$\underline{\mathrm{Rep}(\mathfrak{sl}_2)} \equiv U(\mathfrak{g})\text{-mod}_{f.d.}$

Today $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. $\quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$[e,f] = h \qquad [h,e] = 2e \qquad [h,f] = -2f$

If v is a non-zero vector in some $\mathfrak{g}$-module V such that $hv = \lambda v$ $(\lambda \in \mathbb{C})$

then
$$\left.\begin{aligned} h(ev) &= (\lambda+2)\, ev \\ h(fv) &= (\lambda-2)\, fv \end{aligned}\right\}$$

v is an eigenvector for h of eigenvalue λ

v is a <u>weight vector</u> of <u>weight</u> λ.

$[h,e]v = h(ev) - e(hv)$

$\therefore\ 2ev = h(ev) - \lambda ev$

$\therefore\ h(ev) = (\lambda+2)ev$ ✓

If $0 \neq V$ is any f.d. $\mathfrak{g}$-module, you can always find a (non-zero) weight vector $v \in V$ of weight $\lambda \in \mathbb{C}$.

$$\underset{\lambda}{v}, \quad \underset{\lambda+2}{ev}, \quad \underset{\lambda+4}{e^2 v}, \ \ldots$$

Can't go forever as V is f.d.

Eventually you get to a weight vector killed by e

= highest weight vector

$$\left.\begin{array}{l} 0 \neq v \in V \\ hv = \lambda v \\ ev = 0 \end{array}\right\}$$

We're going to show any f.d. irred. representation is determined uniquely up to isomorphism by its highest weight — the weight of a highest wt vector.

This weight is always $\lambda = n \in \mathbb{N}$

$L(n) \quad (n \in \mathbb{N})$

"highest weight n"

$\dim L(n) = n+1$.

Examples

$\mathbb{C}$ highest weight 0 $L(0)$

trivial module

$$v_1 \underset{e}{\overset{f}{\rightleftarrows}} v_2$$

$V = \mathbb{C}^2$ $L(1)$

natural representation   wt 1 wt -1 highest weight 1

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$S^n V$ $L(n)$, dimension $(n+1)$

nth symmetric power

$$v_1^n \underset{e}{\overset{\frac{1}{n}f}{\rightleftarrows}} v_1^{n-1} v_2 \underset{\frac{1}{2}e}{\overset{\frac{1}{n-1}f}{\rightleftarrows}} v_1^{n-2} v_2^2 \underset{\frac{1}{3}e}{\overset{\frac{1}{n-2}f}{\rightleftarrows}} \cdots \rightleftarrows v_1 v_2^{n-1} \rightleftarrows v_2^n$$

$$n \qquad n-2 \qquad n-4 \qquad \cdots \qquad 2-n \qquad -n$$

Proof that irred. Any weight vector generates

Lemma If $v \in V$ is any vector in a $\mathfrak{g}$-module with $ev = 0$,

then $e f^{n+1} v = (n+1) f^n (h-n) v$ $\quad (n \in \mathbb{N})$.

Proof $e f^{n+1} v = [e, f^{n+1}] v = \sum_{i=0}^{n} f^i [ef] f^{n-i} v$

commutator in $U(\mathfrak{g})$

$$= \sum_{i=0}^{n} f^i h f^{n-i} v = \sum_{i=0}^{n} f^i f^{n-i} (h-2n+2i) v$$

$hf = f(h-2)$

$$= f^n \cdot (n+1)(h-n) v$$

Corollary If v is a vector killed by e then

$$e^{n+1} f^{n+1} v = (n+1)!\, h(h-1) \cdots (h-n)\, v$$

Proof $e^n (e f^{n+1} v) = (n+1) e^n f^n (h-n) v$ then induct

Now take $\lambda \in \mathbb{C}$.

"Borel"

Let $\mathfrak{b} = \mathbb{C}h \oplus \mathbb{C}e < \mathfrak{g}$ $\qquad \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

This is a solvable subalgebra, $\mathfrak{b} \twoheadrightarrow \mathbb{C}h$.

Let $\mathbb{C}_\lambda$ be 1-d. irred $\mathfrak{b}$-module on which h acts as λ, e acts as 0.

As $\mathfrak{b} \leq \mathfrak{g}$, $U(\mathfrak{b}) < U(\mathfrak{g})$ ... form the

<u>Verma module</u>

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

As $U(\mathfrak{g})$ has basis $f^i h^j e^k$ $(i,j,k \geq 0)$ by PBW Theorem, it is free as a right $U(\mathfrak{b})$-module with basis f^i $(i \geq 0)$. So $M(\lambda)$ has basis $f^i \otimes 1$ $(i \geq 0)$ ... denote these by $f^i v_+$

Note also that $f^i v_+$ is a weight vector of weight $\lambda - 2i$.

