Last time: three lemmas

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

<u>Cartan decomposition</u>
<u>of semisimple Lie algebra</u>

$(\cdot,\cdot)$ non-degenerate invariant form

Restriction to $\mathfrak{h}$ non-degenerate

$\mathfrak{h} \longleftrightarrow \mathfrak{h}^*$

$t_\lambda \longleftrightarrow \lambda$

L1 • For $\alpha \in R$, $c\alpha \in R \iff c = \pm 1$

• Each $\mathfrak{g}_\alpha$ is 1-D

L2 • Can pick $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$

and $h_\alpha = \dfrac{2t_\alpha}{(t_\alpha, t_\alpha)} \in \mathfrak{h}$

$\alpha^\vee \in \mathfrak{h}^*$

so $(e_\alpha, h_\alpha, f_\alpha)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ ... span subalgebra $\mathfrak{g}_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$

L3 • For $\alpha, \beta \in R$, $\beta \neq \pm\alpha$, let r, q maximal so $\beta - r\alpha, \beta + q\alpha \in R$

Then $\underbrace{\beta - r\alpha, \dots, \beta + i\alpha, \dots, \beta + q\alpha}_{\alpha\text{-string through } \beta}$ entire string (all $-r \leq i \leq q$) are roots

$(\beta, \alpha^\vee)$

hence $\beta(h_\alpha) = r - q \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in R$.

<u>Notation</u> • Transport form $(\cdot,\cdot)$ on $\mathfrak{h}$ to $\mathfrak{h}^*$ so $(\lambda, \mu) = (t_\lambda, t_\mu)$ $\lambda, \mu \in \mathfrak{h}^*$

• Always write $\alpha^\vee = \dfrac{2\alpha}{(\alpha,\alpha)}$ for $\alpha \in R$. <u>COROOT</u> corresponding to α.

Definition A **root system** is a pair (E, R) such that

Euclidean space
Real vector space with a pos. def. symmetric bilinear form $(\cdot,\cdot)$

subset of E of "roots"

① R is finite and spans E

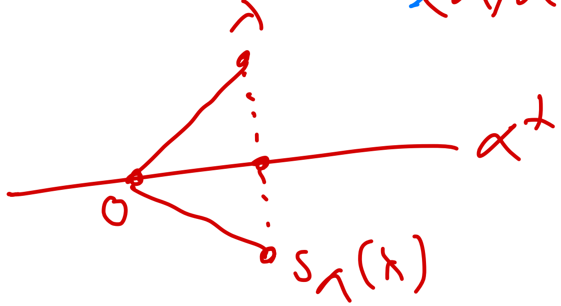
② If $\alpha \in R$, then $c\alpha \in R \iff c = \pm 1$

③ If $\alpha, \beta \in R$ then $\boxed{s_\alpha(\beta)} \in R$

④ If $\alpha, \beta \in R$ then $\cancel{\dfrac{2(\beta,\alpha)}{(\alpha,\alpha)}} \in \mathbb{Z}$
$(\beta, \alpha^\vee)$

$s_\alpha : E \to E$ reflection in $\alpha^\perp$

$$s_\alpha(\lambda) = \lambda - \cancel{\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}}\,\alpha$$
$(\lambda, \alpha^\vee)$



Shorthand $\alpha^\vee = \dfrac{2\alpha}{(\alpha,\alpha)}$ for $\alpha \in R$

coroot

$\mathbb{C}$, $\mathbb{Z}^*$ NEED A REAL SPACE NOT COMPLEX

We want to show our R of roots coming from Cartan decomposition is a root system in this sense. We've seen ②, ③, ④ already!!

Definition Given $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ Cartan decomposition of semisimple Lie algebra

define $E = \mathbb{R} R \subset \mathfrak{t}^*$
(real vector space spanned by R in complex vectorspace $\mathfrak{t}^*$)

Lemma 4 $\underbrace{\dim_{\mathbb{R}} E = \dim_{\mathbb{C}} \mathfrak{t}^*}_{\text{rank}(\mathfrak{g})}$, i.e. $\mathfrak{t}^* = \mathbb{C} \otimes_{\mathbb{R}} E$

"E is an $\mathbb{R}$-lattice in $\mathfrak{t}^*$"

Proof Pick $\alpha_1, \ldots, \alpha_\ell \in R$ so give a basis for $\mathfrak{t}^*$ as a $\mathbb{C}$-v.space.
So $\alpha_1, \ldots, \alpha_\ell$ are lin. independent over $\mathbb{C}$, hence, over $\mathbb{R}$. We need to show $\alpha_1, \ldots, \alpha_\ell$ span E as a $\mathbb{R}$-v.space. So take $\beta \in R$.
Write $\beta = \sum_{i=1}^{\ell} c_i \alpha_i$ for $c_i \in \mathbb{C}$, need to show each $c_i \in \mathbb{R}$.

$$\mathbb{Z} \ni (\beta, \alpha_j^\vee) = \sum_{i=1}^{\ell} c_i (\alpha_i, \alpha_j^\vee) \longrightarrow \left((\alpha_i, \alpha_j^\vee)\right)_{1 \le i,j \le \ell}$$

integer matrix
Invertible as $\alpha_1, \ldots, \alpha_\ell$ and $\alpha_1^\vee, \ldots, \alpha_\ell^\vee$ are bases for $\mathfrak{t}^*$ and $(\cdot,\cdot)$ is non-deg.

