Introduction to Lie Theory
Homework #5

1. Prove the *Clebsch-Gordon rule* for representations of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$:

$$L(\lambda) \otimes L(\mu) \cong \bigoplus_{\substack{|\lambda-\mu| \leq \nu \leq \lambda+\mu \\ \nu \equiv \lambda+\mu \pmod 2}} L(\nu)$$

   for $\lambda, \mu \in \mathbb{N}$. Use this to calculate the dimension of the space $(V^{\otimes 10})^G$ of invariants of $G = SL_2(\mathbb{C})$ acting on the tenth tensor power of its natural representation.

The remaining questions questions are concerned with the *algebra of distributions* $\mathrm{Dist}(G)$ of a connected algebraic group G from L4-3. Recall that this is the subalgebra

$$\mathrm{Dist}(G) = \{\theta \in \Bbbk[G]^* \mid \theta(M_e^{n+1}) = 0 \text{ for } n \gg 0\} = \bigcup_{n \geq 0} \left(M_e^{n+1}\right)^\circ$$

of $\Bbbk[G]^*$ viewed as an algebra via the dual map to the comultiplication m^* on $\Bbbk[G]$ (here, e is the unit element of G). It is a Hopf algebra with comultiplication Δ arising from the dual of the commutative multiplication on $\Bbbk[G]$ and counit $\varepsilon : \mathrm{Dist}(G) \to \Bbbk, \theta \mapsto \theta(1)$ (here, 1 is the identity in the associative algebra $\Bbbk[G]$).

2. The Lie algebra $\mathfrak{g}$ of G may be identified with the subspace

$$(M_e/M_e^2)^* = \left\{\theta \in \left(M_e^2\right)^\circ \;\middle|\; \theta(1) = 0\right\}$$

   of $\mathrm{Dist}(G)$. Verify that this subspace is indeed a Lie subalgebra of $\mathrm{Dist}(G)$, then show that this approach to the definition of $\mathfrak{g}$ is equivalent to the approach taken in L3-1.

In characteristic zero, a theorem of Cartier mentioned in the lectures shows that the Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Dist}(G)$ from Q2 induces an algebra isomorphism $U(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Dist}(G)$.

3. Calculate $\mathrm{Dist}(G)$ explicitly for $G = \mathbb{G}_a$. Recall for this that the coordinate algebra is $\Bbbk[T]$ and $m^*(T) = T \otimes 1 + 1 \otimes T$. You should show

first that $\mathrm{Dist}(G)$ has a basis $\{x_n \mid n \geq 0\}$ such that $x_i(T^j) = \delta_{i,j}$, and then that the algebra structure satisfies

$$x_n x_m = \binom{n+m}{n} x_{n+m}.$$

Finally, assuming $\Bbbk = \mathbb{C}$, show directly that $\mathrm{Dist}(G) \cong U(\mathfrak{g})$. What element of $U(\mathfrak{g}) = \mathbb{C}[x]$ does x_n correspond to under the canonical isomorphism?

4. Let $G = \mathbb{G}_m$ with coordinate algebra $\Bbbk[T, T^{-1}]$. Let R be the ring of *integer-valued polynomials*, that is, the subring of $\mathbb{Q}[x]$ consisting of polynomials $f(x)$ such that $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. Note that R is spanned as a $\mathbb{Z}$-module by the polynomials

$$\binom{x}{n} := x(x-1)\cdots(x-n+1)/n!$$

for $n \geq 0$, and also $x\binom{x}{n} = (n+1)\binom{x}{n+1} + n\binom{x}{n}$.

   (a) Show that $\mathrm{Dist}(G)$ has basis $\{x_n \mid n \geq 0\}$ with $x_i((T-1)^j) = \delta_{i,j}$ and that $x_1 x_n = (n+1)x_{n+1} + n x_n$. Deduce that $\mathrm{Dist}(G) \cong \Bbbk \otimes_{\mathbb{Z}} R$.

   (b) Now assume that $\Bbbk = \mathbb{C}$. Use (a) to verify directly that $\mathrm{Dist}(G) \cong U(\mathfrak{g})$. What element of $U(\mathfrak{g}) = \Bbbk[x]$ does x_n correspond to under your isomorphism?

For a representation V of G, let $\eta : V \to V \otimes \Bbbk[G]$ be its comodule structure map as in HW2-3. You can make V into a $\mathrm{Dist}(G)$-module by defining $\theta v := (\mathrm{id}_V \bar{\otimes} \theta)(\eta(v))$. If you identify $\mathfrak{g}$ with a Lie subalgebra of $\mathrm{Dist}(G)$ as in Q2, this makes V into a $\mathfrak{g}$-module, and this construction agrees with the $\mathfrak{g}$-module structure on V discussed in L3-3.

5. Assuming that $\mathrm{char}\, \Bbbk = 0$ whenever necessary, establish the following equalities):

$$\begin{aligned} V^G &= \{v \in V \mid gv = v \text{ for all } g \in G\} \\ &= \{v \in V \mid \eta(v) = v \otimes 1\} \\ &= \{v \in V \mid \theta v = \varepsilon(\theta) v \text{ for all } \theta \in \mathrm{Dist}(G)\} \\ &= \{v \in V \mid xv = 0 \text{ for all } x \in \mathfrak{g}\} = V^{\mathfrak{g}}. \end{aligned}$$

This gives another approach to showing $V^G = V^{\mathfrak{g}}$; cf. HW4-4.

(*Hint.* $\bigcap_{n \geq 0} M_e^{n+1} = 0$ by Krull's intersection theorem.)