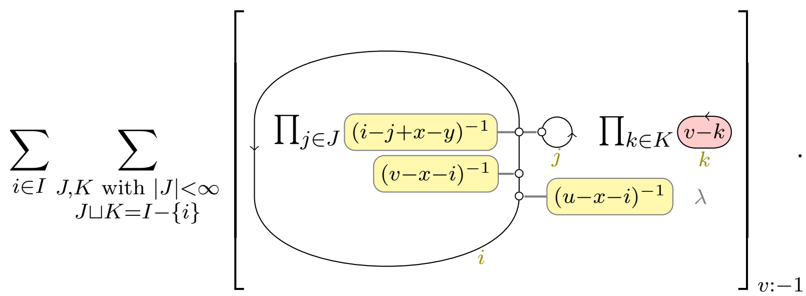
Using this identity, we can simplify (7.16), that is, the right hand side of (7.15), to obtain

$$\sum_{\substack{J,K \text{ with } 0<|J|<\infty\\ J\sqcup K=I}} \sum_{\substack{(r_k)_{k\in K}\\ 0\leq r_k\leq h_k(\lambda)}} \widehat{\theta}_J\left(\sum_{i\in J}\frac{x_i^{\sum_k r_k}}{(u-x_i)\prod_{i\neq j\in J}(x_i-x_j)}\right)\textstyle\prod_{k\in K}\beta_{k,r_k}. \tag{7.17}$$

Now we turn our attention to the left hand side of (7.15). Using (4.13) to (4.15) then the definition (7.9), we get that



Applying (7.6) (replacing u there with v), this equals



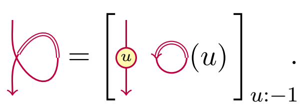
Switching to the fancy notation using a variation on (7.14) and (7.11), this is

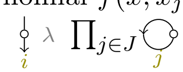
$$\sum_{i\in I}\sum_{\substack{J,K \text{ with } |J|<\infty\\ J\sqcup K=I-\{i\}}} \sum_{\substack{(r_k)_{k\in K}\\ 0\leq r_k\leq h_k(\lambda)}} \widehat{\theta}_{J\cup\{i\}}\left(\left[\frac{v^{\sum_k r_k}}{(u-x_i)(v-x_i)\prod_{j\in J}(x_i-x_j)}\right]_{v:-1}\right)\textstyle\prod_{k\in K}\beta_{k,r_k}.$$

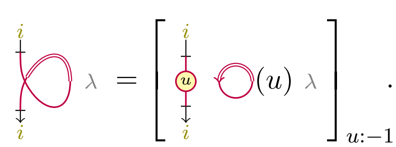
Applying (2.15), this is

$$\sum_{i\in I}\sum_{\substack{J,K \text{ with } |J|<\infty\\ J\sqcup K=I-\{i\}}} \sum_{\substack{(r_k)_{k\in K}\\ 0\leq r_k\leq h_k(\lambda)}} \widehat{\theta}_{J\cup\{i\}}\left(\frac{x_i^{\sum_k r_k}}{(u-x_i)\prod_{j\in J}(x_i-x_j)}\right)\textstyle\prod_{k\in K}\beta_{k,r_k}.$$

Finally, to see that this is equal to (7.17), one just needs to reindex the second summation, replacing J by $J\sqcup\{i\}$, then move the first summation over $i\in I$ to the inside.

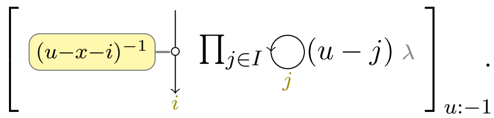
Claim 2. 

Proof. We need to extend slightly the fancy notation introduced in the proof of Claim 1. Suppose we are given a finite subset $J\subseteq I$ and $i\in I$. There is a linear map (usually *not* an algebra homomorphism!) $\theta_{J,i}: A_J[x]\to \operatorname{End}(F_i|_{\mathbf{R}_\lambda})$ mapping the polynomial $f(x,x_j\mid j\in J)$ to the natural transformation obtained by pinning $f(x+i,x_j+j\mid j\in J)$ to  with x corresponding to the dot on the propagating string and x_j corresponding to the dot on the jth bubble. Letting $\widehat{A_J[x]}$ be the completion of $A_J[x]$ at the maximal ideal $(x-i, x_j-j\mid j\in J)$, the map $\theta_{J,i}$ extends to $\widehat{\theta}_{J,i}:\widehat{A_J[x]}\to\operatorname{End}(F_i|_{\mathbf{R}_\lambda})$. Claim 2 follows if we can show that



(7.18)

for all $i \in I$. By the definition of Kac-Moody dots and Claim 1, the right hand side is equal to



Opening some parentheses, using also the fancy notation and (7.12), this expands as

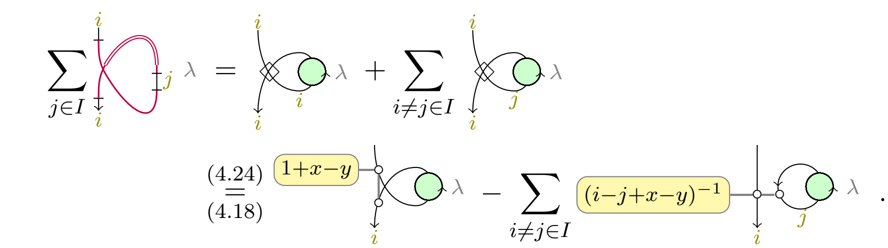
$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ J\sqcup K=I}} \sum_{\substack{(r_k)_{k\in K} \\ 0\le r_k\le h_k(\lambda)}} \theta_{J,i}\left(\left[\frac{u^{\sum_k r_k}}{(u-x)\prod_{j\in J}(u-x_j)}\right]_{u:-1}\right)\prod_{k\in K}\beta_{k,r_k}.$$

