20. Elementary obstruction theory

20.1. **Eilenberg-MacLane spaces and cohomology operations.** Let π and G be abelian groups, n, m be nonnegtive integers.

Definition 20.1. A family of maps

$$\theta_X : H^n(X;\pi) \to H^{n'}(X;\pi')$$

is called a *cohomology operation* θ *of the type* $(\pi, n; \pi', n')$ if it is determined for every topological space X and such that for any map $f : X \to Y$ the diagram

$$\begin{array}{ccc} H^n(X;\pi) & \xrightarrow{\theta_X} & H^{n'}(X;\pi') \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} \\ H^n(Y;\pi) & \xrightarrow{\theta_Y} & H^{n'}(Y;\pi') \end{array}$$

commutes, i.e. $f^*\theta_Y = \theta_X f^*$. In different terms, we say that the operation θ is *natural.* The set of all cohomological operations of the type $(\pi, n; \pi', n')$ is denoted by $\mathcal{O}(\pi, n; \pi', n')$.

Example. For each n and any π the operation $a \mapsto a^2$ is a cohomology operation. Notice that a cohomology operation is not, in general, a homomorphism.

Our next goal is to idendify the set $\mathcal{O}(\pi, n; \pi', n')$ with cohomology groups of the Eilenberg-McLane spaces.

Let X be a space. We recall that there is Hurewicz homomorphism $h : \pi_q(X) \to H_q(X;\mathbf{Z})$ defined as follows. Let $\iota_q \in H_q(S^q)$ be a canonical generator. Then for an element $\varphi \in \pi_q(X)$ and its represenative $f : S^q \to X$, the image $h(\varphi) \in H_q(X;\mathbf{Z})$ is given by $f_*(\iota_q)$.

Now assume that X is $(n-1)$-connected. Then $H_q(X;\mathbf{Z}) = 0$ for $q \leq n-1$ and the Hurewicz homomorphism $h : \pi_n(X) \to H_n(X;\mathbf{Z})$ is isomorphism. Then the universal coefficient formula

$$0 \to \mathrm{Ext}(H_{n-1}(X;\mathbf{Z}), \pi) \to H^n(X;\pi) \to \mathrm{Hom}(H_n(X;\mathbf{Z}), \pi) \to 0$$

shows that $H^n(X;\pi) \cong \mathrm{Hom}(H_n(X;\mathbf{Z}), \pi)$ since $H_{n-1}(X;\mathbf{Z}) = 0$.

Let $\pi = \pi_n(X)$. Thus the group $\mathrm{Hom}(H_n(X;\mathbf{Z}), \pi)$ contains the inverse h^{-1} to the Hurewicz homomorphism h.

Definition 20.2. For an $(n-1)$-connected space X, we denote by ι_X the cohomology class

$$\iota_X := h^{-1} \in \mathrm{Hom}(H_n(X;\mathbf{Z}), \pi) \cong H^n(X;\pi).$$

Sometimes the class ι_X is called as *fundamental class of* $(n-1)$*-connected space* X.

In particular, the Eilenberg-McLane space $K(\pi, n)$ has a canonical class

$$\iota_n \in \mathrm{Hom}(H_n(K(\pi, n);\mathbf{Z}), \pi).$$

Below we will prove the following result.

Theorem 20.3. *There is a bijection*

$$[X, K(\pi, n)] \leftrightarrow H^n(X;\pi).$$

given by the formula $[f] \mapsto f^*\iota_n$.

Here $[f]$ means a homotopy class of a map $f : X \to K(\pi, n)$. Before proving Theorem 20.3, we derive several important corollaries of Theorem 20.3.

Corollary 20.4. *Let* π, π' *be abelian groups. There is a bijection*

$$[K(\pi, n), K(\pi', n)] \leftrightarrow \mathrm{Hom}(\pi, \pi').$$

Proof. We combine the statement of Theorem 20.3 with the universal coefficient theorem and Hurewicz isomorphism to see that

$$[K(\pi, n), K(\pi', n)] \leftrightarrow H^n(K(\pi, n); \pi') \cong \mathrm{Hom}(H_n(K(\pi, n); \mathbf{Z}), \pi') \cong \mathrm{Hom}(\pi, \pi').$$

This proves Corollary 20.4. □

Corollary 20.5. *Let* π *be an abelian group. The homotopy type of the Eilenberg-McLane space* $K(\pi, n)$ *is completely determined by the group* π *and the integer* n.

