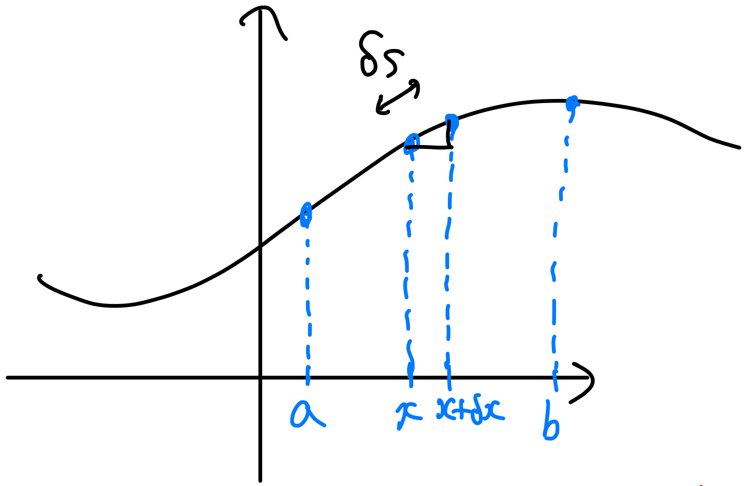
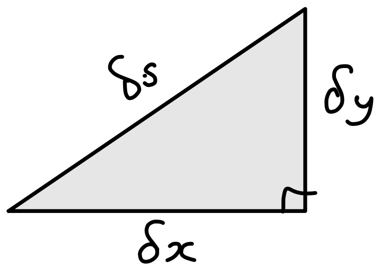
Arc lengths & Surface areas

Let s = arc length





$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

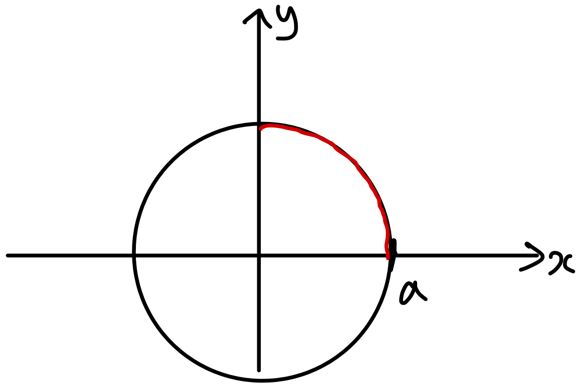
$$= \left[1 + \left(\frac{\delta y}{\delta x}\right)^2\right](\delta x)^2$$

$$\therefore \ \delta s = \sqrt{1 + \left(\frac{\delta y}{dx}\right)^2}\ \delta x$$

$$\text{"} \sum_{x=a}^{x=b} \delta s = \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\ \delta x \text{"}$$

$$\boxed{\text{Arc length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\cdot dx}$$

eg Circumference of circle of radius a



$x^2+y^2=a^2$

$\frac{d}{dx}$: $2x+2y\cdot\frac{dy}{dx}=0$

$\therefore \quad \frac{dy}{dx}=\frac{-x}{y}$

$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{x^2}{y^2} = \frac{y^2+x^2}{y^2} = \frac{a^2}{y^2}$

$$C = 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\int_0^a \frac{a}{y}\,dx$$

