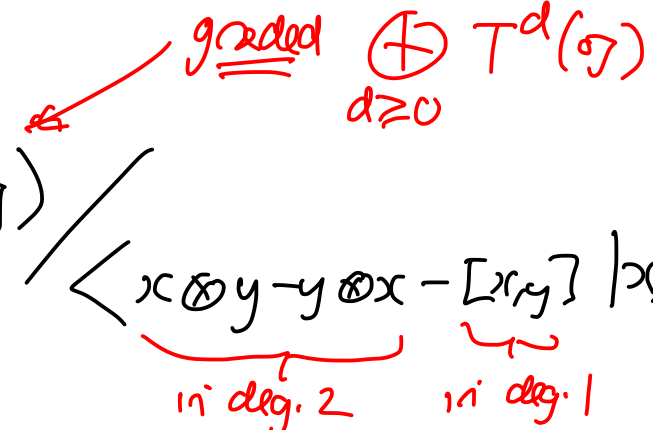


The Poincaré - Birkhoff - Witt Theorem (PBW)

Let \mathfrak{g} be a Lie algebra, $U(\mathfrak{g}) = T(\mathfrak{g}) / \langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle$
 its universal enveloping algebra.



The canonical map $i: \mathfrak{g} \rightarrow U(\mathfrak{g})$ is injective (so we can identify \mathfrak{g} with a subspace of $U(\mathfrak{g})$). Moreover, if $x_i (i \in I)$ is a basis for \mathfrak{g} and some total order on I , then

$$\left\{ x_{i_1} \cdots x_{i_n} \mid n \geq 0, i_1, \dots, i_n \in I, i_1 \leq \dots \leq i_n \right\}$$

is a basis for $U(\mathfrak{g})$. Ordered monomials / PBW monomials

Usually, \mathfrak{g} is f.d. with basis x_1, \dots, x_n . Then PBW basis is

$$\left\{ x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \mid m_1, \dots, m_n \geq 0 \right\}.$$

Comments on proof

You probably saw the first Weyl algebra $A = \mathbb{C}\langle x, y \rangle / \langle xy - yx - 1 \rangle$

This has basis $y^j x^i$ ($i, j \geq 0$).

Usual proof • $y^j x^i$ ($i, j \geq 0$) span A

• define an action of A on $\mathbb{C}[y]$ so y acts as mult.

check relations! $\frac{d}{dy}(yf) = y \frac{df}{dy} + f$
 x acts as $\frac{d}{dy}$

• Then use this action to prove lin. independence, act on $y^k \in \mathbb{C}[y]$.

In fact, A is a filtered deformation of $\mathbb{C}[x, y]$

Note $T = \mathbb{C}\langle x, y \rangle$ is a graded algebra (x, y in degree 1)

But $I = \langle xy - yx - 1 \rangle$ is not homogeneous ... A is merely filtered.

Let $A_{\leq n} = \text{Im} \left(\bigoplus_{d=0}^n T_d \right)$. Then:

- $[k] = A_{\leq 0} \subseteq A_{\leq 1} \subseteq A_{\leq 2} \subseteq \dots$ Subspaces
- $A_{\leq m} \cdot A_{\leq n} \subseteq A_{\leq (m+n)}$ ~~— (*)~~

For a filtered algebra, its associated graded algebra $gr A = \bigoplus_{d \geq 0} (gr A)_d$

where $(gr A)_d := A_{\leq d} / A_{< d}$.

Check well-defined using (*)

For $x \in A_{\leq n}$, let $gr_n x = x + A_{< n} \in (gr A)_n$.

Then $gr A$ is a graded algebra with $(gr_n x) \cdot (gr_m y) = gr_{n+m}(xy)$.

For our A , the first Weyl algebra, $(gr A)_n$ has basis $gr_n(y^j x^i)$ ($i+j=n$).

$$(gr_n y)^j (gr_n x)^i$$

Also $gr_1 x, gr_1 y$ commute in $gr A$.

$gr_2(xy) = gr_2(yx)$ So: $\mathbb{C}[x, y] \hookrightarrow gr A$, algebra ~~homomorphism~~ isomorphism.

$x \mapsto gr_1 x$
 $y \mapsto gr_1 y$

Story for $U(\mathfrak{g})$ is similar!

To prove PBW theorem

- The ordered monomials span $U(\mathfrak{g})$
- Make $S(\mathfrak{g})$ into a left ~~$U(\mathfrak{g})$~~ -module so
$$\overset{\mathfrak{g}}{x_i} \cdot \overset{\mathfrak{g}}{x_{j_1} \cdots x_{j_n}} \equiv \overset{\mathfrak{g}}{x_i x_{j_1} \cdots x_{j_n}} \quad (\text{modulo lower degree terms})$$

Unpleasant step!!! \rightarrow

Induction on degree.

and $x_i \cdot x_{j_1} \cdots x_{j_n} = x_i x_{j_1} \cdots x_{j_n}$ if $i \leq j_1, \dots, j_n$
(unique way to do this)

- Use this action to show lin. independence (action 1)

There's an axiomatization of this type of argument — Bergman's diamond lemma

In fact, $U(\mathfrak{g})$ is a filtered deformation of $S(\mathfrak{g})$.

