History and Applications of Calculus Homework 3

1. The cycloid is defined parametrically by the equations

$$x = a(t - \sin\theta), \qquad y = a(1 - \cos\theta).$$

Calculate the area under one "cycle" of this cycloid and above the x-axis.

2. The cardioid is defined in polar coordinates by the equation

$$r = 2a(1 - \cos\theta).$$

Calculate the area of the cardioid.

3. Suppose F_1 and F_2 are two given foci in the plane which are a distance $2c$ apart, and $0 < a < c$. The *hyperbola* is the locus of points P such that $\big||PF_2| - |PF_1|\big| = 2a$.

(a) Assuming that the foci are at coordinates $(\pm c, 0)$, show that the Cartesian equation for this hyperbola has the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

for some $b > 0$. Write b in terms of a and c and show that $c = ae$ where $e = \sqrt{1 + \frac{b^2}{a^2}}$ is the *eccentricity*.

(b) Find the polar equation for this "left half" of this hyperbola, that is, the curve defined by the equation $|PF_2| - |PF_1| = 2a$ (I've removed the absolute values!). Assume for this that F_1 is the origin and F_2 is the point $(2c, 0)$.

4. In this question you can assume without proof that the general solution of the differential equation

$$f''(x) + f(x) = 0$$

is $f(x) = a\cos x + b\sin x$ where $a = f(0)$ and $b = f'(0)$, i.e., they are constants you can determine from the initial conditions. Use this to prove the multiple angle formulae

$$\sin(x+y) = \sin x\cos y + \cos x\sin y,$$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y,$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}.$$

5. If you have a graph $r = f(\theta)$ in polar coordinates, you can also consider the *inversion* of this graph, which is $r = \frac{1}{f(\theta)}$. Note this transformation switches points inside the unit circle with points outside...

(a) The graphs $r = 1 - \cos\theta$ and $r = \frac{1}{1-\cos\theta}$ are the inversions of each other. Sketch these two graphs on the same axes. What are these two graphs called?

(b) Consider the straight line whose Cartesian equation is $x = c$ for some constant $c > 0$. Show that its inversion is a circle passing through the origin.

(c) Show that the inversion of a circle which does *not* pass through the origin is another such circle.