Alternative: Let $x=a\sin\theta$

$$= 4\int_0^a \frac{a}{\sqrt{a^2-x^2}}\,dx$$

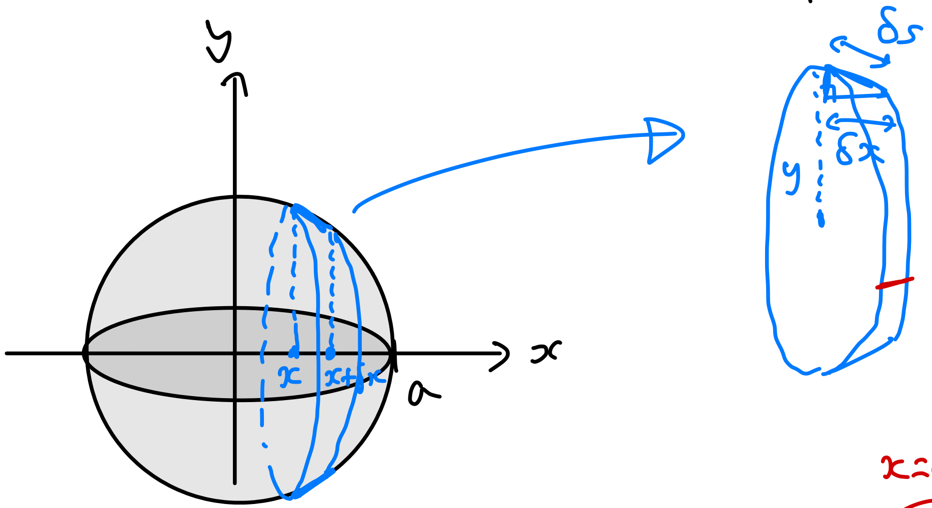
$$= 4\int_0^a \frac{1}{\sqrt{1-(x/a)^2}}\,dx$$

$$= 4a\left[\arcsin(x/a)\right]_0^a$$

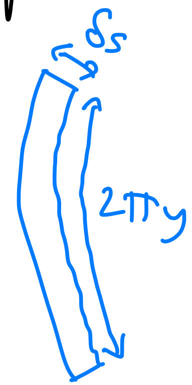
$$= 4a\times\frac{\pi}{2} = \boxed{2\pi a}$$

(eg) Surface area of sphere of radius a

Cut into discs



Surface area of disc ?



$2\pi y \, \delta s$

"$2\sum_{x=0}^{x=a} 2\pi y \sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\,\delta x$"

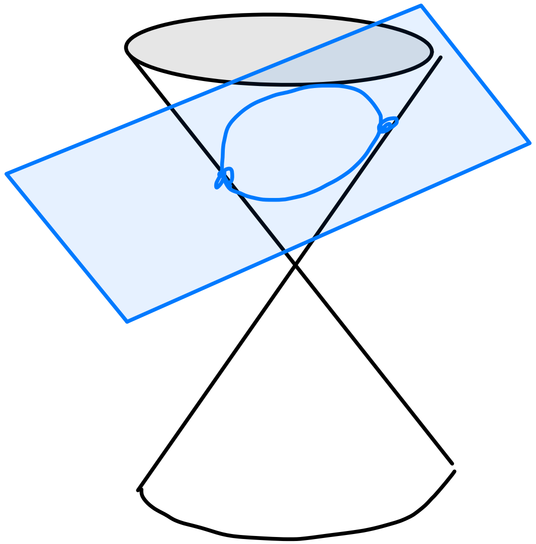
$$A = 2\int_0^a 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

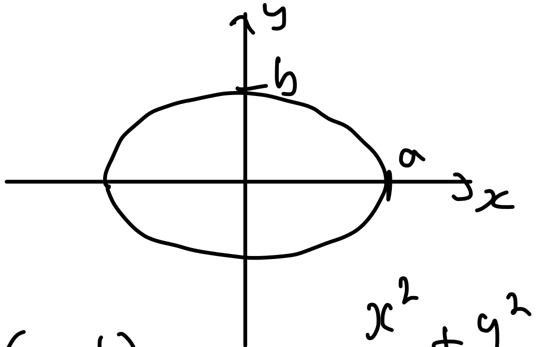
$$= 4\pi \int_0^a \cancel{y}\cdot\frac{a}{\cancel{y}}\,dx = \left[4\pi a x\right]_0^a = \boxed{4\pi a^2}$$

Next up: Conic sections

① ellipse

"defect"