Of course $U(\mathfrak{g})$ is a filtered algebra as a quotient of $T(\mathfrak{g})$

$$U(\mathfrak{g})_{\leq n} = \text{Im} \left(\bigoplus_{d=0}^n T^d(\mathfrak{g}) \right)$$

PBW \Rightarrow $(\text{gr } U(\sigma))_n$ has basis $\text{gr}_n(x_{i_1} \cdots x_{i_n})$

\parallel $i_1 \leq \dots \leq i_n$

Also $\text{gr } U(\sigma)$ is commutative

$(\text{gr}_1 x_{i_1}) \cdots (\text{gr}_1 x_{i_n})$

$$[\text{gr}_1 x, \text{gr}_1 y] = \text{gr}_2(xy - yx) = \text{gr}_2([x, y]) = 0$$

So

$$\begin{array}{ccc} S(\sigma) & \xrightarrow{\sim} & \text{gr } U(\sigma) \\ x & \longmapsto & \text{gr}_1 x \\ \uparrow \text{gr}_1 & & \end{array}$$

$$\sigma \hookrightarrow U(\sigma)$$

Last time: three basic examples of Hopf algebras Δ (coint.) Σ (coint.) and S (antipode)

① kG , G a finite group

$$\Delta(g) = g \otimes g \quad g \in G$$

"group-like elements"

$$\Sigma(g) = 1 \quad g \in G$$

$$S(g) = g^{-1} \quad g \in G$$

② $k[L(G)]$, G algebraic group

$$\Delta = m^*$$

$$\Sigma = \text{ev}_e$$

$$S = i^*$$

③ $U(\mathfrak{g})$, \mathfrak{g} a Lie algebra

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

$$\Sigma(x) = 0 \quad x \in \mathfrak{g}$$

$$S(x) = -x$$

↑
Need to use univ. prop. of $U(\mathfrak{g})$ to see that these extend appropriately.

" $U(\mathfrak{g})$ is to \mathfrak{g} as kG is to G "
(finite group G)

$$\text{Rep}(\mathfrak{g}) = U(\mathfrak{g})\text{-mod}_{\text{fd}} \quad \text{Rep}(G) = kG\text{-mod}_{\text{fd}}$$

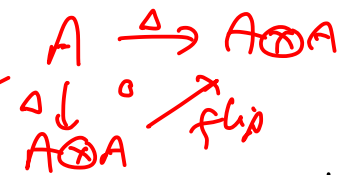
↑
fd. representations of \mathfrak{g}

$$\text{Rep}(G) = \text{comod}_{\text{fd}} - k[L(G)]$$

algebraic group

↙ All three cats. are symmetric tensor cats.

Why is Hopf algebra structure important?



A cocommutative Hopf algebra $\Delta: A \rightarrow A \otimes A$ $\Sigma: A \rightarrow k$ $S: A \rightarrow A$

Then A -mod fd is a Symmetric tensor category

V, W

Symmetric tensor category

- Abelian
- There's a \otimes
- There's a duality $*$
- There's a unit object, trivial module k

$V \otimes W \cong W \otimes V$
 $v \otimes w \mapsto w \otimes v$
 is an A -module isomorphism

$V \otimes W = V \otimes W$ is an $A \otimes A$ -module

$(a \otimes b)(v \otimes w) = av \otimes bw$

So it's an A -module via $\Delta: A \rightarrow A \otimes A$

$V^* = \text{Hom}_{k}(V, k)$, linear dual

$(a \cdot f)(v) = f(S(a)v)$

Trivial module k has action defined by $a \cdot 1_k = \Sigma(a) \cdot 1_k$.

If A is a Hopf algebra,

A^* , linear dual, is always an algebra $A^* \otimes A^* \hookrightarrow (A \otimes A)^* \xrightarrow{\text{dual map to comult. } \Delta \text{ on } A} A^*$

But A^* is NOT necessarily a coalgebra (here, Hopf algebra) in a natural way.

$$A^* \xrightarrow{\quad} (A \otimes A)^* \hookrightarrow A^* \otimes A^*$$

In general the image of this map, dual of mult. on A , needn't be in subspace $A^* \otimes A^* \hookrightarrow (A \otimes A)^*$

If A is f.d., no problems, and A^* is again a Hopf algebra via this construction

(eg) kG , G finite group.

$$(kG)^* \equiv k[G] \text{ as Hopf algebras}$$

$$\text{Rep}(G) \equiv \text{Rep}(G)$$

left kG -modules \equiv right $k[G]$ -comodules

For G connected algebraic group.

Then $[k[G]]^*$ is an algebra but "too big" to be a coalgebra / Hopf algebra

$$M_e = \ker v_e \triangleleft [k[G]]$$

Now you consider

$$\text{Dut}(G) = \left\{ \theta \in [k[G]]^* \mid \theta(M_e^{n+1}) = 0 \text{ for } n \gg 0 \right\}$$

ρ
algebra of distributions of G

$$\cong \bigcup_{n \geq 0} \left([k[G]] / M_e^{n+1} \right)^*$$

It's a subalgebra, even filtered with $\text{Dut}(G)_{\leq n} = \left([k[G]] / M_e^{n+1} \right)^*$.

Even a Hopf algebra with comult. dual to mult. on $[k[G]]$

In char. 0, turns out that $\text{Dut}(G) \cong U(\mathfrak{g})$ as Hopf algs.

$$\sigma = \text{Der}(k[G], k_e) \cong (M_e/M_e^2)^* \subseteq \text{Der}(G)$$

Get Lie alg. hom. $\sigma \longrightarrow \text{Der}(G)$ from this