Using partial fractions, this decomposes as the sum of the following two expressions:

$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ J\sqcup K=I}} \sum_{\substack{(r_k)_{k\in K} \\ 0\le r_k\le h_k(\lambda)}} \theta_{J,i}\left(\frac{x^{\sum_k r_k}}{\prod_{j\in J}(x-x_j)}\right)\prod_{k\in K}\beta_{k,r_k}, \tag{7.19}$$

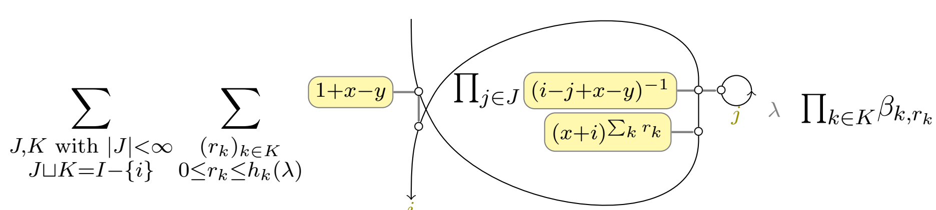
$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ J\sqcup K=I}} \sum_{\substack{(r_k)_{k\in K} \\ 0\le r_k\le h_k(\lambda)}} \theta_{J,i}\left(\sum_{j\in J}\frac{x_j^{\sum_k r_k}}{(x_j-x)\prod_{j\ne h\in J}(x_j-x_h)}\right)\prod_{k\in K}\beta_{k,r_k}. \tag{7.20}$$

Now consider the left hand side of (7.18). It is

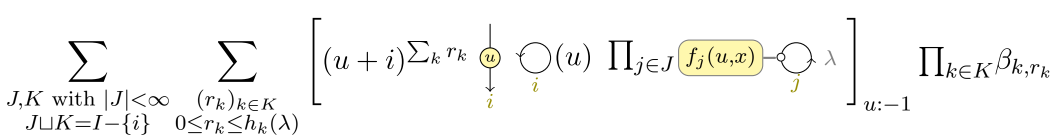


(7.21)

We look at the two terms on the right hand side of this separately. Expanding the definition of the internal bubble like we did in the proof of Claim 1, the first term is

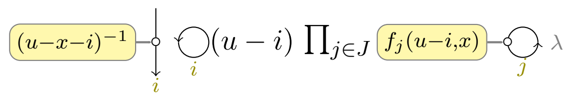


For $j \neq i$, the expression $(i-j+x-y)^{-1}$ here is a formal power series in x and y, which makes sense because they are locally nilpotent. To ensure that that they are actually nilpotent, from now on, we consider this natural transformation evaluated on some fixed choice of indecomposable object $V \in \operatorname{ob}\mathbf{R}_\lambda$ (this is sufficient for the proof). Then, $(i-j+x-y)^{-1}$ can be replaced by a polynomial $f_j(x,y) \in \Bbbk[x,y]$ of some large degree in x and y (depending implicitly on the choice of V). After this reduction we can "open" the curl using (2.15) and (3.16) as in [BSW20a, (3.21)] to obtain



Replacing u by $u-i$, this is

$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ J\sqcup K=I-\{i\}}} \sum_{\substack{(r_k)_{k\in K} \\ 0\le r_k\le h_k(\lambda)}} \Bigg[u^{\sum_k r_k}$$



$$\Bigg]_{u:-1} \prod_{k\in K} \beta_{k,r_k}.$$

Then we expand the bubble generating function using (7.2) to split this as the sum of the following two terms, writing them using fancy notation:

$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ i\in K, J\sqcup K=I}} \sum_{\substack{(r_k)_{k\in K} \\ 0\le r_k\le h_k(\lambda)}} \widehat{\theta}_{J,i}\left(\left[\frac{u^{\sum_k r_k}\prod_{j\in J} f_j(u-i,x_j-j)}{u-x}\right]_{u:-1}\right)\prod_{k\in K}\beta_{k,r_k}, \tag{7.22}$$

$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ i\in J, J\sqcup K=I}} \sum_{\substack{(r_k)_{k\in K} \\ 0\le r_k\le h_k(\lambda)}} \widehat{\theta}_{J,i}\left(\left[\frac{u^{\sum_k r_k}\prod_{i\neq j\in J} f_j(u-i,x_j-j)}{(u-x)(u-x_i)}\right]_{u:-1}\right)\prod_{k\in K}\beta_{k,r_k}, \tag{7.23}$$

We simplify (7.22) by first using (2.15) to obtain

$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ i\in K, J\sqcup K=I}} \sum_{\substack{(r_k)_{k\in K} \\ 0\le r_k\le h_k(\lambda)}} \widehat{\theta}_{J,i}\left(x^{\sum_k r_k}\prod_{j\in J} f_j(x-i,x_j-j)\right)\prod_{k\in K}\beta_{k,r_k},$$

then noting that each $f_j(x-i,x_j-j)$ can be replaced by $(x-x_j)^{-1}$ as they have the same image under $\widehat{\theta}_{J,i}$, so that (7.22) equals

$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ i\in K, J\sqcup K=I}} \sum_{\substack{(r_k)_{k\in K} \\ 0\le r_k\le h_k(\lambda)}} \widehat{\theta}_{J,i}\left(\frac{x^{\sum_k r_k}}{\prod_{j\in J}(x-x_j)}\right)\prod_{k\in K}\beta_{k,r_k}. \tag{7.24}$$