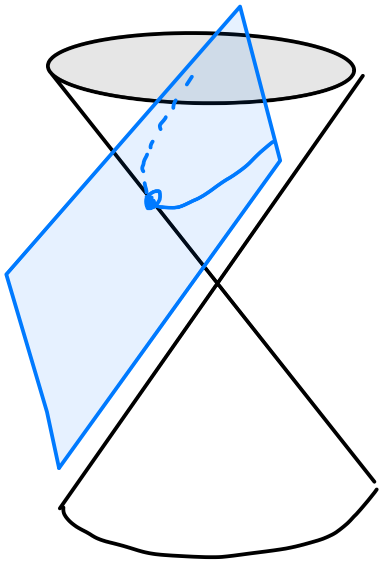


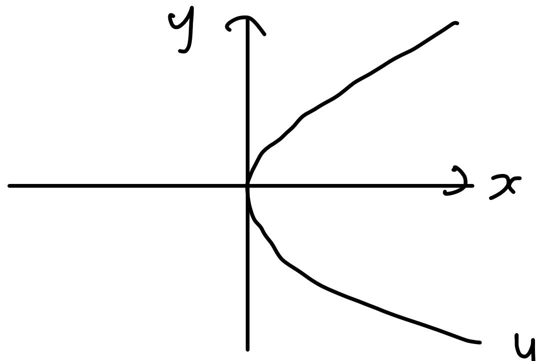
$(a>b)$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

② parabola

"alongside"

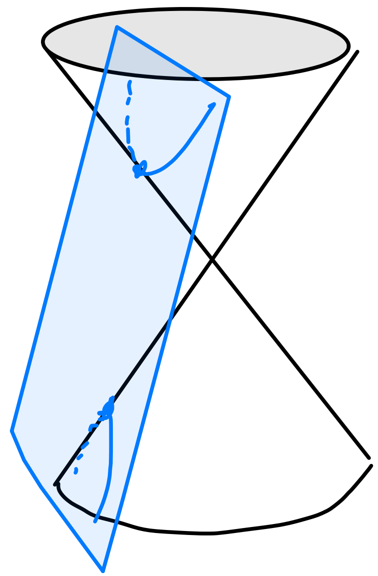


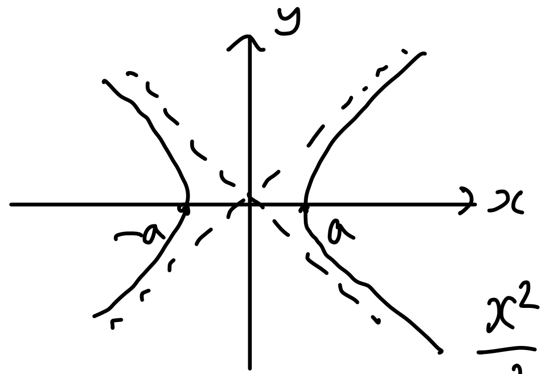


$$y^2=4ax$$

③ hyperbola

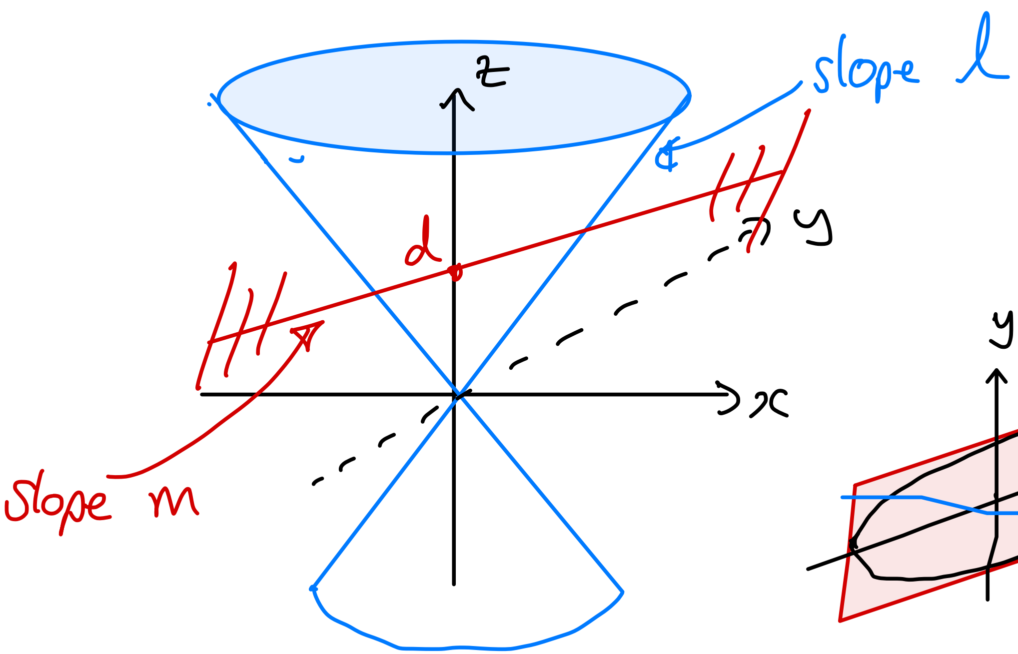
"excess"

