

Rep (\mathfrak{sl}_2) \cong $U(\mathfrak{g})$ -mod_{f.d.}

Today $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$[e, f] = h \quad [h, e] = 2e \quad [h, f] = -2f$$

If v is a non-zero vector in some \mathfrak{g} -module V such that $hv = \lambda v$
($\lambda \in \mathbb{C}$)

then $\left. \begin{aligned} h(ev) &= (\lambda+2)ev \\ h(fv) &= (\lambda-2)fv \end{aligned} \right\}$

v is an eigenvector for h of eigenvalue λ
 v is a weight vector of weight λ .

$$[h, e]v = h(ev) - e(hv)$$

$$\therefore 2ev = h(ev) - \lambda ev$$

$$\therefore h(ev) = (\lambda+2)ev \quad \checkmark$$

If $0 \neq V$ is any f.d. \mathfrak{sl}_2 -module, you can always find a (non-zero) weight vector $v \in V$ of weight $\lambda \in \mathbb{C}$.

v, ev, e^2v, \dots Can't go forever as V is f.d.
 $\lambda \quad \lambda+2 \quad \lambda+4$

Eventually you get to a weight vector killed by e $\left. \begin{array}{l} 0 \neq v \in V \\ hv = \lambda v \\ ev = 0 \end{array} \right\}$
 \equiv highest weight vector

We're going to show any f.d. irred. representation is determined uniquely up to isomorphism by its highest weight — the weight of a highest wt vector.

This weight is always $\lambda = n \in \mathbb{N}$

$L(n)$ $(n \in \mathbb{N})$

"highest weight n "
 $\dim L(n) = n + 1.$

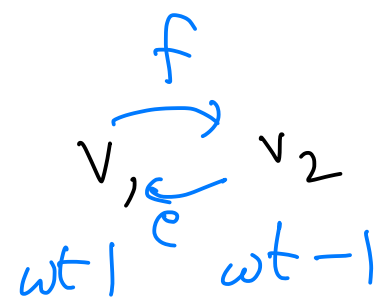
Examples

\mathbb{C} highest weight 0 $L(0)$

trivial module

$V = \mathbb{C}^2$

natural representation



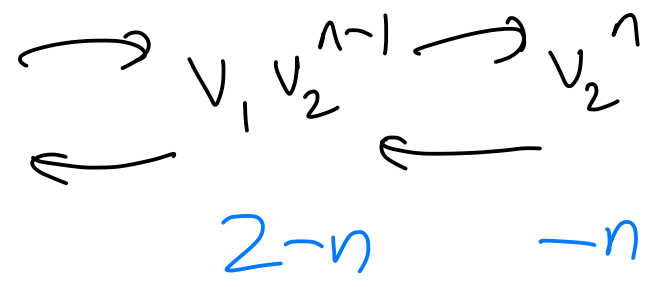
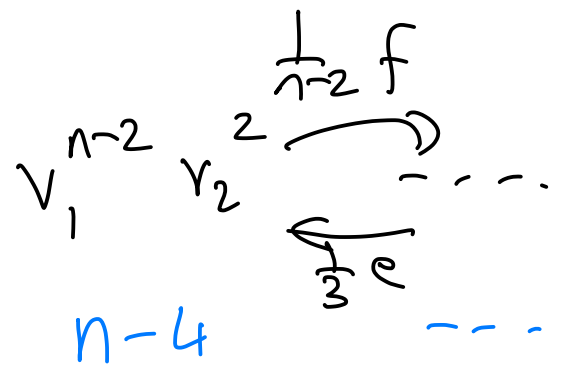
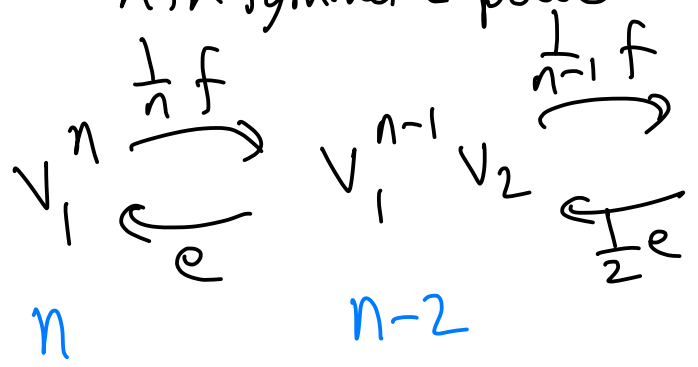
$L(1)$
highest weight 1

$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$S^n V$

$L(n)$, dimension $(n+1)$

nth symmetric power



Proof that is true. Any weight vector generates

Lemma If $v \in V$ is any vector in a \mathfrak{g} -module with $e v = 0$,
 then $e f^{n+1} v = (n+1) f^n (h-n) v$ ($n \in \mathbb{N}$).

Proof $e f^{n+1} v = [e, f^{n+1}] v = \sum_{i=0}^n f^i [e, f] f^{n-i} v$
 commutator in $U(\mathfrak{g})$ $= \sum_{i=0}^n f^i h f^{n-i} v = \sum_{i=0}^n f^i f^{n-i} (h-2n+2i) v$
 $h f = f(h-2)$ $= f^n \cdot (n+1)(h-n) v$

Corollary If v is a vector killed by e then

$$e^{n+1} f^{n+1} v = (n+1)! h(h-1)\dots(h-n) v$$

Proof $e^n (e f^{n+1} v) = (n+1) e^n f^n (h-n) v$ then induct

Now take $\lambda \in \mathbb{C}$.

Let $\mathfrak{b} = \mathbb{C}h \oplus \mathbb{C}e < \mathfrak{g}$ ^{"Borel"}

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

This is a solvable subalgebra, $\mathfrak{b} \longrightarrow \mathbb{C}h$.