For (7.23), we use partial fractions to expand it as the sum of the following two terms:

$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ i\in J, J\sqcup K=I}} \sum_{\substack{(r_k)_{k\in K} \\ 0\le r_k\le h_k(\lambda)}} \widehat{\theta}_{J,i}\left(\frac{x^{\sum_k r_k}\prod_{i\neq j\in J} f_j(x-i,x_j-j)}{x-x_i}\right)\prod_{k\in K}\beta_{k,r_k},$$

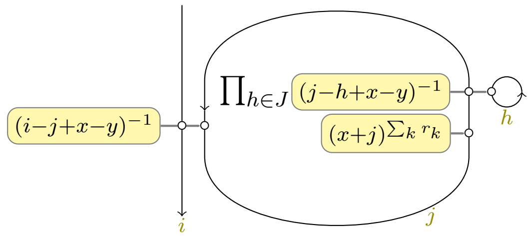
$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ i\in J, J\sqcup K=I}} \sum_{\substack{(r_k)_{k\in K} \\ 0\le r_k\le h_k(\lambda)}} \widehat{\theta}_{J,i}\left(\frac{x_i^{\sum_k r_k}\prod_{i\neq j\in J} f_j(x_i-i,x_j-j)}{x_i-x}\right)\prod_{k\in K}\beta_{k,r_k}.$$

Then we move $f_j(x-i,x_j-j)$ and $f_j(x_i-i,x_j-j)$ in the numerator to $(x-x_j)$ and (x_i-x_j) in the denominator, respectively, to obtain finally the sum of the terms

$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ i\in J, J\sqcup K=I}} \sum_{\substack{(r_k)_{k\in K} \\ 0\le r_k\le h_k(\lambda)}} \widehat{\theta}_{J,i}\left(\frac{x^{\sum_k r_k}}{\prod_{j\in J}(x-x_j)}\right)\prod_{k\in K}\beta_{k,r_k}, \tag{7.25}$$

$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ i\in J, J\sqcup K=I}} \sum_{\substack{(r_k)_{k\in K} \\ 0\le r_k\le h_k(\lambda)}} \widehat{\theta}_{J,i}\left(\frac{x_i^{\sum_k r_k}}{(x_i-x)\prod_{i\neq j\in J}(x_i-x_j)}\right)\prod_{k\in K}\beta_{k,r_k}. \tag{7.26}$$

Finally we look at the second term from (7.21), which expands as

$$-\sum_{i\neq j\in I}\ \sum_{\substack{J,K \text{ with } |J|<\infty \\ J\sqcup K=I-\{j\}}}\ \sum_{\substack{(r_k)_{k\in K}\\ 0\leq r_k\leq h_k(\lambda)}}$$


$$\prod_{k\in K}\beta_{k,r_k}.$$

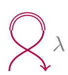
Reindexing the summation and using fancy notation, this is

$$\sum_{\substack{J,K \text{ with } |J|<\infty \\ J\sqcup K=I}}\ \sum_{\substack{(r_k)_{k\in K}\\ 0\leq r_k\leq h_k(\lambda)}}\ \sum_{i\neq j\in J}\widehat{\theta}_{J,i}\left(\frac{x_j^{\sum_k r_k}}{(x_j-x)\prod_{j\neq h\in J}(x_j-x_h)}\right)\prod_{k\in K}\beta_{k,r_k}. \tag{7.27}$$

Now we have expanded the right hand side of (7.18) as the sum of (7.19) and (7.20), and the left hand side as the sum of (7.24) to (7.27). The two sides are equal: the sum of the terms from (7.19) with $i\in J$ is (7.25), the sum of the terms from (7.19) with $i\in K$ is (7.24), the term from (7.20) with $j=i$ is (7.26), and the sum of the remaining terms from (7.20) is (7.27). Thus, (7.18) is proved.

Claim 3. *We have that* λ $=$ λ , *proving the first equality in* (7.9).

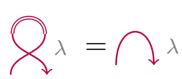
Proof. We first prove this assuming that $\kappa<0$. By the definition of λ as the last entry of the inverse of the matrix M_κ from (3.5), this reduces to checking the following identities:


$$\lambda = 0,$$


$$\lambda = \delta_{n,-\kappa-1} \text{ for } 0\leq n\leq -\kappa-1. \tag{7.28}$$

To check these, Claim 1 plus (2.20) and (4.6) implies that (u) λ $\in u^\kappa+u^{\kappa-1}Z(\mathbf{R}_\lambda)[\![u^{-1}]\!]$. The second of the equalities (7.28) follows immediately from this. For the first one, we attach λ to the top of the identity from Claim 2, noting that the right hand side is zero when $\kappa<0$.

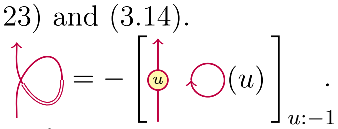
It remains to treat the case that $\kappa=0$. We have that $=$ by a special case of (3.17). If we attach λ to the top of this identity, keeping the definition of λ in mind, the problem in hand reduces to showing that



This is easily checked in a similar way to the proof of the first equality in (7.28).

Claim 4. *Both of the equalities in* (7.10) *hold.*

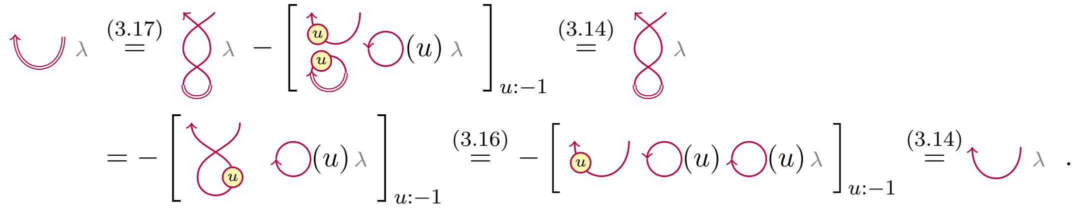
Proof. Claim 3 implies that (u) λ $=$ (u) λ . Consequently, the first of the identities in (7.10) follows from Claim 1. The second of these identities then follows by taking inverses on both sides, using (2.23) and (3.14).

