Introduction to Lie Theory
Homework #6

1. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a direct sum of two Lie algebras. Let V be a finite-dimensional $\mathfrak{g}$-module. Prove that V is completely reducible as a $\mathfrak{g}$-module if and only it is completely reducible both as a $\mathfrak{g}_1$-module and as a $\mathfrak{g}_2$-module.

   (*Hint.* Consider the finite-dimensional algebra A that is the image of $U(\mathfrak{g})$ under the induced algebra homomorphism $\rho : U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(V)$, use Wedderburn theorem.)

2. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra with Killing form κ. We showed in L6-2 that $\mathfrak{g}$ is the direct sum $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ of its simple ideals, with $\mathfrak{g}_i \perp_\kappa \mathfrak{g}_j$ for $i \neq j$. Let β be some other invariant bilinear form on $\mathfrak{g}$. Prove:

   (a) $\beta|_{\mathfrak{g}_i}$ is a multiple of the Killing form $\kappa|_{\mathfrak{g}_i}$ for each $i = 1, \ldots, n$.

   (b) $\mathfrak{g}_i \perp_\beta \mathfrak{g}_i$ for $i \neq j$.

   (c) β is a symmetric bilinear form.

The remaining questions require the notion of a *toral subalgebra* $\mathfrak{t}$ of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$. Recall that this is a subalgebra consisting entirely of semisimple elements of $\mathfrak{g}$. As established in L7-1, such a subalgebra is necessarily Abelian. Hence, there is a *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where $R = \{0 \neq \alpha \in \mathfrak{t}^* \mid \mathfrak{g}_\alpha \neq 0\}$ is the set of *roots* of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Here,

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{t} \mid [t, x] = \alpha(t)x \text{ for all } t \in \mathfrak{t}\}$$

for any $\alpha \in \mathfrak{t}^*$.

3. Suppose that $\mathfrak{t}$ is a toral subalgebra of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ such that $\mathfrak{t}$ is equal to its centralizer $\mathfrak{c}_{\mathfrak{g}}(\mathfrak{t})$. Why does this imply that $\mathfrak{t}$ is actually a *maximal* toral subalgebra of $\mathfrak{g}$? Use this observation to show that the subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ consisting of all diagonal matrices of trace zero is a maximal toral subalgebra.

Now let $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ using the coordinates chosen in L3-2. Let $\mathfrak{t}$ be the subalgebra of all diagonal matrices in $\mathfrak{g}$, and note that any element $t \in \mathfrak{t}$ can be written as $t = \mathrm{diag}(t_1, \ldots, t_n, -t_n, \ldots, -t_1)$ for $t_1, \ldots, t_n \in \mathbb{C}$. Let $\varepsilon_i \in \mathfrak{t}^*$ be the linear map sending $t \mapsto t_i$ for $i = 1, \ldots, n$, so that $\varepsilon_1, \ldots, \varepsilon_n$ give a basis for $\mathfrak{t}^*$.

4. Describe the root space decomposition of $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ with respect to $\mathfrak{t}$ using the notation just introduced. You should show in particular that

$$R = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\},$$

that $\mathfrak{g}_\alpha$ is one-dimensional for each $\alpha \in R$, and that $\mathfrak{g}_0 = \mathfrak{t}$. Deduce that $\mathfrak{t}$ is a maximal toral subalgebra.

5. Let $V = \mathbb{C}^{2n}$ be the natural representation of $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ denoting its standard ordered basis $v_1, \ldots, v_n, v_{-n}, \ldots, v_{-1}$.

(a) We noted in L5-3 that V is an irreducible $\mathfrak{g}$-module. Convince yourself of this again! It follows that $\mathfrak{g}$ is indeed a semisimple Lie algebra.

(b) Show that v_i is of weight ε_i and v_{-i} is of weight $-\varepsilon_i$, where $\varepsilon_1, \ldots, \varepsilon_n \in \mathfrak{t}^*$ are as defined above.

(c) Describe how $S^2(V)$ decomposes into weight spaces with respect to $\mathfrak{t}$. Then show that $S^2(V) \cong \mathfrak{g}$ (the adjoint representation) as $\mathfrak{g}$-modules.

(d) Show that $S^2(V)$ is an irreducible $\mathfrak{g}$-module. Deduce that $\mathfrak{g}$ is a simple Lie algebra.

(*Hint.* For (a) and/or (d) you might find it easier to prove irreducibility as representations of the algebraic group $G = Sp_{2n}(\mathbb{C})$.)