Proof. According to Corollary 20.4, any isomorphism $\pi \to \pi$ is induced by some map $f : K(\pi, n) \to K(\pi, n)$. Since all other groups are trivial, the map f induces isomorphism in all homotopy groups. Then Whitehead Theorem 14.10 implies that f is homotopy equivalence. □

Now let θ be a cohomology operation of the type $(\pi, n; \pi', n')$. Then we have en element $\theta(\iota_n) \in H^{n'}(K(\pi, n), \pi')$.

Theorem 20.6. *There is a bijection*

$$\mathcal{O}(\pi, n; \pi', n') \leftrightarrow H^{n'}(K(\pi, n), \pi')$$

given by the formula $\theta \leftrightarrow \theta(\iota_n)$.

Proof. Let $\varphi \in H^{n'}(K(\pi, n), \pi')$. We define an operation $\varphi \in \mathcal{O}(\pi, n; \pi', n')$ as follows. We should describe the action

$$H^n(X;\pi) \xrightarrow{\varphi_X} H^{n'}(X;\pi')$$

for any space X. Let $u \in H^n(X;\pi)$, then, according to Theorem 20.3, there exists a map $f : X \to K(\pi; n)$ such that $[f] \mapsto f^*(\iota_n) = u$. Then we define

$$\varphi(u) = f^*(\varphi) \in H^{n'}(X;\pi').$$

Thus we have the maps

$$\mathcal{O}(\pi, n; \pi', n') \to H^{n'}(K(\pi, n), \pi'), \quad \varphi \mapsto \varphi(\iota_n)$$

$$H^{n'}(K(\pi, n), \pi') \to \mathcal{O}(\pi, n; \pi', n'), \quad \varphi(u) = f^*(\varphi), \quad \text{where} \quad f^*(\iota_n) = u.$$

Let $X = K(\pi, n)$ and $u = \iota_n$, then $f : K(\pi, n) \to K(\pi, n)$ is homotopic the identity. Thus $\varphi(\iota_n) = f^*(\varphi) = \varphi$. In the other direction, let $\varphi = \theta(\iota_n)$. Then

$$\varphi(u) = f^*(\varphi) = f^*(\theta(\iota_n)) = \theta(f^*(\iota_n)) = \theta(u)$$

for any $u \in H^n(X;\pi)$. □

Theorems 20.3 and 20.6 imply the following result:

Corollary 20.7. *There is a bijection*

$$\mathcal{O}(\pi, n; \pi', n') \leftrightarrow [K(\pi, n), K(\pi', n')].$$

Now we have to prepare some tools to prove Theorem 20.3.

20.2. **Obstruction theory.** Let Y be a space with a base point $y_0 \in Y$. We recall that the fundamental group $\pi_1(Y, y_0)$ acts on the group $\pi_n(Y, y_0)$ for each n. We will say that a space Y is *homotopically simple* if this action is trivial. In the case when the space Y is homotopically simple, we may (and will) ignore a choice of the base point. In particular, any map $f : S^n \to Y$ gives well-defined element in the group $\pi_n(Y)$.

Now let B be a CW-complex and $A \subset B$ be its subcomplex. We denote $X^n = B^{(n)} \cup A$, where $B^{(n)}$ is the n-th skeleton of B. Let $\sigma = e^{n+1}$ be an $(n+1)$-cell of B, which does not belong to A. We denote by $\varphi_\sigma : S^n \to X^n$ be the attaching map corresponding to the cell σ. We consider the cells σ as generators of the cellular chain group $\mathcal{E}_{n+1}(B, A)$.