Claim 5.


Proof. The follows by a similar argument to the proof of Claim 2, using the second of the identities (7.10) established in Claim 4 in place of the identity from Claim 1.

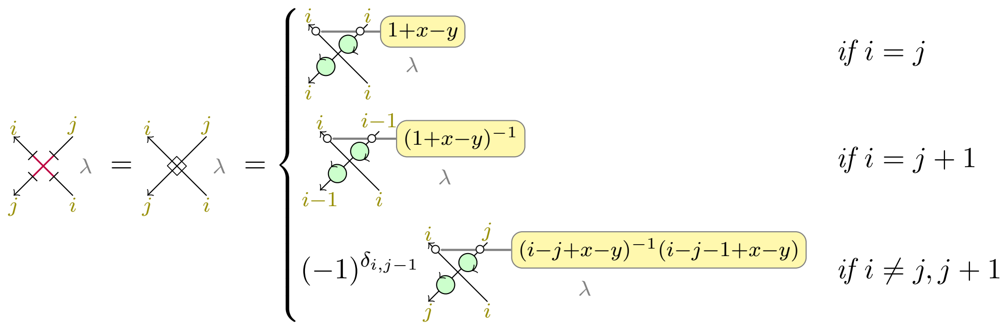
Claim 6. λ $=$ λ , *proving the second equality in* (7.9).

Proof. Using Claim 5 for the unlabelled equality, we have that



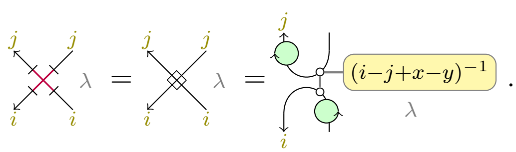
□

Corollary 7.2. *The leftward Heisenberg crossing is related to the leftward Kac-Moody crossings by*



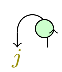
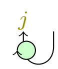
(7.29)

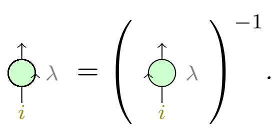
for any $i, j \in I$. For $i \neq j$ in I, we also have that



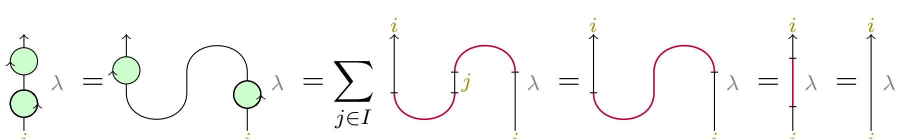
(7.30)

We know already that all other projected leftward Heisenberg crossings are zero.

Proof. The identity (7.29) follows by attaching  to the top left and  to the bottom right of the identity (4.23), then simplifying using other relations including (2.14) and (7.9). The proof of (7.30) is similar, starting from (4.18) instead of (4.23). □

Corollary 7.3. 

Proof. The following checks that the counterclockwise and clockwise internal bubbles are inverses on one side:



A similar calculation with the other leftward zig-zag identity shows that they are inverses on the other side too. □

Remark 7.4. The formulae (7.9), (7.29) and (7.30) are new, but (7.10) was proved already in [BSW20a, (5.37)] by a more indirect method. The proof just explained depends on Theorem 4.2, hence, on [BSW20a, Lem. 4.9, Lem. 4.10], but is independent of all of the results of [BSW20a, Sec. 5].

Theorem 7.1 and Corollary 7.2 explain how to write the leftward Heisenberg cap, cup and crossing in terms of the Kac-Moody ones and internal bubbles. When applying Theorem 4.2, one really wants to be able to go in the other direction. Since the internal bubbles are invertible by Corollary 7.3, it is quite trivial to rearrange the formulae obtained so far to write the leftward Kac-Moody caps, cups and crossings in terms of the Heisenberg ones and internal bubbles. However, one still needs to be able to determine the internal bubbles, which were defined explicitly in terms of the Kac-Mooody dotted bubbles. The final theorem in this section completes the picture by explaining how the Kac-Moody bubble generating functions can be obtained from the Heisenberg bubble generating functions. It is not very explicit, but this is to be expected in this place.

Given an indecomposable $V \in \operatorname{ob}\mathbf{R}$, we denote its endomorphism algebra by Z_V. This is a finite-dimensional commutative local algebra. We denote its unique maximal ideal by J_V.

