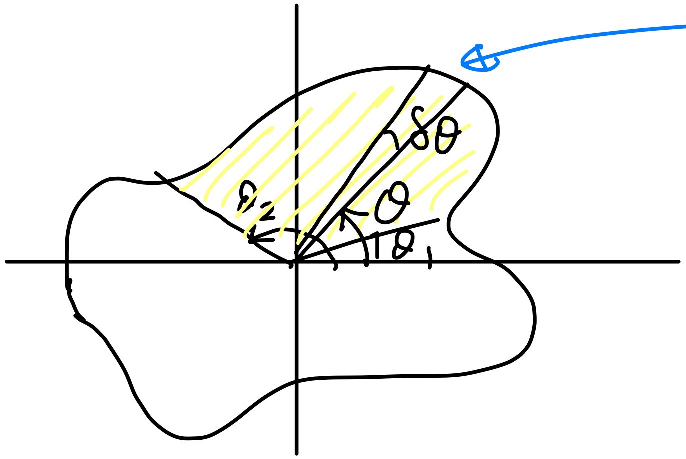
Finding areas in polar coordinates

$\cancel{y=f(x)}$ $\quad r=f(\theta)$



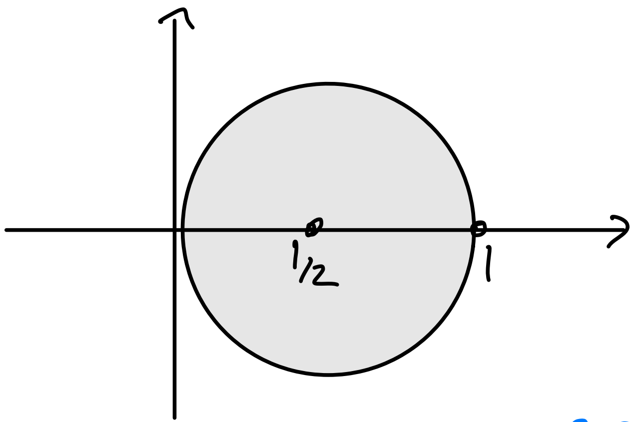
$\delta\theta$-sector has area $= \frac{1}{2} f(\theta)^2 \cdot \delta\theta$

$$\text{"} \sum_{\theta=\theta_1}^{\theta=\theta_2} \frac{1}{2} f(\theta)^2 \cdot \delta\theta \text{"}$$

Area of sector under polar graph from $\theta=\theta_1$ to $\theta=\theta_2$ $= \boxed{\frac{1}{2}\int_{\theta_1}^{\theta_2} f(\theta)^2 \cdot d\theta}$

Example

$r=\cos\theta$

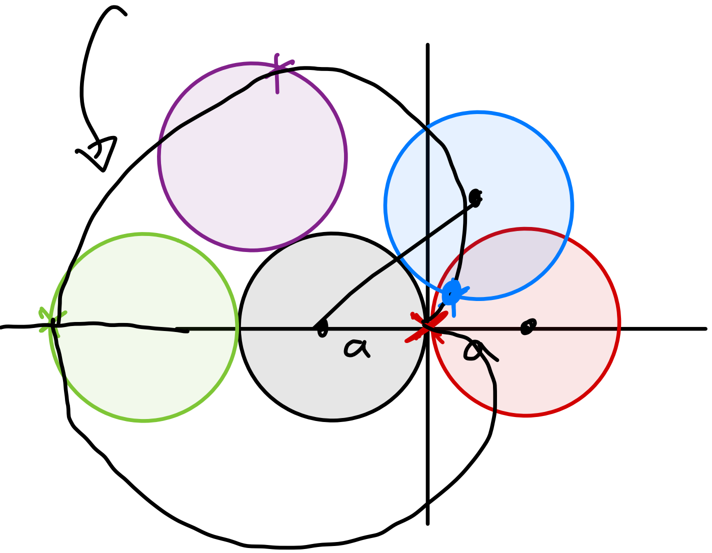


$$A = \frac{1}{2}\int_0^{\pi} \cos^2\theta \cdot d\theta = \frac{1}{2}\int_0^{\pi} \frac{\cos 2\theta + 1}{2} \cdot d\theta$$