For any map $f : X^n \to Y$, where Y is homotopically simple, we define a cochain

$$c(f) \in \mathcal{E}^{n+1}(B, A; \pi_n(Y)) = \mathrm{Hom}(\mathcal{E}_{n+1}(B, A), \pi_n(Y))$$

as follows. The value $c(f)$ on the generator σ is given by

$$c(f)(\sigma) = [f \circ \varphi_\sigma] \in \pi_n(Y), \quad \text{where}$$

$$f \circ \varphi_\sigma : S^n \xrightarrow{\varphi_\sigma} X^n \xrightarrow{f} Y.$$

Lemma 20.8. *The cochain $c(f)$ is a cocycle, i.e. $\delta c(f) = 0$.*

Proof. We recall that if (K, L) is a CW-pair with $\pi_1 K = \pi_1 L = 0$, and $pi_1(K, L) = 0$ for $q = 0, 1, \ldots, n-1$, then the Hurewicz homomomorphism $h : \pi_n(K, L) \to H_n(K, L; \mathbf{Z})$ is an isomorphism. This is the relative version of the Hurewicz Theorem, see Theorem 14.9. We will use this result below. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{E}_{n+2}(B,A) & \xrightarrow{\cong} & H_{n+2}(X^{n+2},X^{n+1};\mathbf{Z}) & \xrightarrow{h^{-1}} & \pi_{n+2}(X^{n+2},X^{n+1}) & & \\
 & & & & \downarrow{\scriptstyle\tilde\partial} & & \\
\downarrow{\scriptstyle\partial_{n+2}} & & \downarrow{\scriptstyle\partial_{n+2}} & & \pi_{n+1}X^{n+1} & & \\
 & & & & \downarrow{\scriptstyle j_*} & & \\
\mathcal{E}_{n+1}(B,A) & \xrightarrow{\cong} & H_{n+1}(X^{n+1},X^{n};\mathbf{Z}) & \xrightarrow{h^{-1}} & \pi_{n+1}(X^{n+1},X^{n}) & & \\
 & \searrow{\scriptstyle i} & & & \downarrow{\scriptstyle\partial} & & \\
 & & & & \pi_n X^n & \xrightarrow{f_*} & \pi_n Y
\end{array} \tag{91}$$

Here the horizontal homomorphisms are given by the inverses to the Hurewicz isomorphisms. By definition, the boundary operator

$$\partial^{b}_{n+2} : H_{n+2}(X^{n+2}, X^{n+1}; \mathbf{Z}) \longrightarrow H_{n+1}(X^{n+1}, X^n; \mathbf{Z})$$

is given by the boundary operator in the long exact sequence of the triple (X^{n+2}, X^{n+1}, X^n) and thus by Hurewicz isomorphism is reduced to the boundary operator in the long exact sequence in homotopies for the same triple:

$$\partial : \pi_{n+2}(X^{n+2}, X^{n+1}) \longrightarrow \pi_{n+1}(X^{n+1}, X^n)$$

which coincides with the composition:

$$\pi_{n+2}(X^{n+2}, X^{n+1}) \xrightarrow{\tilde{\partial}} \pi_{n+1}X^{n+1} \xrightarrow{j_*} \pi_{n+1}(X^{n+1}, X^n).$$

by construction. Here $\tilde{\partial}$ is the boundary operator in the long exact sequence in homotopy groups for the pair (X^{n+2}, X^{n+1}). Then we identify the cochain $c(f) : \mathcal{E}_{n+1}(B, A) \to \pi_n Y$; clearly it coinsides with the composition $f_* \circ i$. Now let $\sigma \in \mathcal{E}_{n+2}(B, A)$. By definition, $\delta_{n+1}c(f)(\sigma) = c(f)(\partial^{b}_{n+2}\sigma)$. On the other hand, we can first take σ to $\bar{\sigma} \in \pi_{n+2}(X^{n+2}, X^{n+1})$ via the Hurewicz isomorphism and then down the right column of the diagram (91). Then we have $\partial \circ f_* \circ \tilde{\partial}(\bar{\sigma}) = 0$ since $\partial \circ f_* = 0$ by exactness. □

Exercise 20.1. *Prove the following Lemma 20.9.*

Lemma 20.9. *The map $f : X^n \to Y$ can be extended to a map $\tilde{f} : X^{n+1} \to Y$ if and only if $c(f) = 0$.*

