From now $\mathfrak{g}$ will be a f.d. semisimple Lie algebra.

Any $x \in \mathfrak{g}$ can be written as $x = \underset{\text{semisimple}}{x_s} + \underset{\text{nilpotent}}{x_n}$, $[x_s, x_n] = 0$

(act semisimply or nilpotently on every f.d. representation of $\mathfrak{g}$)

<u>Definition</u> A <u>toral subalgebra</u> of $\mathfrak{g}$ is a subalgebra consisting of semisimple elts.

<u>Lemma</u> If $\mathfrak{t}$ is a toral subalgebra of $\mathfrak{g}$, then $\mathfrak{t}$ is Abelian.

<u>Proof</u> Take $x \in \mathfrak{t}$. As $\mathrm{ad}\, x : \mathfrak{g} \to \mathfrak{g}$ is semisimple and leaves $\mathfrak{t}$ invariant, its restriction $\mathrm{ad}\, x|_{\mathfrak{t}} : \mathfrak{t} \to \mathfrak{t}$ is semisimple too.

So we can take $y \in \mathfrak{t}$ with $[x, y] = \lambda y$ $(\lambda \in \mathbb{C})$ and ETS that $\lambda = 0$.

$[x,y] = \lambda y$

$\mathrm{ad}\, y|_{\mathfrak{t}} : \mathfrak{t} \to \mathfrak{t}$ also diagonalisable, so can write $x = \sum_{\mu \in \mathbb{C}} x_\mu$

where $[y, x_\mu] = \mu x_\mu$.

Then $[y,[y,x]] = -[y, \lambda y] = 0$

$$\begin{aligned} [y,[y,x]] &= \sum_{\mu\in\mathbb{C}} [y,[y,x_\mu]] = \sum_{\mu\in\mathbb{C}} \mu [y,x_\mu] = \sum_{\mu\in\mathbb{C}} \mu^2 x_\mu \end{aligned}$$

$\therefore \sum_{\mu\in\mathbb{C}} \mu^2 x_\mu = 0$ $\quad \therefore \mu^2 x_\mu = 0 \ \forall \mu$

Shows $x = x_0$ and $[y,x] = 0$

Hence $\lambda = 0$ //

Now take some $\mathfrak{t} < \mathfrak{g}$ toral, hence, Abelian.

As $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ is diagonalisable $\forall x \in \mathfrak{t}$, and these endomorphisms all commute, we can simultaneously diagonalise $\mathfrak{g}$ wrt $\mathfrak{t}$:

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{t}^*} \mathfrak{g}_\lambda \qquad \text{where} \quad \mathfrak{g}_\lambda = \left\{ y \in \mathfrak{g} \;\middle|\; [x,y] = \lambda(x)\, y \ \ \forall x \in \mathfrak{t} \right\}$$

"weight"

Let $R = \{ 0 \neq \alpha \in \mathfrak{t}^* \mid \mathfrak{g}_\alpha \neq 0 \}$ so...

$$\boxed{\mathfrak{g} = \underset{\substack{\| \\ C_{\mathfrak{g}}(\mathfrak{t}) \text{ centraliser}}}{\mathfrak{g}_0} \oplus \bigoplus_{\alpha \in R} \underset{\substack{\nparallel \\ 0}}{\mathfrak{g}_\alpha}}$$

Lemma $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$

Proof Jacobi identity. $[x,[y,z]] = \cdots$ //

Corollary 1 Any $x \in \mathfrak{g}_\alpha$ $(\alpha \in R)$ is nilpotent.

Corollary 2 If $\lambda, \mu \in \mathfrak{h}^*$ with $\lambda+\mu \neq 0$, then $\mathfrak{g}_\lambda \perp_K \mathfrak{g}_\mu$.

Proof $\operatorname{tr}_{\mathfrak{g}}(\underbrace{\operatorname{ad}x \circ \operatorname{ad}y}_{?})$ for $x \in \mathfrak{g}_\lambda, y \in \mathfrak{g}_\mu$

maps $\mathfrak{g}_\alpha \longrightarrow \mathfrak{g}_{\alpha + \underbrace{\lambda+\mu}_{\neq 0}}$ hence $\operatorname{ad}x \circ \operatorname{ad}y$ is nilpotent, so it has trace zero //

Corollary 3 $K|_{\mathfrak{g}_\alpha} = 0$ $(\alpha \in R)$, $K|_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}}$ is non-degenerate

and $K|_{\mathfrak{g}_0}$ is non-degenerate.

Proof Use Corollary 2 and non-degeneracy of K. //

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \qquad \text{———} \qquad (\ast)$$

$\perp$

$\mathfrak{g}_0$

Theorem Let $\mathfrak{t} \leq \mathfrak{g}$ be a toral subalgebra as above. Then $\mathfrak{t}$ is a <u>maximal</u> toral subalgebra if and only if $\mathfrak{t} = c_{\mathfrak{g}}(\mathfrak{t})$

Proof ($\Leftarrow$) If $\mathfrak{t}$ was not maximal, you could find $\hat{\mathfrak{t}} \supsetneq \mathfrak{t}$, a bigger toral subalgebra. But $\hat{\mathfrak{t}}$ Abelian by lemma, hence $\hat{\mathfrak{t}} \leq c_{\mathfrak{g}}(\mathfrak{t})$ $\#$

($\Rightarrow$) Assume $\mathfrak{t}$ is a maximal toral subalgebra.

Let $\mathfrak{h} = c_{\mathfrak{g}}(\mathfrak{t})$.

Show $\mathfrak{h} = \mathfrak{t}$.

