Introduction to Lie Theory
Homework #4

1. For a field $\mathbb{k}$ of characteristic zero, the Lie algebra $\mathfrak{so}_3(\mathbb{k})$ may be defined as the Lie subalgebra of $\mathfrak{gl}_3(\mathbb{k})$ consisting of all skew-symmetric matrices, while $\mathfrak{sl}_2(\mathbb{k})$ is of course the 2×2 matrices of trace zero in $\mathfrak{gl}_2(\mathbb{k})$.

   (a) Show that $\mathfrak{so}_3(\mathbb{R})$ is isomorphic to $\mathbb{R}^3$ viewed as a Lie algebra with Lie bracket being the usual cross product of vectors.

   (b) Show that $\mathfrak{so}_3(\mathbb{R}) \not\cong \mathfrak{sl}_2(\mathbb{R})$.

   (c) Show that $\mathfrak{so}_3(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C})$.

   Hint. For (c) see Q3 below.

2. In L4-2, I explained how the universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra with comultiplication $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, counit $\varepsilon : U(\mathfrak{g}) \to \mathbb{k}$ and antipode $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ defined so that $\Delta(x) = x \otimes 1 + 1 \otimes x, \varepsilon(x) = 0$ and $S(x) = -x$ for all $x \in \mathfrak{g}$. Fill in the details!

3. The comultiplication Δ on $U(\mathfrak{g})$ is important because it means you can define the *tensor product* of two $\mathfrak{g}$-modules: if V and W are $\mathfrak{g}$-modules then the tensor product $V \otimes W$ (over the ground field $\mathbb{k}$) is naturally a $U(\mathfrak{g}) \otimes U(\mathfrak{g})$-module with $(x_1 \otimes x_2)(v \otimes w) = x_1 v \otimes x_2 w$; hence, using the algebra homorphism Δ, it becomes a $\mathfrak{g}$-module.

   (a) Show that the exterior and symmetric powers $\bigwedge^n V$ and $S^n V$ of a $\mathfrak{g}$-module V are $\mathfrak{g}$-module quotients of $\bigotimes^n V$.

   (b) Now suppose that $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ for $n \geq 2$ and let V be the natural n-dimensional representation of column vectors. Show that $\bigwedge^i V$ $(1 \leq i \leq n)$ and $S^j V$ $(j \geq 0)$ are both irreducible $\mathfrak{g}$-modules. Is $S^2 V$ irreducible over the subalgebra $\mathfrak{so}_n(\mathbb{C})$?

   (c) Finally suppose that $n = 2$. Show that V and $S^2 V$ possess non-degenerate bilinear forms which are invariant under the action of $\mathfrak{g}$, i.e., $(xv, w) + (v, xw) = 0$ for all $x \in \mathfrak{g}$ and all vectors v, w. Deduce that $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{sp}_2(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{C})$.

4. Let V and W be $\mathfrak{g}$-modules.

(a) Verify that $\mathrm{Hom}_{\mathbb{k}}(V, W)$ can be made into a $\mathfrak{g}$-module by setting $(xf)(v) := xf(v) - f(xv)$ for $x \in \mathfrak{g}, f \in \mathrm{Hom}_{\mathbb{k}}(V, W)$ and $v \in V$.

(b) If V is finite-dimensional then $\mathrm{Hom}_{\mathbb{k}}(V, W) \cong V^* \otimes W$. What does the $\mathfrak{g}$-module structure from (a) correspond to under this natural isomorphism?

(c) For any $\mathfrak{g}$-module V, let $V^{\mathfrak{g}} := \{v \in V \mid xv = 0 \text{ for all } x \in \mathfrak{g}\}$ denote the submodule of $\mathfrak{g}$-*invariants*. For V, W as in (a), check that

$$\mathrm{Hom}_{\mathbb{C}}(V, W)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, W).$$

(d) Suppose that G is a connected algebraic group for $\mathbb{k}$ of characteristic zero. If V is a representation of G, the submodule of G-*invariants* is $V^G := \{v \in V \mid gv = v \text{ for all } g \in G\}$. Viewing V as a $\mathfrak{g}$-module via the differential, show that $V^G = V^{\mathfrak{g}}$.

(e) Finally let V and W are representations of G. Explain how to make $\mathrm{Hom}_{\mathbb{C}}(V, W)$ into a representation of G so that the $\mathfrak{g}$-module structure from (a) is the naturally induced one. Deduce that $\mathrm{Hom}_{\mathfrak{g}}(V, W) = \mathrm{Hom}_G(V, W)$.

(Part (e) proves that the category of representations of G is a full subcategory of the category of finite-dimensional representations of $\mathfrak{g}$.)

5. Let V be a vector space and $T(V)$ be its tensor algebra. Viewing the associative algebra $T(V)$ as a Lie algebra via the commutator, let $F(V)$ be the Lie subalgebra of $T(V)$ generated by V.

   (a) Prove that $F(V)$ together with the evident linear map $V \hookrightarrow F(V)$ satisfies the appropriate universal property making it the *free Lie algebra on the vector space* V.

   (b) When V is one-dimensional, $F(V) = V$. What can you say about $F(V)$ in the case that V is two-dimensional with basis x, y?

   (c) Now that the free Lie algebra is defined, you can make sense of Lie algebras defined by generators and relations. Let $\mathfrak{g}$ be the Lie algebra with two generators x, y subject only to the relations

   $$[x, [x, y]] = [y, [y, x]] = 0.$$

   Prove that $\mathfrak{g}$ is three-dimensional by identifying it with the Lie algebra of all strictly upper triangular matrices in $\mathfrak{gl}_3(\mathbb{k})$.