Now let $f, g : X^n \to Y$ be two maps which coincide on X^{n-1}, i.e. $f|_{X^{n-1}} = g|_{X^{n-1}}$. Then for each n-cell ω, we define a map $h_\omega : S^n \to Y$ as follows. We decompose S^n as union of the hemispheres: $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$. Then for each n-cell ω, we have the attaching map $\psi_\omega : S^{n-1} \to X^{n-1} = B^{(n-1)} \cup A$ and characteristic map $\Psi_\omega : D^n \to X^n = B^{(n)} \cup A$. Then we define $h_\omega : S^n = D^n_+ \cup_{S^{n-1}} D^n_- \to Y$ by

$$h_\omega|_{D^n_+} = g \circ \Psi_\omega : D^n_+ \xrightarrow{\Psi_\omega} X^n \xrightarrow{g} Y,$$

$$h_\omega|_{D^n_-} = f \circ \Psi_\omega : D^n_- \xrightarrow{\Psi_\omega} X^n \xrightarrow{f} Y.$$

Clearly $h_\omega|_{S^{n-1}} = (f \circ \Psi_\omega)|_{S^{n-1}} = (g \circ \Psi_\omega)|_{S^{n-1}}$ since $f|_{X^{n-1}} = g|_{X^{n-1}}$. This construction defines the *distinguishing cochain* $d(f, g)$ in the cochain group $\mathcal{E}^n(B, A; \pi_n(Y))$.

Lemma 20.10. *There are the following properties of the cochain $d(f, g)$:*

(1) *Let $f, g : X^n \to Y$ be two maps which coincide on X^{n-1}, then*

$$\delta d(f, g) = c(g) - c(f).$$

(2) *Let $f, g, h : X^n \to Y$ be three maps which coincide on X^{n-1}, then*

$$d(f, g) + d(g, h) = d(f, h).$$

Proof. We prove (1) leaving (2) as an exercise. For simplicity, we assume that the maps $f, g : X^n \to Y$ are different only on a single n-cell ω of X^n. Let σ be any $(n+1)$-cell of X^{n+1}. Then, by definition,

$$\delta d(f,g)(\sigma) = d(f,g)(\flat\!\!\partial_{n+1}\sigma),$$

where $\flat\!\!\partial_{n+1} : \mathcal{E}_{n+1}(B,A) \to \mathcal{E}_n(B,A)$ is the boundary operator in the cellular chain complex.

Let Φ_σ a characteristic map and φ_σ be an attaching map coresponding to the cell σ:

$$\begin{array}{ccc} D^{n+1} & \xrightarrow{\ \Phi_\sigma\ } & X^{n+1} \\ \uparrow & & \uparrow \\ S^n & \xrightarrow{\ \varphi_\sigma\ } & X^n \end{array}$$

We consider the following diagram:

$$\begin{array}{ccccc} S^n & \xrightarrow{\ \varphi_\sigma\ } & X^n & \xrightarrow{\ pr\ } & X^n/X^{n-1} \\ & \searrow^{\zeta} & & & \downarrow{=} \\ & & S^n_e & \xleftarrow{\ p_e\ } & \bigvee_j S^n_j \end{array}$$

Here S^n_e is the sphere corresponding to the cell e and p_e the projection on S^n_e. Since f and g are the same on all cells but e, we obtain

$$\delta d(f,g)(\sigma) = d(f,g)(\flat\!\!\partial_{n+1}\sigma) = [\sigma : e] d(f,g)(e),$$

where $[\sigma : e] = \deg \zeta$, where $\zeta : S^n \to S^n_e$ is the map from the above diagram. Now we recall

