Introduction to Lie Theory
Homework #3

Let G be a connected algebraic group over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. Let $\mathfrak{g} := L(G)$ be its Lie algebra.

1. For $x \in G$, let G_x denote the set of fixed points of $\operatorname{Int} x : G \to G$ and $\mathfrak{g}_x$ denote the set of fixed points of $\operatorname{Ad} x : \mathfrak{g} \to \mathfrak{g}$. These are the *centralizers* of x in G and $\mathfrak{g}$, respectively, and may also be denoted $C_G(x)$ and $\mathfrak{c}_{\mathfrak{g}}(x)$. Check the following.

   (a) G_x is a closed subgroup of G and $\mathfrak{g}_x$ is a subalgebra of $\mathfrak{g}$.

   (b) $L(G_x) \subseteq \mathfrak{g}_x$.

   (c) Equality holds in (b) either if $G = GL_n(\Bbbk)$ or if $p = 0$.

   (*Hint.* For (c), first treat $GL_n(\Bbbk)$ by thinking about the equations which define these centralizers, then for the general case embed G as a closed subgroup of $GL_n(\Bbbk)$ and use some of the properties proved in L3-3.)

2. Let $Z(G)$ be the center of G.

   (a) Show that $Z(G) \subseteq \ker \operatorname{Ad}$. Then use the previous question and the lattice correspondence to show that equality holds when $p = 0$.

   (b) Assume that $p > 0$. Let G be the closed subgroup of $GL_3(\Bbbk)$ consisting of all matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a^p & b \\ 0 & 0 & 1 \end{pmatrix}$$

   for $a \in \Bbbk^\times, b \in \Bbbk$. Show that $Z(G) \subsetneq \ker \operatorname{Ad}$.

3. Show that $\operatorname{Aut}(\mathbb{G}_m)$, the group of automorphisms of the algebraic group $\mathbb{G}_m$, is isomorphic to $\mathbb{Z}^\times$. What is $\operatorname{Aut}(\mathbb{G}_m \times \mathbb{G}_m)$?

4. Let $T := \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ (n times). This is the *n-dimensional torus.*

   (a) Show that $X(T) := \operatorname{Hom}(T, \mathbb{G}_m)$, the set of morphisms of algebraic groups from T to $\mathbb{G}_m$ viewed as an Abelian group with the pointwise operation, is isomorphic to $\mathbb{Z}^n$.

(b) You saw in HW2-5 that every representation V of T is completely reducible, indeed, you have that

$$V = \bigoplus_{\lambda \in X(T)} V_\lambda$$

where $V_\lambda := \{v \in V \mid tv = \lambda(t)v \text{ for all } t \in T\}$. Note that T embeds into $G = GL_n(\Bbbk)$ as the diagonal invertible matrices. The adjoint action of G on $\mathfrak{g}$ restricts to make $\mathfrak{g}$ into a representation of T, hence, we get a decomposition of $\mathfrak{g}$ as above. Show that the ij-matrix unit $e_{i,j}$ belongs to $\mathfrak{g}_\lambda$ for some $\lambda \in X(T)$ which you should describe explicitly in terms of i and j.

5. Assume that $p = 0$. We showed in L3-3 that the ideals of $\mathfrak{g}$ are exactly the subspaces which are invariant under the adjoint action of G, and that they are also the Lie algebras of the closed connected normal subgroups of G. What are these subspaces in the case $G = GL_n(\Bbbk)$? Deduce that $SL_n(\Bbbk)$ is a simple algebraic group and $\mathfrak{sl}_n(\Bbbk)$ is a simple Lie algebra.

   (*Hint.* It may be useful to use the previous question! Such a subspace is invariant under the action of T too, hence, using complete reducibility, it decomposes as a direct sum of weight spaces.)