Theorem $\mathfrak{g}$ is semisimple $\iff$ K is non-degenerate

Killing form $K(x,y) = tr_{\mathfrak{g}}(\mathrm{ad}\,x \circ \mathrm{ad}\,y)$

trace form $\tau(x,y) = tr_V(x \circ y)$

Proof reduced to ...

Cartan's Criterion If $\mathfrak{g} \leq \mathfrak{gl}(V)$, f.d. V, and $\tau(x,y) = 0 \ \forall x,y \in \mathfrak{g}$,

then $\mathfrak{g}$ is solvable.

Proof of this needs ...

Lemma Let S be a subspace of $\mathfrak{gl}(V)$, f.d. V, and let

$$\mathfrak{n} = \mathfrak{n}_{\mathfrak{gl}(V)}(S) = \{x \in \mathfrak{gl}(V) \mid \mathrm{ad}\,x(S) \subseteq S\}.$$

If $x \in \mathfrak{n}$ has $\tau(x,y) = 0 \ \forall y \in \mathfrak{n}$, then x is nilpotent.

Proof. Let $x = x_s + x_n$ be Jordan decomposition of $x \in End_{\mathbb{C}}(V)$.

RTP $x_s = 0$.

Pick a basis for V so that matrix of x is upper triangular, hence,

$x_s = \mathrm{diag}(t_1, \dots, t_n) \quad (t_i \in \mathbb{C})$.

Let $F = \mathbb{R}\text{-span}(t_1, \dots, t_n)$ real subspace of $\mathbb{C}$.

RTP $F = 0$.

Take $f \in \mathrm{Hom}_{\mathbb{R}}(F, \mathbb{R})$. RTP $f = 0$.

Let $y = \mathrm{diag}(f(t_1), \dots, f(t_n)) \in \mathrm{End}_{\mathbb{C}}(V)$.

$[x_s, e_{ij}] = (t_i - t_j) e_{ij}$ $\quad [y, e_{ij}] = (f(t_i) - f(t_j)) e_{ij} = f(t_i - t_j) e_{ij}$.

ij-matrix unit

Pick poly $r(t) \in \mathbb{C}[t]$ so $r(t_i - t_j) = f(t_i - t_j)\ \forall i,j$. (Lagrange interpolation)

Then $\mathrm{ad}\, x_s(e_{ij}) = (t_i - t_j) e_{ij}$ so $r(\mathrm{ad}\, x_s)(e_{ij}) = r(t_i - t_j) e_{ij}$

Shows $r(\mathrm{ad}\, x_s) = \mathrm{ad}\, y$ $\quad = f(t_i - t_j)(e_{ij}) = \mathrm{ad}\, y(e_{ij}) \quad \forall\, i,j$

$r(\operatorname{ad} x_s) = \operatorname{ad} y$

Next note that $\operatorname{ad}(x_s) = (\operatorname{ad} x)_s$.

follows because $\operatorname{ad}(x_s)$ is semisimple, $\operatorname{ad}(x_n)$ is nilpotent, they commute, and $\operatorname{ad} x = \operatorname{ad}(x_s) + \operatorname{ad}(x_n)$

$\therefore \operatorname{ad}(x_s) = (\operatorname{ad} x)_s$ and $\operatorname{ad}(x_n) = (\operatorname{ad} x)_n$

by uniqueness of Jordan decomposition.

So $\operatorname{ad}(x_s)$ is a polynomial in $\operatorname{ad} x$

$\implies \operatorname{ad} y = p(\operatorname{ad} x)$ for some poly $p(t) \in \mathbb{C}[t]$

$\operatorname{ad} x (S) \subseteq S$

$\therefore p(\operatorname{ad} x)(S) \subseteq S$

$\therefore \operatorname{ad} y (S) \subseteq S$ Shows $y \in \mathfrak{n}$.