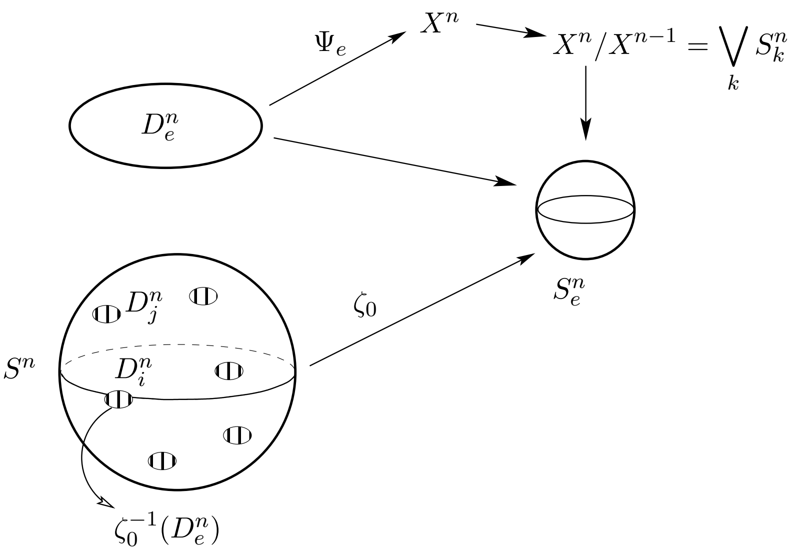

FIGURE 30

that a map $\zeta : S^n \to S^n_e$ of degree $[\sigma : e]$ is homotopic to a map $\zeta_0 : S^n \to S^n_e$ which satisfies the following properties:

(1) there are disjoint disks $D_1^n, \dots, D_s^n \subset S^n$ such that
   (a) $\zeta_0|_{D_j^n} : D_j^n \to S_e^n$ is a map of degree ± 1;
   (b) $\zeta_0|_{S^n \setminus (D_1^n \sqcup \dots \sqcup D_s^n)} : S^n \setminus (D_1^n \sqcup \dots \sqcup D_s^n) \to S_e^n$ is a constant map;
(2) the degree $[\sigma : e]$ is the algebraic number of such disks counting ± 1's.

Now we can take a close look at the cell $e^n \subset X^n$, see Fig. 30. It shows that

$$(c(f) - c(g))(\sigma) = [\sigma : e]d(f, g)(e)$$

This proves the result. □

It turns out that any cochain in $\mathcal{E}^n(B, A; \pi_n(Y))$ could be realized as a distinguishing cochain:

Lemma 20.11. *For any map $f : X^n \to Y$ and a cochain $d \in \mathcal{E}^n(B, A; \pi_n(Y))$ there exists a map $g : X^n \to Y$ such that $f|_{X^{n-1}} = g|_{X^{n-1}}$ and $d(f, g) = d$.*

Exercise 20.2. *Prove Lemma 20.11.*

We denote by $[c(f)] \in H^{n+1}(B, A; \pi_n Y)$ the cohomology class of $c(f)$.

Theorem 20.12. *Let Y be a homotopy simple space, (B, A) a CW-pair and $X^n = B^{(n)} \cup A$ for $n = 0, 1, \dots$. Assume $f : X^n \longrightarrow Y$ is a map. Then there exists a map $g : X^{n+1} \longrightarrow Y$ such that $g|_{X^{n-1}} = f|_{X^{n-1}}$ if and only if $[c(f)] = 0$ in $H^{n+1}(B, A; \pi_n Y)$.*

Proof. Let $\delta d = c(f)$. Then we find $g : X^n \longrightarrow Y$ such that $g|_{X^{n-1}} = f|_{X^{n-1}}$ and $d(f, g) = -d$. Since

$$c(f) = \delta d = -\delta d(f, g) = c(f) - c(g),$$

we obtain that $c(g) = 0$. Thus there exists an extension of g to X^{n+1}. □

Let K be a CW-complex. Then we let $B = K \times I$, $A = K \times \{0, 1\}$. To illustrate the technique, we give another proof of the following simple fact.

Lemma 20.13. *Let Y be n-connected space and K be a CW-complex of dimension n. Then $[K, Y] = *$.*