Lemma 7.5. *Let Z be any finite-dimensional commutative local algebra and J be its unique maximal ideal. Let $\bar{Z} := Z/J \cong \Bbbk$, denoting the canonical homomorphism $Z[x] \to \bar{Z}[x]$ by $f(x) \mapsto \bar{f}(x)$. Suppose that we are given $k \in \mathbb{Z}$ and a rational function $\frac{f(u)}{g(u)}$ in $Z(u) \cap \left(u^k + u^{k-1}Z[\![u^{-1}]\!]\right)$. Let $h_i \in \mathbb{Z}$ be the multiplicity of $i \in \Bbbk$ as a zero or pole of $\frac{\bar{f}(u)}{\bar{g}(u)} \in \bar{Z}(u)$. There are unique rational functions $\frac{f_i(u)}{g_i(u)}$ in $Z(u) \cap \left(u^{h_i} + u^{h_i-1}J[\![u^{-1}]\!]\right)$ such that $\frac{f_i(u)}{g_i(u)} = 1$ for all but finitely many i and*

$$\frac{f(u)}{g(u)} = \prod_{i\in\Bbbk} \frac{f_i(u-i)}{g_i(u-i)}.$$

Proof. This follows from [BSW20a, Cor. 2.4], which is the analogous statement for polynomials rather than rational functions. ☐

Lemma 7.6. *Let $V \in \operatorname{ob}\mathbf{R}_\lambda$ be an indecomposable object. The Heisenberg bubble generating functions $\circlearrowleft(u)$ and $\circlearrowright(u)$ act on V as rational functions belonging to $Z_V(u) \cap \left(u^\kappa + u^{\kappa-1}Z_V[\![u^{-1}]\!]\right)$ and $Z_V(u) \cap \left(-u^{-\kappa} + u^{-\kappa-1}Z_V[\![u^{-1}]\!]\right)$, respectively.*

Proof. It suffices to prove this for the counterclockwise bubble generating function, since the other one is its inverse (up to a sign) by (3.14). Let $f(x) \in \Bbbk[x]$ be the (monic) minimal polynomial of the endomorphism of EV defined by $\overset{\bullet}{\uparrow}$. By [BSW20a, Lem. 4.3(1)], there is a monic polynomial $g(u) \in Z_V[u]$ such that $\circlearrowleft(u)$ acts on V in the same way as the rational function $\frac{g(u)}{f(u)}$. The lemma follows using also that we know the leading term from (3.12). ☐

Theorem 7.7. *The following hold for all $\lambda \in X$:*

(1) The bubble generating functions $\underset{i}{\circlearrowleft}(u)\ \lambda$ are the unique elements of $Z(\mathbf{R}_\lambda)(\!(u^{-1})\!)$ such that $\circlearrowleft(u)\ \lambda = \prod_{i\in I} \underset{i}{\circlearrowleft}(u-i)\ \lambda$ and $\underset{i}{\circlearrowleft}(u)\ \lambda$ acts on V as a rational function in $Z_V(u) \cap \left(u^{h_i(\lambda)} + u^{h_i(\lambda)-1}J_V[\![u^{-1}]\!]\right)$ for each indecomposable $V \in \operatorname{ob}\mathbf{R}_\lambda$.

(2) The bubble generating functions $\underset{i}{\circlearrowright}(u)\ \lambda$ are the unique elements of $Z(\mathbf{R}_\lambda)(\!(u^{-1})\!)$ such that $\circlearrowright(u)\ \lambda = -\prod_{i\in I} \underset{i}{\circlearrowright}(u-i)\ \lambda$ and $\underset{i}{\circlearrowright}(u)\ \lambda$ acts on V as a rational function in $Z_V(u) \cap \left(u^{-h_i(\lambda)} + u^{-h_i(\lambda)-1}J_V[\![u^{-1}]\!]\right)$ for each indecomposable $V \in \operatorname{ob}\mathbf{R}_\lambda$.

Proof. This follows from Lemmas 7.5 and 7.6 using the known formulae (7.10). ☐

8. The quantum case

There is also a quantum version of the Heisenberg category and of Theorem 4.2, which allows further examples of Kac-Moody categorifications to be obtained from various categories of representations of quantum linear groups and associated Hecke algebras (perhaps at roots of unity). We briefly recall the general setup in this section, then write down the analogues of the results of Section 7, omitting the detailed proofs since they are similar in spirit to the degenerate case.

Fix once again an algebraically closed field $\Bbbk$. As well as the central charge $\kappa \in \mathbb{Z}$, the definition of the quantum Heisenberg category depends on an additional parameter $q \in \Bbbk - \{0, \pm 1\}$. It occurs often so we use the shorthand

$$z := q - q^{-1}. \tag{8.1}$$

Then we define q-$\mathbf{Heis}_\kappa$ to be the strict $\Bbbk[t, t^{-1}]$-linear monoidal category with generating objects E and F, whose identity endomorphisms are represented by the oriented strings $\uparrow$ and $\downarrow$, and the following four generating morphisms:

 $: E \to E,$  $: E \otimes E \to E \otimes E,$  $: E \otimes F \to \mathbb{1},$  $: \mathbb{1} \to F \otimes E.$

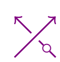
The dot and the crossing, which from now on we will call the *positive* crossing, are required to be invertible. The invertibility of the dot means that now it makes sense to label dots by an arbitrary integer rather than just by $n \in \mathbb{N}$, and it makes sense to pin Laurent polynomials $f(x) \in \Bbbk[x, x^{-1}]$ rather than mere polynomials to them. We denote the inverse of the positive crossing by

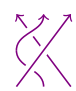
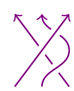
 $: E \otimes E \to E \otimes E,$

and call this the *negative* crossing. We also introduce the negative rightward crossing:

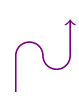
 $:=$  . (8.2)

The other relations are as follows. First, the affine Hecke algebra relations:

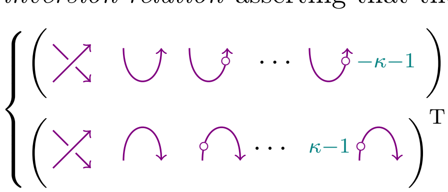
 $=$  ,  $=$  , (8.3)

 $-$  $= z \uparrow \uparrow,$  $=$  . (8.4)

Next, the *right adjunction relations* implying that F is right dual to E:

 $= \downarrow,$  $= \uparrow.$ (8.5)

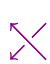
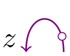
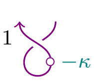
There is an *inversion relation* asserting that the morphism



$: E \otimes F \oplus \mathbb{1}^{\oplus(-\kappa)} \to F \otimes E$ if $\kappa \leq 0$

$: E \otimes F \to F \otimes E \oplus \mathbb{1}^{\oplus \kappa}$ if $\kappa > 0$ (8.6)

is invertible. Then there is one more *bubble relation* which had no counterpart in the degenerate case:

- If $\kappa < 0$, we let  and $t^{-1}z$  be the first and last entries of the inverse of the matrix (8.6), respectively, define  $:= t^{-1}$  , and require that  $= -t^{-1}z^{-1}\,\mathrm{id}_{\mathbb{1}}$.