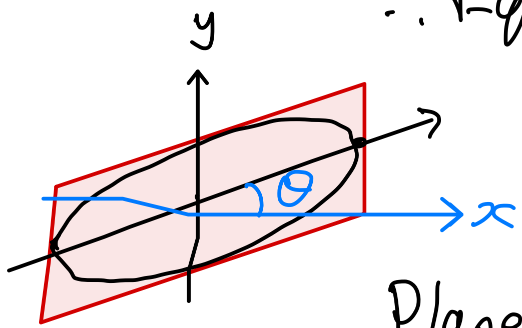




$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

How can we calculate equation of these graphs





y axis going into page

Plane coming straight into page

① $m < l$ ② $m = l$ ③ $m > l$

Should really replace x by $x\cos\theta$

Equation of cone?

Cone is line $z = lx$ rotated around z-axis.

$\therefore$ Equation is $z = \pm l\sqrt{x^2+y^2}$

$$\boxed{z^2 = l^2(x^2+y^2)}$$

Plane has equation $\boxed{z = mx + d}$

Compute intersection:

$$(mx+d)^2 = l^2(x^2+y^2)$$

$$\therefore l^2x^2 + l^2y^2 = m^2x^2 + 2mxd + d^2$$

$$\boxed{\therefore (l^2-m^2)x^2 + 2mxd + l^2y^2 = d^2}$$

$$\boxed{\therefore (\ell^2 - m^2)x^2 + 2mxd + \ell^2 y^2 = d^2}$$

$$(\ell^2 - m^2)\left(x + \frac{md}{\ell^2 - m^2}\right)^2 + \ell^2 y = d^2 + \frac{m^2 d^2}{\ell^2 - m^2} = \frac{\ell^2 d^2}{\ell^2 - m^2}$$

Shifting origin ... get

$$(\ell^2 - m^2)x^2 + \ell^2 y^2 = \frac{\ell^2 d^2}{\ell^2 - m^2}$$

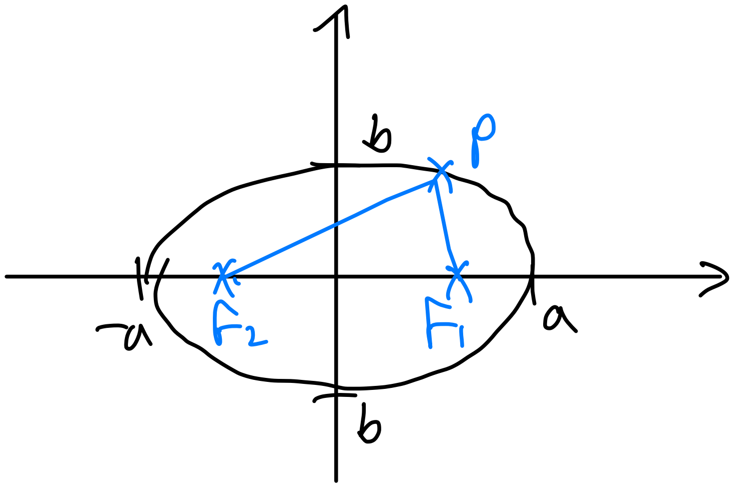
1. $m < \ell$ ... equation has form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ ✓ ellipse

2. $m = \ell$ ... $y^2 = 4ax$ ✓ parabola

3. $m > \ell$ ... $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ ✓ hyperbola.

Right approach ... classical Euclidean geometry.

Explain just case of ellipse.

