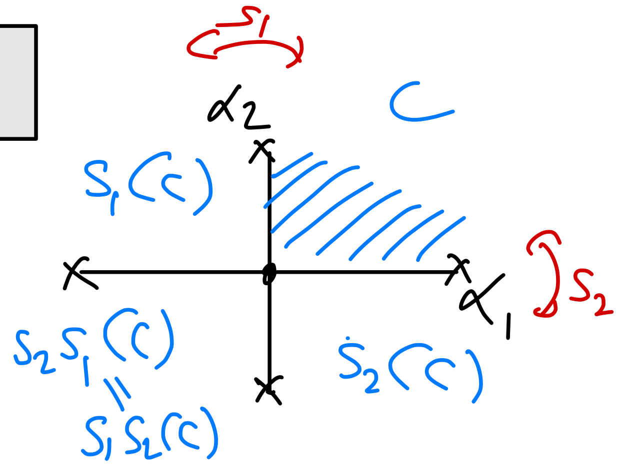
$\{\text{chambers}\} \xrightarrow{\sim} \{\text{bases}\}$, $\quad C \longmapsto \Delta_C = \{\alpha\in R \mid \alpha^\perp \text{ bounds of } C,\ (\alpha,\lambda)>0\ \forall \lambda\in C\}$

connected component of $E - \bigcup_{\alpha\in R} \alpha^\perp$

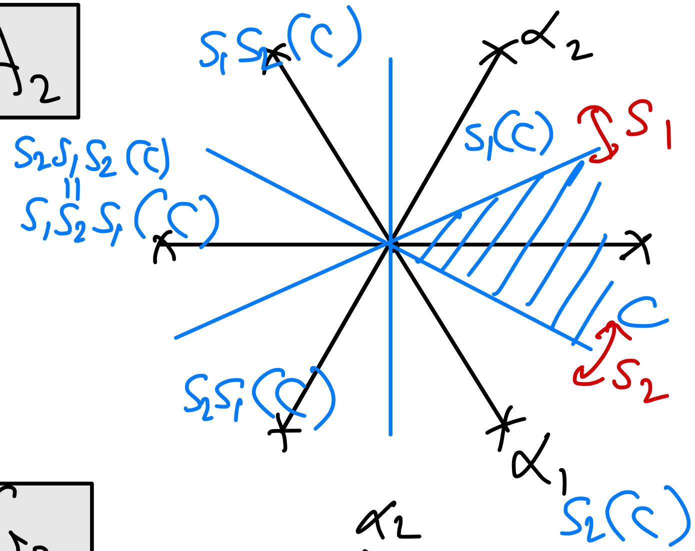
basis $\Delta = \{\alpha_1,\dots,\alpha_\ell\}\subset R$ so every $\alpha\in R$ is either a positive sum or a negative sum of α_i's.

This map is onto (stated last time) it also 1–1. Given $\Delta=\Delta_C$, you recover C by $\bigcap_{\alpha\in\Delta}\{\lambda\in E \mid (\alpha,\lambda)>0\}$