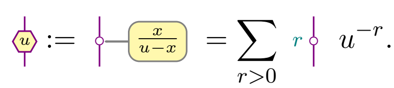
- If $\kappa = 0$, we let := $\left(\right)^{-1}$, := t^{-1}, := t^{-1}, and require that $= (tz^{-1} - t^{-1}z^{-1})\,\mathrm{id}_{\mathbb{1}}$.
- If $\kappa > 0$, we let and $t^{-1}z$ be the first and *second* entries of the inverse of the matrix (8.6), respectively, define := t κ, and require that κ $= -t^{-1}z^{-1}\,\mathrm{id}_{\mathbb{1}}$.

The leftward cup and cap just introduced automatically satisfy

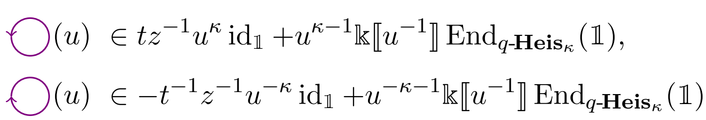

(8.7)

so they define a second adjunction making F into a left dual of E. There is also a downward dot, a positive rightward crossing, another leftward crossing, and positive and negative downward crossings, all defined so that the resulting string diagrams are invariant under planar isotopy. Hence, q-$\mathbf{Heis}_\kappa$ is strictly pivotal. Proofs of these results, plus some other equivalent (and more symmetric) formulations of the definition, can be found in [BSW20b].

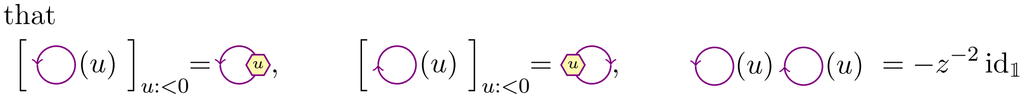
Now that dots are invertible, we find it better to use a different generating function for positive powers of dots, setting


(8.8)

The bubble generating functions in this setting are the unique formal Laurent series


(8.9)
(8.10)

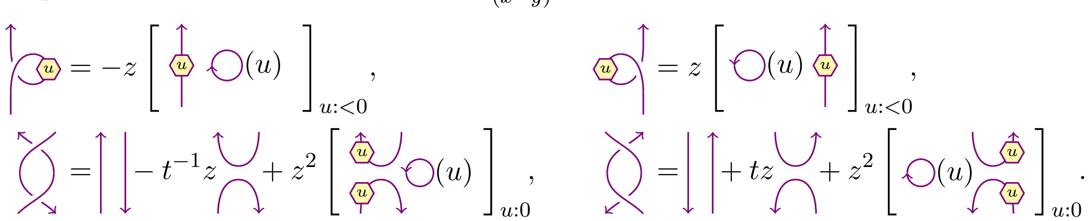
such that


(8.11)

Using the same notation (2.18) for the generating function for non-negative powers of dots, the counterparts of the relations (3.15) to (3.17) are


(8.12)

where $R_q(x,y) := 1 - z^2xy(x-y)^{-2} = \frac{(x-q^2y)(x-q^{-2}y)}{(x-y)^2}$,

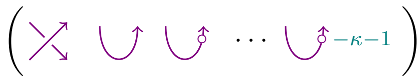

(8.13)
(8.14)

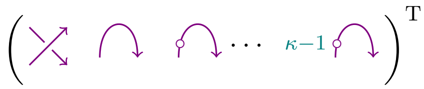
Definition 8.1. A *quantum Heisenberg categorification* of central charge κ and $q \in \mathbb{k} - \{0, \pm 1\}$ is a locally finite $\mathbb{k}$-linear Abelian category $\mathbf{R}$ plus an invertible element $t \in Z(\mathbf{R})^\times$ (usually, a scalar in $\mathbb{k}^\times$) and an adjoint pair (E, F) of $\mathbb{k}$-linear endofunctors such that:

(q-H1) The adjoint pair (E, F) has a prescribed adjunction with unit and counit of adjunction denoted $: \mathrm{id}_{\mathbf{R}} \Rightarrow F \circ E$ and $: E \circ F \Rightarrow \mathrm{id}_{\mathbf{R}}$.

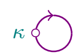
(q-H2) There are given invertible natural transformations $: E \Rightarrow E$ and $: E \circ E \Rightarrow E \circ E$ satisfying the AHA relations (8.3) and (8.4), where := $\left(\right)^{-1}$.

(q-H3) Defining the negative rightward crossing  according to (8.2), the matrix

 $: E \circ F \oplus \mathrm{id}_{\mathbf{R}}^{\oplus(-\kappa)} \Rightarrow F \circ E$ if $\kappa \leq 0$, or

 $: E \circ F \Rightarrow F \circ E \oplus \mathrm{id}_{\mathbf{R}}^{\oplus \kappa}$ if $\kappa > 0$,

is invertible.