$\tau(x,y) = 0$ by assumption

$$\overset{\|}{\sum_{i=1}^{n} t_i f(t_i)} \in F$$

Apply $\mathbb{R}$-linear map f ... get $\sum_{i=1}^{n} f(t_i)^2 = 0$

$\Longrightarrow f(t_i) = 0 \quad \forall i$

$\therefore f = 0$ //

<u>Proof of Cartan's criterion</u> $\mathfrak{g} \leq \mathfrak{gl}(V)$ $\quad \tau(\mathfrak{g}, \mathfrak{g}) = 0$.

Apply lemma to $S = \mathfrak{g}$. $\quad n = n_{\mathfrak{gl}(V)}(\mathfrak{g})$

Take $x \in \mathfrak{g}'$, derived subalgebra.

To apply lemma, need to check

$$\tau(x,y)=0 \quad \forall\, y\in\mathfrak{n}$$

$\mathfrak{g}'$ is spanned by $[x_1,x_2]$, $x_1,x_2\in\mathfrak{g}$

Linearity $\Rightarrow$ STP

$$\tau([x_1,x_2],y)=0 \quad \forall\, y\in\mathfrak{n}.$$

$$\| $$

$$\tau(\underset{\in\,\mathfrak{g}}{x_1},\ \underset{\in\,\mathfrak{g}}{[x_2,y]})=0 \ \checkmark$$

Then lemma $\Rightarrow$ x is nilpotent.

Shows every $x\in\mathfrak{g}'$ is nilpotent,

hence, $\mathfrak{g}'$ is a nilpotent Lie algebra

by Engel's theorem.

So $\mathfrak{g}$ is solvable //

Corollary A f.d. Lie algebra $\mathfrak{g}$ is semisimple if and only if it is a finite direct sum of simple Lie algebras.

non-Abelian

$[\mathfrak{g}_i, \mathfrak{g}_j] = 0$
$i \neq j$

Moreover, if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, $\mathfrak{g}_i$'s simple, then the ideals of $\mathfrak{g}$ are exactly the 2^n subspaces obtained that are $\oplus$'s of some of the $\mathfrak{g}_i$'s. Hence, all ideals and all quotients of a semisimple Lie algebra are semisimple.

Proof ($\Longrightarrow$) Suppose that $\mathfrak{g}$ is semisimple. Let $\mathfrak{n} \trianglelefteq \mathfrak{g}$ be an ideal.

$$\mathfrak{n}^\perp = \{ y \in \mathfrak{g} \mid K(x,y) = 0 \ \forall x \in \mathfrak{n} \} \trianglelefteq \mathfrak{g}$$

and $\mathfrak{n} \cap \mathfrak{n}^\perp \trianglelefteq \mathfrak{g}$ too.

Killing form on $\mathfrak{n}$ is the restriction of Killing form on $\mathfrak{g}$. This follows as for $x \in \mathfrak{n}$

$$\operatorname{ad} x = \begin{array}{cc} \text{basis for } \mathfrak{n} & \text{extend to basis for } \mathfrak{g} \\ \left[\begin{array}{c|c} * & * \\ \hline 0 & 0 \end{array}\right. & \left.\right] \end{array}$$

Radical of Killing form on $\mathfrak{n}$ is $\mathfrak{n} \cap \mathfrak{n}^\perp$, so Killing form on $\mathfrak{n} \cap \mathfrak{n}^\perp$ is zero. Deduce $\mathfrak{n} \cap \mathfrak{n}^\perp$ is solvable by Cartan's criterion. Shows $\mathfrak{n} \cap \mathfrak{n}^\perp = 0$.

$\mathfrak{g}$ semisimple, $\mathfrak{n} \trianglelefteq \mathfrak{g}$, then $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{n}^\perp$

$\oplus$ of Lie algebras too
$[\mathfrak{n}, \mathfrak{n}^\perp] \subseteq \mathfrak{n} \cap \mathfrak{n}^\perp = 0$.

Also Killing forms on $\mathfrak{n}$ and $\mathfrak{n}^\perp$ are non-degenerate, so $\mathfrak{n}$ and $\mathfrak{n}^\perp$ are themselves semisimple Lie algebras.

