$\mathfrak{g}$ f.d. semisimple Lie algebra, max toral subalgebra $\mathfrak{t}$, root R, base $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$, basis $\{e_\alpha, f_\alpha \mid \alpha \in R^+\} \cup \{h_\alpha \mid \alpha \in \Delta\}$ as fixed last time. $\underbrace{e_i, h_i, f_i}$ are $e_{\alpha_i}, h_{\alpha_i}, f_{\alpha_i}$

We showed last time these $\mathfrak{sl}_2$-triples for $i=1,\dots,\ell$ generate $\mathfrak{g}$ as a Lie algebra. RELATIONS?

Let $c_{ij} = (\alpha_i, \alpha_j^\vee)$

$C = (c_{ij})_{i,j=1,\dots,\ell}$ <u>Cartan matrix</u> $\leftarrow$ Note I've followed Humphreys' definition here... modern stuff transpose this matrix!!

<u>Serre's Theorem</u> $\mathfrak{g}$ is generated by e_i, h_i, f_i $(i=1,\dots,\ell)$ subject only to the relations $\forall i,j \in I$:

(S1) $[h_i, h_j] = 0$

(S2) $[e_i, f_j] = \delta_{ij} h_i$

(S3) $[h_i, e_j] = c_{ji} e_j$, $[h_i, f_j] = -c_{ji} f_j$

Serre relations ... depend only on Cartan matrix C $\rightarrow$

$\left.\begin{array}{ll} (S_{ij}^+) & (\operatorname{ad} e_i)^{1-c_{ji}}(e_j) = 0 \\ (S_{ij}^-) & (\operatorname{ad} f_i)^{1-c_{ji}}(f_j) = 0 \end{array}\right\}$ for $i \neq j$

Note here we're talking about generators and relations for Lie algebras. This requires notion of <u>free Lie algebra</u> on a set. See HW 4, Q5.

The theorem gives a presentation for a Lie algebra $\mathfrak{g}(C)$

$$\mathfrak{g}(C) = \frac{F(e_i, h_i, f_i \mid i = 1, \dots, \ell)}{\langle S1, S2, S3, S_{ij}^{f}, S_{ij} \rangle}$$

free Lie algebra on these generators

ideal generated by elements corresponding to relations

<u>Serre's Theorem</u>. Let C be a Cartan matrix coming from a root system R. The Lie algebra $\mathfrak{g}(C)$ defined by the Serre relations is a f.d. semisimple Lie algebra with maximal toral subalgebra $\mathfrak{t} = \langle h_1, \dots, h_\ell \rangle$, root system R with simple roots $\alpha_1, \dots, \alpha_\ell$ corresponding to $h_1, \dots, h_\ell$, and Cartan matrix $C = \left((\alpha_i, \alpha_j^\vee) \right)_{i,j = 1, \dots, \ell}$.

Some rank 2 examples

A_2 $\quad \mathfrak{sl}_3(\mathbb{C})$ $\quad e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

f_1, f_2 transposes, $h_1 = \mathrm{diag}(1,-1,0)$, $h_2 = \mathrm{diag}(0,1,-1)$

Some relations (S_{ij}^+) $\quad [e_1[e_1 e_2]] = [e_2[e_2 e_1]] = 0$ $\quad$ similarly for f_i

Cartan matrix A_2 $\quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$(\mathrm{ad}\, e_i)^{1-c_{ji}}(e_j) = 0$

If in $U(\mathfrak{g})$ you often see $\quad [e_1[e_1 e_2]] = 0 \quad$ turned into

associative algebra relations $\quad e_1(e_1e_2 - e_2e_1) - (e_1e_2 - e_2e_1)e_1 = 0$

$$\boxed{(112) - 2(121) + (112)}$$

B_2 $\quad so_5(\mathbb{C})$ $\quad \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$