Inverse matrix has rational entries $\Longrightarrow c_i \in \mathbb{Q} \ \forall i$ //

Finally we need inner product on E.

Up to now, any invariant non-degenerate form was fine $(\cdot,\cdot)$, not necessarily K. Now to get Euclidean space structure on E, we need to be more careful! Look at HW 6-2 ... all forms are K scaled by non-zero scalars on each simple component of $\mathfrak{g}$.

<u>Lemma 5</u> Assume that $(\cdot,\cdot)$ is the Killing form possibly scaled by positive real numbers on each simple component of $\mathfrak{g}$.
The restriction of $(\cdot,\cdot)$ to $E \subset \mathfrak{h}^*$ is real-valued, positive definite symmetric bilinear form making E into Euclidean space.

<u>Proof</u> RTP

$$\left.\begin{array}{l} (\alpha,\beta) \in \mathbb{R} \quad \forall\, \alpha,\beta \in R \\ \text{and} \quad (\lambda,\lambda) > 0 \quad \forall\, 0 \neq \lambda \in E \end{array}\right\}$$

WLOG $(\cdot,\cdot)$ is the Killing form.

Take $\lambda, \mu \in Z^*$. $(\lambda, \mu) = (t_\lambda, t_\mu) = \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad} t_\lambda \circ \operatorname{ad} t_\mu)$

$$= \sum_{\alpha \in R} \alpha(t_\lambda)\,\alpha(t_\mu) = \sum_{\alpha \in R} (\alpha, \lambda)(\alpha, \mu)$$

$(\dagger)$

Now take $\beta \in R$. $(\beta, \beta) = \sum_{\alpha \in R} (\alpha, \beta)^2$ by $(\dagger)$

$$\therefore \frac{1}{(\beta,\beta)} = \sum_{\alpha \in R} \frac{(\alpha,\beta)^2}{(\beta,\beta)^2} = \sum_{\alpha \in R} \frac{(\alpha, \beta^\vee)^2}{4} \in \mathbb{Q}$$

$$\therefore (\beta, \beta) \in \mathbb{Q} \quad \forall \ \beta \in R$$

$$\therefore (\alpha, \beta) = (\alpha, \beta^\vee) \cdot \frac{(\beta,\beta)}{2} \in \mathbb{Q} \quad \forall \ \alpha, \beta \in \mathbb{Q}$$

Shows $(\cdot,\cdot)$ is real-valued ✓

Finally for $0 \neq \lambda \in E$, by (†)

$$(\lambda, \lambda) = \sum_{\alpha \in R} \underbrace{(\alpha, \lambda)}_{\in \mathbb{R}}^2 \geq 0$$

If it equals zero, $(\alpha, \lambda) = 0 \; \forall \alpha$, hence, $\lambda = 0$ by non-degeneracy. So actually $(\lambda, \lambda) > 0$, and the form is positive definite

Goal next:

1. Classification of root systems
2. Show any semisimple Lie algebra $\mathfrak{g}$ is determined up to $\cong$ by its root system.
3. Show every root system comes from a semisimple Lie algebra.

$$\{\text{semisimple Lie algebras}\}/\cong \quad \hookrightarrow \quad \{\text{root system}\}/\cong$$

Root systems are sums of indecomposable root systems.

Indecomposable root systems $\longleftrightarrow$ indecomposable Cartan matrices $\longleftrightarrow$ Dynkin diagrams

$$\big((\alpha_i, \alpha_j^\vee)\big)_{1 \le i,j \le \ell}$$ for careful choice of basis $\alpha_1, \dots, \alpha_\ell$

$\ell = \operatorname{rank}(\mathfrak{g}) = \dim E$

$\mathfrak{sl}_n(\mathbb{C}) \qquad n = \ell + 1$

$$\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}$$

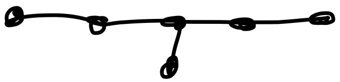
Cartan matrix !!!

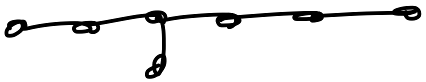
A_ℓ 

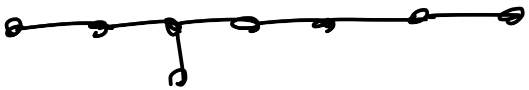
B_ℓ 

C_ℓ 

D_ℓ 

E_6 

E_7 

E_8 

F_4 

G_2