Repeat ... induction on dimension gives that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ simple ideals.

$\mathfrak{g}_1, \mathfrak{g}_2 \ldots$

$$\begin{pmatrix} K_1 & & 0 \\ & K_2 & \\ 0 & & \ddots \\ & & & K_n \end{pmatrix}$$

$(\Leftarrow)$ Suppose $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ simple ideals.

Killing form on $\mathfrak{g}$ is $\oplus$ of Killing forms on each $\mathfrak{g}_i$.

Hence K is non-degenerate, so $\mathfrak{g}$ is semisimple.

"Moreover" $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ simple ideals.

Take $\mathfrak{n} \triangleleft \mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{n}^\perp$ ... repeat deduce that $\mathfrak{n}$ is a $\oplus$ of simple ideals of $\mathfrak{g}$. So all we need to do is show every simple ideal $\mathfrak{n} \triangleleft \mathfrak{g}$ is equal to some $\mathfrak{g}_i$.

But $[\mathfrak{n}, \mathfrak{g}] = [\mathfrak{n}, \mathfrak{g}_1] \oplus \cdots \oplus [\mathfrak{n}, \mathfrak{g}_n] \trianglelefteq \underset{\text{simple}}{\mathfrak{n}}$

Not zero as $\mathfrak{z}(\mathfrak{g}) = 0$

Hence, $[\mathfrak{n}, \mathfrak{g}] = \mathfrak{n}$. Deduce $[\mathfrak{n}, \mathfrak{g}_i]$ is zero for all but one i;

for this i: $[\mathfrak{n}, \mathfrak{g}_i] = \mathfrak{n} \trianglelefteq \underset{\text{simple}}{\mathfrak{g}_i} \implies \mathfrak{n} = \mathfrak{g}_i$

Next up:

<u>Weyl's theorem on complete reducibility</u> $\mathfrak{g}$ f.d. semisimple.

$$\begin{array}{l}\mathrm{Rep}(\mathfrak{g}) \text{ is a semisimple Abelian category.} \\ \quad \text{\rotatebox{90}{$\cong$}} \\ U(\mathfrak{g})\text{-mod}_{\text{f.d.}}\end{array}$$

Ingredient needed in proof of Weyl's theorem: <u>Casimir element</u>

Def. Let $\mathfrak{g}$ be a f.d. semisimple Lie algebra. Its Casimir element is

$$z = \sum_{i=1}^{n} x_i y_i \in U(\mathfrak{g})$$

where $x_1, \ldots, x_n$ is a basis for $\mathfrak{g}$ and $y_1, \ldots, y_n$ is the dual basis wrt Killing form K.

Note: z is well-defined independent of the choice of basis. Ex.

<u>Lemma</u> z, Casimir, lies in the center $\underbrace{Z(U(\mathfrak{g}))}_{}$ of the associative algebra $U(\mathfrak{g})$.

$Z(\mathfrak{g})$ for short

__Proof.__ $z = \sum_{i=1}^{n} x_i y_i \in U(\mathfrak{g})$.

Take $x \in \mathfrak{g}$. We must show $[x, z] = xz - zx = 0$.

Let
$$\left.\begin{aligned} [x_i, x] &= \sum_{j=1}^{n} a_{ij} x_j \\ [x, y_j] &= \sum_{i=1}^{n} b_{ij} y_i \end{aligned}\right\} \quad \text{some } a_{ij}, b_{ij}.$$

$$K([x_i, x], y_j) = a_{ij} \qquad a_{ij} = b_{ij} \quad \forall i, j.$$
$$\shortparallel$$
$$K(x_i, [x, y_j]) = b_{ij}$$

$$[z, x] = \sum_{i=1}^{n} [x_i, x] y_i - \sum_{j=1}^{n} x_j [x, y_j]$$
$$= \sum_{i,j=1}^{n} a_{ij} x_j y_i - \sum_{i,j=1}^{n} x_j b_{ij} y_i = 0$$

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