Let \mathbb{C}_λ be 1-d. mod \mathfrak{b} -module on which h acts as λ
 e acts as 0 .

As $\mathfrak{b} < \mathfrak{g}$, $U(\mathfrak{b}) < U(\mathfrak{g})$... form the

Verma module $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$

As $U(\mathfrak{g})$ has basis $f^i h^j e^k$ ($i, j, k \geq 0$) by PBW Theorem, it is free as a right $U(\mathfrak{b})$ -module with basis f^i ($i \geq 0$). So $M(\lambda)$ has basis $f^i \otimes 1$ ($i \geq 0$) ... denote these by $f^i v_+$

Note also that $f^i v_+$ is a weight vector of weight $\lambda - 2i$.

Theorem 1 For $\lambda \in \mathbb{C} - \mathbb{N}$, $M(\lambda)$ is an irreducible \mathfrak{g} -module.

If $\lambda = n \in \mathbb{N}$, $M(n)$ has a unique submodule isomorphic to

$M(-n-2)$, and unique quotient $L(n) := M(n) / M(-n-2)$

which is a f.d. irreducible representation of highest weight n , dimension $n+1$.

Every f.d. irred. representation of \mathfrak{g} is isomorphic to $L(n)$ for $\exists n \in \mathbb{N}$.

Proof Take $\lambda \in \mathbb{C} - \mathbb{N}$.

Take $0 \neq v \in M(\lambda)$, $v = \sum_{i \in \mathbb{N}} c_i f^i v_+$.

Choose biggest i so $c_i \neq 0$. Consider $e^i v$.

$$e f^{n+1} v_+ = (n+1) \underbrace{(\lambda - n)}_{\neq 0} f^n v_+ \quad \text{by lemma}$$

It follows that $e^i v$ is a non-zero multiple of v_+ , cyclic vector.

Hence v generates and $M(\lambda)$ is irred.

But if $\lambda = n \in \mathbb{N}$, then $f^{n+1} v_+$ is killed by e .

So $\langle f^{n+1} v_+, f^{n+2} v_+, \dots \rangle$ span a submodule of $M(n)$.

weight $n - 2(n+1) = -n - 2 \dots$ copy of $M(-n-2)$, irreducible.

The quotient of $M(\lambda)$ by this submodule has basis

$\overline{v_+}, \overline{f v_+}, \dots, \overline{f^n v_+}$, dim $n+1 \dots$ irreducible ✓

Now take any f.d. irred. \mathfrak{g} -module L

As above, find $0 \neq v$ highest weight vector of weight $\lambda \in \mathbb{C}$.

$$hv = \lambda v, \quad ev = 0$$

$$\text{Hom}_{\mathfrak{b}} \left(\mathbb{C}_{\lambda}, \text{res}_{\substack{U(\mathfrak{g}) \\ U(\mathfrak{b})}} L \right) \neq 0$$

S1)

$$\text{Hom}_{\mathfrak{g}} \left(\underbrace{U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda}}_{M(\lambda)}, L \right)$$

$\therefore \lambda = n \in \mathbb{N}$
and $L \cong L(n)$
where $\dim L = n+1$

Shows there's non-zero $M(\lambda) \twoheadrightarrow L$

\mathfrak{g} -module hom.

So L is a f.d. quotient

$$f^i v_+ \mapsto f^i v$$

of $M(\lambda) \leftarrow$

Theorem 2 Any f.d. rep. of \mathfrak{g} is completely reducible.

Proof Suffices to show $\text{Ext}_{\mathfrak{g}}^1(L(a), L(b)) = 0$ for $a, b \in \mathbb{N}$.

Case one $a > b$

$$0 \rightarrow L(b) \rightarrow V \xrightarrow{\pi} L(a) \rightarrow 0$$

$\begin{matrix} b \\ b-2 \\ \vdots \\ 2-b \\ -b \end{matrix}$
 $\begin{matrix} a \\ a-2 \\ a-4 \\ \vdots \\ 2-a \\ -a \end{matrix}$

The generalized a -eigenspace of h on V is 1-D

So you can find $v \in V$ of weight a mapping to $v_+ \in L(a)$ under π .

By universal property of $M(a)$, get \mathfrak{g} -module hom. $(e \cdot v = 0)$. This gives splitting of π .

$$\begin{matrix} L(a) \\ \cancel{M(a)} \end{matrix} \rightarrow V$$

$v_+ \mapsto v$

Case two $a < b$ $\text{Ext}_{\mathfrak{g}}^1(L(a), L(b)) \cong \text{Ext}_{\mathfrak{g}}^1(L(b)^*, L(a)^*)$

$\begin{matrix} \cong \\ \cong \\ L(b) \end{matrix}$
 $\begin{matrix} \cong \\ \cong \\ L(a) \end{matrix}$

Done.

Case three $a=b$ $0 \rightarrow L(a) \rightarrow V \rightarrow L(a) \rightarrow 0$

Highest weight a --- generalised a -eigenspace for $h \in V$ is 2D.

The argument in case one works fine if h is diagonalizable on this eigenspace. We don't know this -- why not a Jordan block $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$?

Take any $0 \neq v$ in this generalised eigenspace.

$$ev = 0 = f^{a+1} v$$

$$\therefore \underbrace{e^{a+1} f^{a+1}}_{=0} v = (a+1)! h(h-1)\dots(h-a) v \text{ by Corollary.}$$

Show min. poly. of h acting on this eigenspace divides $x(x-1)\dots(x-a)$
This is a product of distinct linear factors, hence, h is indeed diagonalizable //