$$\pi \times \left(\frac{1}{2}\right)^2 = \frac{\pi}{4} \qquad = \frac{1}{4}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\pi} = \frac{\pi}{4} \checkmark$$

$$\cos 2\theta = \frac{2\cos^2\theta - 1}{2}$$

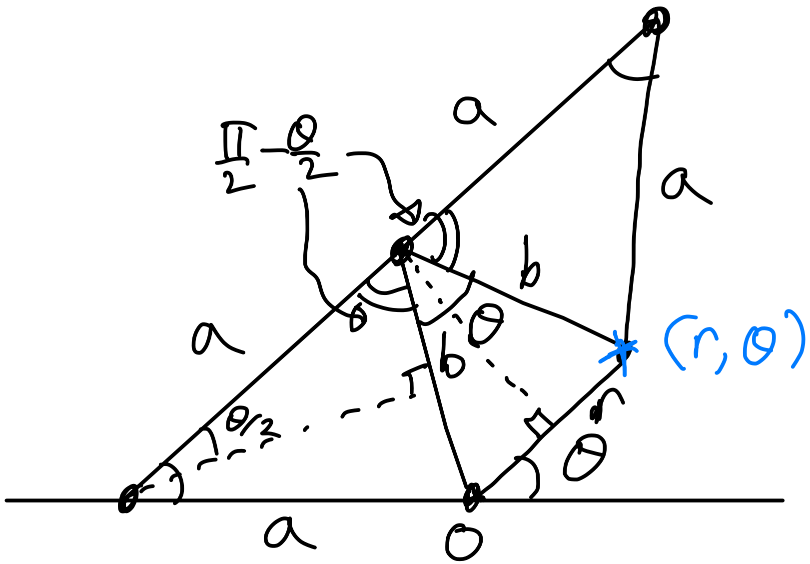
Cardioid



Roll red circle ↻ around fixed grey circle

Look at curve traced out by special point on circumference of rolling circle

Find its equation in polar coordinates.



$$\sin\frac{\theta}{2} = \frac{b}{2a} = \frac{r}{2b} \qquad \therefore\ b^2 = ar$$

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\left(\frac{b}{2a}\right)^2$$

$$\therefore\ \cos\theta = 1 - \frac{b^2}{2a^2} = 1 - \frac{\cancel{a}r}{2a^{\cancel{2}}} = 1 - \frac{r}{2a}$$

$$\therefore\ \frac{r}{2a} = 1 - \cos\theta$$

$$\therefore\ \boxed{r = 2a(1-\cos\theta)}$$

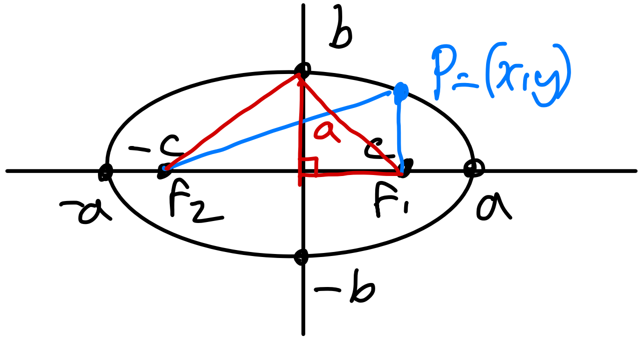
HW: Find area of cardioid.

Back to conic sections

Ellipse

$a \geq b$

$0 \leq e < 1$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \iff |PF_1|+|PF_2|=2a$$