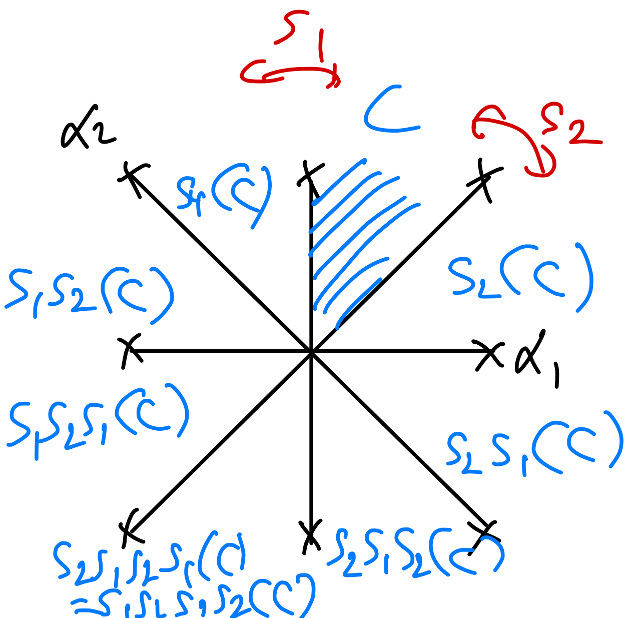
$A_1\times A_1$



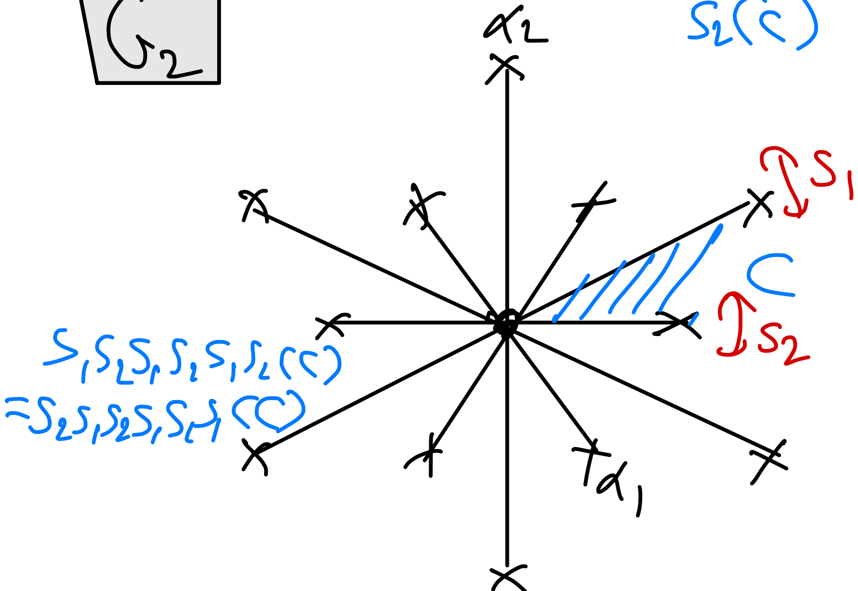
A_2



B_2



G_2



I'm going to assume from now on that we've fixed a choice of <u>fundamental chamber</u> C, hence, a base $\Delta = \Delta_C$.

Write $\Delta = \{\underbrace{\alpha_1, \dots, \alpha_\ell}_{\text{simple roots}}\}$ so $\ell = \operatorname{rank}(R) = \dim E$

Let $I = \{1, \dots, \ell\}$ so $\Delta = \{\alpha_i \mid i \in I\}$.

Let $R = \underset{\text{positive roots}}{R^+} \sqcup \underset{\text{negative roots}}{R^-}$

<u>Lemma</u> 3 Every $\alpha \in R$ belongs to at least one base.

<u>Proof</u> Take $\alpha \in R$. Pick $\gamma \in \alpha^\perp$ that's not on any other $\beta^\perp$ $(\beta \neq \pm\alpha)$

Then move a tiny bit from γ to γ' so $(\gamma', \alpha) = \varepsilon > 0$

and $|(\gamma', \beta)| > \varepsilon$ for $R \ni \beta \neq \pm\alpha$. Then let C' be the chamber containing γ'

This ensures $\alpha^\perp$ is bounding hyperplane for C', so $\alpha \in \Delta_{C'}$. //

<u>Weyl group</u> Let $W < O(E)$ be subgroup generated by all the reflections s_α $(\alpha \in R)$

$s_\alpha^2 = 1$

s_α fixes hyperplane $\alpha^\perp$ pointwise

By axiom ③, $s_\alpha(R) = R$, i.e. $W \curvearrowright R$

It acts faithfully as R spans E. Hence, $W \hookrightarrow \mathrm{Sym}(R)$

finite group!

So W is finite.