① $\mathfrak{h}$ contains the semisimple and nilpotent parts of all of its elements

$x \in \mathfrak{h} \iff \operatorname{ad} x\,(\mathfrak{t}) = 0$

As $\operatorname{ad} x_s$ and $\operatorname{ad} x_n$ are polynomial in $\operatorname{ad} x$ with constant term zero

$(\operatorname{ad} x)_s$ $(\operatorname{ad} x)_n$

It follows that $(\operatorname{ad} x_s)(\mathfrak{t}) = (\operatorname{ad} x_n)(\mathfrak{t}) = 0$, so $x_s, x_n \in \mathfrak{h}$.

② All semisimple element of $\mathfrak{h}$ lie in $\mathfrak{t}$

Take $x \in \mathfrak{h}$ semisimple, $[x, \mathfrak{t}] = 0$

Then $\mathfrak{t} + \mathbb{C}x$ is a toral subalgebra of $\mathfrak{g}$, hence, it equals $\mathfrak{t}$ by the assumed maximality.

Show $x \in \mathfrak{t}$

③ $\operatorname{ad} x|_{\mathfrak{h}}$ is nilpotent $\forall x \in \mathfrak{h}$

Take $x \in \mathfrak{h}$, $x = x_s + x_n$ with $x_s, x_n \in \mathfrak{h}$ by ①.

As $x_s \in \mathfrak{z}$ by ②, $\operatorname{ad} x_s|_{\mathfrak{h}} = 0$.

Also $\operatorname{ad} x_n$ is nilpotent, so $\operatorname{ad} x_n|_{\mathfrak{h}}$ is nilpotent

Now $\operatorname{ad} x_s$, $\operatorname{ad} x_n$ commuting nilpotent endomorphisms of $\mathfrak{h}$

So $\operatorname{ad} x|_{\mathfrak{h}} = \operatorname{ad}(x_s + x_n)|_{\mathfrak{h}} = \operatorname{ad} x_s|_{\mathfrak{h}} + \operatorname{ad} x_n|_{\mathfrak{h}}$ is nilpotent.

④ $K|_{\mathfrak{z}}$ is non-degenerate

We know $K|_{\mathfrak{h}}$ is non-degenerate by Cor. 3.

Take $x \in \mathfrak{z}$ with $K(x, \mathfrak{z}) = 0$. RTP $x = 0$.

It suffices to show $K(x, y) = 0 \quad \forall y \in \mathfrak{h}$

$$\underset{\parallel}{K(x,y)}$$

$$K(x, y_s + y_n) = K(x, \underset{\uparrow \mathfrak{z}}{y_s}) + K(x, y_n)$$

$$\underset{=0}{K(x,y_s)}$$

It remains to show $K(x, y_n) = 0$

As $\underset{\substack{\uparrow \\ \text{semisimple}}}{\operatorname{ad} x}$, $\underset{\substack{\uparrow \\ \text{nilpotent}}}{\operatorname{ad} y_n}$ commute, hence their composition is nilpotent,

its trace is zero, as required.

⑤ $\underline{\mathfrak{h} \text{ is Abelian}}$

Suppose not, then $\mathfrak{h}' \neq 0$. Consider adjoint action of $\mathfrak{h}$ on $\mathfrak{h}'$.

For $x \in \mathfrak{h}$, $\operatorname{ad} x : \mathfrak{h}' \to \mathfrak{h}'$ is nilpotent by ③, so by Engel's theorem we can find some $0 \neq y \in \mathfrak{h}'$ so $[\mathfrak{h}, y] = 0$.

Consider $y = y_s + y_n$ $\quad y_s, y_n \in \mathfrak{h}$ by ①, and $[\mathfrak{h}, y_s] = [\mathfrak{h}, y_n] = 0$

as $\operatorname{ad} y_s$, $\operatorname{ad} y_n$ are polys in $\operatorname{ad} y$ with constant term zero.

$K(y_n, \mathfrak{h}) = 0$ as $\operatorname{ad} y_n \circ \operatorname{ad} x$ is nilpotent $\forall x \in \mathfrak{h}$

As $K|_{\mathfrak{h}}$ is non-deg. by Cor. 3 this gives that $y_n = 0$.

Shown y is semisimple. Hence, $y \in \mathfrak{z}$ by ②.

But then $K(\mathfrak{z}, \underset{\overset{\cup}{y}}{\mathfrak{h}'}) = K(\mathfrak{z}, [\mathfrak{h}, \mathfrak{h}]) = K(\underbrace{[\mathfrak{z}, \mathfrak{h}]}_{0}, \mathfrak{h}) = 0$

As $K|_{\mathfrak{z}}$ is non-degenerate by ④, this shows $y = 0$.

⑥ $\underline{\mathfrak{h} = \mathfrak{z}}$ By ① & ②, it suffices to show $x \in \mathfrak{h}$, x nilpotent has $x = 0$.

But $[x, \mathfrak{h}] = 0$ by ⑤, ad x is nilpotent

So $K(x, \mathfrak{h}) = 0$, which implies $x = 0$ by Cor. 3

$\uparrow$

ad $x \circ$ ad y nilpotent

When you choose $\mathfrak{t}$ to be a maximal toral subalgebra, get decomposition

$\mathfrak{g}_0 = c_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$ by Theorem

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

$\perp$

- CARTAN DECOMPOSITION of $\mathfrak{g}$ wrt $\mathfrak{t}$
- R ROOT SYSTEM
- $\alpha \in R$ ROOT

$K|_{\mathfrak{t}}$ is non-deg.

$K|_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}}$ is non-deg. $\forall \alpha \in R$, $-\alpha$ also lies R