$e_1, e_2, h_1, h_2, f_1, f_2$

$$\underbrace{\begin{aligned} (\mathrm{ad}\, e_1)^2 (e_2) &= 0 \\ (\mathrm{ad}\, e_2)^3 (e_1) &= 0 \end{aligned}}$$

$[e_2 [e_2 [e_2\, e_1]]] = 0$ $\quad$ Serre relation when $c_{ji} = -2$

Expanded: $\boxed{(2221) - 3(2212) + 3(2122) - (1222)}$

G_2 $\quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

$(\mathrm{ad}\, e_1)^4 (e_2) = 0 \leftarrow [e_1 [e_1 [e_1 [e_1\, e_2]]]] = 0$

$(\mathrm{ad}\, e_2)^2 (e_1) = 0$

$\boxed{(11112) - 4(11121) + 6(11211) - 4(12111) \quad (2111)}$

$\hat{A}_1$ $\quad \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ $\quad$ Serre presentation still makes sense

generalized Cartan matrix

$\mathfrak{g}(C)$ $\quad$ Kac-Moody $\quad \widehat{sl_2(\mathbb{C})}^1$

Proof of Serre's Theorem (assuming Serre's Theorem!)

Starting with $\mathfrak{g}$! We've shown $\mathfrak{g}$ is generated by e_i, h_i, f_i's.
Let's check all Serre relations hold in $\mathfrak{g}$.

(S1) $[h_i, h_j] = 0$ ✓ $\mathfrak{h}$ is Abelian

(S2) $[e_i, f_i] = h_i$ ✓ $\quad \underbrace{[e_i, f_j]}_{\text{lies in } (\alpha_i - \alpha_j)\text{-wt space}} = 0 \quad i \neq j$ ✓

lies in $(\alpha_i - \alpha_j)$-wt space which is zero as $\alpha_i - \alpha_j \notin R$.

(S3) $[h_i, e_j] = \underbrace{\alpha_j(h_i)}_{(\alpha_j, \alpha_i^\vee) = c_{ji}} e_j$ ✓

$r = 0$ as $\alpha_j - \alpha_i \notin R$

(S_{ij}^+) L7-2, Lemma 3: α_i-string through α_j. Let $r, q \geq 0$ maximal so that $\alpha_j - r\alpha_i$, $\alpha_j + q\alpha_i$ are roots. Then $\cancel{r} - q = \alpha_j(h_i) = c_{ji}$. $q = -c_{ji}$

$\alpha_j \quad \alpha_j + \alpha_i \quad \cdots \quad \alpha_j - c_{ji}\alpha_i$

$\bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_{\alpha_j + n\alpha_i}$ $\quad S_{\alpha_i}$ acts irreducible representation

$\in$
$e_j \qquad (\operatorname{ad} e_i)^{1 - c_{ji}}(e_j) = 0$ ✓

(S_{ij}^-) Similar

Now it follows that

$$\begin{aligned} \mathfrak{g}(C) &\twoheadrightarrow \mathfrak{g} \\ e_i, h_i, f_i &\mapsto e_i, h_i, f_i \end{aligned}$$

To show this is an isomorphism, remains to see $\underbrace{\dim \mathfrak{g}}_{|R| + |\Delta|} = \dim \mathfrak{g}(C)$

This follows from $\underbrace{\text{Serre Theorem}}_{\text{Main thing left to do (tricky!)}}$! //

Instead of proving that now, let's see the applications.

Serre's Theorem is really an EXISTENCE THEOREM — every root system is realized as the root system of some f.d. semisimple $\mathfrak{g}$.

<u>Lemma</u> Let R and R' be two root systems with bases $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ and $\Delta' = \{\alpha_1', \dots, \alpha_\ell'\}$ and Cartan matrices $C = ((\alpha_i, \alpha_j^\vee))$ and $C' = ((\alpha_i', \alpha_j'^\vee))$. Assume R and R' are <u>NOT</u> isomorphic root systems. Then $\mathfrak{g}(C) \not\cong \mathfrak{g}(C')$.