The representation E of W is called the <u>reflection representation</u> of W

Let $s_i = s_{\alpha_i}$ for short, call $s_1, \dots, s_\ell$ <u>simple reflections</u>

Note finally that $w s_\alpha w^{-1} = s_{w(\alpha)}$ $(\alpha \in R,\ w \in W)$

Def for $\alpha \in R$, let $ht(\alpha) = \sum_{i=1}^{\ell} c_i$ if $\alpha = \sum_{i=1}^{\ell} c_i \alpha_i$.
height

Lemma 4 Every $\alpha \in R^+$ is an $\mathbb{N}$-linear combination of simple roots. In fact,
$\alpha = \alpha_{i_1} + \dots + \alpha_{i_h}$ $(i_1, \dots, i_h \in I)$ in such a way that
$\alpha_{i_1} + \dots + \alpha_{i_k} \in R^+$ $\forall$ $1 \leq k \leq h$.

Proof If α is simple, there's nothing to prove, so take $\alpha \in R^+ - \Delta$.
Note that $(\alpha, \alpha_i) > 0$ for some $i \in I$.
Else, $\alpha = \sum_{i=1}^{\ell} c_i \alpha_i$ and $(\alpha, \alpha_i) \leq 0$ $\forall i$, you'd get

$$0 < (\alpha, \alpha) = \sum_{i=1}^{\ell} \underbrace{c_i}_{\geq 0} \underbrace{(\alpha, \alpha_i)}_{\leq 0} \leq 0 \quad \#$$

Pick such an i. Then $\alpha - \alpha_i \in R$ by Lemma 1, it must be positive as there's some $j \neq i$ so $\alpha - \alpha_i$ has α_j with positive coefficient

Now $ht(\underset{\uparrow \atop R^+}{\alpha-\alpha_i})$ is one smaller than α ... repeat

s_{α_i}

<u>Lemma 5</u> If $i \in I$, the simple reflection s_i permutes $R^+ - \{\alpha_i\}$, and $s_i(\alpha_i) = -\alpha_i$. Hence, $s_i(\rho) = \rho - \alpha_i$

$$E \ni \boxed{\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha}$$

<u>Proof</u> Take $\alpha \in R^+ - \{\alpha_i\}$. So $\alpha \in \mathbb{R}\alpha_i$, so α has some other α_j with positive coefficient. Then $s_i(\alpha) = \alpha - (\alpha, \alpha_i^\vee)\alpha_i$ is a root, it be positive by its α_j-coefficient.

Remains to note $s_i(\alpha) \neq \alpha_i$ as $\alpha \neq s_i(\alpha_i) = -\alpha_i$

Lemma 6 Take $i_1, \ldots, i_t \in I$. If $s_{i_1} \cdots s_{i_{t-1}} (\alpha_{i_t}) \in R^-$, then there exists $1 \leq u < t$ such that $\underbrace{s_{i_1} \cdots s_{i_t}}_{t \text{ reflection}} = \underbrace{s_{i_1} \cdots s_{i_{u-1}} s_{i_{u+1}} \cdots s_{i_{t-1}}}_{t-2 \text{ reflection}}$.

Proof Let u be minimal so $s_{i_{u+1}} \cdots s_{i_{t-1}} (\alpha_{i_t}) \in R^+$

When apply s_{i_u}, this positive root becomes negative, so Lemma 5 tells us that $s_{i_{u+1}} \cdots s_{i_{t-1}} (\alpha_{i_t}) = \alpha_{i_u}$

$$\underbrace{(s_{i_{u+1}} \cdots s_{i_{t-1}})}_{\omega} \underbrace{s_{i_t}}_{\alpha = \alpha_{i_t}} \underbrace{(s_{i_{u+1}} \cdots s_{i_{t-1}})^{-1}}_{\omega} = s_{\omega(\alpha)} = s_{i_u}$$

$$\therefore s_{i_1} \cdots s_{i_{u-1}} s_{i_{u+1}} \cdots s_{i_{t-1}} s_{i_t} = s_{i_1} \cdots s_{i_{u-1}} s_{i_u} s_{i_{u+1}} \cdots s_{i_{t-1}}$$

Now right multiply by s_{i_t} to finish proof //

Theorem The Weyl group W is generated by its simple reflections $s_1, \dots, s_\ell$.