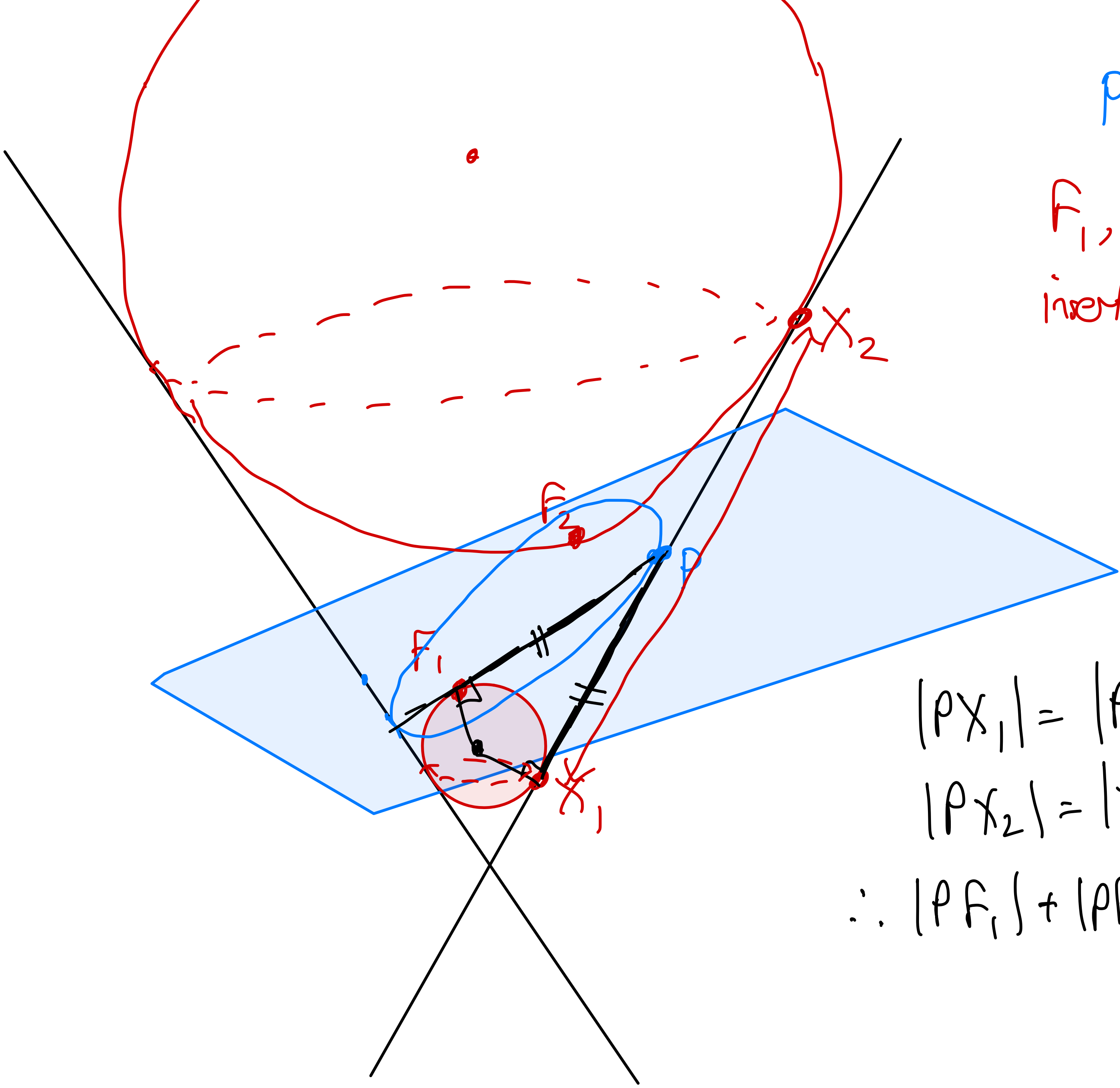


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b)$$

$$|PF_1| + |PF_2| = 2a$$

"Locus of points whose distance to two foci sum to a constant"

Want to prove conic section is an ellipse according to this geometric description



P point on conic section

F_1, F_2 foci calculated by inserting spheres above & below

$$|PX_1| = |PF_1|$$

$$|PX_2| = |PF_2|$$

$$\therefore |PF_1| + |PF_2| = |PX_1| + |PX_2| = \overset{2a}{|X_1X_2|}$$

constant!!