Proof. Let $h : K \to Y$ be a map. We define a map $f : K \times \{0, 1\} \to Y$ as

$$f|_{K \times \{0\}} = h, \quad f|_{K \times \{1\}} = *$$

We choose a CW-structure of $K \times I$ to be a product-structure. In particular, all zero cells of $K \times I$ are located inside of $K \times \{0, 1\}$. Thus the map

$$f^{(0)} : X^0 = (K \times I)^{(0)} \cup K \times \{0, 1\} \to Y$$

is already defined. Assume that its exension

$$f^{(k)} : X^k = (K \times I)^{(k)} \cup K \times \{0, 1\} \to Y$$

to the space $X^k = (K \times I)^{(k)} \cup K \times \{0, 1\}$ for $k = 0, \dots, \ell - 1$ is defined, where $\ell \leq n$. Then the obstruction $c(f^{(\ell)}) \in \mathcal{E}^\ell(K \times I, K \times \{0, 1\}; \pi_\ell(Y))$ vanishes since $\pi_\ell(Y) = 0$. This shows that a homotopy between h and the constant map extends to $K \times I$, i.e. we have proved that $[K, Y] = *$. □

Next, we would like to prove a result concerning exension of a homotopy in more general setting.

Theorem 20.14. *Let $f, g : K \longrightarrow Y$ be two maps, where K is a CW-complex and Y is homotopy-simple space. Assume that $f|_{K^{(n-1)}} = g|_{K^{(n-1)}}$. Then the cohomology class $[d(f,g)] \in H^n(K, \pi_n Y)$ vanishes if and only if there exists a homotopy between the maps $f|_{K^{(n)}}$ and $g|_{K^{(n)}}$ relative to the skeleton $K^{(n-2)}$.*

Proof. We have that $f|_{K^{(n-1)}} = g|_{K^{(n-1)}}$. We would like to construct a homotopy between $f|_{K^{(n)}}$ and $g|_{K^{(n)}}$ relative to the skeleton $K^{(n-2)}$. We consider the pair

$$(B, A) = (K \times I, K \times \{0, 1\}).$$

Here again we choose a standard CW-structure on the interval I: two zero cells ϵ^0_0, ϵ^0_1 and one 1-cell ϵ^1. Then we denote $X^k = (K \times I)^{(k)} \cup (K \times \{0, 1\})$. Since $f|_{K^{(n-1)}} = g|_{K^{(n-1)}}$, and an n-cell of $K \times I$ is a product $e^{n-1} \times \epsilon^1$, where e^{n-1} is an $(n-1)$-cell of K, we have a map

$$H : (K \times I)^{(n)} \cup (K \times \{0, 1\})$$

such that $H|_{K\times\{0\}} = f$, $H|_{K\times\{1\}} = g$, and

$$H|_{K^{(n-1)}\times I} = f|_{K^{(n-1)}} \times Id = g|_{K^{(n-1)}} \times Id.$$

Consider the obstruction cocycle $c(H)$. Again, we notice that every $(n+1)$-cell σ^{n+1} of $(K \times I) \setminus (K \times \{0, 1\})$ has a form $e^n \times \epsilon^1$. Then we can easily identify the obstruction cocycle $c(H) \in \mathcal{E}^{n+1}(K \times I, K \times \{0, 1\}; \pi_n Y)$ with the distinguishing cochain

$$d(f, g) \in \mathcal{E}^n(K; \pi_n Y).$$

Indeed, each n-cell e^n of K gives a map

$$h : S^n = D^n \times \{0\} \cup S^{n-1} \times I \cup D^n \times \{1\} \to Y$$

where $h|_{D^n\times\{0\}}$ is given by f and $h|_{D^n\times\{1\}}$ is given by g. A homotopy class of h gives nothing but the value of $d(f, g)$ on the same cell e^n.

In this case, we have that $c(f|_{K^{(n)}}) = 0$ and $c(g|_{K^{(n)}}) = 0$ since f and g both are defined on all K. Thus we have

$$\delta d(f, g) = c(g) - c(f) = 0.$$

Thus $\delta d(f, g) = 0$ and determines an element in cohomology $[d(f, g)] \in H^n(K, \pi_n Y)$. Now Theorem 20.12 implies the result. □

Exercise 20.3. *Show details that Theorem 20.12 indeed implies the result at the end of the above proof.*