Moreover, there are bijections

$$w(C) \mapsfrom w \qquad w \mapsto w(\Delta)$$

$$\begin{tikzcd} & W \arrow[dl, "\sim"'] \arrow[dr, "\sim"] \arrow[d, phantom, "\circ"] & \\ \{\text{chambers}\} \arrow[rr, "\sim"] & {} & \{\text{bases}\} \end{tikzcd}$$

constructed already, $C \mapsto \Delta_C$

C and Δ are initially chosen fundamental chamber/base

Proof Let $W' = \langle s_1, \dots, s_\ell \rangle \leq W$. Prove $\swarrow$ $\searrow$ with W replaced by W', then show $W = W'$ at the end.

Note that the diagram commutes obviously.

To show $\swarrow$ is onto, take $\gamma \in E - \bigcup_{\alpha \in R} \alpha^\perp$ in some chamber.

We'll show $\exists\, w \in W'$ so $w(\gamma) \in C$. That shows W' acts transitively on chambers, giving $\swarrow$.

Pick $w \in W'$ so $(w(\gamma), \rho)$ is maximal

$\uparrow \frac{1}{2} \sum_{\alpha \in R^+} \alpha$

For $i \in I$:

$$(w(\gamma), \rho) \geq (s_i w(\gamma), \rho) = (w(\gamma), s_i(\rho)) = (w(\gamma), \rho - \alpha_i)$$
$$= (w(\gamma), \rho) - (w(\gamma), \alpha_i)$$

Shows $(w(\gamma), \alpha_i) \geq 0 \quad \forall i$.

Since γ is not on any hyperplanes, nor is $w(\gamma)$, so $(w(\gamma), \alpha_i) > 0$ $\forall i$

Hence $w(\gamma) \in C$.

Now show $\searrow$ is injective. Take $1 \neq w \in W'$ with $w(\Delta) = \Delta$.

Write $w = s_{i_1} \cdots s_{i_r}$ with r minimal.

Then $s_{i_1} \cdots s_{i_r}(\alpha_{i_r}) \in R^+$, hence, $s_{i_1} \cdots s_{i_{r-1}}(\alpha_{i_r}) \in R^-$.

By Lemma 6, we deduce that

$$w = s_{i_1} \cdots s_{i_r} = s_{i_1} \cdots s_{i_{u-1}} s_{i_{u+1}} \cdots s_{i_{r-1}}$$

but that contradicts the minimality of r.

W' acts transitively on bases

Hence $\swarrow^{S}$ $\searrow^{S}$ both bijections.

Finally must show $W = W'$.

Take $\alpha \in R$, need to show $s_\alpha \in W'$. By Lemma 3, there's a base containing α, so get $w \in W'$ with $\alpha \in w(\Delta)$.

So $w^{-1}(\alpha) = \alpha_i$ for some $i \in I$.

$\therefore \; w s_i w^{-1} = s_{w(\alpha_i)} = s_\alpha \in W'$ //

$$W = \langle s_1, \ldots, s_\ell \rangle = \langle s_1, \ldots, s_\ell \mid s_i^2 = 1, \underbrace{s_i s_j \cdots}_{m_{ij}} = \underbrace{s_j s_i \cdots}_{m_{ij}} \quad i \neq j \rangle$$

COXETER GROUP

finite, crystallographic

What are relations between these?

Follows by inspection of rank 2 pictures!

$$s_i^2 = 1 \qquad \forall i$$

Take $i \neq j$, $\underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}$

where

m_{ij}	$(\alpha_i, \alpha_j^\vee)(\alpha_j, \alpha_i^\vee)$	type of rank 2 system
2	0	$A_1 \times A_1$
3	1	A_2
4	2	B_2
6	3	G_2

In fact these give ALL relations for W