<u>Proof</u> WLOG assume R and R' are both indecomposable.

Suppose $\mathfrak{g}(C) \cong \mathfrak{g}(C')$.

Then $\dim \mathfrak{g}(C) = |R| + |\Delta| = |R'| + |\Delta'| = \dim \mathfrak{g}(C')$.

Moreover, the dimension of smallest irreducible non-trivial $\mathfrak{g}(C)$-module is as listed in table ... is the same as for $\mathfrak{g}(C')$.

It remains to note that these two invariants determine the isomorphism type of root R uniquely //

R	dimension of smallest non-trivial irrep.	$\lvert R\rvert + \lvert\Delta\rvert$
$A_\ell^{\ell\geq 1}$	$\ell+1$	$\ell^2+2\ell$
$B_\ell^{\ell\geq 2}$	$2\ell+1$	$2\ell^2+\ell$
$C_\ell^{\ell\geq 3}$	2ℓ	$2\ell^2+\ell$
$D_\ell^{\ell\geq 4}$	2ℓ	$2\ell^2-\ell$
E_6	27	78
E_7	56	133
E_8	248	248
F_4	26	52
G_2	7	14

$\mathfrak{sl}_{\ell+1}(\mathbb{C})$

$\mathfrak{so}_{2\ell+1}(\mathbb{C})$

These numbers follow from the Weyl dimension formula (not proved in this course!).

Conjugacy Theorem Let $\mathfrak{g}$ be a f.d. semisimple Lie algebra. All maximal toral subalgebras of $\mathfrak{g}$ are conjugate under automorphisms of $\mathfrak{g}$.

Proof. Let $\mathfrak{h}$ and $\mathfrak{h}'$ be two max. toral subalgebras of $\mathfrak{g}$

$$\mathfrak{h} \to R, \Delta, C \qquad \mathfrak{h}' \to R', \Delta', C'$$

Serre's Theorem shows

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\sim"] & \mathfrak{g}(C) \arrow[dd, "\wr"] \\
\mathfrak{h} \arrow[r, mapsto] & \langle h_1, \dots, h_l \rangle \\
\mathfrak{g} \arrow[r, "\sim"] & \mathfrak{g}(C') \\
\mathfrak{h}' \arrow[r, mapsto] & \langle h_1', \dots, h_{l'}' \rangle
\end{tikzcd}$$

Lemma: $\mathfrak{g}(C) \cong \mathfrak{g}(C')$

so R and R' are isomorphic.

Pick isomorphism $R \to R'$ taking $\Delta \to \Delta'$ (ok as all bases are conjugate)

This obviously induces $\mathfrak{g}(C) \xrightarrow{\sim} \mathfrak{g}(C')$ taking

$$\langle h_1, \dots, h_l \rangle \longmapsto \langle h_1', \dots, h_{l'}' \rangle.$$

Now [diagram] gives automorphism of $\mathfrak{g}$ taking $\mathfrak{h}$ to $\mathfrak{h}'$ //

Isomorphism Theorem There's mutually inverse bijections

$$\{\text{f.d. semisimple Lie algebras}\}/\cong \;\underset{\sim}{\rightleftarrows}\; \{\text{root systems}\}/\cong$$

In the forward direction, you take $\mathfrak{g}$, pick a maximal toral subalgebra $\mathfrak{t}$, let R be the corresponding root system arising from Cartan decomposition of $\mathfrak{g}$ wrt $\mathfrak{t}$. This is well-defined independent of choice of $\mathfrak{t}$ due to conjugacy theorem.

In reverse directn, take a base Δ for R, hence, Cartan matrix C. Send R to $\mathfrak{g}(C)$, using Serre's theorem'.

Proof $\circlearrowleft = id$ $\quad \circlearrowright = id$ follow from Serre theorem and Serre theorem'

//