20.3. **Proof of Theorem 20.3.** Let $\iota_n \in H^n(K(\pi, n); \pi)$ be the fundamental class. We would like to prove that the map $[f] \mapsto f^*\iota_n$ gives a bijection

$$[X, K(\pi, n)] \leftrightarrow H^n(X; \pi)$$

for a CW-complex X. Let $\alpha \in H^n(X; \pi)$, we have to find a map $f : X \to K(\pi, n)$ such that $f^*\iota_n = \alpha$. We choose a cocycle $a : \mathcal{E}_n(X) \to \pi$ which represents $\alpha \in H^n(X; \pi)$. In particular, a assigns an element $a(\sigma^n_i) \in \pi_n K(\pi, n) = \pi$. We choose representatives $h_i : S^n_i \to K(\pi, n)$ of the elements $a(\sigma^n_i) \in \pi_n K(\pi, n)$. Now we define a map $f^{(n)} : X^{(n)} \to K(\pi, n)$ as follows. We let $f^{(n)}|_{X^{(n-1)}}$ to be a constant map. Then we define $f^{(n)}$ as the composition

$$f^{(n)} : X^{(n)} \to X^{(n)}/X^{(n-1)} = \bigvee_i S^n_i \xrightarrow{\bigvee_i h_i} K(\pi, n).$$

We notice that by construction, a coincides with the distinguishing cochain $d(*, f^{(n)})$. Since a is a cocycle, we have:

$$0 = \delta a = \delta d(*, f^{(n)}) = c(f^{(n)}) - c(*) = c(f^{(n)}).$$

Thus $c(f^{(n)}) = 0$ and there exists an extension of the map $f^{(n)} : X^{(n)} \to K(\pi, n)$ to a map $f^{(n+1)} : X^{(n+1)} \to K(\pi, n)$. Then we notice that the further obstructions to extend the map $f^{(n+1)} : X^{(n+1)} \to K(\pi, n)$ to the skeletons $X^{(n+q)}$ live in the corresponding groups

$$\mathcal{E}^{n+q}(X; \pi_{n+q-1} K(\pi, n)) = 0 \quad \text{for} \quad q \geq 2.$$

This proves that the map $[f] \mapsto f^* \iota_n$ is surjective.

Now we assume that $f, g : X \to K(\pi, n)$ are such that $f^* \iota_n = g^* \iota_n$ in the cohomology group $H^n(X; \pi)$. By Cellular aproximation Theorem, we may assume that $f|_{X^{(n-1)}} = g|_{X^{(n-1)}} = *$. Then as we have seen, the element $f^* \iota_n$ coincides with the cohomology class of the distiguishing cocycle $d(*, f)$. Thus $f^* \iota_n = [d(*, f)]$ and $g^* \iota_n = [d(*, g)]$. Then

$$[d(f, g)] = [d(f, *)] + [d(*, g)] = -f^* \iota_n + g^* \iota_n = 0.$$

Thus by Theorem 20.14, there exists a homotopy $f|_{X^{(n)}} \sim g|_{X^{(n)}}$ relative to the skeleton $X^{(n-2)}$. Clearly all obstructions to extend this homotopy to the skeletons $X^{(n+q)}$ vanish. If X is a CW-complex of infinite dimension, then we should use the intervals

$$\left[\tfrac{2^k-1}{2^k}, \tfrac{2^{k+1}-1}{2^{k+1}}\right] = \left[1 - \tfrac{1}{2^k}, 1 - \tfrac{1}{2^{k+1}}\right]$$

to construct a homotopy between $f|_{X^{(n+k)}}$ and $g|_{X^{(n+k)}}$. This proves Theorem 20.3. □

Theorem 20.15. (Hopf) *Let X be an n-dimensional CW-complex. Then there is a bijection:*

$$H^n(X; \mathbf{Z}) \cong [X, S^n].$$

Exercise 20.4. *Prove Theorem 20.15.*

Consider a k-torus T^k. We identify T^k with the quotient space $\mathbf{R}^k / \sim$, where two vectors $\vec{x} \sim \vec{y}$ if and only if all coordinates of the vector $\vec{x} - \vec{y}$ are integers. It is easy to see that a linear map $\bar{f} : \mathbf{R}^k \to \mathbf{R}^\ell$ given by an $k \times \ell$-matrix A with integral entries descends to a map $f : T^k \to T^\ell$. In that case a map $f : T^k \to T^\ell$ is called *linear*.

Exercise 20.5. *Prove that any map $f : T^k \to T^\ell$ is homotopic to a linear map.*