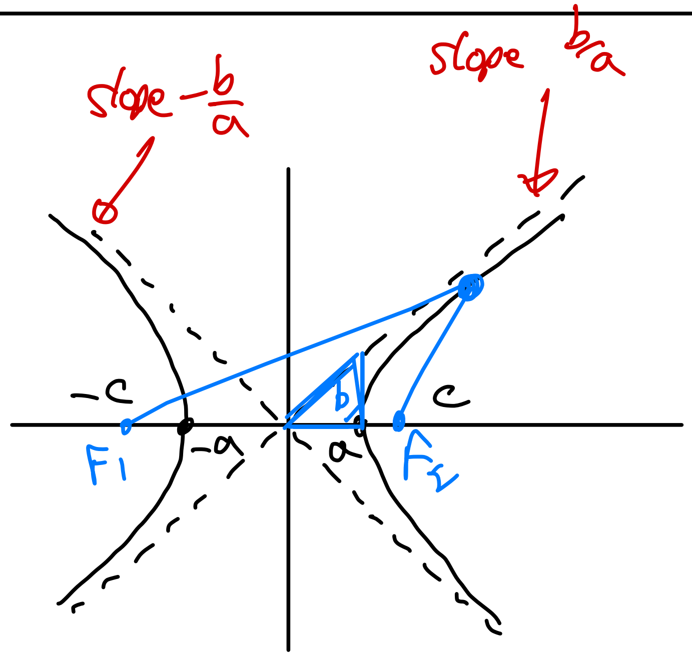
Where are the foci? $a^2=b^2+c^2$

$$\therefore c=\sqrt{a^2-b^2}=a\underbrace{\sqrt{1-b^2/a^2}}_{e,\ \text{eccentricity}}$$