<u>Theorem</u> For $\lambda \in \mathbb{C} - \mathbb{N}$, $M(\lambda)$ is an irreducible $\mathfrak{g}$-module.

If $\lambda = n \in \mathbb{N}$, $M(n)$ has a unique submodule isomorphic to $M(-n-2)$, and unique quotient $L(n) := M(n) / M(-n-2)$

which is a f.d. irreducible representation of highest weight n, dimension $n+1$.

Every f.d. irred. representation of $\mathfrak{g}$ is isomorphic to $L(n)$ for $!\, n \in \mathbb{N}$.

<u>Proof</u> Take $\lambda \in \mathbb{C} - \mathbb{N}$.

Take $0 \neq v \in M(\lambda)$, $v = \sum_{i \in \mathbb{N}} c_i f^i v_+$.

Choose biggest i so $c_i \neq 0$. Consider $e^i v$.

$$e f^{n+1} v_+ = (n+1) \underbrace{(\lambda - n)}_{\neq 0} f^n v_+ \quad \text{by lemma}$$

It follows that $e^i v$ is a non-zero multiple of v_+, cyclic vector.
Hence v generates and $M(\lambda)$ is irred.

But if $\lambda = n \in \mathbb{N}$, then $f^{n+1} v_+$ is killed by e.

So $\langle f^{n+1} v_+, f^{n+2} v_+, \dots \rangle$ span a submodule of $M(n)$.

weight $n - 2(n+1) = -n-2$ ... copy of $M(-n-2)$, irreducible.

The quotient of $M(\lambda)$ by this submodule has basis

$\overline{v_+}, \overline{f v_+}, \dots, \overline{f^n v_+}$, dim $n+1$ ... irreducible ✓

Now take any f.d. irred. $\mathfrak{g}$-module L

As above, find $0 \neq v$ highest weight vector of weight $\lambda \in \mathbb{C}$.

$$hv = \lambda v, \quad ev = 0$$

$$\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, \mathrm{res}^{U(\mathfrak{g})}_{U(\mathfrak{b})} L) \neq 0$$

$$1 \mapsto v$$

$$\cong$$

$$\mathrm{Hom}_{\mathfrak{g}}(\underbrace{U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda}_{M(\lambda)}, L)$$

$\therefore \lambda = n \in \mathbb{N}$

and $L \cong L(n)$

where $\dim L = n+1$

Shows there's non-zero $M(\lambda) \twoheadrightarrow L$ $\mathfrak{g}$-module hom.

$$f^i v_+ \mapsto f^i v$$

So L is a f.d. quotient of $M(\lambda)$

Theorem 2 Any f.d. rep. of $\mathfrak{g}$ is completely reducible.

Proof Suffices to show $\mathrm{Ext}^1_{\mathfrak{g}}(L(a), L(b))$ for $a, b \in \mathbb{N}$.

$$0 \longrightarrow L(b) \longrightarrow V \xrightarrow{\pi} \overset{v_+ \in}{L(a)} \longrightarrow 0$$

$$\begin{array}{cc} b & a \\ b-2 & a-2 \\ \vdots & a-4 \\ 2-b & \vdots \\ -b & 2-a \\ & -a \end{array}$$

Case one $a > b$

The generalized a-eigenspace of h on V is 1-D

So you can find $v \in V$ of weight a mapping to $v_+ \in L(a)$ under π.

$$\overset{L(a)}{\cancel{M(a)}} \longrightarrow V$$
$$v_+ \longmapsto v$$

By universal property of $M(a)$, get $\mathfrak{g}$-module hom. $(ev = 0)$. This gives splitting of π.

Case two $a < b$

$$\mathrm{Ext}^1_{\mathfrak{g}}(L(a), L(b)) \cong \mathrm{Ext}^1_{\mathfrak{g}}(\underset{\substack{\cong \\ L(b)}}{L(b)^*}, \underset{\substack{\cong \\ L(a)}}{L(a)^*})$$

Done.

Case three $a = b$ $\quad 0 \to L(a) \to V \to L(a) \to 0$

Highest weight a --- generalised a-eigenspace for $h \curvearrowright V$ is 2D.

The argument in case one works fine if h is diagonalisable on this eigenspace. We don't know this .. why not a Jordan block $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$?

Take any $0 \neq v$ in this generalised eigenspace.

$$ev = 0 = f^{a+1} v$$

$$\therefore \; e^{a+1} f^{a+1} v = (a+1)!\, h(h-1)\cdots(h-a)\, v \quad \text{by Corollary.}$$

$$\| \\ 0$$

Shows min. poly. of h acting on this eigenspace divides $x(x-1)\cdots(x-a)$

This is a product of distinct linear factors, hence, h is indeed diagonalisable