History and Applications of Calculus Homework 1

1. Some differentiation practice!

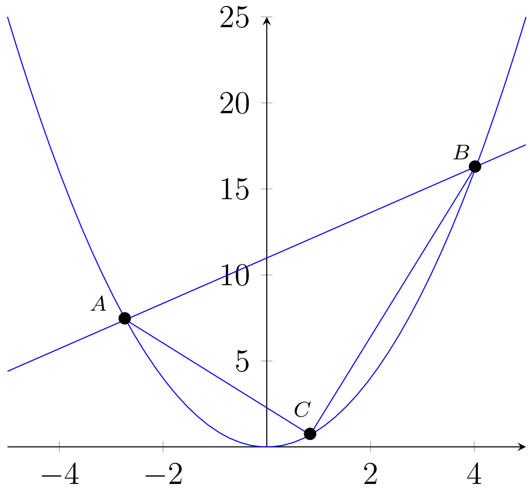
(a) Starting from $\sin' = \cos$ and $\cos' = -\sin$, find the derivatives of $\tan x, \cot x, \sec x$ and $\operatorname{cosec} x$.

(b) Sketch the graphs of $y = \arcsin x$ and $y = \arccos x$ on the same axes. How does this demonstrate that $\arccos' = -\arcsin'$?

(c) You know that $\arcsin' x = \frac{1}{\sqrt{1-x^2}}$. So from (b) you get that $\arccos' x = -\frac{1}{\sqrt{1-x^2}}$. Use implicit differentiation to prove this formula directly.

2. At what angle to the ground should you launch a projectile in order to get it to travel the furthest? (Ignore air resistance, assume the ground is flat, etc...)

3. One of Archimedes' famous accomplishments was the quadrature of the parabola. Here is the result. Take the usual parabola with graph $y = x^2$. Pick two points on the parabola A and B with coordinates (a, a^2) and (b, b^2), respectively, assuming for definiteness that $a < b$. Join these two points up with a straight line AB. Let C be the point on the parabola in between A and B such that the slope of the tangent line at C is parallel to the line AB.



Theorem. *The area of the region below the line AB and above the parabola is equal to $\frac{4}{3}$ times the area of the triangle ABC.*

Give a proof of this theorem using all the tools from calculus at your disposal.

4. Use calculus to prove Snell's Law of Light Refraction: *If you shine a ray of light across a horizontal boundary between two media such that the speed of light in the top medium is u and the speed of light in the bottom medium is v, then the light refracts in such a way that*

$$\frac{\sin\theta}{\sin\phi} = \frac{u}{v},$$

where θ is the angle of incidence to the vertical above and ϕ is the angle of incidence to the vertical below the boundary.