(q-H4) Defining  $: \mathrm{id}_{\mathbf{R}} \Rightarrow E \circ F$ and  $: F \circ E \Rightarrow \mathrm{id}_{\mathbf{R}}$ as explained after (8.6), we have that  $= -t^{-1}z^{-1}$ if $\kappa < 0$,  $= tz^{-1} - t^{-1}z^{-1}$ if $\kappa = 0$, and  $= tz^{-1}$ if $\kappa > 0$.

This data makes $\mathbf{R}$ is a strict left q-$\mathbf{Heis}_\kappa$-module category.

Next we summarize the quantum Heisenberg to Kac-Moody construction from [BSW20a, Sec. 4]. Let $\mathbf{R}$ be a quantum Heisenberg categorification of central charge κ. We fix representatives $L(b)\,(b \in \mathcal{B})$ for the isomorphism classes of irreducible objects in $\mathbf{R}$, and define the (monic) minimal polynomials $m_b(x), n_b(x) \in \Bbbk[x]$ and the central character generating function $\chi_b(u) \in \Bbbk(\!(u^{-1})\!)$ for $b \in \mathcal{B}$ exactly as we did in the degenerate case in Section 4 (using the new bubble generating function (8.9) for $\chi_b(u)$). We also have the scalar $t_b \in \Bbbk$ which is how the central endomorphism t_L acts on L. The analogue of (4.1) proved in [BSW20a, Lem. 4.4] is that

$$\chi_b(u) = t_b z^{-1} n_b(u)/m_b(u). \tag{8.15}$$

Moreover, $t_b^2 = m_b(0)/n_b(0)$ (this makes sense because $m_b(0)$ and $n_b(0)$ are non-zero in the quantum case as the dot is invertible).

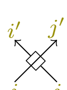
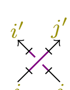
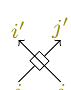
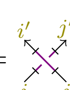
The *spectrum* I is the set of roots of all $m_b(x)$. It is a subset of $\Bbbk^\times$ closed under the operations $i \mapsto q^{\pm 2} i$. The Cartan matrix $A = (a_{i,j})_{i,j \in I}$ is

$$a_{i,j} := \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i = q^{\pm 2} j \text{ and } q^4 \neq 1 \\ -2 & \text{if } i = q^2 j = q^{-2} j \\ 0 & \text{otherwise.} \end{cases} \tag{8.16}$$

The connected components of A are all of type A_∞ if q is not a root of unity or $A_{p-1}^{(1)}$ if q^2 is a primitive pth root of 1 (note now that p is not necessarily prime). We also fix a root datum of this Cartan type in the same way as in Section 4. Then we define $\varepsilon_i(b)$ and $\phi_i(b)$ as in (4.4), and $\mathrm{wt} : \mathbf{B} \to X$ as in (4.5). We obtain the weight decomposition (4.7), letting $\mathbf{R}_\lambda$ be the Serre subcategory of $\mathbf{R}$ generated by the irreducible objects $L(b)$ for $b \in \mathcal{B}_\lambda := \{b \in \mathcal{B} \mid \mathrm{wt}(b) = \lambda\}$ as before. The eigenfunctors $E_i, F_i\ (i \in I)$ are defined as in (4.8). Again, [BSW20a, Lem. 4.7], which depends on (8.12), implies that the restrictions of E_i and F_i are functors

$$E_i|_{\mathbf{R}_\lambda} : \mathbf{R}_\lambda \to \mathbf{R}_{\lambda+\alpha_i}, \qquad F_i|_{\mathbf{R}_{\lambda+\alpha_i}} : \mathbf{R}_{\lambda+\alpha_i} \to \mathbf{R}_\lambda. \tag{8.17}$$

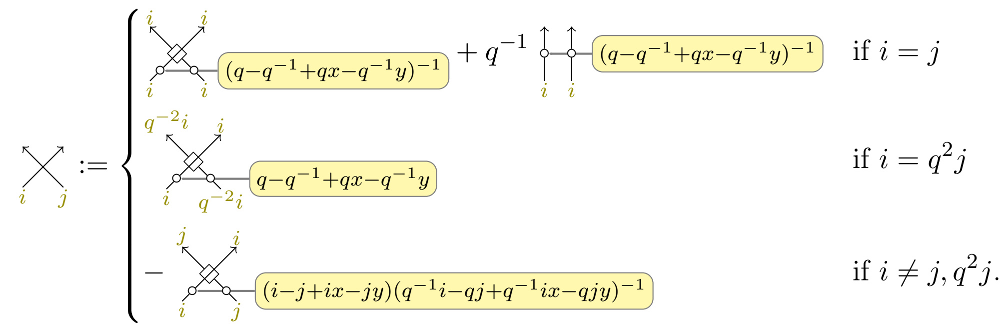
It remains to define the various Kac-Moody natural transformations. We introduce the projectors as in (4.9), and use these to define projected rightward cups and caps, dots and crossings. The only difference is that there are now *two* projected crossings, positive and negative, for each orientation. For example, the positive and negative upward projected crossings are

 $:=$  ,  $:=$  . (8.18)

These have the same vanishing properties as before. Then we define the Kac-Moody rightward cups and caps, the Kac-Moody dots (which are nilpotent), and the upward Kac-Moody crossings by

(8.19)

(8.20)



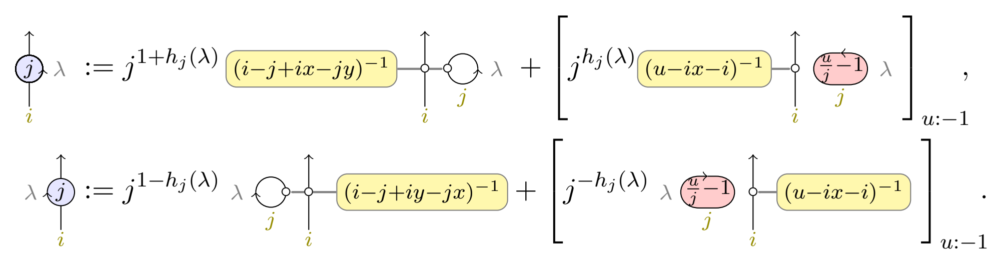
(8.21)

when $i \neq j$. We refer to [BSW20a, Sec. 4] for a more detailed development of the properties of these natural transformations.

