Algebraic groups

- first examples: $GL_n(\mathbb{k})$, $SL_n(\mathbb{k})$, $SO_n(\mathbb{k})$, $Sp_{2n}(\mathbb{k})$
- Every algebraic group is linear

X affine variety

X decomposes uniquely as $X = X_1 \cup \dots \cup X_n$ of <u>irreducible components</u>

maximal irreducible subset (automatically closed)

Makes sense in any topological space that is <u>Noetherian</u> (DCC on closed subsets)

What does irreducible component look like for G?

Lemma 1 G algebraic group.

(1) $e \in G$ belongs to a unique irreducible component of G, denoted G°

(2) G° is closed normal subgroup of G

(3) The irreducible components of G are exactly the cosets of G° in G. In particular, $[G : G^\circ] < \infty$, and G° is also the connected component of G containing e.

G° is the identity component of G.

G is connected $\iff G = G^\circ$

(4) Any closed subgroup $H \leq G$ of finite index contains G°.

If G is finite, $G^\circ = \{e\}$.

$GL_n(\mathbb{k}), SL_n(\mathbb{k}), Sp_{2n}(\mathbb{k}), SO_n(\mathbb{k})$ connected

$$1 \to SO_n(\mathbb{k}) \to O_n(\mathbb{k}) \to \{\pm 1\} \to 1$$

$$1 \to G^\circ \to G \to G/G^\circ \to 1$$

G°: connected alg. gp.

G/G°: finite group (component group of G)

Proof (1) Let X, Y be irred. components of G containing e

$m: \underbrace{X \times Y}_{\text{irreducible}} \longrightarrow G$ continuous

($\Bbbk[X] \otimes_{\Bbbk} \Bbbk[Y]$ is an integral domain ⟵ algebraic closed is crucial)

$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not a domain !!!

$\Rightarrow$ XY is irreducible $\Rightarrow$ $\overline{XY}$ is irreducible

$$\begin{array}{ccc} & \overline{XY} & \\ \cup\!\!/ & & \cup\!\!/ \\ X & & Y \end{array}$$

Maximality $\Rightarrow$ $X = \overline{XY} = Y$ $\Rightarrow$ $X = Y$ //

(2) Apply (1) with $X = Y = G^\circ$, get $G^\circ = \overline{G^\circ G^\circ}$,

hence, G° is closed under mult.

$i : G \to G$ is homeomorphism, so $i(G^0)$ is an irreducible
$g \mapsto g^{-1}$

component containing e, so $i(G^0) = G^0$ by (1)

$\Rightarrow G^0 \leq G$

For $g \in G$, $\mathrm{Int}\, g : G \to G$ a homeomorphism
$x \mapsto gxg^{-1}$

so $g G^0 g^{-1}$ an irreducible cpt containing e, so

$$g G^0 g^{-1} = G^0$$

$\Rightarrow G^0 \trianglelefteq G$.

(3) Let X be any irred. cpt of G, take $x \in X$.

Let translation by x^{-1}, $\begin{array}{l} G \to G \\ g \mapsto x^{-1}g \end{array}$ is a homeomorphism

So $x^{-1}X = G^{\circ}$, so $X = xG^{\circ}$.

Similarly every coset of G° is an irreducible component.

$\Rightarrow [G : G^{\circ}] < \infty$.

(4) $H \leq G$ closed subgroup of finite index

$$\underbrace{H^{\circ} \underset{\text{finite}}{\leq} H \underset{\text{finite}}{\leq} G}_{[G:H^{\circ}] < \infty}$$

$\Rightarrow H^{\circ}$ is open & closed in G

connected

$\Rightarrow H^{\circ}$ is a connected component of G

$\Rightarrow H^{\circ} = G^{\circ} \Rightarrow G^{\circ} \leq H$

Lemma 2 G algebraic group, $U, V \subseteq G$ dense open.

Then $G = UV$

Proof Take $g \in G$. $U^{-1}g$, V both dense in G.

So $U^{-1}g \cap V \neq \emptyset$.

Pick some $v \in V$ so $v \in U^{-1}g$, i.e. $v = u^{-1}g$ for $u \in U$

$\Longrightarrow g = uv$

Lemma 3 Let G be an algebraic group, and H be any subgroup.
Then $\overline{H}$ is also a subgroup. Moreover, if H contains a non-empty subset of $\overline{H}$ then $H = \overline{H}$.

Proof Take $h \in H$. As $g \mapsto hg$ homeomorphism,

$\overline{H} = \overline{hH} = h\overline{H} \implies H\overline{H} = \overline{H}$

Take $h \in \overline{H}$ $\qquad Hh \subseteq \overline{H}$

$\therefore \overline{Hh} \subseteq \overline{H}$

$\quad \overline{H}h = \overline{Hh}$

Shows $\overline{H}$ is closed under mult..

As i is homeomorphism, $\overline{i(H)} = i(\overline{H})$

$\overline{H} = \overline{i(H)}$

Shows $\overline{H}$ is closed under inverse $\implies \overline{H} \leq G$ ✓.

Now suppose $\exists\ \emptyset \neq U \subseteq H \leq \overline{H}$

U open in $\overline{H}$

$$H = \bigcup_{h \in H} hU$$

$\Rightarrow H$ is open in $\overline{H}$.

Now apply lemma 2 with $U, V = H$ to get $\overline{H} = HH = H$

<u>Theorem</u> Let $\varphi : G \to H$ be a morphism of algebraic groups.

Then $\varphi(G)$ is a <u>closed</u> subgroup of H, hence, an

algebraic group in its own right.

Proof Need one more alg. geometry fact:

> The image of any morphism of affine varieties contains a non-empty open subset of its closure.

Now apply Lemma 3 $\varphi(G) \leq H$ to deduce that $\varphi(G) = \overline{\varphi(G)}$

Of course $\ker \varphi = \varphi^{-1}(\{e\})$ is also a closed, normal subgroup of G. Begs question

YES if φ is separable

NO in general

$$G/\ker\varphi \xrightarrow{\sim} \varphi(G)$$

Can one give this the structure of an algebraic group in some intrinsic way?

YES

Is there an iso of algebraic groups?

(eg) $p=2$ $\quad \varphi : SL_2(\mathbb{k}) \twoheadrightarrow PSL_2(\mathbb{k})$

correct definition of $PSL_2(\mathbb{k})$ as algebraic group fixes this!

"usual" definition

$PSL_2(\mathbb{k}) = SL_2(\mathbb{k}) / (\text{scalar matrices of det. } 1)$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Even when $p=2$,

$SL_2(\mathbb{k}) \not\cong PSL_2(\mathbb{k})$

as algebraic group

Good in characteristic not 2

If $p=2$, $PSL_2(\mathbb{k})$ "=" $SL_2(\mathbb{k})$

as abstract group

When $p=2$, φ is not separable.

If you have <u>any</u> closed subgroup H of G, how do you make set G/H of left cosets into a variety??

Possible, but in general it's not <u>affine</u>. Need quasi-projective varieties.

<u>Orbits</u> of algebraic groups $\rho : G \times X \to X$ (any variety)

action which is also a morphism varieties.

Can you make orbits of G on X into varieties too??

$$G/\mathcal{S}_G(x) \longrightarrow G \cdot x \,, \quad g\,\mathcal{S}_G(x) \longmapsto g \cdot x$$

$x \in X$

Iso. of varieties??