$$\boxed{c=ae \quad \text{where} \quad e=\sqrt{1-b^2/a^2}}$$

Hyperbola

$e>1$



$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\Updownarrow$$

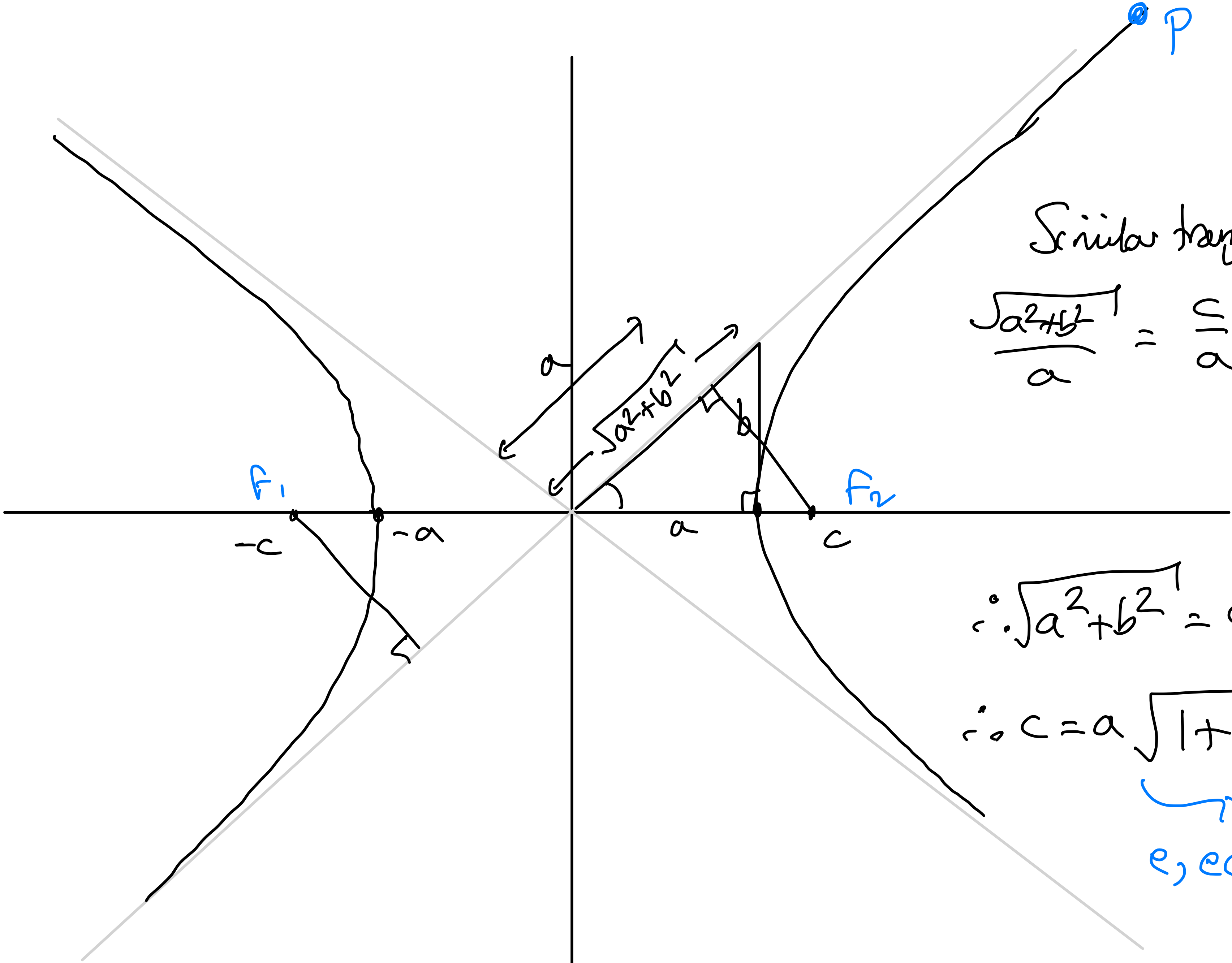
$$\big||PF_2|-|PF_1|\big|=2a$$

Asymptotes: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

$$y=\pm\frac{b}{a}x$$

Where are the foci?

$$\boxed{c=ae \quad \text{where} \quad e=\sqrt{1+b^2/a^2}}$$



Similar triangles:

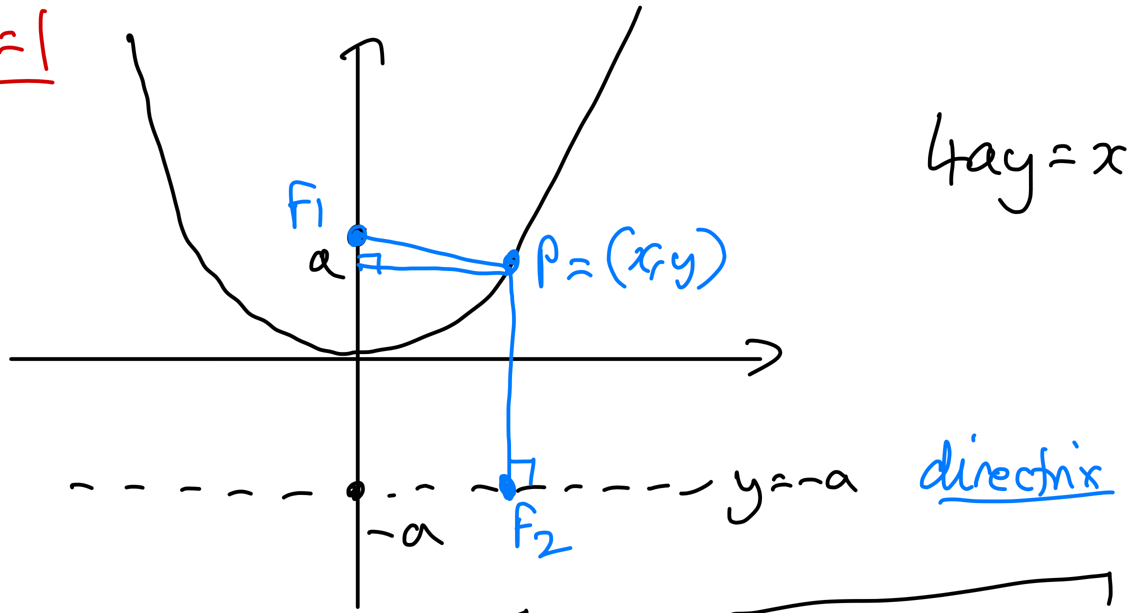
$$\frac{\sqrt{a^2+b^2}}{a} = \frac{c}{a}$$

$$\therefore \sqrt{a^2+b^2} = c$$

$$\therefore c = a\underbrace{\sqrt{1+\frac{b^2}{a^2}}}_{e,\ \text{eccentricity}}$$

Parabola (Conic section of eccentricity 1)

$e = 1$



$$4ay = x^2$$

$$|PF_1| = \sqrt{x^2 + (a-y)^2} = \sqrt{4ay + a^2 - 2ay + y^2}$$

$$= \sqrt{y^2 + 2ay + a^2} = \sqrt{(y+a)^2} = y + a = |PF_2|$$

$\therefore$ $\boxed{|PF_1| = |PF_2|}$ where $F_1 = (0,a)$ focus and F_2 is projection of P onto directrix $y = -a$