Theorem 8.2 ([BSW20a, Th. 4.11])**.** *Let* $\mathbf{R}$ *be a quantum Heisenberg categorification. The definitions summarized in the previous two paragraphs make* $\mathbf{R}$ *into a Kac-Moody categorification with the parameters defined from*

$$Q_{i,j}(x,y) := \begin{cases} 0 & \text{if } i=j \\ y-x & \text{if } i=q^2j\neq q^{-2}j \\ x-y & \text{if } i=q^{-2}j\neq q^2j \\ (y-x)(x-y) & \text{if } i=q^2j=q^{-2}j \\ 1 & \text{if } i\neq j, q^{\pm 2}j. \end{cases}$$

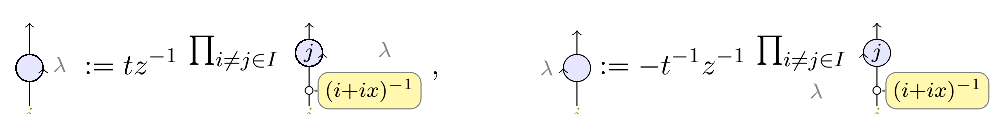
We close by writing down the definition of internal bubbles in the quantum case and formulating the counterpart of Theorem 7.1. It is quite similar to the degenerate case from the previous section, so we will be brief. For $i \neq j$ in I, we define the "quantum" counterclockwise and clockwise *internal bubbles of color* j



(8.22)

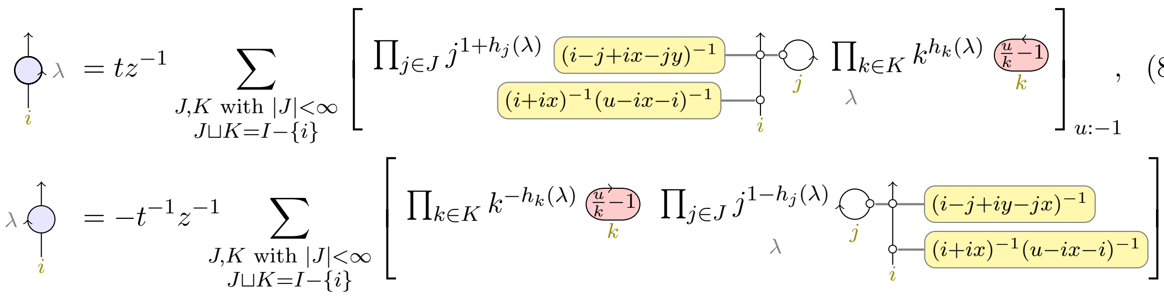
(8.23)

The conventions here are similar to (7.3) and (7.4). For similar reasons to (7.6) and (7.7) in the previous section, the infinite vertical compositions in the next definitions make sense:



(8.24)

Opening some parentheses and using (2.16) gives the following more explicit formulae:



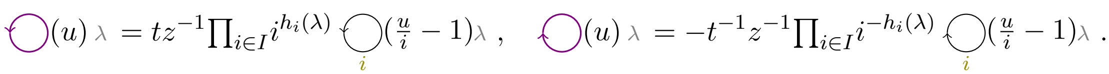
(8.25)

(8.26)

Theorem 8.3. *In the quantum case, the leftward Heisenberg caps and cups are related to the leftward Kac-Moody caps and cups by*

(8.27)

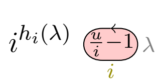
for $i, j \in I$ and $\lambda \in X$. Hence, the Heisenberg bubble generating functions are related to the Kac-Moody bubble generating functions by



(8.28)

Sketch proof. The argument is similar to the proof of Theorem 7.1. We define and so that

Thus, to prove (8.27), we must show that $=$ and $=$. To treat the case $\kappa \leq 0$, we introduce the new counterclockwise bubble generating function $(u)\,\lambda$, which is the formal Laurent series in $Z(\mathbf{R})[\![u^{-1}]\!]$ with leading term $tz^{-1}u^{\kappa}$ such that $\left[(u)\,\lambda\right]_{u:<0} =$. The shorthand $\beta_{i,r}$ now denotes the coefficients in the expansion

 $$= \sum_{r=0}^{h_i(\lambda)} \beta_{i,r}u^r, \tag{8.29}$$

so that $\beta_{i,h_i(\lambda)} = 1$ when $h_i(\lambda) \geq 0$. Now the proof proceeds with analogues of the Claims 1–6 from the proof of Theorem 7.1. The homomomorphism $\widehat{\theta}_J : \widehat{A}_J \to Z(\mathbf{R}_\lambda)$ referred to as "fancy notation" in the proof of Claim 1 is now defined so that $f(x_j \mid j \in J)$ goes to the natural transformation obtained by pinning $f(jx_j + j \mid j \in J)$ to $\prod_{j\in J}$ λ with the variable x_j corresponding to the dot on the jth bubble. ∎

Remark 8.4. The formulae (8.28) already appeared in [BSW20a, (5.38)] (with a less direct proof).

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